

Notes on Abstract Space

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These are notes on abstract spaces, mainly based on the book *An Introduction to Abstract Spaces* by Hu Shigeng and Zhang Xianwen.

1 Basic Concepts

1.1 Sets and Relations

We denote by 2^X the power set of X , i.e., the set of all subsets of X . Some related notation is as follows. For any $\mathcal{A} \subset 2^X$, we set

$$\bigcup \mathcal{A} = \bigcup \{A : A \in \mathcal{A}\},$$

and

$$\bigcap \mathcal{A} = \bigcap \{A : A \in \mathcal{A}\}.$$

In addition, we define

$$\mathcal{A}^* = \{\bigcap \mathcal{B} : \mathcal{B} \subset \mathcal{A} \text{ and } |\mathcal{B}| \text{ is finite}\}.$$

If $\mathcal{A}, \mathcal{B} \subset 2^X$, we write

$$\mathcal{A} \vdash \mathcal{B} \iff \forall B \in \mathcal{B}, \exists A \in \mathcal{A} : A \subset B,$$

and

$$\mathcal{A} \prec \mathcal{B} \iff \forall A \in \mathcal{A}, \exists B \in \mathcal{B} : A \subset B.$$

Definition 1.1. If $\mathcal{A} \subset 2^X$ is nonempty and satisfies

(a) $\emptyset \notin \mathcal{A}$;

(b) For any $A_1, A_2 \in \mathcal{A}$, $A_1 \cap A_2 \in \mathcal{A}$;

11 (c) If $\mathcal{A} \vdash A \subset X$, then $A \in \mathcal{A}$,
 12 then we call \mathcal{A} a **filter** on X . Moreover, if $\mathcal{B} \subset \mathcal{A}$ satisfies $\mathcal{B} \vdash \mathcal{A}$, then we say that \mathcal{B} is
 13 a **base** of the filter \mathcal{A} .

14 It's easy to see that condition (b) in Definition ?? can be replaced by $\mathcal{A} = \mathcal{A}^*$.

15 Any filter has at least one base, for example, itself. Conversely, any nonempty $\mathcal{B} \subset 2^X$
 16 satisfying suitable conditions is a base of a filter.

17 **Theorem 1.1.** *If a nonempty $\mathcal{B} \subset 2^X$ satisfies*

18 (a) $\emptyset \notin \mathcal{B}$;

19 (b) *For any $B_1, B_2 \in \mathcal{B}$, we have $\mathcal{B} \vdash B_1 \cap B_2$,*

20 *then \mathcal{B} is a base of a filter $\mathcal{A} \subset 2^X$.*

Proof. Let

$$\mathcal{A} = \{A \subset X : \mathcal{B} \vdash A\}.$$

21 It is straightforward to verify that \mathcal{A} is a filter and that \mathcal{B} is a base of \mathcal{A} . □

22 Next, we discuss relations between two nonempty sets. For any two nonempty sets X
 23 and Y , we consider elements $x \in X$ and $y \in Y$ as abstract variables, and define how x is
 24 related to y .

25 **Definition 1.2.** Let X and Y be two nonempty sets. Any subset $F \subset X \times Y$ is called a
 26 **relation** from X to Y . When $(x, y) \in F$, we say that x is related to y by the relation F ,
 27 denoted by xFy or $y \in Fx$. If $F \subset X \times X$, we call it a **binary relation** on X .

For any relation F from X to Y and any subset $A \subset X$, define

$$F(A) = \{y \in Y : \exists x \in A, (x, y) \in F\}.$$

Let $Fx = F\{x\}$. Then $F(A) = \bigcup_{x \in A} Fx$. If $G \subset Y \times Z$ is another relation, the
 composition $G \circ F$ is defined by

$$G \circ F = \{(x, z) \in X \times Z : \exists y \in Y, (x, y) \in F \text{ and } (y, z) \in G\}.$$

The inverse relation of F is

$$F^{-1} = \{(y, x) \in Y \times X : (x, y) \in F\}.$$

28 Let us consider any relation $F \subset X \times X$. If $F^{-1} = F$, we say that F is **symmetric**.
 29 If $F \circ F \subset F$, we say that F is **transitive**. If the diagonal $\{(x, x) : x \in X\}$ of $X \times X$ is
 30 contained in F , we say that F is **reflexive**. If F is symmetric, transitive, and reflexive,
 31 we say that F is an **equivalence relation** on X .

Fix a relation $F \subset X \times Y$. We can view F as the correspondence

$$F : X \rightarrow 2^Y, \quad x \mapsto Fx.$$

32 Thus F is a set-valued function. In applications, we often consider the special case that
 33 for each $x \in X$, the set Fx contains exactly one element. In that case we say F is a
 34 **function** from X to Y .

