Lebesgue Measure

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Acknowledgement

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Abstract

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1 Introduction

The goal of this project is to investigate the Lebesgue measure and some of its applications in probability measure.

1.1 Measure

How do we measure a set or an object? If we have a 3-dimensional object, we could find its volume in 2 different ways, we could fill the container with simple objects or *boxes* and find their sum, or we could encase the whole container in a box and start chipping away smaller boxes. These approaches work only when our box measure is a properly defined measure.

The mathematical concept of measure is a generalisation of length in $\mathbb{R}$, area in $\mathbb{R}^2$ or volume in $\mathbb{R}^3$. To avoid splitting into cases depending on the dimension, we shall refer to the *measure* of E, depending on what Euclidean space $\mathbb{R}^n$ we are using and $E \subset \mathbb{R}^n$.

We know the measure or length of an interval, say $[0, 1]$, we also have $(0, 1)$ and $(0, 1]$, as these are subsets of our initial interval, their measures must not exceed that of $[0, 1]$ (monotonic). Let us assume the length of this unit interval be 1. We let the measure to be zero, for a single point or an empty set, and infinite if we consider the real line. Ideally, we'd like to associate a non-negative measure $m(E)$ to every subset E of $\mathbb{R}^n$. It should also obey some reasonable properties like $m(A \cup B) = m(A) + m(B)$ whenever A and B are disjoint, we should have $m(A) \leq m(B)$ whenever $A \subset B$, and $m(x+A) = m(A)$ (translation invariant).

Here, we hit a roadblock in our definition of a measure, from our definition, we can see that two sets having the same number of points need not have the same measure, let $A = [0, 1]$ and $B = [0, 2]$, there exists a bijection from A to B (x $\mapsto$ 2x), but B is twice as long as A.

Remarkably, it turns out such a measure *does not exist*. This is quite a surprising fact, because it is counter-intuitive. These examples tell us that it is impossible to measure every subset of $\mathbb{R}^n$ applying all the above properties.

If we are developing a measure m defined on the subsets of $\mathbb{R}$, we hope that these conditions are met:

1. $m(A)$ is defined for every set A of the real numbers;
2. $0 \leq m(A) \leq \infty$;
3. $m(A) \leq m(B)$ provided $A \subset B$;
4. $m(\phi) = 0$;
5. $m(\{a\}) = 0$ (points are dimensionless);
6. $m(I) = l(I)$, I is an interval (the measure on an interval should be its length)

7. $m(A) = m(x + A)$, translation invariance, (location doesn't affect the length, so it shouldn't affect measure)

8. $m(\bigcup_1^\infty A_k) = \sum_1^\infty m(A_k)$, for any mutually disjoint sequence (A_k) of subsets of real numbers (countable additivity).

Our goal is to construct a measure that satisfies as many conditions as possible.

1.1.1 Length of Intervals

The length of an interval I with end points $a \leq b$, $a, b \in \mathbb{R}$, is defined as:

$$l(I) = b - a$$

For example, $l((0, 1]) = l([0, 1]) = 1$ and $l((1, \infty)) = \infty$.

We immediately conclude that, if I_1 and I_2 are two intervals with $I_1 \subset I_2$, $l(I1) \leq l(I2)$.

1.1.2 Cover

A collection $\{G_\alpha\}$ of open sets covers a set A if $A \subset \bigcup G_\alpha$. And the collection $\{G_\alpha\}$ is called the cover.

1.1.3 Subcover

Let C be a cover of a topological space X. A subcover of C is a subset of C that still covers X.

1.1.4 Compact Set

A set of real numbers is compact if every open cover of the set contains a finite subcover.

1.1.5 Heine-Borel Theorem

A set of real numbers is compact iff it is closed and bounded.

1.1.6 Result

If $I, I_1, I_2, \ldots, I_n$ are bounded open intervals with

$$I \subset \bigcup_1^n I_k, \; then \; l(I) \leq \sum l(I_k)$$

The length of an interval can not exceed the length of a $finite$ cover.

