Notes on Abstract Space

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These are notes on abstract spaces, mainly based on the book *An Introduction to Abstract Spaces* by Hu Shigeng and Zhang Xianwen.

1 Basic Concepts

1.1 Sets and Relations

We denote by 2^X the power set of X, i.e., the set of all subsets of X. Some related notation is as follows. For any $\mathcal{A} \subset 2^X$, we set

$$\bigcup \mathcal{A} = \bigcup \{A : A \in \mathcal{A}\},$$

and

$$\bigcap \mathcal{A} = \bigcap \{A : A \in \mathcal{A}\}.$$

In addition, we define

$$\mathcal{A}^* = \{\bigcap \mathcal{B} : \mathcal{B} \subset \mathcal{A} \text{ and } |\mathcal{B}| \text{ is finite}\}.$$

If $\mathcal{A}, \mathcal{B} \subset 2^X$, we write

$$\mathcal{A} \vdash \mathcal{B} \iff \forall B \in \mathcal{B},\ \exists A \in \mathcal{A}:\ A \subset B,$$

and

$$\mathcal{A} \prec \mathcal{B} \iff \forall A \in \mathcal{A},\ \exists B \in \mathcal{B}:\ A \subset B.$$

?⟨def1.1⟩? **Definition 1.1.** If $\mathcal{A} \subset 2^X$ is nonempty and satisfies

(a) $\emptyset \notin \mathcal{A}$;

(b) For any $A_1, A_2 \in \mathcal{A}$, $A_1 \cap A_2 \in \mathcal{A}$;

(c) If $\mathcal{A} \vdash A \subset X$, then $A \in \mathcal{A}$,

then we call $\mathcal{A}$ a **filter** on X. Moreover, if $\mathcal{B} \subset \mathcal{A}$ satisfies $\mathcal{B} \vdash \mathcal{A}$, then we say that $\mathcal{B}$ is a **base** of the filter $\mathcal{A}$.

It's easy to see that condition (b) in Definition **??** can be replaced by $\mathcal{A} = \mathcal{A}^*$.

Any filter has at least one base, for example, itself. Conversely, any nonempty $\mathcal{B} \subset 2^X$ satisfying suitable conditions is a base of a filter.

Theorem 1.1. *If a nonempty $\mathcal{B} \subset 2^X$ satisfies*

(a) $\emptyset \notin \mathcal{B}$*;*

(b) *For any $B_1, B_2 \in \mathcal{B}$, we have $\mathcal{B} \vdash B_1 \cap B_2$,*

then $\mathcal{B}$ is a base of a filter $\mathcal{A} \subset 2^X$.

Proof. Let

$$\mathcal{A} = \{A \subset X : \mathcal{B} \vdash A\}.$$

It is straightforward to verify that $\mathcal{A}$ is a filter and that $\mathcal{B}$ is a base of $\mathcal{A}$. □

Next, we discuss relations between two nonempty sets. For any two nonempty sets X and Y, we consider elements $x \in X$ and $y \in Y$ as abstract variables, and define how x is related to y.

?⟨def1.3⟩?

Definition 1.2. Let X and Y be two nonempty sets. Any subset $F \subset X \times Y$ is called a **relation** from X to Y. When $(x, y) \in F$, we say that x is related to y by the relation F, denoted by xFy or $y \in Fx$. If $F \subset X \times X$, we call it a **binary relation** on X.

For any relation F from X to Y and any subset $A \subset X$, define

$$F(A) = \{y \in Y : \exists x \in A,\ (x, y) \in F\}.$$

Let $Fx = F\{x\}$. Then $F(A) = \bigcup_{x \in A} Fx$. If $G \subset Y \times Z$ is another relation, the composition $G \circ F$ is defined by

$$G \circ F = \{(x, z) \in X \times Z : \exists y \in Y,\ (x, y) \in F \text{ and } (y, z) \in G\}.$$

The inverse relation of F is

$$F^{-1} = \{(y, x) \in Y \times X : (x, y) \in F\}.$$

Let us consider any relation $F \subset X \times X$. If $F^{-1} = F$, we say that F is **symmetric**. If $F \circ F \subset F$, we say that F is **transitive**. If the diagonal $\{(x,x) : x \in X\}$ of $X \times X$ is contained in F, we say that F is **reflexive**. If F is symmetric, transitive, and reflexive, we say that F is an **equivalence relation** on X.

