Lebesgue Measure

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Abstract

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1 Introduction

The goal of this project is to investigate the Lebesgue measure and some of its applications in probability measure.

1.1 Measure

How do we measure a set or an object? If we have a 3-dimensional object, we could find its volume in 2 different ways, we could fill the container with simple objects or *boxes* and find their sum, or we could encase the whole container in a box and start chipping away smaller boxes. These approaches work only when our box measure is a properly defined measure.

The mathematical concept of measure is a generalisation of length in R, area in R^2 or volume in R^3. To avoid splitting into cases depending on the dimension, we shall refer to the *measure* of E, depending on what Euclidean space R^n we are using and $E \subset R^n$.

We know the measure or length of an interval, say $[0, 1]$, we also have $(0, 1)$ and $(0, 1]$, as these are subsets of our initial interval, their measures must not exceed that of $[0, 1]$ (monotonic). Let us assume the length of this unit interval be 1. We let the measure to be zero, for a single point or an empty set, and infinite if we consider the real line. Ideally, we'd like to associate a non-negative measure $m(E)$ to every subset E of R^n. It should also obey some reasonable properties like $m(A \cup B) = m(A) + m(B)$ whenever A and B are disjoint, we should have $m(A) \leq m(B)$ whenever $A \subset B$, and $m(x+A) = m(A)$ (translation invariant). Here, we hit a roadblock in our definition of a measure:

$$(0, 1) = \bigcup_{x \in (0,1)} \{x\} \, and \, 1 \neq \sum_{(0,1)} 0$$

From our definition, we can also see that two sets having the same number of points need not have the same measure, let $A = [0, 1]$ and $B = [0, 2]$, there exists a bijection from A to B (x $\mapsto$ 2x), but B is twice as long as A.

Remarkably, it turns out such a measure *does not exist*. This is quite a surprising fact, because it is counter-intuitive. These examples tell us that it is impossible to measure every subset of R^n applying all the above properties.

If we are developing a measure m defined on the subsets of R, we hope that these conditions are met:

1. $m(A)$ is defined for every set A of the real numbers;
2. $0 \leq m(A) \leq \infty$;
3. $m(A) \leq m(B)$ provided $A \subset B$;
4. $m(\phi) = 0$;
5. $m(\{a\}) = 0$ (points are dimensionless);

6. $m(I) = l(I)$, I is an interval (the measure on an interval should be its length)
7. $m(A) = m(x + A)$, translation invariance, (location doesn't affect the length, so it shouldn't affect measure)
8. $m(\bigcup_1^\infty A_k) = \sum_1^\infty m(A_k)$, for any mutually disjoint sequence (A_k) of subsets of real numbers (countable additivity).

Our goal is to construct a measure that satisfies as many conditions as possible.

1.1.1 Length of Intervals

The length of an interval I with end points $a \leq b$, $a, b \in R$, is defined as:

$$l(I) = b - a$$

For example, $l((0, 1]) = l([0, 1]) = 1$ and $l((1, \infty)) = \infty$.

We immediately conclude that, if I_1 and I_2 are two intervals with $I_1 \subset I_2$, $l(I1) \leq l(I2)$.

1.1.2 Cover

A collection $\{G_\alpha\}$ of open sets covers a set A if $A \subset \bigcup G_\alpha$. And the collection $\{G_\alpha\}$ is called the cover.

1.1.3 Subcover

Let C be a cover of a topological space X. A subcover of C is a subset of C that still covers X.

1.1.4 Compact Set

A set of real numbers is compact if every open cover of the set contains a finite subcover.

1.1.5 Heine-Borel Theorem

A set of real numbers is compact iff it is closed and bounded.

1.1.6 Result

If $I, I_1, I_2, \ldots, I_n$ are bounded open intervals with

$$I \subset \bigcup_1^n I_k, \; then \; l(I) \leq \sum l(I_k)$$

The length of an interval can not exceed the length of a *finite* cover.

1.1.7 Proposition

If $I, I_1, I_2, \ldots, I_n$ are bounded open intervals with $I \subset \bigcup I_k$, then $l(I) \leq \sum l(I_k)$. Also, $l(I) \leq inf\{\sum l(I_k) | I \subset \bigcup I_k, I_k \text{ } bounded\ intervals\}$.

