Introduction to ARMA processes, Causality

Exercise 1 (Moving average processes)

Consider the MA(1) process

$$y_t = \varepsilon_t + \beta_1 \varepsilon_{t-1}, \tag{1}$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \text{IID}(0, \sigma^2)$ is a white noise process.

(a) Calculate the mean and variance of the process.

$$\begin{aligned}
E(y_t) &= E(\varepsilon_t) + \beta_1 E(\varepsilon_{t-1}) \\
&= 0 \\
\text{Var}(y_t) &= \text{Var}(\varepsilon_t + \beta_1 \varepsilon_{t-1}) \\
&= \underbrace{\text{Var}(\varepsilon_t)}_{=\sigma^2} + \beta_1^2 \underbrace{\text{Var}(\varepsilon_{t-1})}_{=\sigma^2} + 2\beta_1 \underbrace{\text{Cov}(\varepsilon_t, \varepsilon_{t-1})}_{=0} \\
&= \sigma^2 + \beta_1^2 \sigma^2 \\
&= \sigma^2 (1 + \beta_1^2)
\end{aligned}$$

(b) Calculate the autocovariance and autocorrelation functions of the process.

$$\begin{aligned}
\gamma(h) = \text{Cov}(y_t, y_{t-h}) &= \text{Cov}(\varepsilon_t + \beta_1 \varepsilon_{t-1}, \varepsilon_{t-h} + \beta_1 \varepsilon_{t-h-1}) \\
&= \text{Cov}(\varepsilon_t, \varepsilon_{t-h}) + \beta_1 \text{Cov}(\varepsilon_t, \varepsilon_{t-h-1}) + \beta_1 \text{Cov}(\varepsilon_{t-1}, \varepsilon_{t-h}) + \beta_1^2 \text{Cov}(\varepsilon_{t-1}, \varepsilon_{t-1-h}) \\
&= \begin{cases} \sigma^2(1+\beta_1^2), & h = 0, \\ \beta_1 \sigma^2, & |h| = 1, \\ 0, & |h| > 1. \end{cases}
\end{aligned}$$

It follows that the stationarity assumption was correct. Hence

$$\rho(h) = \gamma(h)/\gamma(0) = \begin{cases} 1, & h = 0, \\ \beta_1/(1+\beta_1^2), & |h| = 1, \\ 0, & |h| > 1. \end{cases}$$

(c) Now consider the more general MA(q) process given by

$$y_t = \varepsilon_t + \beta_1 \varepsilon_{t-1} + \cdots + \beta_q \varepsilon_{t-q}. \tag{2}$$

Show that its autocovariance function cuts off after q lags. That is, the ACovF becomes zero after q lags.

First, rewrite y_t

$$y_t = \varepsilon_t + \beta_1 \varepsilon_{t-1} + ... + \beta_q \varepsilon_{t-q}. \tag{3}$$

Defining $\beta_0 := 1$ and write the equation above as a sum, i.e.

$$y_t = \beta_0 \varepsilon_t + \beta_1 \varepsilon_{t-1} + ... + \beta_q \varepsilon_{t-q} = \sum_{i=0}^{q} \beta_i \varepsilon_{t-i} \tag{4}$$

Now rewrite y_{t-h}

$$y_{t-h} = \varepsilon_{t-h} + \beta_1 \varepsilon_{t-h-1} + ... + \beta_q \varepsilon_{t-h-q} = \varepsilon_{t-h} + \sum_{i=1}^{q} \beta_i \varepsilon_{t-h-i}. \tag{5}$$

$$\begin{aligned}
\mathrm{Cov}(y_t, y_{t-h}) &= \mathrm{Cov}\Big(\sum_{i=0}^{q} \beta_i \varepsilon_{t-i}, \varepsilon_{t-h} + \sum_{i=1}^{q} \beta_i \varepsilon_{t-h-i}\Big) \\
&= \underbrace{\mathrm{Cov}\Big(\sum_{i=0}^{q} \beta_i \varepsilon_{t-i}, \varepsilon_{t-h}\Big)}_{(i)} + \underbrace{\mathrm{Cov}\Big(\sum_{i=0}^{q} \beta_i \varepsilon_{t-i}, \sum_{i=1}^{q} \beta_i \varepsilon_{t-h-i}\Big)}_{(ii)}
\end{aligned}$$

We exploit the fact that $\mathrm{Cov}(\varepsilon_k, \varepsilon_l) = 0$, $k \neq l$ and $\mathrm{Var}(\varepsilon_t) = \sigma^2$.