Fix a relation $F \subset X \times Y$. We can view F as the correspondence

$$F : X \to 2^Y, \quad x \mapsto Fx.$$

Thus F is a set-valued function. In applications, we often consider the special case that for each $x \in X$, the set Fx contains exactly one element. In that case we say F is a **function** from X to Y.

2 Abstract Space

2.1 Uniform Space

The definition of uniform space is similar to that of topological space. The difference is that in a uniform space, the neighborhoods of points are defined by relations instead of subsets.

?⟨def2.1⟩?

Definition 2.1. Let X be a nonempty set. A nonempty family $\mathcal{U}$ of relations on X is called a **uniform structure** on X if it satisfies the following conditions:

(a) For any $U \in \mathcal{U}$, $\Delta_X \subset U$, where $\Delta_X = \{(x,x) : x \in X\}$ is the diagonal of $X \times X$;

(b) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;

(c) For any $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$;

(d) $\mathcal{U}$ is a filter on $X \times X$.

The space X or the pair $(X,\mathcal{U})$ is called a **uniform space**.

For any base of the filter $\mathcal{U}$, we call it a **base** of the uniform structure $(X,\mathcal{U})$. For any $U \in \mathcal{U}$, $U \cap U^{-1} = (U \cap U^{-1})^{-1} \in \mathcal{U}$. Moreover, we have an element $V \in \mathcal{U}$ such that $V \circ V \subset U$. By the fact that $V = V \circ \Delta \subset V \circ V$, we have $V \circ V \in \mathcal{U}$. Thus the set $\{U \cap U^{-1} : U \in \mathcal{U}\}$ or $\{V \circ V : V \in \mathcal{U}\}$ either forms a base of $\mathcal{U}$.

The condition (b) in Definition (**??**) means that for any $x, y \in X$,

$$x = y \iff (x,y) \in U \text{ for all } U \in \mathcal{U}.$$

We can say that the more $U \in \mathcal{U}$ "small", the closer x and y are. The uniform structure $\mathcal{U}$ provides a set of criteria that measure the closeness between two points in the space X. Those facts naturally lead to the definition of uniform neighborhood, i.e.,

$$\mathcal{N}_x = \{U(x) : U \in \mathcal{U}\}, \quad \text{for any } x \in X.$$

Next, we are going to prove that, there exists a unique topology τ on X such that for any $x \in X$, $\mathcal{N}_x$ is the neighborhood system of x with respect to the topology τ. We call τ the **uniform topology** induced by the uniform structure $\mathcal{U}$.

?⟨thm2.1⟩? **Theorem 2.1.** *Suppose that $(X,\mathcal{U})$ is a uniform space. Then the following conclusions holds:*

(a) *There exists a unique hausdorff topology τ on X such that for any $x \in X$, $\mathcal{N}_x$ is the neighborhood system of x with respect to τ.*

(b) *Suppose that $\mathcal{B} \subset \mathcal{U}$ is a base of $\mathcal{U}$. Define*

$$\mathcal{B}_x = \{B(x) : B \in \mathcal{B}\}.$$

Then for any $x \in X$, $\mathcal{B}_x$ is a base of the neighborhood system $\mathcal{N}_x$ with respect to the uniform topology τ. Thus if $\mathcal{B}$ is countable, the uniform topology τ is first countable.

(c) *For any $A \in X$ and $M \in X \times X$, we have the following closure formulas with respect to the uniform topology τ:*

$$\bar{A} = \bigcap_{U \in \mathcal{U}} U(A)$$

and

$$\bar{M} = \bigcap_{U \in \mathcal{U}} (U \circ M \circ U).$$

The uniform structure $\mathcal{U}$ can be replaced by any base of $\mathcal{U}$.

Proof. (a) define

$$\tau = \{A \subset X : \forall x \in A,\ \exists U \in \mathcal{N}_x,\ U \subset A\}.$$

First we prove that τ is a topology on X. We have that $X, \emptyset \in \tau$. For any $A, B \in \tau$, we have $A \cap B \in \tau$. In fact, for any $x \in A \cap B$, there exist $U, V \in \mathcal{N}_x$ such that $U \subset A$ and $V \subset B$. Then $U \cap V \in \mathcal{N}_x$ and $U \cap V \subset A \cap B$. Finally, for any family $\{A_i\}_{i \in I} \subset \tau$, we have $\bigcup_{i \in I} A_i \in \tau$. In fact, for any $x \in \bigcup_{i \in I} A_i$, there exists $j \in I$ such that $x \in A_j$. Since $A_j \in \tau$, there exists $U \in \mathcal{N}_x$ such that $U \subset A_j \subset \bigcup_{i \in I} A_i$. Thus τ is a topology on X.