1.1.7 Proposition

If $I, I_1, I_2, \dots, I_n$ are bounded open intervals with $I \subset \bigcup I_k$, then $l(I) \leq \sum l(I_k)$. Also, $l(I) \leq inf\{\sum l(I_k) | I \subset \bigcup I_k, I_k \; bounded \; intervals\}$.

Proof: Assume $I = (a, b)$ and let $\epsilon > 0$. The intervals $(a - \epsilon, a + \epsilon), (b + \epsilon, b - \epsilon), I_1, \dots, I_n$ form an open cover of the compact set $[a, b]$. And by Heine-Borel Theorem, a finite sub collection will cover [a,b] and thus (a,b). Using 1.1.6:

$$l(I) \leq 4\epsilon + \sum l(I_k)$$

As this holds for any ϵ, $l(I) \leq \sum l(I_k)$, the proof is complete.

1.2 Lebesgue Outer Measure

We have seen that every set of real numbers can be covered with a countable collection of open intervals.

1.2.1 Definition

A is any subset of $\mathbb{R}$. Form the collection of all countable covers of A by open intervals. The Lebesgue outer measure of A, $m^*(A)$, is given by

$$m^*(A) = inf\Big\{\sum_1^\infty l(I_k) | A \subset \bigcup_1^\infty I_k \; , \; I_k \; open \; intervals\Big\}$$

1.2.2 Results

1. m^* is a set function, whose domain is all subsets of $\mathbb{R}$, and range is $[0, \infty]$, the non-negative extended real numbers. Note that the Lebesgue Outer Measure is defined for every subset of $\mathbb{R}$.

2. Outer Measure is monotonic, i.e. , if $A \subseteq B$, then $m^*(A) \leq m^*(B)$, any cover of B by open intervals is also a cover of A, and the latter infimum is taken over a larger collection than the former.

3. If (I_k) is any countable cover of A by open intervals, since infimum is a lower bound,
$$0 \leq m^*(A) \leq \sum l(I_k)$$

1.2.3 The outer measure of any interval is it's length.

Proof: For a closed interval [a,b], let $\epsilon > 0$. Then $(a - \epsilon, b + \epsilon)$ covers $[a, b]$ and length of this interval is $b - a + \epsilon$. Since ϵ is arbitrary $m^*([a, b]) \leq b - a = l([a, b])$.

Next let I_n be a covering of [a,b] by bounded open intervals. By the Heine-Borel Theorem, there exists a finite subset A of I_n's covering [a,b]. So $a \in I_1$ for some $I_1 = (a_1, b_1) \in A$. Also, if $b_1 \leq b$, then $b_1 \in I_2$ for some $I_2 = (a_2, b_2) \in A$. Similarly we can construct $I_1, I_2, \ldots, I_k$. Then

$$\sum l(I_n) \geq \sum_{i=1}^{k} l(I_i) = \sum_{i=1}^{k} (b_i - a_i)$$

$$= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \ldots + (b_1 - a_1)$$

$$> b_k - a_1$$

Since $a_1 < a$ and $b_k > b$, then $\sum l(I_n) > b_k - a_1$. So, $m^*([a,b]) = b - a = l([a,b])$.

If I is an unbounded interval, then given any natural number $n \in N$, there is a closed interval $J \subset I$ with $l(J) = n$. Hence $m^*(I) \geq m^*(J) = l(J) = n$. Since $m^*(I) \geq n\, and\, n \in N$ is arbitrary, then $m^*(I) = \infty = l(I)$.

1.2.4 Outer Measure is translation invariant

$m^*(A) = m^*(A + y)$

Proof: Suppose $m^*(A) = M < \infty$. Then for all $\epsilon > 0$ there exist I_n bounded open intervals, such that they cover A. And from sec 1.1.7. $\sum l(I_n) < M + \epsilon$, so if $y \in \mathbb{R}$, then $I_n + y$ is a covering of A+y and so $m^*(A+y) \leq \sum l(I_n + y) = \sum l(I_n) < M + \epsilon$ .Therefore $m^*(A + y) \leq M$.

