Lebesgue Measure

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Acknowledgement

Abstract

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1 Introduction

The goal of this project is to investigate the Lebesgue measure and some of its applications in probability measure.

2 Measure

How do we measure a set or an object? If we have a 3-dimensional object, we could find its volume in 2 different ways, we could fill the container with simple objects or *boxes* and find their sum, or we could encase the whole container in a box and start chipping away smaller boxes. These approaches work only when our box measure is a properly defined measure.

The mathematical concept of measure is a generalisation of length in $\mathbb{R}$, area in $\mathbb{R}^2$ or volume in $\mathbb{R}^3$. To avoid splitting into cases depending on the dimension, we shall refer to the *measure* of E, depending on what Euclidean space $\mathbb{R}^n$ we are using and $E \subset \mathbb{R}^n$.

We know the measure or length of an interval, say $[0, 1]$, we also have $(0, 1)$ and $(0, 1]$, as these are subsets of our initial interval, their measures must not exceed that of $[0, 1]$ (monotonic). Let us assume the length of this unit interval be 1. We let the measure to be zero, for a single point or an empty set, and infinite if we consider the real line. Ideally, we'd like to associate a non-negative measure $m(E)$ to every subset E of $\mathbb{R}^n$. It should also obey some reasonable properties like $m(A \cup B) = m(A) + m(B)$ whenever A and B are disjoint, we should have $m(A) \leq m(B)$ whenever $A \subset B$, and $m(x+A) = m(A)$ (translation invariant).

Here, we hit a roadblock in our definition of a measure, from our definition, we can see that two sets having the same number of points need not have the same measure, let $A = [0, 1]$ and $B = [0, 2]$, there exists a bijection from A to B ($x \mapsto 2x$), but B is twice as long as A. Also, if we consider the measure of a point to be zero, it is logical that a collection of points must measure zero and therefore all sets must have a zero measure.

Remarkably, it turns out such a measure *does not exist*. This is quite a surprising fact, because it is counter-intuitive. These examples tell us that it is impossible to measure every subset of $\mathbb{R}^n$ applying all the above properties.

If we are developing a measure m defined on the subsets of $\mathbb{R}$, we hope that these conditions are met:

1. $m(A)$ is defined for every set A of the real numbers;

2. $0 \leq m(A) \leq \infty$;

3. $m(A) \leq m(B)$ provided $A \subset B$;

4. $m(\phi) = 0$;

5. $m(\{a\}) = 0$ (points are dimensionless);

6. $m(I) = l(I)$, I is an interval (the measure on an interval should be its length)

7. $m(A) = m(x + A)$, translation invariance, (location doesn't affect the length, so it shouldn't affect measure)

8. $m(\bigcup_1^\infty A_k) = \sum_1^\infty m(A_k)$, for any mutually disjoint sequence (A_k) of subsets of real numbers (countable additivity).

Our goal is to construct a measure that satisfies as many conditions as possible, the alternative would be to define a measure only for specific sets.

2.1 Length of Intervals

The length of an interval I with end points $a \le b$, $a, b \in \mathbb{R}$, is defined as:

$$l(I) = b - a$$

For example, $l((0, 1]) = l([0, 1]) = 1$ and $l((1, \infty)) = \infty$.

We immediately conclude that, if I_1 and I_2 are two intervals with $I_1 \subset I_2$, $l(I1) \le l(I2)$.

2.2 Cover

A collection $\{G_\alpha\}$ of open sets covers a set A if $A \subset \bigcup G_\alpha$. And the collection $\{G_\alpha\}$ is called the cover.

2.2.1 Subcover

Let C be a cover of a topological space X. A subcover of C is a subset of C that still covers X.

2.2.2 Compact Set

A set of real numbers is compact if every open cover of the set contains a finite subcover.

2.3 Heine-Borel Theorem

A set of real numbers is compact iff it is closed and bounded.

2.4 Result

If $I, I_1, I_2, \ldots, I_n$ are bounded open intervals with

$$I \subset \bigcup_1^n I_k, \; then \; l(I) \le \sum l(I_k)$$

The length of an interval can not exceed the length of a $finite$ cover.

2.5 Proposition

If $I, I_1, I_2, \ldots, I_n$ are bounded open intervals with $I \subset \bigcup I_k$, then $l(I) \leq \sum l(I_k)$. Also, $l(I) \leq inf\{\sum l(I_k)|I \subset \bigcup I_k, I_k \ bounded \ intervals\}$.

