Extra Problems

1. Find the minimum value of the area of the region under the curve $y = x + \frac{1}{x}$ from $x = a$ to $x = a + 1.5$, for all $a > 0$.

2. If $f(x) = \int_0^{g(x)} \frac{1}{\sqrt{1+t^3}}\, dt$, where $g(x) = \int_0^{\cos(x)} \left[1 + \sin(t^2)\right] dt$, find $f'\left(\frac{\pi}{2}\right)$.

3. If $\int_0^6 f(x)\, dx = 10$ and $\int_0^4 f(x)\, dx = 7$, find $\int_4^6 f(x)\, dx$.

4. Evaluate $\int_0^1 \left(x + \sqrt{1-x^2}\right) dx$ by interpreting it in terms of areas.

5. Express $\lim_{n\to\infty} \left[\sum_{i=1}^{n} \sin(x_i)\Delta x\right]$ as a definite integral on the interval $[0, \pi]$ and then evaluate the integral.

6. If f is continuous and $\int_0^4 f(x)\, dx = 10$, find $\int_0^2 f(2x)\, dx$.

7. If f is continuous on $[0, \pi]$, use the substitution $u = \pi - x$ to show that

$$\int_0^{\pi} x f(\sin(x))\, dx = \frac{\pi}{2} \int_0^{\pi} f(\sin(x))\, dx$$

8. If $f(x) = \int_0^x x^2 \sin(t^2)\, dt$, find $f'(x)$.

9. Suppose f is continuous on $[-a, a]$. Prove the following.

   (a) If f is even, then $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$

   (b) If f is odd, then $\int_{-a}^{a} f(x)\, dx = 0$.

10. Evaluate $\lim_{x\to 3} \left(\frac{x}{x-3} \int_3^x \frac{\sin(t)}{t}\, dt\right)$.

11. Suppose the average value of f over $[a, b]$ is 1 and the average value of f over $[b, c]$ is 1 where $a < c < b$. Show that the average value of f over $[a, c]$ is also 1.

12. Given that both $f(x)$ and $f'(x)$ are continuous everywhere, if $f(1) = 10$ and $\int_{-3}^{1} f'(x)\, dx = 12$. Find $f(-3)$.

Solutions

1. Let $F(a) = \int_a^{a+1.5} x + \dfrac{1}{x}\, dx$. To find the minimum area, we must find $F'(a)$ and set $F'(a) = 0$.

   So from $F(a) = \int_a^{a+1.5} x + \dfrac{1}{x}\, dx$, we can separate the integral into two more integrals as follows,

   $F(a) = \int_a^0 x + \dfrac{1}{x}\, dx + \int_0^{a+1.5} x + \dfrac{1}{x}\, dx.$

   $\implies F(a) = -\int_0^a x + \dfrac{1}{x}\, dx + \int_0^{a+1.5} x + \dfrac{1}{x}\, dx$

   $\implies F(a) = \int_0^{a+1.5} x + \dfrac{1}{x}\, dx - \int_0^a x + \dfrac{1}{x}\, dx$

   Now applying the Fundamental Theorem of Calculus Part 1,

   $F'(a) = (a+1.5) + \dfrac{1}{a+1.5} - \left(a + \dfrac{1}{a}\right) = 1.5 + \dfrac{1}{a+1.5} - \dfrac{1}{a}.$

   $\implies F'(a) = \dfrac{1.5a(a+1.5) + a - (a+1.5)}{a(a+1.5)} = \dfrac{1.5a^2 + 2.25a - 1.5}{a(a+1.5)}$

   Now we set $F'(a) = 0$ to find the minimum.

   $F'(a) = 0 = \dfrac{1.5a^2 + 2.25a - 1.5}{a(a+1.5)}$. Since $a > 0$, so is $a(a+1.5)$. Thus, we can multiply both sides by $a(a+1.5)$.

   $\implies 0 = 1.5a^2 + 2.25a - 1.5 = 6a^2 + 9a - 6$ after multiplying both sides by 4

   $\implies 0 = 2a^2 + 3a - 2$ by factoring out a 3.

