1.2 The Definite Integral

1 *Definition and Notation*

We saw that in section 1.1 we defined the area under a curve in terms of Riemann Sums:

$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x$ where x_i^* is any sample point in the subinterval $[x_{i-1}, x_i]$. However, this definition is satisfied if $f(x)$ is a continuous and non-negative function. But, this doesn't model all real-world situations.

Definition:

If $f(x)$ is a function defined on the closed interval $[a, b]$, the definite integral of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided that the limit exists.

Theorem 1.1:

If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

1.1 *Evaluating an Integral Using the Definition*

1. Use the definition of the definite integral to evaluate $\int_0^1 \left(x^3 - 3x^2\right)\,dx$.

2 *Area and the Definite Integral*

2.1 *Net Signed and Total Area*

Net Signed Area

If f takes on positive and negative values, then the Riemann sum is the sum of the areas of the rectangles that lie above the $x-$axis and the sum of the areas of rectangles lie below the $x-$axis. A definite integral can be interpreted as a net area, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2,$$

where A_1 is the area of region above the $x-$axis and below the graph of f, and A_2 is the area of region below the $x-$axis and above the graph of f.

Total Area

Similar to Net Signed Area, Total Area converts the negative areas into positive areas. So the total area between $f(x)$ and the $x-$axis is given by,

$$\int_a^b |f(x)|\,dx = A_1 + A_2.$$

2.2 *Net Signed and Total Area Practice Question*

1. Find the total area between $f(x) = x - 2$ and the x-axis over the interval $[0, 6]$.

2. Evaluate the following integral by interpreting each in terms of area.

   $\int_0^3 (x - 1)\ dx.$

3 *Properties of the Definite Integral*

1. $\int_a^a f(x)\,dx = 0$ (a singular point has no width, thus no area)
2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
3. $\int_a^b [f(x) \pm g(x)\,dx] = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
4. $\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx$
5. for $c \in [a,b]$, $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
6. if C is some constant, $\int_a^b C\,dx = C(b-a)$

3.1 *Comparison Properties of Integrals*

Theorem 1.2: Comparison Theorem

1. If $f(x) \geq 0$ for $a \leq x \leq b$, then

   $\int_a^b f(x)\,dx \geq 0$
2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

   $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
3. If m and M are constants such that $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

   $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

3.2 *Properties of Integrals Practice*

1. If $\int_1^5 f(x)\,dx = 12$ and $\int_4^5 f(x)\,dx = 3.6$, find $\int_1^4 f(x)\,dx$.

3.3 *Comparison Theorem Practice*

1. Show that $\int_0^1 \sqrt{1+x^3}\,dx \le \int_0^1 \sqrt{1+x^2}\,dx$

4 *Solutions to Practice Questions*

1.1.1 ***Solution:***

Recall that by definition $\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*)\Delta x\right]$.

So $\int_0^1 \left(x^3 - 3x^2\right) = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*)\Delta x\right]$. We will now find $\Delta x, x_i^*, f(x_i^*)$.

$\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$

$x_i^* = x_i = a + i\Delta x = 0 + \dfrac{i}{n} = \dfrac{i}{n}$

$f(x_i^*) = \left(\dfrac{i^3}{n^3} - \dfrac{3i^2}{n^2}\right)$.

$\implies \int_0^1 \left(x^3 - 3x^2\right) = \lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(\dfrac{i^3}{n^3} - \dfrac{3i^2}{n^2}\right)\dfrac{1}{n}\right]$

$\implies \int_0^3 \left(x^3 - 3x^2\right)\,dx = \lim_{n\to\infty}\left[\dfrac{1}{n}\sum_{i=1}^{n}\dfrac{i^3}{n^3} - \dfrac{3}{n}\sum_{i=1}^{n}\dfrac{i^2}{n^2}\right]$

$\implies \int_0^3 \left(x^3 - 3x^2\right)\,dx = \lim_{n\to\infty}\left[\dfrac{1}{n^4}\sum_{i=1}^{n} i^3 - \dfrac{3}{n^3}\sum_{i=1}^{n} i^2\right]$

$\implies \int_0^3 \left(x^3 - 3x^2\right)\,dx = \lim_{n\to\infty}\left[\dfrac{1}{n^4}\dfrac{n^2(n+1)^2}{4} - \dfrac{3}{n^3}\dfrac{n(n+1)(2n+1)}{6}\right]$

$\implies \int_0^3 \left(x^3 - 3x^2\right)\,dx = \lim_{n\to\infty}\left[\dfrac{1}{n^2}\dfrac{(n+1)^2}{4} - \dfrac{1}{n^2}\dfrac{(n+1)(2n+1)}{2}\right]$

$\implies \int_0^3 \left(x^3 - 3x^2\right)\,dx = \lim_{n\to\infty}\left[\dfrac{(n+1)^2}{4n^2} - \dfrac{(n+1)(2n+1)}{2n^2}\right]$

$\implies \int_0^3 \left(x^3 - 3x^2\right)\,dx = \lim_{n\to\infty}\left[\dfrac{(n+1)^2 - 2(n+1)(2n+1)}{4n^2}\right]$

$\implies \int_0^3 \left(x^3 - 3x^2\right)\,dx = \lim_{n\to\infty}\left[\dfrac{-3n^2 - 4n - 1}{4n^2}\right]$

$\implies \int_0^3 \left(x^3 - 3x^2\right)\,dx = \dfrac{-3}{4}$.