35 2 Abstract Space

36 2.1 Uniform Space

37 The definition of uniform space is similar to that of topological space. The difference is
 38 that in a uniform space, the neighborhoods of points are defined by relations instead of
 39 subsets.

40 **Definition 2.1.** Let X be a nonempty set. A nonempty family \mathcal{U} of relations on X is
 41 called a **uniform structure** on X if it satisfies the following conditions:

- 42 (a) For any $U \in \mathcal{U}$, $\Delta_X \subset U$, where $\Delta_X = \{(x, x) : x \in X\}$ is the diagonal of $X \times X$;
- 43 (b) If $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;
- 44 (c) For any $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$;
- 45 (d) \mathcal{U} is a filter on $X \times X$.

46 The space X or the pair (X, \mathcal{U}) is called a **uniform space**.

47 For any base of the filter \mathcal{U} , we call it a **base** of the uniform structure (X, \mathcal{U}) . For
 48 any $U \in \mathcal{U}$, $U \cap U^{-1} = (U \cap U^{-1})^{-1} \in \mathcal{U}$. Moreover, we have an element $V \in \mathcal{U}$ such
 49 that $V \circ V \subset U$. By the fact that $V = V \circ \Delta \subset V \circ V$, we have $V \circ V \in \mathcal{U}$. Thus the
 50 set $\{U \cap U^{-1} : U \in \mathcal{U}\}$ or $\{V \circ V : V \in \mathcal{U}\}$ either forms a base of \mathcal{U} .

51 The condition (b) in Definition (??) means that for any $x, y \in X$,

$$52 \quad x = y \iff (x, y) \in U \text{ for all } U \in \mathcal{U}.$$

We can say that the more $U \in \mathcal{U}$ "small", the closer x and y are. The uniform structure \mathcal{U} provides a set of criteria that measure the closeness between two points in the space X . Those facts naturally lead to the definition of uniform neighborhood, i.e.,

$$\mathcal{N}_x = \{U(x) : U \in \mathcal{U}\}, \quad \text{for any } x \in X.$$

Next, we are going to prove that, there exists a unique topology τ on X such that for any $x \in X$, \mathcal{N}_x is the neighborhood system of x with respect to the topology τ . We call τ the **uniform topology** induced by the uniform structure \mathcal{U} .

Theorem 2.1. *Suppose that (X, \mathcal{U}) is a uniform space. Then the following conclusions holds:*

(a) *There exists a unique hausdorff topology τ on X such that for any $x \in X$, \mathcal{N}_x is the neighborhood system of x with respect to τ .*

(b) *Suppose that $\mathcal{B} \subset \mathcal{U}$ is a base of \mathcal{U} . Define*

$$\mathcal{B}_x = \{B(x) : B \in \mathcal{B}\}.$$

Then for any $x \in X$, \mathcal{B}_x is a base of the neighborhood system \mathcal{N}_x with respect to the uniform topology τ . Thus if \mathcal{B} is countable, the uniform topology τ is first countable.

(c) *For any $A \in X$ and $M \in X \times X$, we have the following closure formulas with respect to the uniform topology τ :*

$$\bar{A} = \bigcap_{U \in \mathcal{U}} U(A)$$

and

$$\bar{M} = \bigcap_{U \in \mathcal{U}} (U \circ M \circ U).$$

The uniform structure \mathcal{U} can be replaced by any base of \mathcal{U} .

Proof. (a) define

$$\tau = \{A \subset X : \forall x \in A, \exists U \in \mathcal{N}_x, U \subset A\}.$$

First we prove that τ is a topology on X . We have that $X, \emptyset \in \tau$. For any $A, B \in \tau$, we have $A \cap B \in \tau$. In fact, for any $x \in A \cap B$, there exist $U, V \in \mathcal{N}_x$ such that $U \subset A$ and $V \subset B$. Then $U \cap V \in \mathcal{N}_x$ and $U \cap V \subset A \cap B$. Finally, for any family $\{A_i\}_{i \in I} \subset \tau$, we have $\bigcup_{i \in I} A_i \in \tau$. In fact, for any $x \in \bigcup_{i \in I} A_i$, there exists $j \in I$ such that $x \in A_j$. Since $A_j \in \tau$, there exists $U \in \mathcal{N}_x$ such that $U \subset A_j \subset \bigcup_{i \in I} A_i$. Thus τ is a topology on X .

Next we prove that for any $x \in X$, \mathcal{N}_x is the neighborhood system of x with respect to the topology τ . Let \mathcal{A}_x be the neighborhood system of x with respect to τ . For any $U \in \mathcal{N}_x$, by the definition of τ , we have $U \in \mathcal{A}_x$ and thus $\mathcal{N}_x \subset \mathcal{A}_x$. Conversely, for any $A \in \mathcal{A}_x$, by the definition of τ , there exists $U \in \mathcal{N}_x$ such that $U \subset A$. Thus we have $A \in \mathcal{N}_x$ and $\mathcal{A}_x \subset \mathcal{N}_x$. Therefore, \mathcal{N}_x is the neighborhood system of x with respect to the topology τ .

