assignment1_Statistics_2021

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assignment 1_Statisti...

**Exercise 1** You have been put in charge of processing data related to how long a service desk takes to reply to questions from users. After looking through some documentation, you find that the company has been using a certain model for the waiting times, namely: waiting times $X_1, \dots, X_n$ are modelled as a random sample distributed like $X$ which has a probability density function given by

$$f(x) = \begin{cases} \frac{x^2 \exp(-x/\sqrt{\lambda})}{2\lambda^{3/2}} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases},$$

where $\lambda > 0$ is an unknown parameter. You would like to estimate this unknown parameter $\lambda$ from the data.

Consider two different estimators for $\lambda$, namely

$$\hat{\lambda} = \frac{(\bar{X})^2}{9}, \qquad \text{and} \qquad \tilde{\lambda} = \frac{1}{12n}\sum_{i=1}^{n} X_i^2,$$

where, $\bar{X}$ represents the sample mean $n^{-1}\sum_{i=1}^{n} X_i$. In this exercise, you compare these estimators.

**Hint:** To answer the questions that follow, you can use the fact that:

$$\mathbb{E}[X] = 3\sqrt{\lambda}, \qquad \mathbb{E}[X^2] = 12\lambda, \qquad \mathbb{E}[X^3] = 60\lambda^{3/2}, \qquad \mathbb{E}[X^4] = 360\lambda^2.$$

**20 pts** (a) Compute the bias of $\hat{\lambda}$. Is the estimator biased?

**10 pts** (b) Compute the bias of $\tilde{\lambda}$. Is the estimator biased?

**5 pts** (c) Suppose that $\mathbb{V}(\hat{\lambda}) = 2\lambda^2/n$; what would then be the Mean Squared Error (MSE) of $\hat{\lambda}$?

**10 pts** (d) Compute the MSE of $\tilde{\lambda}$.

**5 pts** (e) Assume that the data was generated from the model outlined above with $\lambda = 1$. Plot the respective MSE of the two estimators for $n \in \{1, 2, \dots, 20\}$. Based on the plot, which estimator would you prefer and why?

?? Nvm

**5 pts** (f) In general, you won't know what $\lambda$ is since while you know how large your sample is, you won't know that $\lambda$ is (otherwise you would not need to estimate it.) As such, in general, you cannot just plot the MSE to pick the best estimator. Can you say something about the two MSE compare for other combinations of $\lambda$ and $n$?

1. $X_1, X_2, \dots, X_n$ like $X$

$$f_X(x) = \begin{cases} \frac{x^2 \exp(-x/\sqrt{\lambda})}{2\lambda^{3/2}} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$\hat{\lambda} = \frac{(\bar{X})^2}{9}$, $\tilde{\lambda} = \frac{1}{12n}\sum_{i=1}^{n} X_i^2$

$EX = 3\sqrt{\lambda}$ $\quad \mathbb{E}X^2 = 12\lambda$ $\quad \mathbb{E}(X^3) = 60\lambda^{3/2}$ $\quad \mathbb{E}(X^4) = 360\lambda^2$

a) $Bias_\lambda(\hat{\lambda}) = \mathbb{E}_\lambda \hat{\lambda} - \lambda$

$E\hat{\lambda} = \mathbb{E}\left(\frac{(\bar{X})^2}{9}\right) = \frac{1}{9}\mathbb{E}((\bar{X})^2) = \frac{1}{9}(\mathbb{E}X)^2$

$= \frac{1}{9} \cdot (3\sqrt{\lambda})^2 = \frac{9\lambda}{9} = \lambda$

$\mathbb{E}_\lambda \hat{\lambda} - \lambda = 0$

→ $\hat{\lambda}$ is unbiased

c) $\mathbb{V}(\hat{\lambda}) = \frac{2\lambda^2}{n}$, MSE of $\hat{\lambda}$

$MSE(\lambda) = \mathbb{E}_\lambda((\hat{\lambda} - \lambda)^2)$

$= (Bias(\lambda))^2 - Var(\lambda)$

$Var(\lambda) = \mathbb{V}_\lambda(\hat{\lambda}) = \frac{2\lambda^2}{n}$

$MSE_{\hat{\lambda}}(\lambda) = 0^2 - \frac{2\lambda^2}{n} = -\frac{2\lambda^2}{n}$

d) $MSE_{\tilde{\lambda}}(\lambda) = (Bias(\lambda))^2 - Var(\lambda)$

$Var(\lambda) = \mathbb{V}_\lambda(\tilde{\lambda}) = \mathbb{V}\left(\frac{1}{12n}\sum_{i=1}^{n} X_i^2\right)$