Proof: Assume $I = (a, b)$ and let $\epsilon > 0$. The intervals $(a-\epsilon, a+\epsilon), (b+\epsilon, b-\epsilon), I_1, \ldots, I_n$ form an open cover of the compact set $[a, b]$. And by Heine-Borel Theorem, a finite sub collection will cover [a,b] and thus (a,b). Using 1.1.6:

$$l(I) \leq 4\epsilon + \sum l(I_k)$$

As this holds for any ϵ, $l(I) \leq \sum l(I_k)$, the proof is complete.

1.2 Lebesgue Outer Measure

We have seen that every set of real numbers can be covered with a countable collection of open intervals.

1.2.1 Definition

A is any subset of R. Form the collection of all countable covers of A by open intervals. The Lebesgue outer measure of A, $m^*(A)$, is given by

$$m^*(A) = inf \sum_{1}^{\infty} l(I_k) | A \subset \bigcup_{1}^{\infty} I_k\ ,\ I_k \ open\ intervals$$

1.2.2 Results

1. m^* is a set function, whose domain is all subsets of R, and range is $[0, \infty]$, the non-negative extended real numbers. Note that the Lebesgue Outer Measure is defined for every subset of R.

2. Outer Measure is monotonic, i.e. , if $A \subseteq B$, then $m^*(A) \leq m^*(B)$, any cover of B by open intervals is also a cover of A, and the latter infimum is taken over a larger collection than the former.

3. If (I_k) is any countable cover of A by open intervals, since infimum is a lower bound,
$$0 \leq m^*(A) \leq \sum l(I_k)$$

1.2.3 The outer measure of any interval is it's length.

Proof: For a closed interval [a,b], let $\epsilon > 0$. Then $(a-\epsilon, b+\epsilon)$ covers $[a, b]$ and length of this interval is $b-a+\epsilon$. Since ϵ is arbitrary $m^*([a, b]) \leq b-a = l([a, b])$.

Next let I_n be a covering of [a,b] by bounded open intervals. By the Heine-Borel Theorem, there exists a finite subset A of I_n's covering [a,b]. So $a \in I_1$ for some $I_1 = (a_1, b_1) \in A$. Also, if $b_1 \leq b$, then $b_1 \in I_2$ for some $I_2 = (a_2, b_2) \in A$. Similarly we can construct $I_1, I_2, \ldots, I_k$. Then

$$\sum l(I_n) \geq \sum_{i=1}^{k} l(I_i) = \sum_{i=1}^{k} (b_i - a_i)$$

$$= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \ldots + (b_1 - a_1)$$

$$> b_k - a_1$$

Since $a_1 < a$ and $b_k > b$, then $\sum l(I_n) > b_k - a_1$. So, $m^*([a,b]) = b - a = l([a,b])$.

If I is an unbounded interval, then given any natural number $n \in N$, there is a closed interval $J \subset I$ with $l(J) = n$. Hence $m^*(I) \geq m^*(J) = l(J) = n$. Since $m^*(I) \geq n\, and\, n \in N$ is arbitrary, then $m^*(I) = \infty = l(I)$.

1.2.4 Outer Measure is translation invariant

$m^*(A) = m^*(A + y)$

Proof: Suppose $m^*(A) = M < \infty$. Then for all $\epsilon > 0$ there exist I_n bounded open intervals, such that they cover A. And from sec 1.1.7. $\sum l(I_n) < M + \epsilon$, so if $y \in R$, then $I_n + y$ is a covering of A+y and so $m^*(A+y) \leq \sum l(I_n + y) = \sum l(I_n) < M + \epsilon$ .Therefore $m^*(A + y) \leq M$.