Next we prove that for any $x \in X$, $\mathcal{N}_x$ is the neighborhood system of x with respect to the topology τ. Let $\mathcal{A}_x$ be the neighborhood system of x with respect to τ. For any $U \in \mathcal{N}_x$, by the definition of τ, we have $U \in \mathcal{A}_x$ and thus $\mathcal{N}_x \subset \mathcal{A}_x$. Conversely, for any $A \in \mathcal{A}_x$, by the definition of τ, there exists $U \in \mathcal{N}_x$ such that $U \subset A$. Thus we have $A \in \mathcal{N}_x$ and $\mathcal{A}_x \subset \mathcal{N}_x$. Therefore, $\mathcal{N}_x$ is the neighborhood system of x with respect to the topology τ.

Finally, we prove that the topology τ is hausdorff. For any two distinct points $x, y \in X$ there exists $U \in \mathcal{U}$ such that $(x, y) \notin U$. Then $y \notin U(x)$. Since $U(x) \in \mathcal{N}_x$, then $U(x)$ is a neighborhood of x not containing y. Similarly, there exists a neighborhood of y not containing x.

(b) It's straightforward by conclusion (a) of Theorem (**??**).

(c) By the definition of closure, we have

$$x \in \bar{A} \iff \forall U \in \mathcal{U},\ U(x) \cap A \neq \emptyset,$$

which is equivalent to

$$\bar{A} = \bigcap_{U \in \mathcal{U}} U(A).$$

The second closure formula can be proved similarly.

□

From the construction of the neighborhood system $\mathcal{N}_x$, we can see that the local structure of a uniform space is uniform everywhere. In other words, for any two points $x, y \in X$, there exists a natural correspondence between the neighborhood systems $\mathcal{N}_x$ and $\mathcal{N}_y$.

As we have mentionded that the uniform structure $\mathcal{U}$ provides a set of criteria that measure the closeness between two points in the space X, we hope to use some numerical values to describe the closeness. Basicaly, we expect that there exists a metric that induces the uniform structure $\mathcal{U}$. However, this is not always possible.

?⟨thm2.2⟩?

Theorem 2.2. *Let $(X, \mathcal{U})$ be a uniform space. There exists a metric d on X inducing the uniform structure $\mathcal{U}$ if and only if $\mathcal{U}$ has a countable base.*

Proof. The only if part is trivial. We only prove the if part. Suppose that $\mathcal{U}$ has a countable base $\{U_n : n \in \mathbb{N}\}$. Without loss of generality, we can assume that U_n is symmetric and

$$U_n \circ U_n \circ U_n \subset U_{n-1} \quad \text{for all } n \geq 1,$$

where we set $U_0 = X \times X$. For any $x, y \in X$, define

$$f(x,y) = \begin{cases} 2^{-n}, & \text{if } (x,y) \in U_n \text{ and } (x,y) \notin U_{n+1} \text{ for some } n \in \mathbb{N}, \\ 0, & \text{if } x = y, \end{cases}$$

and

$$d(x,y) = \inf \left\{ \sum_{i=1}^{m} f(x_{i-1}, x_i) : m \in \mathbb{N}, x_0 = x, x_m = y, x_i \in X \text{ for } i = 1, \ldots, m-1 \right\}.$$

For simplicity, we denote the sum in the above formula by $S(x,y)$. We first prove that d is a metric on X. By the symmetric property of U_n, we know that f and d are symmetric. It's obvious that $d(x,y) = 0$ is equivalent to $x = y$. For any $x, y, z \in X$, we have

$$\begin{aligned} d(x,y) &= \inf S(x,y) \\ &\le \inf(S(x,z) + S(z,y)) \\ &= d(x,z) + d(z,y). \end{aligned}$$

Thus d is a metric on X. Then we prove that the metric d induces the uniform structure $\mathcal{U}$. Define

$$V_r = \{(x,y) \in X \times X : d(x,y) < r\}, \quad \text{for any } r > 0.$$

We only need to prove that

$$V_{1/2^{n+1}} \subset U_n \subset V_{1/2^{n-1}} \text{ for all } n \in \mathbb{N}.$$