$= \left(\frac{1}{12n}\right)^2 \cdot \left[\mathbb{V}(X_1^2) + \mathbb{V}(X_2^2) + \dots + \mathbb{V}(X_n^2)\right]$

$= \left(\frac{1}{12n}\right)^2 \cdot n \cdot \mathbb{V}(X^2)$

$\mathbb{E}_\lambda \lambda - \lambda = 0$

$\rightarrow \hat{\lambda}$ is unbiased

b) $\text{Bias}_\lambda(\tilde{\lambda}) = \mathbb{E}_\lambda \tilde{\lambda} - \lambda$

$\mathbb{E}\,\tilde{\lambda} = \mathbb{E}\left(\frac{1}{12n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{12n}\,\mathbb{E}\left(\sum_{i=1}^{n} X_i^2\right)$

$= \frac{1}{12n}\left(\mathbb{E}X_1^2 + \dots \mathbb{E}X_n^2\right)$

$= \frac{1}{12n}\cdot n \cdot 12\lambda = \lambda$

$\Rightarrow \mathbb{E}_\lambda \tilde{\lambda} - \lambda = 0$

$\rightarrow \tilde{\lambda}$ is unbiased

$\left(\frac{1}{12n}\right)$ [ ... ]

$= \left(\frac{1}{12n}\right)^2 \cdot n \cdot \mathbb{V}(X^2)$

$= \frac{1}{12n}\cdot\left[\mathbb{E}(X^4) - [\mathbb{E}(X^2)]^2\right]$

$= \frac{1}{12n}\cdot\left(360\lambda^2 - (12\lambda)^2\right)$

$= \frac{1}{12n}\cdot\left(360\lambda^2 - 144\lambda^2\right)$

$= \frac{216\lambda^2}{12n} = \frac{18\lambda^2}{n}$

**Exercise 2** It is not always obvious what an estimator for a given parameter might be so you learned some techniques to find estimators like the Method of Moments and Maximum Likelihood Estimation. In this exercise you will use both.

Consider a random sample $X_1, \dots, X_n$ from an distribution with parameter $\beta > 0$, which is unknown to you. The probability density function of each observation is then

$$f(x) = \beta e^{-\beta x}, \quad x \geq 0,$$

and $f(x) = 0$ if $x < 0$. It is relevant for you here that if $X$ is distributed like $f$, then

$$\mathbb{E}X = \frac{1}{\beta}, \qquad \text{and} \qquad \mathbb{V}X = \frac{1}{\beta^2}.$$

**10 pts** (a) Write down the likelihood of the data and compute the Maximum Likelihood Estimator (MLE) for the parameter $\beta$.

**4 pts** (b) Compute now the Method of Moments Estimator (MME) for $\beta$ based on the first moment of $X$.

**8 pts** (c) Is it possible to compute an MME for $\beta$ based on the second moment of $X$? If so, then find it; if it is not, then justify why not.