   $\implies 0 = (2a-1)(a+2)$

   $\implies a = \dfrac{1}{2}$ or $a = -2$. But since $a > 0$, it must be the case that $a = \dfrac{1}{2}$.

   To confirm that $a = \dfrac{1}{2}$ is indeed the minimum, we will use the second derivative test.

   So $F''(a) = \dfrac{1}{a^2} - \dfrac{1}{(a+1.5)^2}$.

   Plugging $a = \dfrac{1}{2}$ into the second derivative, we see that $F''\left(\dfrac{1}{2}\right) > 0$ which means the minimum value of the area of the region under the curve $y = x + \dfrac{1}{x}$ on the interval $[a, a+1.5]$ is $\dfrac{1}{2}$.

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2. Let's define some other function $H(x)$, such that $H(x) = \int_0^x \frac{1}{\sqrt{1+t^3}}\,dt$.

   Then by the fundamental theorem of calculus part 1, $H'(x) = \frac{1}{\sqrt{1+x^3}}$.

   Now notice that $f(x) = H(g(x))$, so by the chain rule, $f'(x) = H'(g(x))g'(x)$.

   So now we need to find $g'(x)$. So let's define another function, $K(x)$,

   such that $K(x) = \int_0^x \left[1 + \sin(t^2)\right]\,dt$.

   Then by the fundamental theorem of calculus part 1, $K'(x) = 1 + \sin(x^2)$.

   Now notice once more that $g(x) = K(\cos(x))$.

   Then applying the chain rule again, $g'(x) = -\sin(x)K'(\cos(x)) = -\sin(x)\left(1 + \sin(\cos^2(x)\right)$.

   Therefore, $f'(\frac{\pi}{2}) = H'(g(\frac{\pi}{2}))g'(\frac{\pi}{2}) = H'(0)\left(-\sin(\frac{\pi}{2}) \cdot \left(1 + \sin(\cos^2(\frac{\pi}{2})\right)\right)$.

   This now simplifies to $f'(\frac{\pi}{2}) = 1(-1(1)) = 1$.

   Therefore, $f'(\frac{\pi}{2}) = -1$ as needed.

$\blacksquare$

3. Notice that $\int_0^6 f(x)\,dx = \int_0^4 f(x)\,dx + \int_4^6 f(x)\,dx$.

   $\implies \int_4^6 f(x)\,dx = \int_0^6 f(x)\,dx - \int_0^4 f(x)\,dx = 10 - 7 = 3$.

$\blacksquare$

4. Splitting the integral, $\int_0^1 \left(x + \sqrt{1-x^2}\right)\,dx = \int_0^1 x\,dx + \int_0^1 \sqrt{1-x^2}\,dx$.

   Notice that the area under the curve $y = x$ is just a triangle and so $\int_0^1 x\,dx = \dfrac{1}{2}$. Now observe that the area under the curve $y = \sqrt{1-x^2}$ on $[0,1]$ is just a quarter circle with radius 1 which is half of the area of a regular circle. So $\int_0^1 \sqrt{1-x^2}\,dx = \dfrac{\pi(1)^2}{4} = \dfrac{\pi}{4}$.

   Therefore, $\int_0^1 \left(x + \sqrt{1-x^2}\right)\,dx = \dfrac{\pi+2}{4}$.

$\blacksquare$

5. Recall that if f is a continuous function on the closed interval $[a, b]$, then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*)\Delta x]$. So on the interval $[0, \pi]$, $\lim_{n\to\infty} \sum_{i=1}^{n} \sin(x_i)\Delta x = \int_0^\pi \sin(x)\,dx$. Now we need to evaluate this integral. Recall that the antiderivative of $\sin(x)$ is $-\cos(x)+C$, so $\int_0^\pi \sin(x)\,dx = [-\cos(x)]_0^\pi = -\cos(\pi) - [-\cos(0)] = 1 - (-1) = 2$.

