1.5 Substitution

Due to the Fundamental Theorem of Calculus, we can evaluate integrals by finding antiderivatives which is a huge improvement over using Riemann Sums. For example, $\int x^2\,dx = \dfrac{x^3}{3} + C$ by the reverse power rule.
But what if we had to evaluate $\int 2x(x^2-3)^3\,dx$? We could expand $(x^2-3)^3$ but what if the exponent was larger, say 50 or 100? We certainly don't want to expand $(x^2-3)^{50}$. In order to solve more complex integrals, we can use integration by substitution. This technique is where we let some variable, most commonly u, equal to some function in the integrand, $u = g(x)$. We then differentiate u with respect to x and substitute it into the integral and hope that some other factor of $g'(x)$ is also in the integral to cancel out and make the integral easier to compute.

Theorem 1.7: Substitution with indefinite Integrals

Let $u = g(x)$ where $g'(x)$ is continuous over an interval and let $f(x)$ be continuous over the corresponding range of g. Let $F(x)$ be the antiderivative of f. Then,

$$\int f[g(x)]g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C.$$

Now returning to our original problem $\int 2x(x^2-3)^3\,dx$, with Theorem 1.7 in mind. Let $u = x^2 - 3$. Then, $du = 2xdx$. Thus our integral becomes, $\int 2x(u)^3\,dx = \int u^3 2xdx = \int u^3 du$. Now using the reverse power rule, $\int u^3\,du = \dfrac{u^4}{4} + C = \dfrac{(x^2-3)^4}{4} + C$.

Problem Solving Strategy

1. Look carefully at the integrand and select an expression $u = g(x)$ such that $\frac{du}{dx} = g'(x)$ is also in the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)dx$ into the integral
3. We should now be able to evaluate the integral in terms of u
4. Rewrite the result in terms of x

Practice Questions Part 1

1. Find $\int x^3 \cos(x^4 + 2)\, dx$
2. Find $\int \frac{x}{\sqrt{1 - 4x^2}}\, dx$
3. Find $\int \sqrt{1 + x^2} \cdot x^5\, dx$
4. Calculate $\int \tan(x)\, dx$

Substitution for definite Integrals

Now that we know how to deal with more complicated integrals, what if we wanted to evaluate the area on the interval $[a, b]$? Such as, $\int_0^4 \sqrt{2x+1}\, x$.

Theorem 1.8: Substitution with definite Integrals Let $u = g(x)$ and let $g'(x)$ be continuous over an interval $[a, b]$, and let f be continuous over the range of $u = g(x)$ and let F be the antiderivative of f. Then,

$$\int_a^b f[g(x)]g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du = [F(g(x))]_a^b = F(g(b)) - F(g(a)).$$

Example: Evaluate $\int_0^4 \sqrt{2x+1}\, dx$.

Solution.

Let $u = 2x + 1$. Then $du = 2dx$. When $x = 0, u = 1$ and when $x = 4, u = 9$. So, $\int_0^4 \sqrt{2x+1}\, dx = \dfrac{1}{2} \int_1^9 u^{\frac{1}{2}}\, du = \dfrac{1}{2} \left[u^{\frac{3}{2}} \cdot \dfrac{2}{3} \right]_1^9 = \dfrac{26}{3}$.

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Notice that when evaluating definite integrals via substitution, we don't return to the variable x are inetgraing. We simply evaluate the definite integral in terms of u.

Practice Questions Part 2

1. Find $\int \dfrac{a+bx^2}{\sqrt{3ax+bx^3}}\,dx$

2. Evaluate $\int_0^{\frac{\pi}{2}} \cos(x)\sin(\sin(x))\,dx$

3. If f is continuous on $\mathbb{R}$, prove that,

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx$$

4. If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 xf(x^2)\,dx$

5. If $f(x) = g(h(x))$, when reversing the chain rule, $\dfrac{d}{dx}\left[g(h(x))\right] = g'(h(x))h'(x)$, should you take $u = g(x)$ or $u = h(x)$?