Proof: Assume $I = (a, b)$ and let $\epsilon > 0$. The intervals $(a - \epsilon, a + \epsilon), (b + \epsilon, b - \epsilon), I_1, \ldots, I_n$ form an open cover of the compact set $[a, b]$. And by Heine-Borel Theorem, a finite sub collection will cover [a,b] and thus (a,b). Using 1.1.6:

$$l(I) \leq 4\epsilon + \sum l(I_k)$$

As this holds for any ϵ, $l(I) \leq \sum l(I_k)$, the proof is complete.

3 Lebesgue Outer Measure

We have seen that every set of real numbers can be covered with a countable collection of open intervals.

3.1 Definition

A is any subset of $\mathbb{R}$. Form the collection of all countable covers of A by open intervals. The Lebesgue outer measure of A, $m^*(A)$, is given by

$$m^*(A) = inf\left\{\sum_1^\infty l(I_k)|A \subset \bigcup_1^\infty I_k \ , \ I_k \ open \ intervals\right\}$$

3.2 Results

1. m^* is a set function, whose domain is all subsets of $\mathbb{R}$, and range is $[0, \infty]$, the non-negative extended real numbers.

2. The Lebesgue Outer Measure is defined for **every** subset of $\mathbb{R}$.

3. Outer Measure is monotonic, i.e. , if $A \subseteq B$, then $m^*(A) \leq m^*(B)$, any cover of B by open intervals is also a cover of A, and the latter infimum is taken over a larger collection than the former.

4. If (I_k) is any countable cover of A by open intervals, since infimum is a lower bound,
$$0 \leq m^*(A) \leq \sum l(I_k)$$

3.3 Jordan Measure

If we had approximated the length of a set A by using covers by finite sets of intervals then we would have gotten Jordan outer measure instead of the Lebesgue outer measure.

We could show that the Jordan outer measure of $\mathbb{Q} \cap [0,1]$ is 1 even though the Jordan outer measure of any point, or any finite set of points, is 0. Therefore it isn't countably sub-additive, more importantly not defined for many sets, limiting the usage of the Jordan measure.

3.4 Outer Measure of any interval is it's length

Proof: For a closed interval [a,b], let $\epsilon > 0$. Then $(a - \epsilon, b + \epsilon)$ covers $[a, b]$ and length of this interval is $b - a + \epsilon$. Since ϵ is arbitrary $m^*([a,b]) \le b - a = l([a,b])$.

Next let I_n be a covering of [a,b] by bounded open intervals. By the Heine-Borel Theorem, there exists a finite subset A of I_n's covering [a,b]. So $a \in I_1$ for some $I_1 = (a_1, b_1) \in A$. Also, if $b_1 \le b$, then $b_1 \in I_2$ for some $I_2 = (a_2, b_2) \in A$. Similarly we can construct $I_1, I_2, \dots, I_k$. Then

$$\sum l(I_n) \ge \sum_{i=1}^{k} l(I_i) = \sum_{i=1}^{k} (b_i - a_i)$$

$$= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \cdots + (b_1 - a_1)$$

$$> b_k - a_1$$

Since $a_1 < a$ and $b_k > b$, then $\sum l(I_n) > b_k - a_1$. So, $m^*([a,b]) = b - a = l([a,b])$.

If I is an unbounded interval, then given any natural number $n \in N$, there is a closed interval $J \subset I$ with $l(J) = n$. Hence $m^*(I) \ge m^*(J) = l(J) = n$. Since $m^*(I) \ge n \, and \, n \in N$ is arbitrary, then $m^*(I) = \infty = l(I)$.

3.5 Outer Measure is translation invariant

$$m^*(A) = m^*(A + y)$$

Proof: Suppose $m^*(A) = M < \infty$. Then for all $\epsilon > 0$ there exist I_n bounded open intervals, such that they cover A. And from Sec 2.5 $\sum l(I_n) < M + \epsilon$, so if $y \in \mathbb{R}$, then $I_n + y$ is a covering of A+y and so $m^*(A + y) \le \sum l(I_n + y) = \sum l(I_n) < M + \epsilon$ .Therefore $m^*(A + y) \le M$.

Now, let J_n be a collection of bounded open intervals such that $\cup J_n \supset A + y$.Assume that $\sum l(J_n) < M$. Then $J_n - y$ is a covering of A and $\sum l(J_n - y) = \sum l(J_n) < M$, a contradiction. So, $\sum l(J_n) \ge M$ and hence $m^*(A) \ge M$. So $m^*(A) = m^*(A + y) = M$.

Suppose $m^*(A) = \infty$.Then for any$\{I_n\}_{n=1}^{\infty}$ a set of bounded open intervals such that $A \subset \cup I_n$, we must have$\sum l(I_n) = \infty$. Consider A+y. For any $\{J_n\}_{n=1}^{\infty}$ a set of bounded open intervals such that $A+y \subset \cup J_n$, the collection $\{J_n - y\}_{n=1}^{\infty}$ is a set of bounded open intervals such that $A + y \subset \cup J_n - y$.