Now, let J_n be a collection of bounded open intervals such that $\cup J_n \supset A + y$.Assume that $\sum l(J_n) < M$. Then $J_n - y$ is a covering of A and $\sum l(J_n - y) = \sum l(J_n) < M$, a contradiction. So, $\sum l(J_n) \geq M$ and hence $m^*(A) \geq M$. So $m^*(A) = m^*(A + y) = M$

Suppose $m^*(A) = \infty$.Then for any$\{I_n\}_{n=1}^{\infty}$ a set of bounded open intervals such that $A \subset \cup I_n$, we must have$\sum l(I_n) = \infty$ .Consider A+y. For any $\{J_n\}_{n=1}^{\infty}$ a set of bounded open intervals such that $A+y \subset \cup J_n$, the collection $\{J_n - y\}_{n=1}^{\infty}$ is a set of bounded open intervals such that $A + y \subset \cup J_n - y$. So $\sum l(J_n - y) = \infty$. But l(J_n) =l($J_n - y$), so we must have $\sum l(J_n) = \infty$. Since $\{J_n\}_{n=1}^{\infty}$ is an arbitrary collection of bounded open intervals covering A+y, we must have $m^*(A) = m^*(A + y) = \infty$.

1.2.5 Outer Measure is countably sub-additive

That is, if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets, then

$$m^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m^*(E_k)$$

Proof: If one of the E_k's has infinite outer measure, the inequality holds trivially. We therefore suppose each of the E_k's has finite outer measure. Let $\epsilon > 0$. For each natural number k, there is a countable collection $\{I_{k,i}\}_{i=1}^{\infty}$ of open bounded intervals for which

$$E_k \subseteq \bigcup_{i=1}^{\infty} \{I_{k,i}\} \; and \; \sum_{i=1}^{\infty} l(I_{k,i}) < m^*(E_k) + \frac{\epsilon}{2^k}$$

Now, $\{I_{k,i}\}_{1\leq k,i\leq\infty}$ is a countable collection of open, bounded intervals that covers $\cup_{k=1}^{\infty}E_k$: the collection is countable since it is a countable collection of countable collections. Thus,

$$m^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{1\leq k,i<\infty} l(I_{k,i}) = \sum_{k=1}^{\infty}[\sum_{i=1}^{\infty} l(I_{k,i})]$$
$$< \sum_{k=1}^{\infty}[m^*(E_k) + \frac{\epsilon}{2^k}]$$
$$m^*(\bigcup_{k=1}^{\infty} E_k) \leq [\sum_{k=1}^{\infty} m^*(E_k)] + \epsilon$$

Since this holds for each $\epsilon > 0$, it also holds for $\epsilon = 0$. The proof is complete.

The finite sub-additivity property follows from countable sub-additivity by taking $E_k = \phi$ for $k > n$.

1.2.6 $[0,1]$ is not countable

As seen in Sec.1.2.3 $[0,1]$ has an outer measure of 1, whereas all countable sets have a measure 0, (refer to Royden and Fitzpatrick pg.31), therefore $[0,1]$ is not countable.

1.3 σ-Algebra of Lebesgue Measurable sets

Unfortunately the outer measure fails to be countably additive on all subsets of $\mathbb{R}$, it is not even finitely additive. A famous example is the Vitali Set, found by Giuseppe Vitali in 1905. However, we can salvage matters by measuring a certain class of sets in $\mathbb{R}$, henceforth called the measurable sets. Once we restrict ourselves to measurable sets, we recover all the properties again.

1.3.1 Carathedory's Measurability Criteria

E is any set of real numbers. If

$$m^*(X) = m^*(X \cap E) + m^*(x \cap E^c)$$

for every set X of real numbers, then the set E is said to be Lebesgue Measurable set of real numbers.

1.3.2 Results

1. Any set of outer measure is measurable, in other words, any countable set is measurable.

2. The union of a finite collection of measurable sets is measurable. This can be inferred using set identities and finite sub-additivity property of outer measure, then applying induction.

