Important Math Review Points

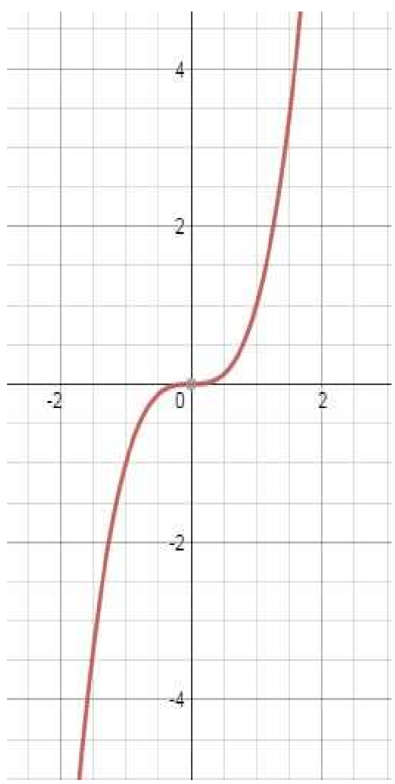
October 25, 2015

1 Increasing/Nondecreasing functions

A function is increasing if its graph climbs steadily upward. More precisely:

Defintion. A function f on the real line is *increasing* (*nondecresing*) if, whenever $x_1 < x_2$ ($x_1 \leq x_2$), $f(x_1) < f(x_2)$ ($f(x_1) \leq f(x_2)$).

Example: $f(x) = x^3$ is an increasing function.

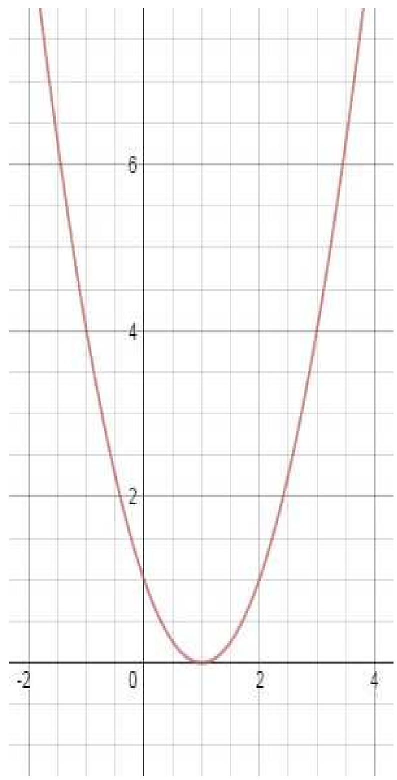


Question Is $f(x) = x^2$ increasing?

2 Eventually Nondecreasing Functions

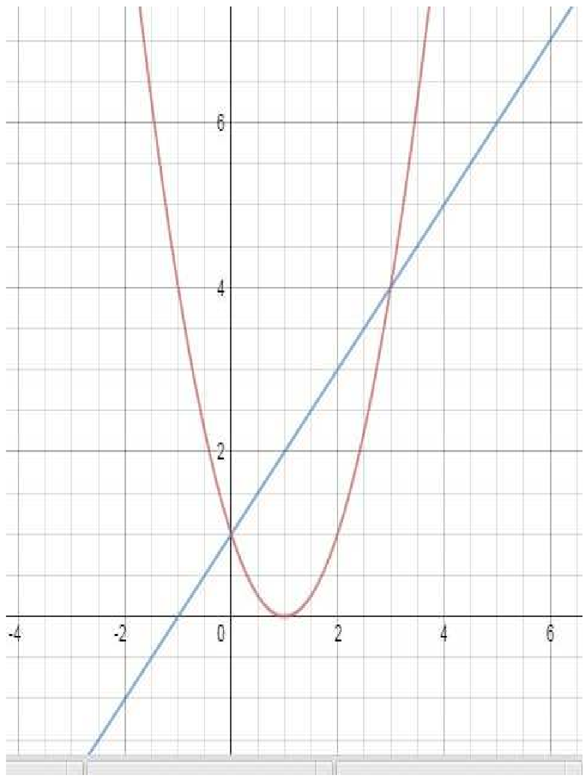
A function is eventually nondecreasing if for all values beyond a certain point on the x-axis, the graph steadily climbs. More precisely,
Definition. A function f is *eventually nondecreasing* if for some real number x_0, f is increasing on $[x_0, \infty)$. In other words, for some x_0 we have that whenever $x_0 \leq x_1 \leq x_2$, then $f(x_0) \leq f(x_1) \leq f(x_2)$.

Example: $f(x) = (x-1)^2$.



3 Growth Rates of Functions

Some functions grow faster than others. Example: $f(x) = (x-1)^2$ and $g(x) = x+1$. Notice when $x = 3$, the quadratic function overtakes the linear function. We say f *is asymptotically greater than* g. These ideas will be used to evaluate and compare running times of algorithms.