So $\sum l(J_n - y) = \infty$. But l(J_n) =l($J_n - y$), so we must have $\sum l(J_n) = \infty$. Since $\{J_n\}_{n=1}^{\infty}$ is an arbitrary collection of bounded open intervals covering A+y, we must have $m^*(A) = m^*(A + y) = \infty$.

3.6 Outer Measure is countably sub-additive

That is, if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets, then

$$m^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m^*(E_k)$$

Proof: If one of the E_k's has infinite outer measure, the inequality holds trivially. We therefore suppose each of the E_k's has finite outer measure. Let $\epsilon > 0$. For each natural number k, there is a countable collection $\{I_{k,i}\}_{i=1}^{\infty}$ of open bounded intervals for which

$$E_k \subseteq \bigcup_{i=1}^{\infty} \{I_{k,i}\} \; and \; \sum_{i=1}^{\infty} l(I_{k,i}) < m^*(E_k) + \frac{\epsilon}{2^k}$$

Now, $\{I_{k,i}\}_{1 \leq k, i \leq \infty}$ is a countable collection of open, bounded intervals that covers $\cup_{k=1}^{\infty} E_k$: the collection is countable since it is a countable collection of countable collections. Thus,

$$m^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{1 \leq k, i < \infty} l(I_{k,i}) = \sum_{k=1}^{\infty} [\sum_{i=1}^{\infty} l(I_{k,i})]$$

$$< \sum_{k=1}^{\infty} [m^*(E_k) + \frac{\epsilon}{2^k}]$$

$$m^*(\bigcup_{k=1}^{\infty} E_k) \leq [\sum_{k=1}^{\infty} m^*(E_k)] + \epsilon$$

Since this holds for each $\epsilon > 0$, it also holds for $\epsilon = 0$. The proof is complete.

The finite sub-additivity property follows from countable sub-additivity by taking $E_k = \phi$ for $k > n$.

3.6.1 $[0, 1]$ is not countable

As seen in Sec.3.3 $[0, 1]$ has an outer measure of 1, whereas all countable sets have a measure 0, (refer to Royden and Fitzpatrick pg.31), therefore $[0, 1]$ is not countable.

4 σ-Algebra of Lebesgue Measurable sets

Unfortunately the outer measure fails to be countably additive on all subsets of $\mathbb{R}$, it is not even finitely additive. A famous example is the Vitali Set, found by Giuseppe Vitali in 1905 (Refer to Sec. result 2 of 3.2 & 5.3).

However, we can salvage matters by measuring a certain class of sets in $\mathbb{R}$, henceforth called the measurable sets. Once we restrict ourselves to measurable sets, we recover all the properties again.

4.1 Carathedory's Measurability Criteria

E is any set of real numbers. If

$$m^*(X) = m^*(X \cap E) + m^*(x \cap E^c)$$

for every set X of real numbers, then the set E is said to be Lebesgue Measurable set of real numbers.

4.2 Results

1. Any set of outer measure is measurable, in other words, any countable set is measurable.

2. The union of a finite collection of measurable sets is measurable. This can be inferred using set identities and finite sub-additivity property of outer measure, then applying induction.

3. The union of countable collection of measurable sets is measurable.

4.3 σ-Algebra

A σ-algebra (or σ-field) Σ on a set Ω is a subset of $P(\Omega)$ (the power set of Ω) such that it satisfies the following properties -

1. $\Omega \in \Sigma$ *(property 1)*

2. If A is an element of Σ then its complement also is an element of Σ i.e A $\in \Sigma \Rightarrow A^c \in \Sigma$ *(property 2)*

3. If $A_i \in \Sigma$, $i \in \mathbb{N}$, then $\bigcup\limits_{i=1}^{n} A_i \in \Sigma$*(property 3)*

4.3.1 Examples

1. Consider $\Sigma = \{\phi, \Omega\}$. The given set is a σ-algebra on Ω as it satisfies all the properties needed. Observe that ϕ is also included in Σ by *property 2*. *Property 1* is also satisfied by the given σ-algebra. $\phi \cup \Omega = \Omega \in \Sigma$, so *property 3* is also satisfied.

2. Consider $\Sigma = P(\Omega)$. Then, Σ satisfies *property 1* as $\Omega \in \Sigma$. Furthermore, $X^c = \phi \in \Sigma$ and hence it satisfies *property 2*. As Σ contains all the subsets of Ω, it is guaranteed that the union of any number of subsets of Ω exists in Σ, and hence satisfying *property 3*. Hence $\Sigma = P(\Omega)$ is a σ-algebra.