3. The union of countable collection of measurable sets is measurable.

1.3.3 Definition of σ-Algebra

A σ-algebra (or σ-field) Σ on a set Ω is a subset of $P(\Omega)$ (the power set of Ω) such that it satisfies the following properties -

1. $\Omega \in \Sigma$ *(property 1)*
2. If A is an element of Σ then its complement also is an element of Σ i.e A $\in \Sigma \Rightarrow$ A$^c \in \Sigma$ *(property 2)*
3. If $A_i \in \Sigma$, $i \in \mathbb{N}$, then $\bigcup\limits_{i=1}^{n} A_i \in \Sigma$*(property 3)*

1.3.4 Examples

1. Consider $\Sigma = \{\phi, \Omega\}$. The given set is a σ-algebra on Ω as it satisfies all the properties needed. Observe that ϕ is also included in Σ by *property 2. Property 1* is also satisfied by the given σ-algebra. $\phi \cup \Omega = \Omega \in \Sigma$, so *property 3* is also satisfied.
2. Consider $\Sigma = P(\Omega)$. Then, Σ satisfies *property 1* as $\Omega \in \Sigma$. Furthermore, $X^c = \phi \in \Sigma$ and hence it satisfies *property 2.* As Σ contains all the subsets of Ω, it is guaranteed that the union of any number of subsets of Ω exists in Σ, and hence satisfying *property 3.* Hence $\Sigma = P(\Omega)$ is a σ-algebra.
3. Consider $\Sigma = \{\phi, \{1\}, \{2\}, \{1, 2\}, \Omega\}$ and $\Omega = \{1, 2, 3\}$. Here, Σ is **not** a σ-algebra on Ω as $\{1, 2\}^c = \{3\}$ is not an element of Σ, and hence Σ does not satisfy *property 2.*

1.3.5 Proposition 1

If Σ is a σ-algebra on Ω, and A$_1$, A$_2$, A$_3$,....A$_n \in \Sigma$, then $\bigcap\limits_{i=1}^{n}$ A$_i \in \Sigma$.

Proof: If A$_1$, A$_2$, A$_3$,....A$_n \in \Sigma$, then A$_1^c$, A$_2^c$, A$_3^c$,....A$_n^c \in \Sigma$ and $\bigcup\limits_{i=1}^{n}$ A$_i^c \in \Sigma$. Then by *property 2,* $(\bigcup\limits_{i=1}^{n} \mathrm{A}_i^c)^c = \bigcap\limits_{i=1}^{n} \mathrm{A}_i \in \Sigma$.

1.3.6 Proposition 2

If Σ is a σ-algebra on Ω and A, B $\in \Sigma$, then A$\backslash$B = A - B $\in \Sigma$.

Proof: If B $\in \Sigma$, then B$^c \in \Sigma$ by *property 2.* So, A $\cap$ B^c = A$\backslash$B $\in \Sigma$ (As seen in proposition 1).

1.3.7 Proposition 3

If Σ_i is a σ-algebra on Ω, i $\in \mathbb{N}$, then $\bigcap\limits_{i=1}^{n} \Sigma_i$ is also a σ-algebra on Ω.

Proof: Consider the σ-algebras Σ_1, Σ_2, Σ_3,... Σ_n defined on Ω, then the smallest intersection of all such σ-algebras would be $\{\phi, \Omega\}$, which is a σ-algebra by definition. Also, if A is an element of the intersection, then A^c would also be an element of the intersection, which satisfies *property 2.* Also, A $\cup$ A^c = Ω $\in \Sigma$ which satisfies properties 1 and 3.

1.3.8 Proposition 4

For $Y \subseteq$ P(Ω), there exists a smallest σ-algebra that contains Y, called the σ-algebra generated by Y, given by the intersection of all such σ-algebras Σ such that $\Sigma \supseteq Y$. That is, $\sigma(Y) = \bigcap\limits_{\Sigma \supseteq Y} \Sigma$.