4 Mathematical Induction

The idea: Suppose you wish to prove that some statement $\phi(n)$, which asserts something about each whole number n, is true for every n. For example, to prove that for all $n \geq 0$, $n < 2^n$, we would use "$n < 2^n$" as our statement $\phi(n)$. We wish to show that this statement holds for every n. Suppose now that we can prove two things:

(A) that $\phi(0)$ is true (in our example, this would mean that we can prove $0 < 2^0$);

(B) that, for any n, if $\phi(n)$ happens to be true, then $\phi(n+1)$ must also be true (in our example, this would mean that, if it happens to be true that $n < 2^n$, then it must be true that $n + 1 < 2^{n+1}$).

Mathematical Induction says that, if you can prove both (1) and (2), then you have proven that, for every n, $\phi(n)$ is indeed true.

5 Standard Induction

Suppose $\phi(n)$ is a statement depending on n. If

1. $\phi(0)$ is true, and
2. under the assumption that $n \geq 0$ and $\phi(n)$ is true, you can prove that $\phi(n+1)$ is also true,

then $\phi(n)$ holds true for all natural numbers n.

Here is a slight generalization:

General Induction Suppose $\phi(n)$ is a statement depending on n and suppose $k \geq 0$ is an integer. If

1. $\phi(k)$ is true, and
2. under the assumption that $n \geq k$ and $\phi(n)$ is true, you can prove that $\phi(n+1)$ is also true,

then $\phi(n)$ holds true for all natural numbers $n \geq k$.

In General Induction, the step in the proof where $\phi(k)$ is verified is called the *Basis Step*. The second step, where $\phi(n+1)$ is proved assuming $\phi(n)$, is called the *Induction Step*. As we reason during this second step, we will typically need to make use of $\phi(n)$ as an assumption; in this context, $\phi(n)$ is called the *induction hypothesis*.

6 Example of Mathematical Induction

We will prove that for every positive integer n,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

We take this formula to be the formula $\phi(n)$ that we will use in the induction; that is, we let $\phi(n)$ be the statement

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

For the Basis Step, notice that $\phi(1)$ is the statement

$$\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$$

which is obviously true. For the Induction Step, we assume $\phi(n)$ is true, and we prove $\phi(n+1)$. $\phi(n+1)$ is the following statement:

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

To prove $\phi(n+1)$ is true, we follow these steps:

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= \Big(\sum_{i=1}^{n} i\Big) + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \qquad \text{(by Induction Hypothesis)} \\
&= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}$$

7 The Division Algorithm

(1) Suppose m, n are positive integers. Dividing n by m gives a quotient and remainder.

(2) Example: Divide 17 by 3: Quotient is 5 and remainder is 2. Using integer division and Java mod notation, we can write:

1. quotient $= 17/3$
2. remainder $= 17\,\%\,3$

Using mathematical notation:

1. quotient $= \lfloor 17/3 \rfloor$
2. remainder $= 17 \mod 3$

We can write:

$$17 = \text{quotient} \cdot 3 + \text{remainder} = \lfloor 17/3 \rfloor \cdot 3 + 17 \bmod 3.$$

(3) In general, for any positive integers m, n, there are unique q, r so that

$$n = mq + r \qquad \text{and} \qquad 0 \leq r < m.$$

In other words

$$n = m \cdot \lfloor \frac{n}{m} \rfloor + n \bmod m.$$

8 Calculus

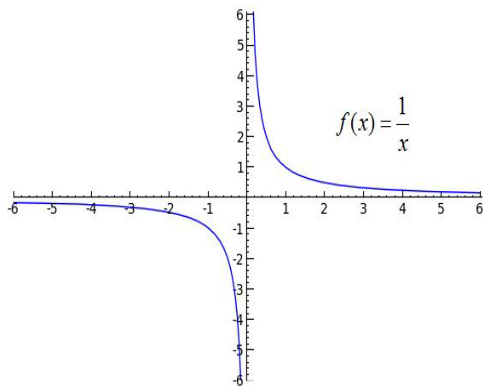
For this Algorithms course, it is not necessary to have an in-depth understanding of calculus, but it *is* important to know a few of the simple concepts and formulas, which we review here. The two concepts to be familiar with are:

(1) Limits at infinity. Example: $\lim_{n\to\infty}(n+1)/n^2 = 0$.

(2) Derivative formulas. Example: $\frac{d}{dx}x^2 - x + 1 = 2x - 1$.