3. Consider $\Sigma = \{\phi, \{1\}, \{2\}, \{1, 2\}, \Omega\}$ and $\Omega = \{1, 2, 3\}$. Here, Σ is **not** a σ-algebra on Ω as $\{1, 2\}^c = \{3\}$ is not an element of Σ, and hence Σ does not satisfy *property 2*.

4.3.2   Proposition 1

If Σ is a σ-algebra on Ω, and A_1, A_2, A_3,....$A_n \in \Sigma$, then $\bigcap\limits_{i=1}^{n} A_i \in \Sigma$.

Proof: If A_1, A_2, A_3,....$A_n \in \Sigma$, then A_1^c, A_2^c, A_3^c,....$A_n^c \in \Sigma$ and $\bigcup\limits_{i=1}^{n} A_i^c \in \Sigma$. Then by *property 2*, $(\bigcup\limits_{i=1}^{n} A_i^c)^c = \bigcap\limits_{i=1}^{n} A_i \in \Sigma$.

4.3.3   Proposition 2

If Σ is a σ-algebra on Ω and A, $B \in \Sigma$, then $A\backslash B = A - B \in \Sigma$.

Proof: If $B \in \Sigma$, then $B^c \in \Sigma$ by *property 2*. So, $A \cap B^c = A\backslash B \in \Sigma$ (As seen in proposition 1).

4.3.4   Proposition 3

If Σ_i is a σ-algebra on Ω, $i \in \mathbb{N}$, then $\bigcap\limits_{i=1}^{n} \Sigma_i$ is also a σ-algebra on Ω.

Proof: Consider the σ-algebras Σ_1, Σ_2, Σ_3,... Σ_n defined on Ω, then the smallest intersection of all such σ-algebras would be $\{\phi, \Omega\}$, which is a σ-algebra by definition. Also, if A is an element of the intersection, then A^c would also be an element of the intersection, which satisfies *property 2*. Also, $A \cup A^c = \Omega \in \Sigma$ which satisfies properties 1 and 3.

4.3.5   Proposition 4

For $Y \subseteq P(\Omega)$, there exists a smallest σ-algebra that contains Y, called the σ-algebra generated by Y, given by the intersection of all such σ-algebras Σ such that $\Sigma \supseteq Y$. That is, $\sigma(Y) = \bigcap\limits_{\Sigma \supseteq Y} \Sigma$.

Proof: Consider σ-algebras Σ_1, Σ_2, Σ_3,.... $\Sigma_n \supseteq Y$ defined on Ω. Then $\bigcap\limits_{i=1}^{n} \Sigma_i \supseteq Y$. As $Y \in \Sigma_i$, $Y^c \in \Sigma_i$ for i = 1, 2, 3...n which satisfies *property 2*. Furthermore, $\Omega \in \Sigma_i$ which satisfies *property 1*. $Y \cup Y^c = \Omega$ which satisfies *property 3*. Hence $\sigma(Y)$ is a σ-algebra.

For example, consider $\Omega = \{1, 2, 3, 4\}$ and $Y = \{\{1\}, \{2\}\}$. Then the smallest σ-algebra on Ω containing Y is $\sigma(Y) = \{\phi, \{1\}, \{2\}, \{1, 2\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \Omega\}$. Also note that if Y is a σ-algebra, then $\sigma(Y) = Y$.

4.4   Every interval is measurable

From the preceding propositions, we immediately conclude that the collection of measurable sets is a σ-algebra. Also, Lebesgue outer measure m^*, written as m when restricted to the σ-algebra E subsets of R satisfying Carathedory's

condition, is countably additive on E, in other words, m is countably additive on Lebesgue measurable subsets of R.

It can further be shown Intervals satisfy Caratheodory's condition, using the Heine-Borel theorem. But we can use Borel sets to extend our applicability of Lebesgue measure to a much larger group of subsets.

4.5 Borel Sets

The σ-algebra generated by the collection of all open intervals of R is called the Borel σ-algebra $\mathscr{B}$.

Therefore, since the measurable sets are a σ-algebra containing all open sets, every Borel set of real numbers is Lebesgue measurable. $\mathscr{B}$ contains about every set that comes up in analysis.

4.5.1 Contents of Borel Sets

- Open intervals

- Open sets

- Closed intervals

- Closed sets

- Compact sets

- Left open, right closed sets

- Right open, left closed sets

- All intervals

Every Borel set is Lebesgue measurable, but not vice-versa, there exist sets that are Lebesgue measurable which are not Borel sets. The collection of all Lebesgue measurable sets is a σ-algebra $\mathscr{M}$, and $\mathscr{B}$, the collection of Borel sets, then
$$\mathscr{B} \subset \mathscr{M} \subset 2^{\mathbb{R}}$$

5 Lebesgue Measure

By restricting outer measure of to a class of measurable sets, we can define the Lebesgue measure. It is denoted by m, if E is a measurable set, it's Lebesgue measure is defined to be
$$m(E) = m^*(E)$$

We can use all our previous results to determine the Lebesgue measure of specific sets of real numbers, note that all following sets are presumed to be Lebesgue measurable sets of real numbers.