Finally, we prove that the topology τ is hausdorff. For any two distinct points $x, y \in X$ there exists $U \in \mathcal{U}$ such that $(x, y) \notin U$. Then $y \notin U(x)$. Since $U(x) \in \mathcal{N}_x$, then $U(x)$ is a neighborhood of x not containing y . Similarly, there exists a neighborhood of y not containing x .

(b) It's straightforward by conclusion (a) of Theorem (??).

(c) By the definition of closure, we have

$$x \in \bar{A} \iff \forall U \in \mathcal{U}, U(x) \cap A \neq \emptyset,$$

which is equivalent to

$$\bar{A} = \bigcap_{U \in \mathcal{U}} U(A).$$

The second closure formula can be proved similarly.

□

From the construction of the neighborhood system \mathcal{N}_x , we can see that the local structure of a uniform space is uniform everywhere. In other words, for any two points $x, y \in X$, there exists a natural correspondence between the neighborhood systems \mathcal{N}_x and \mathcal{N}_y .

As we have mentioned that the uniform structure \mathcal{U} provides a set of criteria that measure the closeness between two points in the space X , we hope to use some numerical values to describe the closeness. Basically, we expect that there exists a metric that induces the uniform structure \mathcal{U} . However, this is not always possible.

?(thm2.2)?

Theorem 2.2. *Let (X, \mathcal{U}) be a uniform space. There exists a metric d on X inducing the uniform structure \mathcal{U} if and only if \mathcal{U} has a countable base.*

Proof. The only if part is trivial. We only prove the if part. Suppose that \mathcal{U} has a countable base $\{U_n : n \in \mathbb{N}\}$. Without loss of generality, we can assume that U_n is symmetric and

$$U_n \circ U_n \circ U_n \subset U_{n-1} \quad \text{for all } n \geq 1,$$

where we set $U_0 = X \times X$. For any $x, y \in X$, define

$$f(x, y) = \begin{cases} 2^{-n}, & \text{if } (x, y) \in U_n \text{ and } (x, y) \notin U_{n+1} \text{ for some } n \in \mathbb{N}, \\ 0, & \text{if } x = y, \end{cases}$$

and

$$d(x, y) = \inf \left\{ \sum_{i=1}^m f(x_{i-1}, x_i) : m \in \mathbb{N}, x_0 = x, x_m = y, x_i \in X \text{ for } i = 1, \dots, m-1 \right\}.$$

95 For simplicity, we denote the sum in the above formula by $S(x, y)$. We first prove that d
 96 is a metric on X . By the symmetric property of U_n , we know that f and d are symmetric.
 97 It's obvious that $d(x, y) = 0$ is equivalent to $x = y$. For any $x, y, z \in X$, we have

$$\begin{aligned} 98 \quad d(x, y) &= \inf S(x, y) \\ 99 \quad &\leq \inf (S(x, z) + S(z, y)) \\ 100 \quad &= d(x, z) + d(z, y). \end{aligned}$$

Thus d is a metric on X . Then we prove that the metric d induces the uniform structure \mathcal{U} . Define

$$V_r = \{(x, y) \in X \times X : d(x, y) < r\}, \quad \text{for any } r > 0.$$

We only need to prove that

$$V_{1/2^{n+1}} \subset U_n \subset V_{1/2^{n-1}} \text{ for all } n \in \mathbb{N}.$$

By the definition of d , it's straightforward to see that $U_n \subset V_{1/2^{n-1}}$. Then we prove that $f(x, y) \leq 2d(x, y)$. We fix a finite sequence $\{x_i\}_{i=0}^m$ with $x_0 = x$ and $x_m = y$ and let $a = \sum_{i=1}^m f(x_{i-1}, x_i)$. We are going to show that $f(x, y) \leq 2a$ by induction on n . We choose $m \in \mathbb{N}$ such that $2^{-m} \leq a < 2^{-m+1}$. There exists the largest integer k such that $\sum_{i=1}^{k-1} f(x_{i-1}, x_i) \leq \frac{a}{2}$. Thus we know

$$\sum_{i=k+1}^m f(x_{i-1}, x_i) = a - \sum_{i=1}^k f(x_{i-1}, x_i) < \frac{a}{2}.$$

By the induction hypothesis, we have

$$f(x, x_{k-1}) \leq 2 \sum_{i=1}^{k-1} f(x_{i-1}, x_i) \leq a,$$

and

$$f(x_k, y) \leq 2 \sum_{i=k+1}^m f(x_{i-1}, x_i) < a.$$

101 It's trivial that $f(x_{k-1}, x_k) \leq a$. Thus $(x, y) \in U_m \circ U_m \circ U_m \subset U_{m-1}$, which implies
 102 that $f(x, y) \leq 2^{-m+1} < 2a$. \square

103 Theorem (??) shows that a uniform space (X, \mathcal{U}) cannot be metrizable in general.
 104 However, we can use a function similar to a metric to describe the closeness between two
 105 points in X . This leads to the following definition of pseudometric.