By the definition of d, it's straightforward to see that $U_n \subset V_{1/2^{n-1}}$. Then we prove that $f(x,y) \le 2d(x,y)$. We fix a finite sequence $\{x_i\}_{i=0}^m$ with $x_0 = x$ and $x_m = y$ and let $a = \sum_{i=1}^m f(x_{i-1}, x_i)$. We are going to show that $f(x,y) \le 2a$ by induction on n. We choose $m \in \mathbb{N}$ such that $2^{-m} \le a < 2^{-m+1}$. There exists the largest integer k such that $\sum_{i=1}^{k-1} f(x_{i-1}, x_i) \le \frac{a}{2}$. Thus we know

$$\sum_{i=k+1}^{m} f(x_{i-1}, x_i) = a - \sum_{i=1}^{k} f(x_{i-1}, x_i) < \frac{a}{2}.$$

By the induction hypothesis, we have

$$f(x, x_{k-1}) \le 2 \sum_{i=1}^{k-1} f(x_{i-1}, x_i) \le a,$$

and

$$f(x_k, y) \le 2 \sum_{i=k+1}^{m} f(x_{i-1}, x_i) < a.$$

It's trivial that $f(x_{k-1}, x_k) \le a$. Thus $(x, y) \in U_m \circ U_m \circ U_m \subset U_{m-1}$, which implies that $f(x, y) \le 2^{-m+1} < 2a$. □

Theorem (**??**) shows that a uniform space $(X, \mathcal{U})$ cannot be metrizable in general. However, we can use a function similar to a metric to describe the closeness between two points in X. This leads to the following definition of pseudometric.

?⟨def2.2⟩? **Definition 2.2.** Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}_+$ is called a **pseudometric** on X if it satisfies the following conditions:

(a) For any $x \in X$, $d(x, x) = 0$;

(b) For any $x, y \in X$, $d(x, y) = d(y, x)$;

(c) For any $x, y, z \in X$, $d(x, z) \le d(x, y) + d(y, z)$.

The pair (X, d) is called a **pseudometric space**.

It's possible that $\mathcal{U}$ can be induced by a family of pseudometrics.

?⟨thm2.3⟩? **Theorem 2.3.** *Suppose that $(X, \mathcal{U})$ is a uniform space. Let P be the set of all uniformly continous pseudometrics on X. Then the uniform structure $\mathcal{U}$ is induced by the family P. Moreover, P is the largest family of pseudometrics on X inducing the uniform structure $\mathcal{U}$.*

We call the family P mentioned in Theorem (**??**) a **gage**.

Proof. Since $d = 0$ is an element of P, P is nonempty. We write $\mathcal{U}_P$ for the uniform structure induced by P and will prove that $\mathcal{U} = \mathcal{U}_P$. For any $d \in P$ and $r > 0$, define

$$V_{d,r} = \{(x, y) \in X \times X : d(x, y) < r\}.$$

By the uniform continuity of d, we have $V_{d,r} \in \mathcal{U}$. In addition, we know that $\mathcal{U}_{\mathcal{P}}$ is the filter generated by the set

$$\mathcal{A} = \{V_{d,r} : d \in P, r > 0\}.$$

Thus we have $\mathcal{U}_P \subset \mathcal{U}$. To prove the converse inclusion, we only need to prove $\mathcal{A} \vdash \mathcal{U}$. For any $U \in \mathcal{U}$, choose $U_1 = U \cap U^{-1}$. There exists a sequence $\{U_n\}$ such that U_n is symmetric and

$$U_n \circ U_n \circ U_n \subset U_{n-1}, \quad n \ge 2,$$

where we set $U_0 = X \times X$. The same as the proof of Theorem (**??**), there exists a pseudometric d on X such that

$$V_{d,1/2^{n+1}} \subset U_n \subset V_{d,1/2^{n-1}} \text{ for all } n \in \mathbb{N}.$$

This also implies that d is uniformly continuous. Thus $d \in P$ and $\mathcal{A} \vdash U$. □

An interesting question is that, given a topological space (X, τ), does there exist a uniform structure $\mathcal{U}$ on X such that τ is the uniform topology induced by $\mathcal{U}$? We can answer the question by using Theorem (**??**).

?⟨cor2.4⟩? **Corollary 2.1.** *Let (X, τ) be a topological space. Then there exists a uniform structure $\mathcal{U}$ on X such that τ is the uniform topology induced by $\mathcal{U}$ if and only if (X, τ) is completely regular.*

Proof. $\square$