We can infer the following properties for Lebesgue measurable sets-

5.1 Properties

1. $m(\phi)=m(\{a\}) = 0$, $m(\mathbb{R}) = \infty$.

2. $m(I) = l(I)$.

3. $m(\text{countable set}) = 0$, $m(\text{subset of a set with measure } 0) = 0$.

4. If E is measurable, then E^c is also measurable.

5. m is translation invariant i.e $m(x + E) = m(E)$ for all $x \in \mathbb{R}$

6. $m(\bigcup\limits_{i=1}^{\infty} A_i) = \sum\limits_{i=1}^{\infty} m(A_i)$ for $A_i \cap A_j = \phi$ for $i \neq j$ for all $A_i \in \Sigma$, a σ-algebra (σ-additivity).

7. $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$.

8. If E_1, E_2, E_3,.... E_n are measurable, then $\bigcup\limits_{i=1}^{n} E_i$ and $\bigcap\limits_{i=1}^{n} E_i$ are measurable.

9. $m(E_1) \leq m(E_2)\, if\, E_1 \subset E_2$. If in addition, $m(E_2) < \infty$, then $m(E_2) - m(E_1) = m(E_2 - E_1)$.

10. If $E_1 \subset E_2 \subset E_3 \ldots$ then $m(\cup E_k) = m(\lim E_k) = \lim m(E_k)$

11. If $E_1 \supset E_2 \supset E_3 \ldots$ and $m(E_1) < \infty$, then $m(\cap E_k) = m(\lim E_k) = \lim m(E_k)$

12. If $m(\cup E_k) < \infty$, then $\lim sup\, m(E_k) \leq m(\lim sup E_k)$

13. $m(\lim inf E_k) \leq \lim inf m(E_k)$

14. If $\lim inf E_k = \lim sup E_k$ and $m(\cup E_k) < \infty$, then $m(\lim E_k) = \lim m(E_k)$

5.2 Not all sets are Lebesgue measurable

Proof: We define an equivalence relation $x \sim y$ as $x - y \in \mathbb{Q}$ for $x, y \in \mathbb{R}$. This establishes equivalence classes $[x] = \{x + r : r \in \mathbb{Q}\}$. We define $A \subset [0, 1]$ with the following properties-

- For each $[x]$, there is a $p \in A$ with $p \in [x]$.

- For all $p, q \in A$, if $p, q \in [x]$, then $p = q$.

Let $A_n = A + r_n$, where $r_n (n \in \mathbb{N})$ is an enumeration of $\mathbb{Q} \cap [-1, 1]$. Then we can claim that $n \neq m \Rightarrow A_n \cap A_m = \phi$. We prove this by contradiction. Assume that there exists $x \in A_n \cap A_m$ for $n \neq m$.

Let $x = r_n + a_n$, $a_n \in A$ and $x = r_m + a_m$, $a_m \in A$. Then,

$$a_n + r_n = a_m + r_m$$

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$$\Rightarrow a_n - a_m = r_m - r_n \in \mathbb{Q}$$

Then, a_n and a_m satisfies the equivalence relation $a_n \sim a_m$. So,

$$a_n, a_m \in [a_m]$$

$$\Rightarrow a_n = a_m$$

$$\Rightarrow r_n = r_m$$

$$\Rightarrow n = m$$

Which is a contradiction to our assumption. So, we can claim that

$$[0,1] \subset \bigcup_{n=1}^{\infty} A_n \subset [-1,2]$$

Assume that m is a measure on $P(\mathbb{R})$. Then,

$$m([0,1]) \leq m(\bigcup_{n=1}^{\infty} A_n) \leq m([-1,2])$$

$$\Rightarrow 1 \leq m(\bigcup_{n=1}^{\infty} A_n) \leq 3$$

By σ-additivity of Lebesgue measures,

$$1 \leq \sum_{n=1}^{\infty} m(A_n) \leq 3$$

By translational invariance of Lebesgue measures,

$$1 \leq \sum_{n=1}^{\infty} m(A) \leq 3$$

$$\Rightarrow 1 \leq \lim_{n \to \infty} nm(A) \leq 3$$

Which is not possible, as $\lim_{n \to \infty} nm(A)$ can take only one of two values - either 0 or ∞. Hence, the set A is not Lebesgue measurable.