Limits at Infinity

Consider the following graph of $f(x) = \frac{1}{x}$:



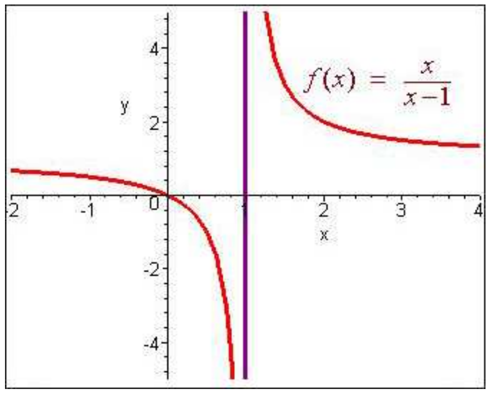
As x gets bigger and bigger, $f(x)$ gets closer and closer to 0. We write

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Since we will be working with the set $\mathbf{N}$ of natural numbers, instead of the set $\mathbf{R}$ of real numbers, we will express this limit as

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

The following is the graph of $f(x) = \frac{x}{x-1}$:



Here, as x gets large, the graph approaches the line $y = 1$. We write:

$$\lim_{x\to\infty} \frac{x}{x-1} = 1$$

or when we are dealing only with natural numbers:

$$\lim_{n\to\infty} \frac{n}{n-1} = 1$$

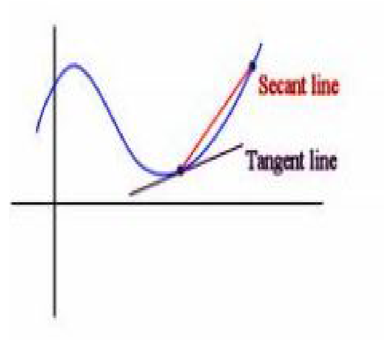
We can compute this limit algebraically by factoring from numerator and denominator the reciprocal of the highest power of x that occurs in the expression:

$$\begin{aligned}
\lim_{n\to\infty} \frac{n}{n-1} &= \lim_{n\to\infty} \left(\frac{n}{n-1} \cdot \frac{1/n}{1/n}\right) \\
&= \lim_{n\to\infty} \frac{1}{1-\frac{1}{n}} \\
&= 1.
\end{aligned}$$

Derivatives

The derivative of a function $f(x)$, which is written in any of these ways: $f'(x)$, $\frac{d}{dx}f(x)$, $\frac{dy}{dx}$, represents the *slope of the line tangent to the graph of* f *at the point* (x, y).

For example:



There are a number of convenient formulas for computing derivatives of familiar functions:

(1) $\frac{d}{dx}\, a = 0$ for any real number a.

(2) $\frac{d}{dx}\, x^r = rx^{r-1}$, for any real number $r \neq 0$.

(3) $\frac{d}{dx}\, 2^x = 2^x \, \ln 2$

(4) $\frac{d}{dx}\, \log x = \frac{1}{x} \cdot \log e$

(5) For any functions $f(x), g(x)$ (whose derivatives exist) and real numbers a, b:

  (a) (Linearity Rule) $\frac{d}{dx}(af(x) + bg(x)) = a\frac{d}{dx}\, f(x) + b\frac{d}{dx}\, g(x)$

  (b) (Product Rule) $\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot \frac{d}{dx}\, g(x) + g(x) \cdot \frac{d}{dx}\, f(x)$

  (c) (Reciprocal Rule) $\frac{d}{dx}(\frac{1}{f(x)}) = \frac{-f'(x)}{[f(x)]^2}$

For example:

(a) $\frac{d}{dx}\, ax = a$

(b) $\frac{d}{dx}\, ax^2 = 2ax$

(c) $\frac{d}{dx}\, ax^3 = 3ax^2$

(d) $\frac{d}{dx}\, \frac{1}{x} = \frac{-1}{x^2}$.

(e) $\frac{d}{dx}\, \sqrt{x} = \frac{d}{dx}\, x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$.

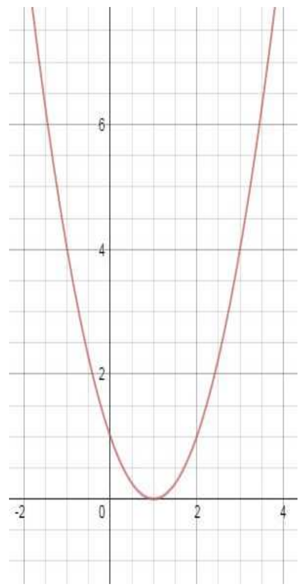
Eventually Increasing Functions and Derivatives

Fact. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function (and assume its derivative exists everywhere). For any interval (a, b) (where a could possibly be $-\infty$, b could be ∞):

(1) if $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b)

(2) if $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b).

Example: The following is the graph of $f(x) = (x - 1)^2$.



We can see from the graph that f is increasing when $x > 1$ and f is decreasing when $x < 1$; in particular, $f(x)$ is eventually increasing.

These observations could also be made by examining the derivative of f: Since $f(x) = x^2 - 2x + 1$, we have

$$f'(x) = 2x - 2 = 2(x - 1).$$

Now $2(x - 1)$ equals 0 when $x = 1$, is positive when $x > 1$ and is negative when $x < 1$. By the Fact, we may conclude:

(1) $f(x)$ is increasing on $(1, \infty)$

(2) $f(x)$ is decreasing on $(-\infty, 1)$.