Proof: Consider σ-algebras Σ_1, Σ_2, Σ_3,.... $\Sigma_n \supseteq Y$ defined on Ω. Then $\bigcap\limits_{i=1}^{n} \Sigma_i \supseteq Y$. As $Y \in \Sigma_i$, $Y^c \in \Sigma_i$ for i = 1, 2, 3...n which satisfies *property 2.* Furthermore, $\Omega \in \Sigma_i$ which satisfies *property 1.* $Y \cup Y^c = \Omega$ which satisfies *property 3.* Hence $\sigma(Y)$ is a σ-algebra.

For example, consider $\Omega = \{1, 2, 3, 4\}$ and $Y = \{\{1\}, \{2\}\}$. Then the smallest σ-algebra on Ω containing Y is $\sigma(Y) = \{\phi, \{1\}, \{2\}, \{1, 2\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \Omega\}$. Also note that if Y is a σ-algebra, then $\sigma(Y) = Y$.

1.3.9 Every interval is measurable

From the preceding propositions, we immediately conclude that the collection of measurable sets is a σ-algebra. Also, Lebesgue outer measure m^*, written as m when restricted to the σ-algebra E subsets of R satisfying Carathedory's condition, is countably additive on E, in other words, m is countably additive on Lebesgue measurable subsets of R.

It can further be shown Intervals satisfy Caratheodory's condition, using the Heine-Borel theorem. But we can use Borel sets to extend our applicability of Lebesgue measure to a much larger group of subsets.

1.4 Borel Sets

The σ-algebra generated by the collection of all open intervals of R is called the Borel σ-algebra $\mathscr{B}$.

Therefore, since the measurable sets are a σ-algebra containing all open sets, every Borel set of real numbers is Lebesgue measurable. $\mathscr{B}$ contains about every set that comes up in analysis.

1.4.1 Contents of Borel Sets

- Open intervals

- Open sets
- Closed intervals
- Closed sets
- Compact sets
- Left open, right closed sets
- Right open, left closed sets
- All intervals

Every Borel set is Lebesgue measurable, but not vice-versa, there exist sets that are Lebesgue measurable which are not Borel sets. The collection of all Lebesgue measurable sets is a σ-algebra $\mathscr{M}$, and $\mathscr{B}$, the collection of Borel sets, then

$$\mathscr{B} \subset \mathscr{M} \subset 2^{\mathbb{R}}$$

1.5 Lebesgue Measure

By restricting outer measure of to a class of measurable sets, we can define the Lebesgue measure. It is denoted by m, if E is a measurable set, it's Lebesgue measure is defined to be

$$m(E) = m^*(E)$$

We can use all our previous results to determine the Lebesgue measure of specific sets of real numbers, note that all following sets are presumed to be Lebesgue measurable sets of real numbers.

We can infer the following properties for Lebesgue measurable sets-

1.5.1 Properties

1. $m(\phi)$=$m(\{a\}) = 0$, $m(\mathbb{R}) = \infty$.
2. $m(I) = l(I)$.
3. $m(\text{countable set}) = 0$, $m(\text{subset of a set with measure 0}) = 0$.
4. If E is measurable, then E^c is also measurable.
5. m is translation invariant i.e $m(x + E) = m(E)$ for all $x \in \mathbb{R}$
6. $m(\bigcup\limits_{i=1}^{\infty} A_i) = \sum\limits_{i=1}^{\infty} m(A_i)$ for $A_i \cap A_j = \phi$ for $i \neq j$ for all $A_i \in \Sigma$, a σ-algebra (σ-additivity).
7. $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$.

8. If E_1, E_2, E_3,.... E_n are measurable, then $\bigcup\limits_{i=1}^{n} E_i$ and $\bigcap\limits_{i=1}^{n} E_i$ are measurable.

9. $m(E_1) \leq m(E_2)\, if\, E_1 \subset E_2$. If in addition, $m(E_2) < \infty$, then $m(E_2) - m(E_1) = m(E_2 - E_1)$.