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6. Suppose f is continuous and $\int_0^4 f(x)\,dx = 10$. We want to find the value of $\int_0^2 f(2x)\,dx$. To do so we will use a substitution.
Let $u = 2x$. This means if $x = 0, u = 0$ and if $x = 2, u = 4$ and $du = 2 \cdot dx \implies dx = \frac{du}{2}$. Now making the substituion, $\int_0^2 f(2x)\,dx = \int_0^4 f(u)\frac{du}{2} = \frac{1}{2}\int_0^4 f(u)\,du = \frac{1}{2} \cdot \int_0^4 f(x)\,dx = \frac{1}{2} \cdot 10 = 5$. Therefore, $\int_0^2 f(2x)\,dx = 5$.

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7. *Proof.* Suppose f is continuous on the closed interval $[0, \pi]$. We want to show that $\int_0^\pi x f(\sin(x))\,dx = \frac{\pi}{2}\int_0^\pi f(\sin(x))\,dx$. We will proceed via u substitution where we will let $u = \pi - x$. So if $x = 0, u = \pi$ and if $x = \pi, u = 0$ and $du = -dx \implies dx = -du$. Making this substitution we have,
$\int_0^\pi x f(\sin(x))\,dx = -\int_\pi^0 (\pi - u) f(\sin(\pi - u))\,du$
$\implies \int_0^\pi x f(\sin(x))\,dx = \int_0^\pi [\pi f(\sin(u)) - u f(\sin(u))]\,du$
since $\sin(\pi - u) = \sin(u)$ (Can be shown using compound angle formulas)
$\implies \int_0^\pi x f(\sin(x))\,dx = \pi\int_0^\pi f(\sin(u))\,du - \int_0^\pi u f(\sin(u))\,du$. Now notice that $\int_0^\pi x f(\sin(x))\,dx = \int_0^\pi u f(\sin(u))\,du$.
So now we have $\int_0^\pi x f(\sin(x))\,dx = \pi\int_0^\pi f(\sin(u))\,du - \int_0^\pi x f(\sin(x))\,dx$
$\implies 2\int_0^\pi x f(\sin(x))\,dx = \pi\int_0^\pi f(\sin(u))\,du$

After dividing both sides by 2 we have,

$\implies \int_0^\pi x f(\sin(x))\,dx = \frac{\pi}{2}\int_0^\pi f(\sin(u))\,du = \frac{\pi}{2}\int_0^\pi f(\sin(\pi - x))\,du = \frac{\pi}{2}\int_0^\pi f(\sin(x))\,dx$ as required. $\square$

8. Suppose $f(x) = \int_0^x x^2 \sin(t^2)\, dt$. We want to find $f'(x)$. So notice that x^2 is just a constant, so we can take it outside the integral.

$\implies f(x) = x^2 \int_0^x \sin(t^2)\, dt$. Now notice that we have two functions in terms of x. To see this better, let $g(x) = x^2$ and $h(x) = \int_0^x \sin(t^2)\, dt$. By the fundamental theorem of calculus part 1, $h'(x) = \sin^2(x)$.

Therefore, $f(x) = g(x)h(x)$. Now to find $f'(x)$, we need to apply the product rule.

So $f'(x) = g'(x)h(x) + h'(x)g(x) = 2x \int_0^x \sin(t^2)\, dt + x^2 \sin(x^2)$ as needed.

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9(a) *Proof.* Suppose f is a continuous even function on the closed interval $[-a, a]$. Recall that if f is even, then $f(x) = f(-x)$. We want to show that $\int_{-a}^a f(x)\, dx = 2\int_0^a f(x)\, dx$. So working with the left hand side, we will split the integral into two more integrals as follows,

$\int_{-a}^a f(x)\, dx = \int_{-a}^0 f(x)\, dx + \int_0^a f(x)\, dx$.

