Notes on Abstract Space

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These are notes on abstract spaces, mainly based on the book *An Introduction to Abstract Spaces* by Hu Shigeng and Zhang Xianwen.

1 Basic Concepts

1.1 Sets and Relations

We denote by 2^X the power set of X, i.e., the set of all subsets of X. Some related notation is as follows. For any $\mathcal{A} \subset 2^X$, we set

$$\bigcup \mathcal{A} = \bigcup \{A : A \in \mathcal{A}\},$$

and

$$\bigcap \mathcal{A} = \bigcap \{A : A \in \mathcal{A}\}.$$

In addition, we define

$$\mathcal{A}^* = \{\bigcap \mathcal{B} : \mathcal{B} \subset \mathcal{A} \text{ and } |\mathcal{B}| \text{ is finite}\}.$$

If $\mathcal{A}, \mathcal{B} \subset 2^X$, we write

$$\mathcal{A} \vdash \mathcal{B} \iff \forall B \in \mathcal{B},\ \exists A \in \mathcal{A} :\ A \subset B,$$

and

$$\mathcal{A} \prec \mathcal{B} \iff \forall A \in \mathcal{A},\ \exists B \in \mathcal{B} :\ A \subset B.$$

?⟨def1.1⟩? **Definition 1.1.** If $\mathcal{A} \subset 2^X$ is nonempty and satisfies

(a) $\emptyset \notin \mathcal{A}$;

(b) For any $A_1, A_2 \in \mathcal{A}$, $A_1 \cap A_2 \in \mathcal{A}$;

(c) If $\mathcal{A} \vdash A \subset X$, then $A \in \mathcal{A}$,

then we call $\mathcal{A}$ a **filter** on X. Moreover, if $\mathcal{B} \subset \mathcal{A}$ satisfies $\mathcal{B} \vdash \mathcal{A}$, then we say that $\mathcal{B}$ is a **base** of the filter $\mathcal{A}$.

It's easy to see that condition (b) in Definition **??** can be replaced by $\mathcal{A} = \mathcal{A}^*$.

Any filter has at least one base, for example, itself. Conversely, any nonempty $\mathcal{B} \subset 2^X$ satisfying suitable conditions is a base of a filter.

Theorem 1.1. *If a nonempty $\mathcal{B} \subset 2^X$ satisfies*

(a) $\emptyset \notin \mathcal{B}$*;*

(b) *For any $B_1, B_2 \in \mathcal{B}$, we have $\mathcal{B} \vdash B_1 \cap B_2$,*

then $\mathcal{B}$ is a base of a filter $\mathcal{A} \subset 2^X$.

Proof. Let

$$\mathcal{A} = \{A \subset X : \mathcal{B} \vdash A\}.$$

It is straightforward to verify that $\mathcal{A}$ is a filter and that $\mathcal{B}$ is a base of $\mathcal{A}$. $\square$

Next, we discuss relations between two nonempty sets. For any two nonempty sets X and Y, we consider elements $x \in X$ and $y \in Y$ as abstract variables, and define how x is related to y.

?⟨def1.3⟩?

Definition 1.2. Let X and Y be two nonempty sets. Any subset $F \subset X \times Y$ is called a **relation** from X to Y. When $(x, y) \in F$, we say that x is related to y by the relation F, denoted by xFy or $y \in Fx$. If $F \subset X \times X$, we call it a **binary relation** on X.

For any relation F from X to Y and any subset $A \subset X$, define

$$F(A) = \{y \in Y : \exists x \in A,\ (x, y) \in F\}.$$

Let $Fx = F\{x\}$. Then $F(A) = \bigcup_{x \in A} Fx$. If $G \subset Y \times Z$ is another relation, the composition $G \circ F$ is defined by

$$G \circ F = \{(x, z) \in X \times Z : \exists y \in Y,\ (x, y) \in F \text{ and } (y, z) \in G\}.$$

The inverse relation of F is

$$F^{-1} = \{(y, x) \in Y \times X : (x, y) \in F\}.$$

Let us consider any relation $F \subset X \times X$. If $F^{-1} = F$, we say that F is **symmetric**. If $F \circ F \subset F$, we say that F is **transitive**. If the diagonal $\{(x,x) : x \in X\}$ of $X \times X$ is contained in F, we say that F is **reflexive**. If F is symmetric, transitive, and reflexive, we say that F is an **equivalence relation** on X.

Fix a relation $F \subset X \times Y$. We can view F as the correspondence

$$F : X \to 2^Y, \quad x \mapsto Fx.$$

Thus F is a set-valued function. In applications, we often consider the special case that for each $x \in X$, the set Fx contains exactly one element. In that case we say F is a **function** from X to Y.

2 Abstract Space

2.1 Uniform Space

The definition of uniform space is similar to that of topological space. The difference is that in a uniform space, the neighborhoods of points are defined by relations instead of subsets.

?⟨def2.1⟩? **Definition 2.1.** Let X be a nonempty set. A nonempty family $\mathcal{U}$ of relations on X is called a **uniform structure** on X if it satisfies the following conditions:

(a) For any $U \in \mathcal{U}$, $\Delta_X \subset U$, where $\Delta_X = \{(x,x) : x \in X\}$ is the diagonal of $X \times X$;

(b) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;

(c) For any $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$;

(d) $\mathcal{U}$ is a filter on $X \times X$.

The space X or the pair $(X,\mathcal{U})$ is called a **uniform space**.