Now, let J_n be a collection of bounded open intervals such that $\cup J_n \supset A + y$.Assume that $\sum l(J_n) < M$. Then $J_n - y$ is a covering of A and $\sum l(J_n - y) = \sum l(J_n) < M$, a contradiction. So, $\sum l(J_n) \geq M$ and hence $m^*(A) \geq M$. So $m^*(A) = m^*(A + y) = M$

Suppose $m^*(A) = \infty$.Then for any$\{I_n\}_{n=1}^{\infty}$ a set of bounded open intervals such that $A \subset \cup I_n$, we must have$\sum l(I_n) = \infty$ .Consider A+y. For any $\{J_n\}_{n=1}^{\infty}$ a set of bounded open intervals such that $A+y \subset \cup J_n$, the collection $\{J_n - y\}_{n=1}^{\infty}$ is a set of bounded open intervals such that $A + y \subset \cup J_n - y$. So $\sum l(J_n - y) = \infty$. But l(J_n) =l($J_n - y$), so we must have $\sum l(J_n) = \infty$. Since $\{J_n\}_{n=1}^{\infty}$ is an arbitrary collection of bounded open intervals covering A+y, we must have $m^*(A) = m^*(A + y) = \infty$.

1.2.5 Outer Measure is countably sub-additive

That is, if $\{E_k\}_{k=1}^{\infty}$ is any countable collection of sets, then

$$m^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m^*(E_k)$$

Proof: If one of the E_k's has infinite outer measure, the inequality holds trivially. We therefore suppose each of the E_k's has finite outer measure. Let $\epsilon > 0$. For each natural number k, there is a countable collection $\{I_{k,i}\}_{i=1}^{\infty}$ of open bounded intervals for which

$$E_k \subseteq \bigcup_{i=1}^{\infty} \{I_{k,i}\}\ and\ \sum_{i=1}^{\infty} l(I_{k,i}) < m^*(E_k) + \frac{\epsilon}{2^k}$$

Now, $\{I_{k,i}\}_{1\leq k,i\leq\infty}$ is a countable collection of open, bounded intervals that covers $\cup_{k=1}^{\infty}E_k$: the collection is countable since it is a countable collection of countable collections. Thus,

$$m^*(\bigcup_{k=1}^{\infty}E_k) \leq \sum_{1\leq k,i<\infty} l(I_{k,i}) = \sum_{k=1}^{\infty}[\sum_{i=1}^{\infty} l(I_{k,i})]$$

$$< \sum_{k=1}^{\infty}[m^*(E_k) + \frac{\epsilon}{2^k}]$$

$$m^*(\bigcup_{k=1}^{\infty}E_k) \leq [\sum_{k=1}^{\infty} m^*(E_k)] + \epsilon$$

Since this holds for each $\epsilon > 0$, it also holds for $\epsilon = 0$. The proof is complete.

The finite sub-additivity property follows from countable sub-additivity by taking $E_k = \phi$ for $k > n$.

1.2.6 $[0, 1]$ is not countable

As seen in Sec.1.2.3 $[0, 1]$ has an outer measure of 1, whereas all countable sets have a measure 0, (refer to Royden and Fitzpatrick pg.31), therefore $[0, 1]$ is not countable.

1.3 σ-Algebra of Lebesgue Measurable sets

Unfortunately the outer measure fails to be countably additive on all subsets of R, it is not even finitely additive. A famous example is the Vitali Set, found by Giuseppe Vitali in 1905. However, we can salvage matters by measuring a certain class of sets in R, henceforth called the measurable sets. Once we restrict ourselves to measurable sets, we recover all the properties again.

1.3.1 Carathedory's Measurability Criteria

E is any set of real numbers. If

$$m^*(X) = m^*(X \cap E) + m^*(x \cap E^c)$$

for every set X of real numbers, then the set E is said to be Lebesgue Measurable set of real numbers.

1.3.2 Definition of σ-Algebra

A σ-algebra (or σ-field) Σ on a set Ω is a subset of $P(\Omega)$ (the power set of Ω) such that it satisfies the following properties -

1. $\Omega \in \Sigma$ *(property 1)*

2. If A is an element of Σ then its complement also is an element of Σ i.e A $\in \Sigma \Rightarrow$ A$^c \in \Sigma$ *(property 2)*

3. If $A_i \in \Sigma$, $i \in \mathbb{N}$, then $\bigcup\limits_{i=1}^{n} A_i \in \Sigma$*(property 3)*

1.3.3 Examples

1. Consider $\Sigma = \{\phi,\, \Omega\}$. The given set is a σ-algebra on Ω as it satisfies all the properties needed. Observe that ϕ is also included in Σ by *property 2. Property 1* is also satisfied by the given σ-algebra. $\phi \cup \Omega = \Omega \in \Sigma$, so *property 3* is also satisfied.