Therefore, $\int_0^1 \left(x^3 - 3x^2\right)\,dx = \dfrac{-3}{4}$.

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2.2.1 ***Solution:***

In order to find the total area, we need to evaluate $\int_0^6 |x-2|\,dx$. Let's visualize the graph of $y = |x-2|$.

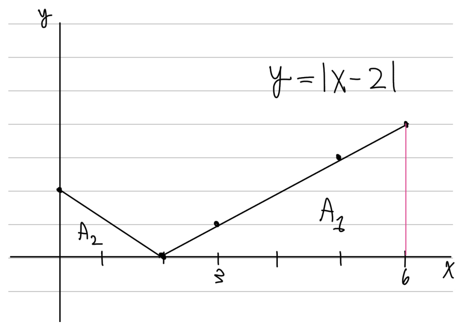

Figure 1: The graph of $y = |x-2|$ on $[0, 6]$

Now to evaluate $\int_0^6 |x-2|\,dx$, we just need to add the areas $A_1 + A_2$. So $A_1 = \dfrac{4 \cdot 4}{2} = 8$ and $A_2 = \dfrac{2 \cdot 2}{2} = 2$.

$\Longrightarrow \int_0^6 |x-2|\,dx = A_1 + A_2 = 8 + 2 = 10.$

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2.2.2 ***Solution:***

Now notice that in Figure 1, $A_2 < 0$ and $A_1 > 0$.

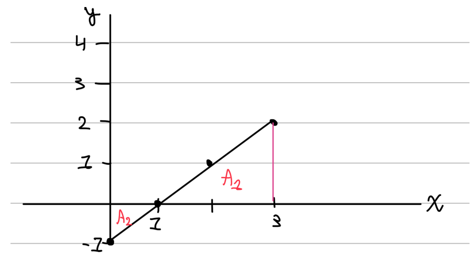

Figure 2: The graph of $y = x - 1$ on $[0, 3]$

Thus, $\int_0^3 (x-1)\ dx = A_1 - A_2$. In order to figure out the areas of A_1 and A_2 we can use the area of a triangle.

So $A_1 = \dfrac{2 \cdot 2}{2} = 2$ and $A_2 = \dfrac{1 \cdot 1}{2} = \dfrac{1}{2}$, which means, $A_1 - A_2 = 2 - \dfrac{1}{2} = \dfrac{3}{2}$.

$\implies \int_0^3 (x-1)\ dx = \dfrac{3}{2}$.

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3.2.1 ***Solution:***

Using property 5 of integrals, we can rewrite $\int_1^5 f(x)\,dx$ as the following,

$\int_1^5 f(x)\,dx = \int_1^4 f(x)\,dx + \int_4^5 f(x)\,dx$. rearranging for $\int_1^4 f(x)\,dx$,

$\int_1^4 f(x)\,dx = \int_1^5 f(x)\,dx - \int_4^5 f(x)\,dx$.

$\implies \int_1^4 f(x)\,dx = 12 - 3.6 = 8.4$ as needed.

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3.3.1 *Proof.* Let $f(x) = \sqrt{1+x^3}$ and $g(x) = \sqrt{1+x^2}$ both be functions over the closed interval $[0, 1]$.

Let's first show that $f(x) \leq g(x)$ for all $x \in [0, 1]$.

So we have the following inequality, $\sqrt{1+x^3} \leq \sqrt{1+x^2}$.

$\implies 1 + x^3 \leq 1 + x^2$ by squaring both sides since $0 \leq x \leq 1$

$\implies x^3 \leq x^2$

$\implies x^2\,(x-1) \leq 0.$

Notice that for this inequality to hold $x-1 \leq 0$ for all $x \in [0,1]$ since $x^2 \geq 0$ for all x. Since for all $x \in [0,1]$, the product is indeed less than or equal to zero, we conclude that $\sqrt{1+x^3} \leq \sqrt{1+x^2}$ for all $x \in [0,1]$. Thus, by the comparison theorem, we can conclude that $\int_0^1 \sqrt{1+x^3}\,dx \leq \int_0^1 \sqrt{1+x^2}$ as required. $\square$