For any base of the filter $\mathcal{U}$, we call it a **base** of the uniform structure $(X,\mathcal{U})$. For any $U \in \mathcal{U}$, $U \cap U^{-1} = (U \cap U^{-1})^{-1} \in \mathcal{U}$. Moreover, we have an element $V \in \mathcal{U}$ such that $V \circ V \subset U$. By the fact that $V = V \circ \Delta \subset V \circ V$, we have $V \circ V \in \mathcal{U}$. Thus the set $\{U \cap U^{-1} : U \in \mathcal{U}\}$ or $\{V \circ V : V \in \mathcal{U}\}$ forms a base of $\mathcal{U}$.

The condition (b) in Definition (**??**) means that for any $x, y \in X$,

$$x = y \iff (x,y) \in U \text{ for all } U \in \mathcal{U}.$$

We can say that the more $U \in \mathcal{U}$ "small", the closer x and y are. The uniform structure $\mathcal{U}$ provides a set of criteria that measure the closeness between two points in the space X. Those facts naturally lead to the definition of uniform neighborhood, i.e.,

$$\mathcal{N}_x = \{U(x) : U \in \mathcal{U}\}, \quad \text{for any } x \in X.$$

Next, we are going to prove that, there exists a unique topology τ on X such that for any $x \in X$, $\mathcal{N}_x$ is the neighborhood system of x with respect to the topology τ. We call τ the **uniform topology** induced by the uniform structure $\mathcal{U}$.

?⟨thm2.1⟩? **Theorem 2.1.** *Suppose that $(X,\mathcal{U})$ is a uniform space. Then the following conclusions holds:*

(a) *There exists a unique hausdorff topology τ on X such that for any $x \in X$, $\mathcal{N}_x$ is the neighborhood system of x with respect to τ.*

(b) *Suppose that $\mathcal{B} \subset \mathcal{U}$ is a base of $\mathcal{U}$. Define*

$$\mathcal{B}_x = \{B(x) : B \in \mathcal{B}\}.$$

Then for any $x \in X$, $\mathcal{B}_x$ is a base of the neighborhood system $\mathcal{N}_x$ with respect to the uniform topology τ. Thus if $\mathcal{B}$ is countable, the uniform topology τ is first countable.

(c) *For any $A \in X$ and $M \in X \times X$, we have the following closure formulas with respect to the uniform topology τ:*

$$\bar{A} = \bigcap_{U \in \mathcal{U}} U(A)$$

and

$$\bar{M} = \bigcap_{U \in \mathcal{U}} (U \circ M \circ U).$$

The uniform structure $\mathcal{U}$ can be replaced by any base of $\mathcal{U}$.

Proof. (a) define

$$\tau = \{A \subset X : \forall x \in A,\ \exists U \in \mathcal{N}_x,\ U \subset A\}.$$

First we prove that τ is a topology on X. We have that $X, \emptyset \in \tau$. For any $A, B \in \tau$, we have $A \cap B \in \tau$. In fact, for any $x \in A \cap B$, there exist $U, V \in \mathcal{N}_x$ such that $U \subset A$ and $V \subset B$. Then $U \cap V \in \mathcal{N}_x$ and $U \cap V \subset A \cap B$. Finally, for any family $\{A_i\}_{i\in I} \subset \tau$, we have $\bigcup_{i\in I} A_i \in \tau$. In fact, for any $x \in \bigcup_{i\in I} A_i$, there exists $j \in I$ such that $x \in A_j$. Since $A_j \in \tau$, there exists $U \in \mathcal{N}_x$ such that $U \subset A_j \subset \bigcup_{i\in I} A_i$. Thus τ is a topology on X.

Next we prove that for any $x \in X$, $\mathcal{N}_x$ is the neighborhood system of x with respect to the topology τ. Let $\mathcal{A}_x$ be the neighborhood system of x with respect to τ. For any $U \in \mathcal{N}_x$, by the definition of τ, we have $U \in \mathcal{A}_x$ and thus $\mathcal{N}_x \subset \mathcal{A}_x$. Conversely, for any $A \in \mathcal{A}_x$, by the definition of τ, there exists $U \in \mathcal{N}_x$ such that $U \subset A$. Thus we have $A \in \mathcal{N}_x$ and $\mathcal{A}_x \subset \mathcal{N}_x$. Therefore, $\mathcal{N}_x$ is the neighborhood system of x with respect to the topology τ.

Finally, we prove that the topology τ is hausdorff. For any two distinct points $x, y \in X$, there exists $U \in \mathcal{U}$ such that $(x, y) \notin U$. Then $y \notin U(x)$. Since $U(x) \in \mathcal{N}_x$, then $U(x)$ is a neighborhood of x not containing y. Similarly, there exists a neighborhood of y not containing x.

(b) It's straightforward by conclusion (a) of Theorem (**??**).

(c) By the definition of closure, we have

$$x \in \bar{A} \iff \forall U \in \mathcal{U},\ U(x) \cap A \neq \emptyset,$$

which is equivalent to

$$\bar{A} = \bigcap_{U \in \mathcal{U}} U(A).$$

The second closure formula can be proved similarly.

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From the construction of the neighborhood system $\mathcal{N}_x$, we can see that the local structure of a uniform space is uniform everywhere. In fact, for any $x, y \in X$ and any $U \in \mathcal{U}$, we have

$$U(y) = U \circ \{y\} = U \circ \Delta_X(y) \supset U \circ U(x).$$

By the condition (c) in Definition (**??**), there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$. Thus we have $V(x) \subset U(y)$. By symmetry, we also have $V(y) \subset U(x)$.