2.1 Areas Between Curves

Area of a Region Between Two Curves

In the previous sections we were able to calculate the area under a singular curve by using Riemann Sums initially then moving towards integrals. But what if we have two curves? How can we find the area between two curves?

The process is quite similar. Let $f(x)$ and $g(x)$ be continuous functions over the closed interval $[a, b]$ such that $f(x) \geq g(x)$ on $[a, b]$. We then want to find the area between the two functions.

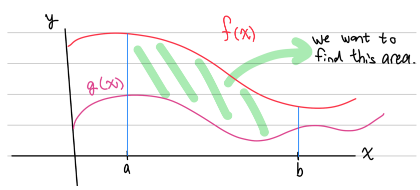

Figure 1: Area between $f(x)$ and $g(x)$ over $[a, b]$

To find the area shown in Figure 1, as before, we divide the interval $[a, b]$ into n sub-intervals of equal width and then we approximate the ith strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$.

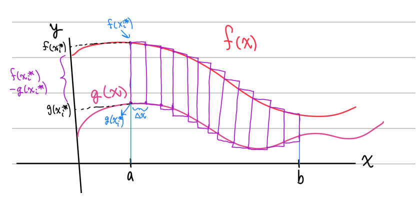

Figure 2: Caption

Adding up all the areas of the rectangles, we form a Riemann Sum:

$$A \approx \sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*)\right] \Delta x.$$

This approximation appears to become better and better as $n \to \infty$.

Therefore, the area between two curves is as follows,

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*)\right] \Delta x = \int_a^b \left[f(x) - g(x)\right]\, dx.$$

Areas of Compound Regions

So far we have only covered the case where $f(x) \geq g(x)$ over the entire interval. But that's not always the case. It's possible to have one function cross the other function. For instance, for some $c \in [a, b]$, we could have that $f(x) \geq g(x)$ over $[a, c]$ but on $[c, b]$ we can have that $g(x) \geq f(x)$. In that case, we modify the process we developed and introduce the absolute value function.

Theorem: Finding the Area of a Region between Curves That Cross

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$. Let R denote the region between the graphs $f(x)$ and $g(x)$, and be bounded on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by,

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

Regions Defined with Respect to y

Notice that we've been finding areas between two curves along the x-axis. But we can also find the area between two curves along the y-axis.

Theorem: Finding Area Between Two Curves along the y-axis

Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geq v(y)$ for all $y \in [c, d]$. Let R denote the region bounded on the right by the graph $u(y)$, on the left by the graph of $v(y)$, and above and below by the lines $y = d$ and $y = c$ respectively. Then the area of R is given by,

$$A = \int_c^d [u(y) - v(y)]\, dy.$$

Exercises

1. Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

2. Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

3. Find the area of the region enclosed by the functions $x^2 = y^3$ and $x = 3y$.

4. The curve $y^2 = x^2(x + 3)$ is called **Tschirn-hausen's cubic**. If you graph this curve you will see that part of the curve forms a loop. Find the area enclosed by the loop.

5. Find the number of b such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area.

Solutions

1. We first find where the two functions intersect so we can get our bounds of integration. Thus, setting $x^2 = 2x - x^2$, we see that $x = 0, 1$. So we integrate on the interval $[0, 1]$. Now to determine which function is larger, we take any point in $[0, 1]$ and evaluate that point in each function. So take $x = \frac{1}{2}$. Then $y = x^2 = \frac{1}{4}$ and $y = 2x - x^2 = \frac{3}{4}$. Thus, $2x - x^2 \geq x^2$ for all $x \in [0, 1]$.
   Therefore, $A = \displaystyle\int_0^1 (2x - 2x^2)\,dx$.
   $\implies A = \dfrac{1}{3}$.

   $\blacksquare$

2. We will want to integrate along the y-axis, so we solve these curves in terms of x. So if $y = x - 1$, then $x = y + 1$ and if $y^2 = 2x + 6$, then $x = \dfrac{y^2}{2} - 3$. Now we find points of intersection.
   So set $y + 1 = \dfrac{y^2}{2} - 3 \implies y = -2, 4$. Thus, we integrate on the interval $-2 \leq y \leq 4$. Taking $y = 0 \in [-2, 4]$ we see that the curve $x = y + 1$ is further to the right than $x = \dfrac{y^2}{2} - 3$. Hence we evaluate the following integral,
   $$A = \int_{-2}^{4} \left(y + 1 - \frac{y^2}{2} + 3 \right) dy.$$
   $\implies A = 18.$

   $\blacksquare$

3. We will integrate along the x-axis. So since $x^2 = y^3, y = \sqrt[3]{x^2}$ and since $x = 3y, y = \dfrac{x}{3}$. Finding the points of intersection, we set $\sqrt[3]{x^2} = \dfrac{x}{3}$.
   After some algebra, $x = 0, 27$. So we integrate on the interval $[0, 27]$.
   Now taking $x = 1 \in [0, 27]$, we see that $\sqrt[3]{x^2} \geq \dfrac{x}{3}$.
   Thus we evaluate the following integral,

$$A = \int_0^{27} \left(x^{2/3} - \frac{x}{3}\right) \, dx$$

$\implies A = \dfrac{243}{10}$

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4. Taking the positive half of the curve, $y = \sqrt{x^2(x+3)}$. We can find the x-intercepts, so setting $y = 0$, we see that $x = 0, -3$. Thus, we integrate on the interval $[-3, 0]$. It is obvious that $\sqrt{x^2(x+3)} \geq -\sqrt{x^2(x+3)}$. So $A = \int_{-3}^{0} \left[(\sqrt{x^2(x+3)} - (-\sqrt{x^2(x+3)})\right] \, dx = 2\int_{-3}^{0} \sqrt{x^2(x+3)} \, dx$.
$\implies A = 2\int_{-3}^{0} \sqrt{x^2} \cdot \sqrt{x+3} \, dx$.
Now recall that $\sqrt{x^2} = |x|$ and since we are on the interval $[-3, 0]$, $\sqrt{x^2} = |x| = -x$. Thus, $A = -2\int_{-3}^{0} x\sqrt{x+3} \, dx$.
Now let $u = x + 3$. Then $du = dx, x = u - 3$ and when $x = -3, u = 0$ and when $x = 3, u = 3$.
Therefore, $A = -2\int_0^3 (u-3)x^{1/2} \, du$.
$\implies A = \dfrac{20\sqrt{27} - 4\sqrt{243}}{5}$

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5. First, let's evaluate the area between $y = x^2$ and $y = 4$. Setting $x^2 = 4$, we see that we will integrate on the interval $[-2, 2]$. Now we evaluate the following integral,
$$A = \int_{-2}^{2} 4 - x^2 \, dx.$$
Notice that $y = 4 - x^2$ is an even function. Thus, $A = 2\int_0^2 4 - x^2 \, dx$.
$\implies A = \dfrac{32}{3}$.
Now we want to find such number b such that the line $y = b$ divides the area into two equal regions. So when $x^2 = b, x = \pm\sqrt{b}$.

Now we set $\int_{-\sqrt{b}}^{\sqrt{b}}(b - x^2) = \frac{32}{3} \cdot \frac{1}{2} = \frac{16}{3}$. Since $b - x^2$ is an even function, $\frac{16}{3} = 2\int_0^{\sqrt{b}} (b - x^2)\, dx$. After evaluating this integral, we get $b = 4^{2/3} = \sqrt[3]{16}$.

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