Introduction to ARMA processes, Causality

Exercise 1 (Moving average processes)

Consider the MA(1) process

$$y_t = \varepsilon_t + \beta_1 \varepsilon_{t-1}, \tag{1}$$

where $\{\varepsilon_t\}_{t \in \mathbb{Z}} \sim \mathrm{IID}(0, \sigma^2)$ is a white noise process.

(a) Calculate the mean and variance of the process.

(b) Calculate the autocovariance and autocorrelation functions of the process.

(c) Now consider the more general MA(q) process given by

$$y_t = \varepsilon_t + \beta_1 \varepsilon_{t-1} + \cdots + \beta_q \varepsilon_{t-q}. \tag{2}$$

Show that its autocovariance function cuts off after q lags. That is, the ACovF becomes zero after q lags.

(d) Under which conditions is the process weakly stationary?

Exercise 2 (Autoregressive processes)

Consider the process

$$y_1 = \varepsilon_1$$
$$y_t = \alpha_1 y_{t-1} + \varepsilon_t, \qquad t = 2, 3, \ldots$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \mathrm{IID}(0, \sigma^2)$ is white noise and $|\alpha_1| < 1$ is a constant parameter.

(a) Show that the process can be rewritten in the following form

$$y_t = \sum_{j=0}^{h-1} \alpha_1^j \varepsilon_{t-j} + \alpha_1^h y_{t-h} = \sum_{j=0}^{t-1} \alpha_1^j \varepsilon_{t-j} \tag{3}$$

for $0 \leq h < t$.

(b) Calculate the mean and variance of the process.

(c) Can you say whether $\{y_t\}_t$ is weakly stationary?

(d) Argue that for large t,

$$\mathrm{Var}(y_t) \approx \frac{\sigma^2}{1 - \alpha_1^2} \tag{4}$$

and

$$\mathrm{Corr}(y_t, y_{t-h}) \approx \alpha_1^h, \qquad 0 \leq h < t \tag{5}$$

holds, in a sense that $\{y_t\}_t$ is "asymptotically stationary". Use the fact that

$$\mathrm{Corr}(y_t, y_{t-h}) = \alpha_1^h \left[\frac{\mathrm{Var}(y_{t-h})}{\mathrm{Var}(y_t)} \right]^{1/2} \tag{6}$$

(e) Now suppose $y_1 = \varepsilon_1/\sqrt{1 - \alpha_1^2}$. Is this process stationary?

Exercise 3 (Autoregressive moving average processes)

Combining the processes known from the previous two exercises, we obtain the general class of *autoregressive moving average* or ARMA(p,q) processes, i.e.

$$y_t = \alpha_1 y_{t-1} + \cdots + a_p y_{t-p} + \beta_1 \varepsilon_{t-1} + \cdots + \beta_q \varepsilon_{t-q} + \varepsilon_t. \tag{7}$$

A more convenient notation can be achieved by using the lag operator L. The same process can then be written as

$$\alpha(L)y_t = \beta(L)\varepsilon_t$$

where

$$\alpha(L) = 1 - \alpha_1 L - \cdots - a_p L^p$$
$$\beta(L) = 1 + \beta_1 L + \cdots + \beta_q L^q.$$

(a) Consider the ARMA(p,q) process in equation (10). On which parameters does the stationarity or non-stationarity of the process depend? (No calculations are required)

(b) For now consider an ARMA(1,1) process, i.e.

$$y_t = \alpha_1 y_{t-1} + \beta_1 \varepsilon_{t-1} + \varepsilon_t \tag{8}$$

Compute the autocovariance and autocorrelation functions of the process.

Hint: Use an infinite sum to represent the AR-commponent since y_0 is not defined. You may assume that y_t is weakly stationary for the computation of the ACF.

Exercise 4 (Causality)

(a) Give the definition of causal ARMA process?

(b) Now, Consider the AR(2) process

$$y_t = 0.7y_{t-1} - 0.1y_{t-2} + \varepsilon_t, \quad \varepsilon_t \overset{iid}{\sim} GWN(0, \sigma^2), \quad t \in \mathbb{Z}. \tag{9}$$

Is it a causal process?