2. Consider $\Sigma = P(\Omega)$. Then, Σ satisfies *property 1* as $\Omega \in \Sigma$. Furthermore, $X^c = \phi \in \Sigma$ and hence it satisfies *property 2.* As Σ contains all the subsets of Ω, it is guaranteed that the union of any number of subsets of Ω exists in Σ, and hence satisfying *property 3.* Hence $\Sigma = P(\Omega)$ is a σ-algebra.

3. Consider $\Sigma = \{\phi,\, \{1\},\, \{2\},\, \{1,\, 2\},\, \Omega\}$ and $\Omega = \{1,\, 2,\, 3\}$. Here, Σ is **not** a σ-algebra on Ω as $\{1,\, 2\}^c = \{3\}$ is not an element of Σ, and hence Σ does not satisfy *property 2.*

1.3.4 Proposition 1

If Σ is a σ-algebra on Ω, and A$_1$, A$_2$, A$_3$,....A$_n \in \Sigma$, then $\bigcap\limits_{i=1}^{n}$ A$_i \in \Sigma$.

Proof: If A$_1$, A$_2$, A$_3$,....A$_n \in \Sigma$, then A$_1^c$, A$_2^c$, A$_3^c$,....A$_n^c \in \Sigma$ and $\bigcup\limits_{i=1}^{n}$ A$_i^c \in \Sigma$. Then by *property 2,* $(\bigcup\limits_{i=1}^{n}$ A$_i^c)^c = \bigcap\limits_{i=1}^{n}$ A$_i \in \Sigma$.

1.3.5 Proposition 2

If Σ is a σ-algebra on Ω and A, B $\in Sigma$, then A$\backslash$B = A - B $\in \Sigma$.

Proof: If B $\in \Sigma$, then B$^c \in \Sigma$ by *property 2.* So, A $\cap$ B^c = A$\backslash$B $\in \Sigma$ (As seen in proposition 1).

1.3.6 Proposition 3

If Σ_i is a σ-algebra on Ω, i $\in \mathbb{N}$, then $\bigcap\limits_{i=1}^{n} \Sigma_i$ is also a σ-algebra on Ω.

Proof: Consider the σ-algebras Σ_1, Σ_2, Σ_3,... Σ_n defined on Ω, then the smallest intersection of all such σ-algebras would be $\{\phi,\, \Omega\}$, which is a σ-algebra by definition. Also, if A is an element of the intersection, then A^c would also be an element of the intersection, which satisfies *property 2.* Also, A $\cup$ A^c = Ω $\in \Sigma$ which satisfies properties 1 and 3.

1.3.7 Proposition 4

For $Y \subseteq \mathrm{P}(\Omega)$, there exists a smallest σ-algebra that contains Y, called the σ-algebra generated by Y, given by the intersection of all such σ-algebras Σ such that $\Sigma \supseteq Y$. That is, $\sigma(Y) = \bigcap_{\Sigma \supseteq Y} \Sigma$.

Proof: Consider σ-algebras Σ_1, Σ_2, Σ_3,.... $\Sigma_n \supseteq Y$ defined on Ω. Then $\bigcap_{i=1}^{n} \Sigma_i \supseteq Y$. As $Y \in \Sigma_i$, $Y^c \in \Sigma_i$ for i = 1, 2, 3...n which satisfies *property 2.* Furthermore, $\Omega \in \Sigma_i$ which satisfies *property 1.* $Y \cup Y^c = \Omega$ which satisfies *property 3.* Hence $\sigma(Y)$ is a σ-algebra.

For example, consider $\Omega = \{1, 2, 3, 4\}$ and $Y = \{\{1\}, \{2\}\}$. Then the smallest σ-algebra on Ω containing Y is $\sigma(Y) = \{\phi, \{1\}, \{2\}, \{1, 2\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \Omega\}$. Also note that if Y is a σ-algebra, then $\sigma(Y) = Y$.