6. If $h(a) = h(b)$ in $\int_a^b g'(h(x))h'(x)\,dx$, what can you say about the integral?

7. Show that the average value of $f(x)$ over $[a,b]$ is the same average value of $f(cx)$ over $\left[\dfrac{a}{c}, \dfrac{b}{c}\right]$ for $c > 0$.

8. Find the area under the graph of $f(t) = \dfrac{t}{(1+t^2)^a}$ between $t = 0$ and $t = x$ where $a > 0$ and $a \neq 1$ is fixed, and evaluate the limit as $x \to \infty$

Solutions for Questions Part 1

1. Let $u = x^4 + 2$. Then $du = 4x^3 dx$.

   $\implies \int x^3 \cos(x^4 + 2)\, dx = \int x^3 \cos(u)\, \frac{du}{4x^3} = \frac{1}{4} \int \cos(u)\, du = \frac{\sin(u)}{4} = \frac{\sin(x^4+2)}{4} + C$

2. Let $u = 1 - 4x^2$. Then $du = -8x dx$.

   $\implies \int \frac{x}{\sqrt{1-4x^2}}\, dx = \int \frac{x}{\sqrt{u}}\, \frac{du}{-8x} = \frac{-1}{8} \int u^{-2}\, du = \frac{-1}{8} \cdot \frac{-1}{u} = \frac{1}{8(1-4x^2)} + C$

3. We can rewrite the integral as follows,

   $$\int \sqrt{1+x^2} \cdot x^5\, dx = \int \sqrt{1+x^2} \cdot x \cdot x^4\, dx.$$

   Now let $u = 1 + x^2$. Then $du = 2x dx$. Making the substitution, we now have, $\int \sqrt{u} \cdot x \cdot x^4 \frac{du}{2x} = \frac{1}{2} \int \sqrt{u}(u^2 - 2u + 1)\, du$ since $x^2 = u - 1 \implies x^4 = (u-1)^2$.

   $\implies \frac{1}{2} \int \sqrt{u}(u^2 - 2u + 1)\, du = \frac{1}{2} \int u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}} = \frac{1}{2}\left[\frac{2u^{\frac{7}{2}}}{7} - \frac{4u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3}\right] + C$

4. $\int \tan(x)\, dx = \int \frac{\sin(x)}{\cos(x)}\, dx$. Now let $u = \cos(x)$. Then $du = -\sin(x) dx$.

   $\implies \int \frac{\sin(x)}{\cos(x)}\, dx = \int \frac{\sin(x)}{u} \frac{du}{-\sin(x)} = -\int \frac{1}{u}\, du = -\ln|\cos(x)| + C$

Solutions for Questions Part 2

1. Let $u = 3ax + bx^3$. Then $du = 3(a + bx^2)dx$.
   $\implies \int \frac{a + bx^2}{\sqrt{3ax + bx^3}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{3}(2\sqrt{u}) = \frac{2\sqrt{3ax + bx^3}}{3} + C$

2. Let $u = \sin(x)$. Then $du = \cos(x)dx$. Observe that when $x = 0, u = 0$ and when $x = \frac{\pi}{2}, u = 1$.
   $\implies \int_0^{\frac{\pi}{2}} \cos(x)\sin(\sin(x))\,dx = \int_0^1 \cos(x)\sin(u)\,\frac{du}{\cos(x)} = \int_0^1 \sin(u)\,du =$ $\sin(1)$

3. *Proof.* Suppose f is continuous on $\mathbb{R}$.
   We want to show that $\int_a^b f(x + c)\,dx = \int_{a+c}^{b+c} f(x)\,dx$. So let $u = x + c$. Then $du = dx$. And so our integral becomes, $\int_{a+c}^{b+c} f(u)\,du = \int_{a+c}^{b+c} f(x)\,dx$ as needed. $\square$

4. Suppose f is continuous and $\int_0^9 f(x)\,dx = 4$. We want to find $\int_0^3 xf(x^2)\,dx$. We will do so via integration by substitution. So let $u = x^2$. Then $du = 2xdx$ and when $x = 0, u = 0$ and when $x = 3, u = 9$. Thus we have, $\frac{1}{2}\int_0^9 xf(u)\,\frac{du}{x} = \frac{1}{2}\int_0^9 f(u)\,dx = \frac{1}{2}\cdot 4 = 2$.
   Therefore, $\int_0^3 xf(x^2)\,dx = 2$.

