# Notes on Abstract Space

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These are notes on abstract spaces, mainly based on the book *An Introduction to Abstract Spaces* by Hu Shigeng and Zhang Xianwen.

# 1 Basic Concepts

## 1.1 Sets and Relations

We denote by $2^X$ the power set of $X$, i.e., the set of all subsets of $X$. Some related notation is as follows. For any $\mathcal{A} \subset 2^X$, we set

$$\bigcup \mathcal{A} = \bigcup \{A : A \in \mathcal{A}\},$$

and

$$\bigcap \mathcal{A} = \bigcap \{A : A \in \mathcal{A}\}.$$

In addition, we define

$$\mathcal{A}^* = \{\bigcap \mathcal{B} : \mathcal{B} \subset \mathcal{A} \text{ and } |\mathcal{B}| \text{ is finite}\}.$$

If $\mathcal{A}, \mathcal{B} \subset 2^X$, we write

$$\mathcal{A} \vdash \mathcal{B} \Leftrightarrow \forall B \in \mathcal{B},\ \exists A \in \mathcal{A} :\ A \subset B,$$

and

$$\mathcal{A} \prec \mathcal{B} \Leftrightarrow \forall A \in \mathcal{A},\ \exists B \in \mathcal{B} :\ A \subset B.$$

⟨def1.1⟩ **Definition 1.1.** If $\mathcal{A} \subset 2^X$ is nonempty and satisfies

(a) $\emptyset \notin \mathcal{A}$;

(b) For any $A_1, A_2 \in \mathcal{A}$, $A_1 \cap A_2 \in \mathcal{A}$;

(c) If $\mathcal{A} \vdash A \subset X$, then $A \in \mathcal{A}$,

then we call $\mathcal{A}$ a **filter** on $X$. Moreover, if $\mathcal{B} \subset \mathcal{A}$ satisfies $\mathcal{B} \vdash \mathcal{A}$, then we say that $\mathcal{B}$ is a **base** of the filter $\mathcal{A}$.

It's easy to see that condition (b) in Definition 1.1 can be replaced by $\mathcal{A} = \mathcal{A}^*$.

Any filter has at least one base, for example, itself. Conversely, any nonempty $\mathcal{B} \subset 2^X$ satisfying suitable conditions is a base of a filter.

**Theorem 1.1.** *If a nonempty $\mathcal{B} \subset 2^X$ satisfies*

(a) $\emptyset \notin \mathcal{B}$*;*

(b) *For any $B_1, B_2 \in \mathcal{B}$, we have $\mathcal{B} \vdash B_1 \cap B_2$,*

*then $\mathcal{B}$ is a base of a filter $\mathcal{A} \subset 2^X$.*

**Proof.** Let

$$\mathcal{A} = \{A \subset X : \mathcal{B} \vdash A\}.$$

It is straightforward to verify that $\mathcal{A}$ is a filter and that $\mathcal{B}$ is a base of $\mathcal{A}$. □

Next, we discuss relations between two nonempty sets. For any two nonempty sets $X$ and $Y$, we consider elements $x \in X$ and $y \in Y$ as abstract variables, and define how $x$ is related to $y$.

?⟨def1.3⟩?

**Definition 1.2.** Let $X$ and $Y$ be two nonempty sets. Any subset $F \subset X \times Y$ is called a **relation** from $X$ to $Y$. When $(x, y) \in F$, we say that $x$ is related to $y$ by the relation $F$, denoted by $xFy$ or $y \in Fx$. If $F \subset X \times X$, we call it a **binary relation** on $X$.

For any relation $F$ from $X$ to $Y$ and any subset $A \subset X$, define

$$F(A) = \{y \in Y : \exists x \in A,\ (x, y) \in F\}.$$

Let $Fx = F\{x\}$. Then $F(A) = \bigcup_{x \in A} Fx$. If $G \subset Y \times Z$ is another relation, the composition $G \circ F$ is defined by

$$G \circ F = \{(x, z) \in X \times Z : \exists y \in Y,\ (x, y) \in F \text{ and } (y, z) \in G\}.$$

The inverse relation of $F$ is

$$F^{-1} = \{(y, x) \in Y \times X : (x, y) \in F\}.$$

Let us consider any relation $F \subset X \times X$. If $F^{-1} = F$, we say that $F$ is **symmetric**. If $F \circ F \subset F$, we say that $F$ is **transitive**. If the diagonal $\{(x,x) : x \in X\}$ of $X \times X$ is contained in $F$, we say that $F$ is **reflexive**. If $F$ is symmetric, transitive, and reflexive, we say that $F$ is an **equivalence relation** on $X$.

Fix a relation $F \subset X \times Y$. We can view $F$ as the correspondence

$$F : X \to 2^Y, \quad x \mapsto Fx.$$

Thus $F$ is a set-valued function. In applications, we often consider the special case that for each $x \in X$, the set $Fx$ contains exactly one element. In that case we say $F$ is a **function** from $X$ to $Y$.