5.3 Examples

- Consider an interval $[2,5]$ on $\mathbb{R}$. Then, the Lebesgue measure of the interval $m([2,5]$ is given by $l([2,5]) = 5 - 2 = 3$. Note that the Lebesgue measures of $(2,5)$, $(2,5]$ and $[2,5)$ are also 3.

- The set of all rational numbers given by $\mathbb{Q}$ has Lebesgue measure equal to zero.

- The set $\{1,2,3,4\}$ has Lebesgue measure equal to zero.

- The Cantor ternary set is an example of an uncountable set that has Lebesgue measure equal to 0. The cantor ternary set $\mathcal{C}$ is given by

$$\mathcal{C} = \bigcap_{n=1}^{\infty} C_n$$

where
$$C_n = \frac{C_{n-1}}{3} \cup (\frac{2}{3} + \frac{C_{n-1}}{3}) \ \forall \ n \geq 1 \ and \ C_0 = [0, 1]$$

- Vitali sets are **not** Lebesgue measurable (see the proof of existence of non-measurable sets given previously in Sec.5.2).

5.4 Outer and Inner approximation of Lebesgue measure

Measurable sets possess the following excision property: If A is a measurable set of finite outer measure that is contained in B, then

$$m^*(B \sim A) = m^*(B) - m^*(A).$$

This follows from Sec 4.1

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^C) = m^*(A) + m^*(B \sim A).$$

Theorem
For an arbitrary subset E of $\mathbb{R}$, following assertions are equivalent to measurability of E

1. For each $\epsilon > 0$, there is an open set O containing E for which $m^*(O \sim E) < \epsilon$.(Outer approximation by open sets)

2. There is a G_δ set G containing E for which $m*(G \sim E) = 0$.

3. For each $\epsilon > 0$, there is an closed set F contained in E for which $m^*(E \sim F) < \epsilon$.(Inner approximation by closed sets)

4. There is a F_σ set F contained in E for which $m*(E \sim F) = 0$.

Here, G_δ denotes a subset that is a countable intersection of open sets and F_σ is a countable union of closed sets.

5.5 Extension to $\mathbb{R}^n$

The modern construction of Lebesgue Measure is an application of Caratheodory's extension theorem (not discussed here).

Fix $n \in \mathbb{N}$. A *box* is a set of the form:

$$B = \prod_{i=1}^{n} [a_i, b_i]$$

where $b_i \geq a_i$, and the product symbol represents a Cartesian product. The volume of this box is defined to be

$$vol(B) = \prod_{i=1}^{n} l([a_i, b_i]).$$

For any subset A of $\mathbb{R}^n$, we can define its outer measure $\lambda^*(A)$ by:

$$\lambda^*(A) = inf\Big\{ \sum_{B \in C} vol(B) : C \text{ is a countable collection of boxes whose union covers } A\Big\}$$

We then define the set A to be Lebesgue-measurable if for every subset S of $\mathbb{R}^n$,

$$\lambda^*(S) = \lambda^*(S \cap A) + \lambda^*(S \cap A^C)$$

These Lebesgue measurable sets form a σ-algebra, and the Lebesgue measure is defined by $\lambda(A) = \lambda^*(A)$ for any Lebesgue measurable set A.

- The Lebesgue measure of the Cartesian product of intervals $[2, 5]$ and $[2, 4]$ is $(5 - 2)(4 - 2) = 6$, which is equal to the area of the corresponding rectangle.

5.6 Banach-Tarski Paradox

Banach-Tarski paradox is a famous example to be demonstrated here, the Banach-Tarski theorem tells us that given a sphere in 3-dimensional space, there exists a decomposition of the ball into a number of finite number of disjoint subsets, which can be put back in a different way to yield two identical copies of the original sphere. This can be done using as few as 5 pieces.

The reason this theorem is called a paradox is that it contradicts our intuition about volume.

While the reassembly process involves only moving the pieces around and rotating them without changing their shape which are closed under the Lebesgue measure, some of the five or more sets in which you divide the sphere are not Lebesgue measurable.

Unlike most theorems in geometry, the proof for the Banach-Tarski theorem itself requires using the axiom of choice, which allows the construction of non-measurable sets.

As some of the sets are non-measurable, it's not possible to define a volume to such sets as seen before. It is therefore absurd to add these 'volumes' in order to double their initial volume. However, note that the initial sphere and the final spheres obtained are measurable.

6 Probability and Measures

We'll start from the very basics of the probability theory and work our ways up to learn more complex concepts and the involvement of Measure theory in probability.

6.1 Random Experiment

An experiment whose outcome cannot be predicted with certainty even if the experiment is conducted several times under identical conditions.