$\implies \int_{-a}^a f(x)\, dx = -\int_0^{-a} f(x)\, dx + \int_0^a f(x)\, dx$. We will now let $u = -x$ so that when $x = 0, u = 0$ and if $x = -a, u = a$ and $dx = -du$. Making the substitution we see that

$\int_{-a}^a f(x)\, dx = \int_0^a f(-u)\, du + \int_0^a f(x)\, dx$. Since f is even, $f(-u) = f(u)$ and so, $\int_{-a}^a f(x)\, dx = \int_0^a f(u)\, du + \int_0^a f(x)\, dx$. Note that $\int_0^a f(u)\, du = \int_0^a f(x)\, dx$.

$\implies \int_{-a}^a f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(x)\, dx = 2\int_0^a f(x)\, dx$ as required.

□

(b) *Proof.* Suppose f is a continuous odd function on the closed interval $[-a, a]$. Recall that a function is odd if $f(-x) = -f(x)$. We want to show that $\int_{-a}^{a} f(x)\,dx = 0$.

So let's separate the integral into two more integrals.

$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$. We will now let $u = -x$. So if $x = 0, u = 0$ and if $x = -a, u = a$ and $dx = -du$.

$\implies \int_{-a}^{a} f(x)\,dx = \int_{a}^{0} f(-u)\,(-du) + \int_{0}^{a} f(x)\,dx$.

$\implies \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx$.

Since f is odd, $f(-u) = -f(u)$.

$\implies \int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$ as required. $\square$

10. Let's define a function F such that $F(x) = \int_{3}^{x} \frac{\sin(t)}{t}\,dt$. Using the fundamental theorem of calculus part 1, $F'(x) = \frac{\sin(x)}{x}$. But let's also recall the limit definition of a derivative, $F'(a) = \lim\limits_{x\to a} \dfrac{F(x) - F(a)}{x - a}$. Letting $a = 3$,

$$F'(3) = \lim_{x\to 3} \frac{F(x) - F(3)}{x - 3} = \lim_{x\to 3} \frac{\int_{3}^{x} \frac{\sin(t)}{t}\,dt - \int_{3}^{3} \frac{\sin(t)}{x}\,dt}{x - 3} = \lim_{x\to 3} \frac{\int_{3}^{x} \frac{\sin(t)}{t}\,dt}{x - 3}.$$

This looks oddly familiar to our original question. So using limit laws,

$$\lim_{x\to 3} \left(\frac{x}{x-3} \int_{3}^{x} \frac{\sin(t)}{x}\,dt \right) = \lim_{x\to 3}(x) \cdot \lim_{x\to 3} \left(\frac{\int_{3}^{x} \frac{\sin(t)}{t}\,dt}{x - 3} \right) = 3 \cdot F'(3) =$$

$3 \cdot \dfrac{\sin(3)}{3} = \sin(3)$ as needed.

$\blacksquare$

11. *Proof.* Suppose the average value of f over $[a, b]$ is 1 and the average value of f over $[b, c]$ is 1 where $a < c < b$. We want to show that the average value of f over $[a, c]$ is also 1.

So since the average value of f over $[a, b]$ and $[b, c]$ is 1, we have the following equations,

$$\frac{1}{b-a}\int_a^b f(x)\,dx = 1 \text{ and } \frac{1}{c-b}\int_b^c f(x)\,dx = 1.$$

Recall that $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for $c \in [a, b]$. Using this property we have, $\frac{1}{b-a}\left[\int_a^c f(x)\,dx + \int_c^b f(x)\,dx\right] = 1$.

$\implies b - a = \int_a^c f(x)\,dx - \int_b^c f(x)\,dx$

$\implies b - a = \int_a^c f(x)\,dx - (c - b)$

$\implies b - a + c - b = \int_a^c f(x)\,dx$

$\implies c - a = \int_a^c f(x)\,dx \implies 1 = \frac{1}{c-a}\int_a^c f(x)\,dx$ as required. $\square$

12. Using the Fundamental Theorem of Calculus Part 2, $\int_{-3}^{1} f'(x)\,dx = f(1) - f(-3) = 12$.

$\implies f(-3) = f(1) - 12$.

Since $f(1) = 10$ we have, $f(-3) = 10 - 12 = -2$.

Hence, $f(-3) = -2$.