106 **Definition 2.2.** Let X be a nonempty set. A function $d : X \times X \rightarrow \mathbb{R}_+$ is called a
 107 **pseudometric** on X if it satisfies the following conditions:

- 108 (a) For any $x \in X$, $d(x, x) = 0$;
- 109 (b) For any $x, y \in X$, $d(x, y) = d(y, x)$;
- 110 (c) For any $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

111 The pair (X, d) is called a **pseudometric space**.

112 It's possible that \mathcal{U} can be induced by a family of pseudometrics.

113 **Theorem 2.3.** Suppose that (X, \mathcal{U}) is a uniform space. Let P be the set of all uniformly
 114 continuous pseudometrics on X . Then the uniform structure \mathcal{U} is induced by the family P .
 115 Moreover, P is the largest family of pseudometrics on X inducing the uniform structure \mathcal{U} .

116 We call the family P mentioned in Theorem (??) a **gage**.

Proof. Since $d = 0$ is an element of P , P is nonempty. We write \mathcal{U}_P for the uniform structure induced by P and will prove that $\mathcal{U} = \mathcal{U}_P$. For any $d \in P$ and $r > 0$, define

$$V_{d,r} = \{(x, y) \in X \times X : d(x, y) < r\}.$$

By the uniform continuity of d , we have $V_{d,r} \in \mathcal{U}$. In addition, we know that \mathcal{U}_P is the filter generated by the set

$$\mathcal{A} = \{V_{d,r} : d \in P, r > 0\}.$$

Thus we have $\mathcal{U}_P \subset \mathcal{U}$. To prove the converse inclusion, we only need to prove $\mathcal{A} \vdash \mathcal{U}$. For any $U \in \mathcal{U}$, choose $U_1 = U \cap U^{-1}$. There exists a sequence $\{U_n\}$ such that U_n is symmetric and

$$U_n \circ U_n \circ U_n \subset U_{n-1}, \quad n \geq 2,$$

where we set $U_0 = X \times X$. The same as the proof of Theorem (??), there exists a pseudometric d on X such that

$$V_{d,1/2^{n+1}} \subset U_n \subset V_{d,1/2^{n-1}} \text{ for all } n \in \mathbb{N}.$$

117 This also implies that d is uniformly continuous. Thus $d \in P$ and $\mathcal{A} \vdash U$. \square

118 An interesting question is that, given a topological space (X, τ) , does there exist a
 119 uniform structure \mathcal{U} on X such that τ is the uniform topology induced by \mathcal{U} ? We can
 120 answer the question by using Theorem (??).

?<cor2.4>?

121 **Corollary 2.1.** *Let (X, τ) be a topological space. Then there exists a uniform structure \mathcal{U}
 122 on X such that τ is the uniform topology induced by \mathcal{U} if and only if (X, τ) is completely
 123 regular.*

Proof. (a) Suppose that τ is induced by a uniform structure \mathcal{U} and P is the gage of
 (X, \mathcal{U}) . Fixing $x \in X$ and $d \in P$, define $f_{d,x}(y) = d(x, y)$ for all $y \in X$. Then $f_{d,x}$ is
 continuous on X and

$$V_{d,r} = \{(y \in X : f_{d,x}(y) < r)\}.$$

124 The family $\{V_{d,r}\}$ forms a base of topology τ , which implies that (X, τ) is completely
 125 regular.

126 (b) Suppose that (X, τ) is completely regular. There exists a topological embedding F
 127 from (X, τ) into a product space $Y = \prod_{\alpha \in I} [0, 1]_{\alpha}$, where I is a nonempty index
 128 set. Then FX is a uniform space. We denote its uniform structure by \mathcal{V} . Thus
 129 $\mathcal{U} = F^{-1}(\mathcal{V})$ is a uniform structure on X and

$$130 \quad F : (X, \mathcal{U}) \rightarrow (FX, \mathcal{V})$$

131 is a uniform isomorphism. We denote by τ' the uniform topology induced by \mathcal{U} .
 132 Then $F : (X, \tau') \rightarrow (FX, \tau'')$ is a homeomorphism, where τ'' is the uniform topology
 133 induced by \mathcal{V} . Thus $\tau = \tau'$, which means that τ is a uniform topology.

134

□