6.2 Sample Space (Ω)

It's defined as the set of all possible outcomes of an experiment.

For example:

1. Throwing a dice : $\Omega = \{1, 2, 3, 4, 5, 6\}$.

2. Tossing a coin twice : $\Omega = \{HH, TT, HT, TH\}$, where 'H' represents that the heads and 'T' represents the tails.

In probability theory, an outcome is denoted by ω and from the definition of the sample space we can say that $\omega \, \epsilon \, \Omega$. So an outcome is nothing but an element of the sample space.

Observe that in probability outcome is the source of randomness and is not of our control in a random experiment.

NOTE - The sample space of an experiment can be countable or uncountable. The sample space of an experiment like tossing a coin n times is countable with 2^n outcomes whereas if we want to determine the speed with which the coin tossed would land back is a set of positive real numbers and the sample space would be uncountable. Another example of uncountable sample space could be the set of all points on the circumference of a circle.

One more thing to discuss before going ahead is that often while discussing probability theory, we are not interested in a particular outcome but a subset of the sample space. For example - when a dice is rolled, we might be interested if the outcome is an even number $\{2, 4, 6\}$ or a prime number $\{2, 3, 5\}$ rather than the outcome being a particular number like 2 or 5.

6.3 Events and event space

The subset of the sample space Ω which is of our interest is referred to as an event. The subset is referred to as an event space. We understand by taking an example of coin tossed thrice and our outcome being at least 2 tails, so our event space is $\{TTH, HTT, THT, TTT\}$ which as we stated is a subset of a sample space.

We say that an event P $\subseteq \Omega$ has occurred or taken place iff $\omega \, \epsilon$ P.

6.4 Algebra and probability theory

Now, before writing down everything mathematically, lets discuss how we are going to link the two concepts.

If we say that an event P is in our interest we can always say that the non-occurrence of P would interest us. Thus P^c, that is the complement of P also would interest us.

If two events M and N are of our interest, then the event M∪N is also of our interest as M∪N will have our favourable outcome.

Also when any outcome is obtained, the Ω has always occurred and thus always interests us.

An algebra as we have stated earlier, it is a collection of subsets of Ω with certain properties as we have already discussed.

Now we are sufficiently equipped to expand our ideas of algebra to probability.

We denote the sample space by Ω, the σ-algebra(the collection of subsets) on Ω by Σ.

6.5   Measurable space

(Ω, Σ) is referred to as a measurable space. Probability is a special case of Measure theory and we'll see how.

Like measure is the function $\mu : \Sigma \to [0, \infty]$, such that-

1. $\mu(\phi) = 0$

2. If $A_1, A_2, A_3....$ are a countable collection of disjoint measurable sets, then

$$\mu(\bigcup_1^\infty A_k) = \sum_1^\infty \mu(A_k).$$

This is referred to as the countable additivity axiom.

Measure is nothing but a mapping of the subsets included in Σ to $[0, \infty]$.

We say that

1. If $\mu(\Omega) < \infty$, μ is called a finite measure.

2. If $\mu(\Omega) = \infty$, μ is called a infinite measure.

3. If $\mu(\Omega) = 1$, μ is called a probability measure.

So a probability measure is a special case of finite measure with the measure of the Ω being 1.

6.6 Probability Measure

A probability measure on S can be expressed on X as

$$P(A) = \frac{\mu(A)}{\mu(X)}, \ A \in \Sigma$$

Where $0 < \mu(\Sigma) < \infty$ is a measure.

Consider that m is a Lebesgue measure on $\mathbb{R}^n$. If $X \subseteq \mathbb{R}^n$ and $0 < m(X) < \infty$, then

$$P(A) = \frac{m(A)}{m(X)}, \ A \subseteq X$$

Then we can define a probability measure(P) on (Ω, Σ) as a function $P: \Sigma \to [0,1]$ such that -

1. $P(\phi) = 0$

2. $P(\Sigma) = 1$

3. If $A_1, A_2....$ are Σ measurable sets then $P(\bigcup_1^\infty A_k) = \sum_1^\infty P(A_k)$.

6.7 Probability Space

The triple (Ω, Σ, P) is referred to as a probability space where
$\quad \Omega$: Sample space
$\quad \Sigma$: Event Space
$\quad P$: Probability Measure
$\quad$ So, in probability, Σ -measurable sets are called events.

