1.6 Integrals of Exponential and Logarithmic Functions

Rule: Integrals of Exponential Functions

1. $\int e^x \, dx = e^x + C$
2. $\int a^x \, dx = \dfrac{a^x}{ln(a)} + C$

Example: Evaluate $\int_0^1 \dfrac{e^z + 1}{e^z + z} \, dz$

Solution.

Let $u = e^z + z$. Then $du = (e^z + 1)dx$. When $z = 0, u = 1$ and when $z = 1, u = e+1$. So $\int_0^1 \dfrac{e^z + 1}{e^z + z} \, dz = \int_1^{e+1} \dfrac{e^z + 1}{u} \dfrac{du}{e^z + 1} = \int_0^{e+1} \dfrac{1}{u} \, du = ln(e+1)$.

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Practice Questions Part 1

1. An oil storage tank ruptures at time $t = 0$ and oil leaks from the tank at a rate of $r(t) = 100e^{-0.01t}$ litres per minute. How much oil leaks out during the first hour?
2. A bacteria population starts with 400 bacteria and grows at a rate of $r(t) = 450e^{1.5t}$ bacteria per hours. How many bacteria will be there after three hours?
3. Evaluate:

(a) $\int_0^1 \dfrac{d}{dx} \left(e^{\arctan(x)}\right) \, dx$

(b) $\dfrac{d}{dx} \int_0^1 e^{\arctan(x)} \, dt$

(c) $\dfrac{d}{dx} \int_0^x e^{\arctan(t)} \, dt$

4. Find $\int \dfrac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$

Solutions Part 1

1. To find how much oil leaks out during the first hour, we must evaluate $\int_0^{60} r(t)\,dt = \int_0^{60} 100e^{-0.01t}\,dt$.
   $\implies \int_0^{60} 100e^{-0.01t}\,dt = \left.\dfrac{-100e^{-0.01t}}{0.01}\right]_0^{60} \approx 4511.88$ litres.

2. Let $f'(t) = r(t)$. Since the population of bacteria start at 400, $f(0) = 400$. We want to find how many bacteria there will be after three hours. To do so, we must evaluate $\int_0^3 f'(t)\,dt = \int_0^3 450e^{1.5t}\,dt$
   $\implies \int_0^3 450e^{1.5t}\,dt = \left.\dfrac{450e^{1.5t}}{1.5}\right]_0^3 \approx 26705$ bacteria.

3. (a) Let $f(x) = e^{\arctan(x)}$. Then, $\int_0^1 \dfrac{d}{dx}[f(x)]\,dx = \int_0^1 f'(x)\,dx$. By the fundamental theorem of calculus,
   $\int_0^1 f'(x)\,dx = f(1) - f(0) = e^{\arctan(1)} - e^{\arctan(0)} = e^{\pi/4} - 1$

   (b) $\dfrac{d}{dx}\int_0^1 e^{\arctan(x)}\,dx = 0$ since the derivative of a constant is always zero.

   (c) By the fundamental theorem of calculus part 1, $\dfrac{d}{dx}\int_0^x e^{\arctan(x)} = e^{\arctan(x)}$.

4. Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}} \cdot dx \implies \dfrac{du}{2} = \dfrac{dx}{\sqrt{x}}$.
   So $\int \dfrac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \dfrac{1}{2}\int e^u\,du = \dfrac{e^u}{2} + C = \dfrac{e^{\sqrt{u}}}{2} + C$.

Integrals Involving Logarithmic Functions

Rule: Integrals of Logarithmic Functions

1. $\int \frac{1}{x}\,dx = \ln|x| + C$
2. $\int \ln(x)\,dx = x(\ln(x) - 1) + C$
3. $\int \log_a(x)\,dx = \frac{x}{\ln(a)}(\ln(x) - 1) + C$

Example: Evaluate $\int \frac{ln(x)}{x}\,dx$

Solution.

Let $u = ln(x)$. Then $du = \frac{dx}{x}$. So $int \frac{ln(x)}{x}\,dx = \int u\,du = \frac{u^2}{2} = \frac{\ln^2(x)}{2} + C$.

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Practice Questions

1. Evaluate $\displaystyle\int \frac{\ln(\sin(x))}{\tan(x)}\,dx$
2. Evaluate $\displaystyle\int e^{\cos(x)} \sin(x)\,dx$
3. Evaluate $\displaystyle\int \frac{dx}{x \ln(x) \ln(\ln(x))}$
4. Use the identity $\ln(x) = \displaystyle\int_1^x \frac{dt}{t}$ to derive the identity $\ln(\frac{1}{x}) = -\ln(x)$
5. Use a change in variable in the integral $\displaystyle\int_1^{xy} \frac{1}{t}\,dt$ to show that $\ln(xy) = \ln(x) + \ln(y)$ for $x, y > 0$.

Solutions Part 2

1. We can rewrite the integral as follows,

$$\int \ln(\sin(x)) \cot(x)\, dx.$$

Now we let $u = \ln(\sin(x))$. Then $du = \cot(x)dx$. Thus we have, $\int u\, du = \frac{u^2}{2} + C$. Changing everything back in terms of x, $\int \frac{\ln(\sin(x))}{\tan(x)}\, dx = \frac{\left[\ln(\sin(x))^2\right]}{2} + C$.

2. Let $u = \cos(x)$. Then $du = -\sin(x)dx$. So making the substitution we have the following integral,

$$\int e^{\cos(x)} \sin(x)\, dx = \int e^u \sin(x) \frac{-du}{\sin(x)}$$

$$\implies -\int e^u\, dx = -e^u + C = -e^{\cos(x)} + C.$$

3. Let $u = \ln(\ln(x))$. Then $du = \frac{dx}{x\ln(x)}$.

So our integral becomes, $\int \frac{1}{u} \cdot \frac{dx}{x\ln(x)} = \int \frac{1}{u}\, du = \ln|u| + C = \ln|\ln(\ln(x))| + C$ as required.

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4. By definition $\ln\left(\frac{1}{x}\right) = \int_1^{\frac{1}{x}} \frac{1}{t}\, dt$.

Let $u = \frac{1}{t}$. Then $t = \frac{1}{u} \implies dt = \frac{-1}{u^2}\, du$.

When $t = 1, u = 1$ and when $t = \frac{1}{x}, u = x$. Thus we have, $\ln\left(\frac{1}{x}\right) = \int_1^x u \cdot \frac{-1}{u^2}\, du = -\int_1^x \frac{1}{u}\, du = -\ln(x)$ as required.

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5. By definition $\ln(xy) = \int_1^{xy} \frac{1}{t}\, dt$. Now let $u = \frac{t}{x}$. Then $t = xu \implies dt = x \cdot du$. So when $t = 1, u = \frac{1}{x}$ and when $t = xy, u = y$. Thus we have the following integral,

$\ln(xy) = \int_{\frac{1}{x}}^{y} \frac{1}{xu}\, x \cdot du = \int_{\frac{1}{x}}^{y} \frac{1}{u}\, du$. And so we have $\ln(xy) = \ln(y) - \ln\left(\frac{1}{x}\right) = \ln(xy) = \ln(y) + \ln(x)$ since $\ln\left(\frac{1}{x}\right) = -\ln(x)$ from the previous question.

Therefore we have $\ln(xy) = \ln(y) + \ln(x)$ as required.

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