1.7 Integrals Resulting in Inverse Trigonometric Functions

Rules: Integrals of Trig Functions

Assume $a > 0$.

1. $\displaystyle\int \frac{1}{\sqrt{a^2 - u^2}}\, du = \sin^{-1}\left(\frac{u}{a}\right) + C$
2. $\displaystyle\int \frac{1}{a^2 + u^2}\, du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
3. $\displaystyle\int \frac{1}{u\sqrt{u^2 - a^2}}\, du = \frac{1}{|a|}\sec^{-1}\left(\frac{|u|}{a}\right) + C$

Example: Evaluate $\int \frac{1}{16 + x^2}\, dx$

Solution.

$$\int \frac{1}{16 + x^2}\, dx = \int \frac{1}{(4)^2 + x^2}\, dx = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C$$

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Practice Questions

1. Explain what is wrong with the following integral:
$$\int_{-1}^{1} \frac{1}{|t|\sqrt{t^2 - 1}}\, dt.$$
2. Compute $\displaystyle\int \frac{e^t}{\sqrt{1 - e^{2t}}}\, dt$
3. For $A > 0$, compute $I(A) = \displaystyle\int_{-A}^{A} \frac{dt}{1 + t^2}$ and evaluate $\lim\limits_{a\to\infty} I(A)$, the area under the graph of $\frac{1}{1 + t^2}$ on $(-\infty, \infty)$.

Solutions

1. The problem is that when $t = -1$, $\sqrt{t^2 - 1} = 0$ which makes the integrand undefined. Thus, $\dfrac{1}{|t|\sqrt{t^2-1}}$ is not continuous on $[-1, 1]$.

2. Let $u = e^t$. Then $du = e^t dt$. $\implies \displaystyle\int \frac{e^t}{\sqrt{1-e^{2t}}}\,dt = \int \frac{1}{\sqrt{1-u^2}}\,du =$ $\sin^{-1}(u) + C = \arcsin(e^t) + C$.

3. Notice that the function $\dfrac{1}{1+t^2}$ is an even function. Thus, $I(A) = \displaystyle\int_{-A}^{A} \frac{dt}{1+t^2} =$ $2\displaystyle\int_0^A \frac{1}{1+t^2}\,dt$. Evaluating this integral we get, $I(A) = 2\tan^{-1}(A)$.
   Now we consider $\displaystyle\lim_{a\to\infty} I(A)$.
   $\implies \displaystyle\lim_{a\to\infty} I(A) = 2\lim_{a\to\infty}\tan^{-1}(A) = 2\cdot\frac{\pi}{2} = \pi$.