3.1 Integration by Parts

The Integration by Parts Formula

Every differentiation rule has a corresponding integration rule. For instance, the substitution rule corresponds to the chain rule for differentiation. The rule that corresponds to the product rule for differentiation is called Integration by Parts.

Recall the product rule: If f and g are differentiable functions, then
$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + g'(x)f(x)$.
Using the notation of indefinite integrals, the equation becomes,

$$\int \left(f'(x)g(x) + g'(x)f(x)\right) dx = f(x)g(x).$$

or

$$\int f'(x)g(x)\, dx + \int g'(x)f(x)\, dx = f(x)g(x).$$

We can rearrange this equation as

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx \tag{1}$$

Now let $u = f(x)$ and $dv = g'(x)dx$. Then $du = f'(x)dx$ and $v = g(x)$. Therefore, making the substitution, the equation now becomes,

$$\int u\, dv = uv - \int v\, du \tag{2}$$

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Equation (1) is the formula for integration by parts. It is perhaps easier to remember the formula in the notation of equation (2).

Integration by Parts for Definite Integrals

Theorem

Let $u = f(x)$ and $v = g(x)$ be functions continuous derivatives on $[a, b]$.

Then

$$\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du.$$

Exercises

1. Find $\displaystyle\int x\sin(x)\,dx$
2. Prove that $\displaystyle\int \ln(x)\,dx = x(\ln(x) - 1) + C$
3. Prove the reduction formula

$$\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

   where $n \geq 2$ is an integer.

4. Calculate the average value of $f(x) = x\sec^2(x)$ on the interval $\left[0, \dfrac{\pi}{4}\right]$.

Solutions

1. Let $u = x$ and $dv = \sin(x)dx$. Then $du = dx$ and $v = -\cos(x)$. Now integrating by parts we have that

$$\int x\sin(x)\,dx = -x\cos(x) - \int -\cos(x)\,dx = -x\cos(x) + \int \cos(x)\,dx.$$
$$\implies \int x\sin(x)\,dx = -x\cos(x) + \sin(x) + C.$$

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2. *Proof.* Let $u = \ln(x)$ and $dv = dx$. Then $du = \frac{1}{x}dx$ and $v = x$. Thus, integrating by parts we have

$$\int \ln(x)\,dx = x\ln(x) - \int x \cdot \frac{1}{x}\,dx = x\ln(x) - \int dx.$$
$$\implies \int \ln(x)\,dx = x\ln(x) - x + C = x(\ln(x) - 1) + C \text{ as required.}$$ □

3. *Proof.* Let $n \geq 2$ be an integer. We will use integration by parts. So let $u = \sin^{n-1}(x)$ and $dv = \sin(x)dx$. Then $du = (n-1)\sin^{n-2}(x)\cos(x)dx$ and $v = -\cos(x)$. Integrating by parts we have

$$\int \sin^n(x)\,dx = -\cos(x)\sin^{n-1}(x) + (n-1)\int \cos^2(x)\sin^{n-2}(x)\,dx$$

Using the identity that $\cos^2(x) = 1 - \sin^2(x)$ the equation becomes,

$$-\cos(x)\sin^{n-1}(x) + (n-1)\int (1 - \sin^2(x))\sin^{n-2}(x)\,dx =$$
$$-\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x) - \sin^n(x)\,dx$$

Splitting the integral we have,

$$\int \sin^n(x)\,dx =$$
$$-\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)\int \sin^n(x)\,dx$$
$$\implies \int \sin^n(x)\,dx = \frac{-\cos(x)\sin^{n-1}(x)}{n-1} + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx \text{ after}$$

rearranging the equation.

Therefore, $\displaystyle\int \sin^n(x)\,dx = -\frac{1}{n}\cos(x)\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$ for $n \geq 2$ as required. $\square$

4. Recall the average value of a function on $[a, b]$,

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

In the context of this problem, we want to find $f_{\text{ave}} \dfrac{1}{\frac{\pi}{4}-0}\displaystyle\int_0^{\pi/4} x\sec^2(x)\,dx = \frac{4}{\pi}\int_0^{\pi/4} x\sec^2(x)\,dx$. We will evaluate this integral via integrating by parts.

So let $u = x$ and $dv = \sec^2(x)dx$. Then $du = dx$ and $v = \tan(x)$. So $\displaystyle\int_0^{\pi/4} x\sec^2(x)\,dx = x\tan(x)|_0^{\pi/4} - \int_0^{\pi/4} \tan(x)\,dx$.

$\displaystyle\implies \int_0^{\pi/4} x\sec^2(x)\,dx = \frac{\pi}{4} - \ln|\cos(x)||_0^{\pi/4} = \frac{\pi}{4} + \frac{1}{2}\ln(2)$.

Therefore, $f_{\text{ave}} = \dfrac{4}{\pi}\left(\dfrac{\pi}{4} + \dfrac{1}{2}\ln(2)\right) = 1 + \dfrac{2}{\pi}\ln(2)$.

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