For $h = 0:$ $\quad$ (i)$= \sigma^2 = \beta_0^2 \sigma^2$ $\quad$ and $\quad$ (ii)$= \sum_{i=1}^{q} \beta_i^2 \sigma^2$ $\quad \Rightarrow \quad \sigma^2 \sum_{i=0}^{q} \beta_i^2$

For $h = 1:$ $\quad$ (i)$= \beta_1 \sigma^2 = \beta_0 \beta_1 \sigma^2$ $\quad$ and $\quad$ (ii)$= \sum_{i=2}^{q} \beta_i \beta_{i-1} \sigma^2$ $\quad \Rightarrow \quad \sigma^2 \sum_{i=1}^{q} \beta_i \beta_{i-1}$

...

Continuing this process yields:

$$\begin{aligned}
\mathrm{Cov}(y_t, y_{t-h}) &= \begin{cases} \sigma^2 \sum_{i=0}^{q} \beta_i^2, & h = 0, \\ \sigma^2 \sum_{i=1}^{q} \beta_i \beta_{i-1}, & |h| = 1, \\ \vdots & \\ \sigma^2 \sum_{i=|h|}^{q} \beta_i \beta_{i-|h|}, & |h| \leq q, \\ 0, & |h| > q. \end{cases} \\
&= \begin{cases} \sigma^2 \sum_{i=|h|}^{q} \beta_i \beta_{i-|h|}, & |h| \leq q, \\ 0, & |h| > q. \end{cases} \\
&= \begin{cases} \sigma^2 \sum_{i=0}^{q-|h|} \beta_i \beta_{i+|h|}, & |h| \leq q, \\ 0, & |h| > q. \end{cases}
\end{aligned}$$

(d) Under which conditions is the process weakly stationary?

The MA(q) process is stationary for any finite $q \in \mathbb{N}$ and finite parameters $\beta_1, \ldots, \beta_q$.

Exercise 2 (Autoregressive processes)

Consider the process

$$
\begin{aligned}
y_1 &= \varepsilon_1 \\
y_t &= \alpha_1 y_{t-1} + \varepsilon_t, \qquad t = 2, 3, \ldots
\end{aligned}
$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}} \sim \mathrm{IID}(0, \sigma^2)$ is white noise and $|\alpha_1| < 1$ is a constant parameter.

(a) Show that the process can be rewritten in the following form

$$
y_t = \sum_{j=0}^{h-1} \alpha_1^j \varepsilon_{t-j} + \alpha_1^h y_{t-h} = \sum_{j=0}^{t-1} \alpha_1^j \varepsilon_{t-j} \tag{6}
$$

for $0 \leq h < t$.

$$
\begin{aligned}
y_t &= \alpha_1 y_{t-1} + \varepsilon_t \\
&= \alpha_1(\alpha_1 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\
&= \alpha_1^2 y_{t-2} + \alpha_1 \varepsilon_{t-1} + \varepsilon_t \\
&= \alpha_1^3 y_{t-3} + \alpha_1^2 \varepsilon_{t-2} + \alpha_1 \varepsilon_{t-1} + \varepsilon_t \\
&= \cdots \\
&= \sum_{j=0}^{h-1} \alpha_1^j \varepsilon_{t-j} + \alpha_1^h y_{t-h} \\
&= \cdots \\
&= \sum_{j=0}^{t-1} \alpha_1^j \varepsilon_{t-j}.
\end{aligned}
$$

(b) Calculate the mean and variance of the process.

Using (a) it follows

$$
\begin{aligned}
E(y_t) &= E\left(\sum_{j=0}^{t-1} \alpha_1^j \varepsilon_{t-j}\right) \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Var}(y_t) &= \mathrm{Var}\left(\sum_{j=0}^{t-1} \alpha_1^j \varepsilon_{t-j}\right) \\
&= \sum_{j=0}^{t-1} \alpha_1^{2j} \underbrace{\mathrm{Var}(\varepsilon_{t-j})}_{\sigma^2} \\
&= \sigma^2 \sum_{j=0}^{t-1} \alpha_1^{2j} \\
&\overset{(*)}{=} \sigma^2 \frac{1 - \alpha_1^{2t}}{1 - \alpha_1^2}.
\end{aligned}
$$

(c) Can you say whether $\{y_t\}_t$ is weakly stationary?