5. We should let $u = h(x)$. Then $du = h'(x)dx \implies dx = \frac{du}{h'(x)}$ and so our integral becomes, $\int g'(h(x))h'(x)\,dx = \int g'(u)\,du$. Now by the fundamental theorem of calculus, $\int g'(u)\,du = g(u) = g(h(x)) = f(x)$.

6. Let $u = h(x)$. Then $du = h'(x)dx$ and when $x = a, u = h(a)$ and when $x = b, u = h(b)$ and so we have $\int_a^b g'(h(x))h'(x)\,dx = \int_{h(a)}^{h(b)} g'(u)\,du = \int_{h(a)}^{h(a)} g'(u)\,du$ since $h(a) = h(b)$ which evaluates to 0. Therefore, if $h(a) = h(b), \int_a^b g'(h(x))h'(x)\,dx = 0$.

7. *Proof.* Recall that if f is continuous on the closed interval $[a, b]$, then the average value of the function, f_{ave} on $[a, b]$ is given by $f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$. So now we want to show that $\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\frac{b}{c}-\frac{a}{c}}\int_{a/c}^{b/c} f(cx)\,dx$ for $c > 0$. So working with the right hand side we have $c\cdot\frac{1}{b-a}\int_{a/c}^{b/c} f(cx)\,dx$. Now let $u = cx$. Then $du = c\,dx$ and our bounds change to $u = a$ to $u = b$. So now our integral is as follows,

$c \cdot \frac{1}{b-a} \cdot \frac{1}{c}\int_a^b f(u)\,du = \frac{1}{b-a}\int_a^b f(u)\,du = \frac{1}{b-a}\int_a^b f(x)\,dx = f_{ave}$ as required. $\square$

8. To find the area under the graph of $f(t) = \frac{t}{(1+t^2)^a}$ between $t = 0$ and $t = x$, we need to evaluate the following integral,

$\int_0^x \frac{t}{(1+t^2)^a}\,dt$ So let $u = 1 + t^2$. Then $du = 2t\,dt$ and our bounds will change from $u = 1$ to $u = 1 + x^2$. Thus our integral becomes,

$\int_1^{1+x^2} \frac{t}{u^a}\frac{du}{2t} = \frac{1}{2}\int_1^{1+x^2} u^{-a}\,du$. Since $a \neq -1$, $\frac{1}{2}\left[\frac{u^{-a+1}}{-a+1}\right]_1^{1+x^2} =$

$\frac{1}{2}\left[\frac{(1+x^2)^{-a+1}}{-a+1} - \frac{1}{-a+1}\right] = \frac{1}{2}\left[\frac{(1+x^2)^{-a+1}-1}{-a+1}\right] = \frac{1}{2}\left[\frac{(1+x^2)^{-(a-1)}}{-a+1} + \frac{1}{a-1}\right]$.

Now we want to take $\frac{1}{2}\lim_{x\to\infty}\left[\frac{(1+x^2)^{-(a-1)}}{-a+1} + \frac{1}{a-1}\right]$. Using limit laws we have,

$\frac{1}{2}\left(\lim_{x\to\infty}\left[\frac{1}{(-a+1)(1+x^2)^{a-1}}\right] + \lim_{x\to\infty}\left[\frac{1}{a-1}\right]\right) = \frac{1}{2}\left(\lim_{x\to\infty}\left[\frac{1}{(-a+1)(1+x^2)^{a-1}}\right] + \frac{1}{a-1}\right) = \frac{1}{2(a-1)}$.

Thus, $\lim_{x\to\infty}\int_0^x \frac{t}{(1+t^2)^a} = \frac{1}{2(a-1)}$ as required.

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