1 Probability Space

There are often various approaches to probability each with its own advantages and disadvantages.

Experiment is a procedure that can be infinitely repeated and has a well-defined set of possible outcomes, known as the sample space.

The observation/result of the experiment are termed as outcomes.

1.1 Classical Approach

Probability of an event E is defined to be:

$$P(E) = \frac{Number\ of\ outcomes\ in\ E}{Total\ number\ of\ outcomes}$$

Some examples are tossing a coin or rolling a die. Disadvantages:

- Unable to model biases. It says nothing about cases where no physical symmetry exists.

- Doesn't deal with cases where total outcomes are infinite.

1.2 Frequentist Approach

Also known as the relative frequency approach or frequentism. It defines an event's probability as the limit of its relative frequency in many trials.

Probability is defined to be:

$$P(E) = \lim_{n \to \infty} \frac{n_E}{n}$$

where an experiment is conducted n times and event E occurs n_E times. Disadvantages:

- It isn't efficient to conduct an experiment multiple times just to find the probability of an event occuring.

- It is unable to deal with subjective belief. Eg: Suppose a cricket expert says there is a 50% of RCB winning the IPL this year. It doesn't mean that the RCB has won half the titles in the past.

1.3 Axiomatic Approach

1.3.1 Probability Space

The triple $(S,\ F,\ P)$ is referred to as a probability space where:

- S : Sample space, set of all possible outcomes of the experiment.

- F : Event Space, subset of the sample space

- P : Probability Measure

1.3.2 Sample Space

S can either be finite or countably infinite or uncountably infinite.

Examples for S:

- Finite Sample Space: Single coin toss $S = \{H, T\}$ and two coin tosses $S = \{HH, HT, TT, TH\}$.

- Countably Infinite Sample Space: Keep tossing a coin till you get a head $S = \{H, TH, TTH...\}$.

- Uncountably Infinite Sample Space: We have a circular dart board and we are measuring the angle at which a dart hits the board. $S = [0, 2\pi]$

1.4 Event Space

Collection of events is called an event space, there are some properties to be satisfied such as: it has to be a "Sigma Field".

1.4.1 Sigma Field

A sigma field (or sigma algebra) F is a collection of subsets of S which satisfies the following properties:

- $S \in F$

- If $E \in F$, then $E^C \in F$

- If $E_1, E_2, E_3 \cdots \in F$, then $\bigcup_{i=1}^{\infty} E_i \in F$

1.4.2 Examples for Event Space

- Smallest possible event space:

$$F = \{\phi, S\}$$

- Next non-trivial event space:

$$F = \{\phi, E, E^c, S\}$$

- If $E_1 \in F$ and $E_2 \in F$, then $E_1 \cap E_2 \in F$ (Proof given later for general case)

- For S=1,2,3,4,5,6 , $E_1 = 1, 2$ and $E_2 = 3, 4$. The smallest event space containing E_1 and E_2 is:

$$F = \phi, S, E_1, E_1^C, E_2, E_2^C, E_1 \bigcup E_2, E_1 \bigcup E_2^{\,C}$$

1.4.3   Proposition 1

If Σ is a σ-algebra on Ω, and A_1, A_2, A_3,....$A_n \in \Sigma$, then $\bigcap\limits_{i=1}^{n} A_i \in \Sigma$.

Proof: If A_1, A_2, A_3,....$A_n \in \Sigma$, then A_1^c, A_2^c, A_3^c,....$A_n^c \in \Sigma$ and $\bigcup\limits_{i=1}^{n} A_i^c \in \Sigma$. Then by property 2, $(\bigcup\limits_{i=1}^{n} A_i^c)^c = \bigcap\limits_{i=1}^{n} A_i \in \Sigma$.

1.4.4   Proposition 2

If Σ is a σ-algebra on Ω and A, B $\in \Sigma$, then $A\backslash B$ = A - B $\in \Sigma$.

Proof: If B $\in \Sigma$, then $B^c \in \Sigma$ by property 2. So, $A \cap B^c = A\backslash B \in \Sigma$ (As seen in proposition 1).

1.5   Probability Space

P has to satisfy the following 3 axioms:

- $P(E) \geq 0$

- $P(S) = 1$

- If $E_1, E_2 \cdots \in F$ such that $E_i \cap E_j = \phi$ then:

$$P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$$

$P(E_1 \cup E_2) = P(E_1) + P(E_2) + \sum$

1.6   Derived Properties of Probability

1.
$$P(E^C) = 1 - P(E)$$

Proof:
$$E \cup E^C = S$$
$$P(E) + P(E^C) = 1$$

2. For any two events E_1 and E_2,
$$P(E_1 \cup E_2) = P(E_1) + P(E_2) + P(E_1 \cap E_2)$$

Proof:
$$P(E_1 \cup E_2) = P(E_1) + P(E_2 \cap E_1^C)$$
Now,
$$E_2 = (E_2 \cap E_1) \cup (E_2 \cap E_1^C)$$

$$P(E_1 \cup E_2) =$$

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Question: $S = \{1, 2, 3, 4, 5, 6\}$. 1 and 5 are equally likely and probability of getting a 6 is one-third.

Find minimum and maximum probability that we get an even number.

Answer:

$$Minimum\ Prob. = \frac{1}{3}\ \ when\ \ P_2 = P_4 = 0$$

$$Maximum\ Prob. = 1,\ \ P_2 + P_4 = \frac{2}{3}$$