10. If $E_1 \subset E_2 \subset E_3 \dots$ then $m(\cup E_k) = m(\lim E_k) = \lim m(E_k)$

11. If $E_1 \supset E_2 \supset E_3 \dots$ and $m(E_1) < \infty$, then $m(\cap E_k) = m(\lim E_k) = \lim m(E_k)$

12. If $m(\cup E_k) < \infty$, then $\lim supm(E_k) \leq m(\lim supE_k)$

13. $m(\lim infE_k) \leq \lim infm(E_k)$

14. If $\lim infE_k = \lim supE_k$ and $m(\cup E_k) < \infty$, then $m(\lim E_k) = \lim m(E_k)$

1.5.2 Not all sets are Lebesgue measurable

Proof: We define an equivalence relation $x \sim y$ as $x - y \in \mathbb{Q}$ for $x, y \in \mathbb{R}$. This establishes equivalence classes $[x] = \{x + r : r \in \mathbb{Q}\}$. We define $A \subset [0, 1]$ with the following properties-

- For each $[x]$, there is a $p \in A$ with $p \in [x]$.
- For all $p, q \in A$, if $p, q \in [x]$, then $p = q$.

Let $A_n = A + r_n$, where $r_n (n \in \mathbb{N})$ is an enumeration of $\mathbb{Q} \cap [-1, 1]$. Then we can claim that $n \neq m \Rightarrow A_n \cap A_m = \phi$. We prove this by contradiction. Assume that there exists $x \in A_n \cap A_m$ for $n \neq m$.

Let $x = r_n + a_n$, $a_n \in A$ and $x = r_m + a_m$, $a_m \in A$. Then,

$$a_n + r_n = a_m + r_m$$

$$\Rightarrow a_n - a_m = r_m - r_n \in \mathbb{Q}$$

Then, a_n and a_m satisfies the equivalence relation $a_n \sim a_m$. So,

$$a_n, a_m \in [a_m]$$

$$\Rightarrow a_n = a_m$$

$$\Rightarrow r_n = r_m$$

$$\Rightarrow n = m$$

Which is a contradiction to our assumption. So, we can claim that

$$[0, 1] \subset \bigcup_{n=1}^{\infty} A_n \subset [-1, 2]$$

Assume that m is a measure on $P(\mathbb{R})$. Then,

$$m([0,1]) \leq m(\bigcup_{n=1}^{\infty} A_n) \leq m([-1,2])$$

$$\Rightarrow 1 \leq m(\bigcup_{n=1}^{\infty} A_n) \leq 3$$

By σ-additivity of Lebesgue measures,

$$1 \leq \sum_{n=1}^{\infty} m(A_n) \leq 3$$

By translational invariance of Lebesgue measures,

$$1 \leq \sum_{n=1}^{\infty} m(A) \leq 3$$

$$\Rightarrow 1 \leq \lim_{n\to\infty} nm(A) \leq 3$$

Which is not possible, as $\lim_{n\to\infty} nm(A)$ can take only one of two values - either 0 or ∞. Hence, the set A is not Lebesgue measurable.

1.5.3 Outer and Inner approximation of Lebesgue measure

Measurable sets possess the following excision property: If A is a measurable set of finite outer measure that is contained in B, then

$$m^*(B \sim A) = m^*(B) - m^*(A).$$

This follows from Sec 1.3.1

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^C) = m^*(A) + m^*(B \sim A).$$

Theorem

For an arbitrary subset E of $\mathbb{R}$, following assertions are equivalent to measurability of E

1. For each $\epsilon > 0$, there is an open set O containing E for which $m^*(O \sim E) < \epsilon$.(Outer approximation by open sets)
2. There is a G_δ set G containing E for which $m * (G \sim E) = 0$.
3. For each $\epsilon > 0$, there is an closed set F contained in E for which $m^*(E \sim F) < \epsilon$.(Inner approximation by closed sets)
4. There is a F_σ set F containing E for which $m * (E \sim F) = 0$.

Here, G_δ denotes a subset that is a countable intersection of open sets and F_σ is a countable union of closed sets.