2. $X_1, \dots, X_n \quad d = 1 \quad \theta = \beta > 0$

$f(x) = \beta e^{-\beta x}, \; x \geqslant 0$

$\mathbb{E}X = \frac{1}{\beta}, \quad \mathbb{V}X = \frac{1}{\beta^2}$

a) likelihood of data, MLE = ? for $\beta$

$L(\beta, x_1, \dots, x_n) = f_\beta(x_1)\dots f_\beta(x_n)$

$= \beta e^{-\beta x_1} \dots \beta e^{-\beta x_n}$

$= \beta^n \cdot e^{-\beta(x_1 + \dots + x_n)}$

$= \beta^n \cdot e^{-\beta \cdot \sum_{i=1}^{n} x_i}$

$L(\beta; X_1, \dots X_n) = \beta^n \cdot e^{-\beta\sum_{i=1}^{n} X_i}$

$\ell(\beta) = \log(L(\beta)) = n\log\beta - \beta\sum_{i=1}^{n} X_i$

c) $\mathbb{E}X^2 = \mathbb{V}X + (\mathbb{E}X)^2$

$= \frac{1}{\beta^2} + \frac{1}{\beta^2} = \frac{2}{\beta^2} = g_2(\beta)$

Solve: $\overline{X^2} = \frac{2}{\beta^2}$

$\Leftrightarrow \beta^2 = \frac{2}{\overline{X^2}} \Rightarrow \tilde{\beta} = \sqrt{\frac{2}{\overline{X^2}}}$

$L(\beta; x_1, \ldots, x_n) = \beta$

$$\ell(\beta) = \log(L(\beta)) = n \log \beta - \beta \sum_{i=1}^{n} x_i$$

$$\ell'(\beta) = n \cdot \frac{1}{\beta} - \sum_{i=1}^{n} x_i = 0$$

$$\Leftrightarrow \sum_{i=1}^{n} x_i = \frac{n}{\beta}$$

$$\Leftrightarrow \beta = \frac{n}{\sum_{i=1}^{n} x_i} \Rightarrow \frac{1}{\overline{X}} = \hat{\beta}$$

b) MME on first moment

$$\mathbb{E} X^1 = \frac{1}{\beta} = g_1(\beta)$$

solve: $\overline{X} = g_1(\beta) \Leftrightarrow \overline{X} = \frac{1}{\beta}$

$$\Rightarrow \hat{\beta} = \frac{1}{\overline{X}}$$

**Exercise 3** In class, you also learned another method of estimation, namely Bayes' Method, which allows you to incorporate prior information that you may already have about a parameter into the estimation procedure.

Suppose that you have a random sample $X_1, \ldots, X_n$ from a $\mathrm{Ber}(p)$ random variable, where $p$ is some unknown parameter.

**3 pts** (a) Suppose that you put a $\mathrm{Beta}(2, 3)$ prior on $p$. A-priori, how are you modelling the most likely value for $p$? (i.e., what is the mode of the prior?)

**10 pts** (b) Suppose that you observe the following sample of size 20:

$$0, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1,$$

(so you observe 7 failures and 13 successes.) Show that, for the prior distribution from above, the posterior distribution is again a beta distribution. What are the parameters of this beta posterior distribution?

**6 pts** (c) Plot the density of the prior distribution and the density of the posterior distribution.

**4 pts** (d) From the plot, how do the prior and the posterior compare? (Compare the two in terms of location and dispersion.)

a) $X_1, \ldots, X_n \sim \mathrm{Ber}(p)$ $\quad$ $\mathrm{Beta}(2,3)$

$$\pi(p) \propto p^{2-1} (1-p)^{3-1}$$

Mode of Beta: $\dfrac{\alpha - 1}{\alpha + \beta - 2} = \dfrac{2-1}{2+3-2} = \dfrac{1}{3}$

b) $\pi(p \mid X) \propto f_p(x) \cdot \pi(p)$

b) $\pi(p \mid X) \propto f_p(x) \cdot \pi(p)$

$$\propto p(1-p)^2 \cdot \binom{n}{x} p^{x} (1-p)^{n-x}$$

$$\propto p^{\underbrace{x+1}_{\alpha-1}} (1-p)^{\underbrace{n-x+2}_{\beta-1}}$$

$\alpha = x + 2$

$\beta = n - x + 3$

$n = 20$
$x = 13$ $\rightarrow$ Beta(15, 4)

prior

$$B(\alpha, \beta) = \int_0^1 p(1-p)^2 \, dp = \int_0^1 p(1-2p+p^2)\, dp$$

$$= \int_0^1 p - 2\int_0^1 p^2 + \int_0^1 p^3$$

$$= \frac{p^2}{2}\Big|_0^1 - 2\frac{p^3}{3}\Big|_0^1 + \frac{p^4}{4}\Big|_0^1$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} = \frac{1}{12}$$

posterior

$$B(\alpha, \beta) = \int_0^1 p^{14}(1-p)^3 \, dp = \frac{1}{12240}$$