The variance is obviously time-dependent, i.e. a function of t. Therefore, the process is **not** weakly stationary!

(d) Argue that for large t,

$$\operatorname{Var}(y_t) \approx \frac{\sigma^2}{1-\alpha_1^2} \tag{7}$$

and

$$\operatorname{Corr}(y_t, y_{t-h}) \approx \alpha_1^h, \qquad 0 \leq h < t \tag{8}$$

holds, in a sense that $\{y_t\}_t$ is "asymptotically stationary". Use the fact that

$$\operatorname{Corr}(y_t, y_{t-h}) = \alpha_1^h \left[\frac{\operatorname{Var}(y_{t-h})}{\operatorname{Var}(y_t)}\right]^{1/2} \tag{9}$$

This can be obtained using the well-known convergence result for the geometric series, i.e.,

$$\begin{aligned}
\lim_{t\to\infty} \operatorname{Var}(y_t) &= \lim_{t\to\infty} \sigma^2 \sum_{j=0}^{t-1} \alpha_1^{2j} \\
&= \sigma^2 \sum_{j=0}^{\infty} \alpha_1^{2j} \\
&\overset{(**)}{=} \frac{\sigma^2}{1-\alpha_1^2}
\end{aligned}$$

For the correlation it follows

$$\begin{aligned}
\lim_{t\to\infty} \operatorname{Corr}(y_t, y_{t-h}) &= \lim_{t\to\infty} \alpha_1^h \sqrt{\frac{\operatorname{Var}(y_{t-h})}{\operatorname{Var}(y_t)}} \\
&\overset{(b)}{=} \alpha_1^h \underbrace{\lim_{t\to\infty} \sqrt{\frac{1-\alpha^{2(t-h)}}{1-\alpha^{2t}}}}_{=1} \\
&= \alpha_1^h
\end{aligned}$$

Under the assumption of (in principle) infinite past, our process is called an AR(1) process.

(e) Now suppose $y_1 = \varepsilon_1/\sqrt{1-\alpha_1^2}$. Is this process stationary?

From (a) we get

$$\begin{aligned}
y_t &= \sum_{j=0}^{t-2} \alpha_1^j \varepsilon_{t-j} + \alpha_1^{t-1} y_1 \\
&= \sum_{j=0}^{t-2} \alpha_1^j \varepsilon_{t-j} + \frac{\alpha_1^{t-1}}{\sqrt{1-\alpha_1^2}} \varepsilon_1.
\end{aligned}$$

For the expectation and the variance it follows

$$
\begin{aligned}
E(y_t) &= 0 \\
\mathrm{Var}(y_t) &= \mathrm{Var}\left(\sum_{j=0}^{t-2} \alpha_1^j \varepsilon_{t-j} + \frac{\alpha_1^{t-1}}{\sqrt{1-\alpha_1^2}}\varepsilon_1\right) \\
&= \sum_{j=0}^{t-2} \alpha_1^{2j} \mathrm{Var}(\varepsilon_{t-j}) + \frac{\alpha_1^{2(t-1)}}{1-\alpha_1^2}\mathrm{Var}(\varepsilon_1) \\
&= \sigma^2 \left[\sum_{j=0}^{t-2} \alpha_1^{2j} + \frac{\alpha_1^{2(t-1)}}{1-\alpha_1^2}\right] \\
&\overset{(*)}{=} \sigma^2 \left[\frac{1-\alpha_1^{2(t-1)}}{1-\alpha_1^2} + \frac{\alpha_1^{2(t-1)}}{1-\alpha_1^2}\right] \\
&= \sigma^2 \frac{1}{1-\alpha_1^2},
\end{aligned}
$$

which is constant and does not depend on t. Furthermore, for the covariance it follows (like in (d))

$$
\begin{aligned}
\mathrm{Cov}(y_t, y_{t-h}) &\overset{(a)}{=} \mathrm{Cov}\left(\sum_{j=0}^{h-1} \alpha_1^j \varepsilon_{t-j} + \alpha_1^h y_{t-h}, y_{t-h}\right) \\
&= \sum_{j=0}^{h-1} \alpha_1^j \mathrm{Cov}(\varepsilon_{t-j}, y_{t-h}) + \mathrm{Cov}(\alpha_1^h y_{t-h}, y_{t-h}) \\
&= \alpha_1^h \mathrm{Var}(y_{t-h}) \\
&= \sigma^2 \frac{\alpha_1^h}{1-\alpha_1^2},
\end{aligned}
$$

which only depends on h but not on t. Note that the process $(y_t)_t$ is not defined for $t < 0$. If we ignore this, we can call it stationary.