6.8 Properties of probability measure

1. $0 \le P(A) \le 1$

2. $P(A^c) = 1 - P(A), \ \forall \ A \in \Sigma$
   Proof: A and A^c are disjoint. So,

   $$P(A \cup A^c) = P(A) + P(A^c)$$

   $A \cup A^c$ gives the entire sample space. So, $P(A \cup A^c) = 1$

   $$\Rightarrow 1 = P(A) + P(A^c)$$

   $$\Rightarrow P(A^c) = 1 - P(A)$$

3. if $A = B$, then $P(A) = P(B)$

4. Monotonicity: if $A \subset B \Rightarrow P(A) \le P(B)$

5. $P(B \cap A^c) = P(B) - P(A \cap B)$

6. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

In general, by the inclusion-exclusion principle,

$$P(\bigcup_{i=1}^{n} A_i) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + ...(-1)^{n+1} P(A_1 \cap A_2 \cap A_3.... \cap A_n)$$

7. Finite additivity: If $A_1, A_2...A_n \epsilon \Sigma$ and suppose they are disjoint. Then,

$$P(\bigcup_{1}^{n} A_k) = \sum_{1}^{n} P(A_k)$$

6.9   Propositions

1. If $E_1 \subset E_2 \subset E_3 .... \subset$ and $E = \bigcup_{i=1}^{\infty} E_i$ lies in Σ, then

$$lim_{n\to\infty} P(E_n) = P(E)$$

Proof: It is obvious that this limit does exist and is ≤ 1. Let $F_i = E_i - E_{i-1}$ for $i \geq 2$ and $F_1 = E_1$. Then, $F_j \cap F_k = \phi$ for all $j \neq k$. So, we can write,

$$A_n = \bigcup_{i=1}^{\infty} B_i \text{ and } A = \bigcup_{i=1}^{\infty} B_i$$

By countable additivity,

$$P(A) = \sum_{i=1}^{\infty} P(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i) = \lim_{n\to\infty} P(A_n)$$

2. If $E_1 \supset E_2 \supset E_3 .... \supset$ and $E = \bigcap_{i=1}^{\infty} E_i$ lies in Σ, then

$$lim_{n\to\infty} P(E_n) = P(E)$$

Proof: Given $E_1 \supset E_2 \supset E_3 .... \supset$. So,

$$E_1^c \subset E_2^c \subset E_3^c .... \subset \text{ and } E^c = (\bigcap_{i=1}^{\infty} E_i)^c = \bigcup_{i=1}^{\infty} E_i^c$$

So, from proposition 1,

$$P(E^c) = \lim_{n\to\infty} P(E_n^c)$$

$$P(E) = 1 - P(E^c) = 1 - \lim_{n\to\infty} P(E_n^c) = \lim_{n\to\infty} (1 - P(E_n^c)) = \lim_{n\to\infty} P(E_n)$$

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7 Lebesgue Integration

The Lebesgue integral is a powerful tool, and is a further extension of the Lebesgue measure, which can be used to integrate more functions when compared to the Riemann integral or even the Riemann–Stieltjes integral. This integral plays an important role in probability theory.

The Lebesgue integral approximates the area under a curve using horizontal strips, or using the measure of an interval whereas the Riemann integral uses vertical strips.

This topic will not be covered in full, as it requires a deeper discussion into Measurable functions, but we can see some applications of the integral.

7.1 Explanation

"I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral.
But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral."

This explanation was given by Henri Lebesgue himself, it is futile to attempt for a better one.

7.2 Simple Functions

As the name suggests, a simple function is a function that takes on only finitely many distinct values.

Example: The Dirichlet function over the real line takes the value 1 if x is rational, 0 otherwise.

A simple function can be written as a linear combination of indicator functions of sets, such as

$$f = c_1 \chi A_1 + c_2 \chi A_2 + \cdots + c_n \chi A_n$$

Where the indicator function denotes whether or not a element is present in the given set.

7.3 Lebesgue Integral

The Lebesgue integral for a simple function defined above is:

$$\int_{\mathbb{R}} f d\lambda = c_1 \lambda(A_1) + c_2 \lambda(A_2) + \cdots + c_n \lambda(A_n)$$

Here, $\lambda(A)$ denotes the Lebesgue measure.

If f takes on both positive and negative values, the integral can be evaluated separately for each part.

Finally, the Lebesgue integral of an arbitrary function f is defined by considering approximations of simple functions.

$$\int_{\mathbb{R}} f d\lambda = \sup_{g \leq f \, : \, g \, simple} \int_{\mathbb{R}} g d\lambda$$

7.4 Properties

7.5 Example

The Dirichlet function,

$$1_{\mathbb{Q}}(x) = \begin{cases} 1, x \text{ is rational} \\ 0, x \text{ is not rational} \end{cases}$$

It is nowhere continuous, and not Riemann-integrable.

The Dirichlet function is Lebesgue-integrable on $\mathbb{R}$ and its integral over $\mathbb{R}$ is zero, as the measure of rational numbers is 0.

Conclusion

Add conclusion here

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