Exercise 3 (Autoregressive moving average processes)

Combining the processes known from the previous two exercises, we obtain the general class of *autoregressive moving average* or ARMA(p,q) processes, i.e.

$$y_t = \alpha_1 y_{t-1} + \cdots + a_p y_{t-p} + \beta_1 \varepsilon_{t-1} + \cdots + \beta_q \varepsilon_{t-q} + \varepsilon_t. \tag{10}$$

A more convenient notation can be achieved by using the lag operator L. The same process can then be written as

$$\alpha(L) y_t = \beta(L) \varepsilon_t$$

where

$$\begin{aligned} \alpha(L) &= 1 - \alpha_1 L - \cdots - a_p L^p \\ \beta(L) &= 1 + \beta_1 L + \cdots + \beta_q L^q. \end{aligned}$$

(a) Consider the ARMA(p,q) process in equation (10). On which parameters does the stationarity or non-stationarity of the process depend? (No calculations are required)

The stationarity does only depend on $\alpha(L)$ and not on $\beta(L)$ (for finite $\beta_1, \ldots, \beta_q$ and q). If all roots solving $a(\lambda) = 1 - \alpha_1\lambda - \alpha_2\lambda^2 - \ldots - \alpha_p\lambda^p = 0$ are outside the unit circle, then the process is stationary.

(b) For now consider an ARMA(1,1) process, i.e.

$$y_t = \alpha_1 y_{t-1} + \beta_1 \varepsilon_{t-1} + \varepsilon_t \tag{11}$$

Compute the autocovariance and autocorrelation functions of the process.

Hint: Use an infinite sum to represent the AR-commponent since y_0 is not defined. You may assume that y_t is weakly stationary for the computation of the ACF.

$$E(y_t) = E\left(\varepsilon_t + \sum_{j=1}^{\infty} \alpha_1^{j-1}(\alpha_1 + \beta_1)\varepsilon_{t-j}\right) = 0.$$

To obtain the autocovariance, we multiply the model equation for y_t by y_{t-h} and take expectations, yielding

$$E(y_t y_{t-h}) = \alpha_1 \underbrace{E(y_{t-1} y_{t-h})}_{=:(i)} + \beta_1 \underbrace{E(y_{t-h}\varepsilon_{t-1})}_{=:(ii)} + \underbrace{E(\varepsilon_t y_{t-h})}_{=:(iii)}.$$

For $h = 0$:

$$\begin{aligned} (i) &= E(y_{t-1} y_t) = \gamma(1), \\ (ii) &= E(y_t \varepsilon_{t-1}) = E((\alpha_1 y_{t-1} + \beta_1 \varepsilon_{t-1} + \varepsilon_t)\varepsilon_{t-1}) \\ &= E(\alpha_1(\alpha_1 y_{t-2} + \beta_1 \varepsilon_{t-2} + \varepsilon_{t-1})\varepsilon_{t-1}) + \beta_1 \sigma^2 \\ &= \beta_1 \sigma^2 + \alpha_1 \sigma^2, \\ (iii) &= E(y_t \varepsilon_t) = E((\alpha_1 y_{t-1} + \beta_1 \varepsilon_{t-1} + \varepsilon_t)\varepsilon_t) = \mathrm{Var}(\varepsilon_t) = \sigma^2, \end{aligned}$$

yielding

$$\gamma(0) = \alpha_1\gamma(1) + (1 + \alpha_1\beta_1 + \beta_1^2)\sigma^2.$$

For $h = 1$:

$$\begin{aligned}
(i) &= E(Y_{t-1}y_{t-1}) = \gamma(0),\\
(ii) &= E(y_{t-1}\varepsilon_{t-1}) = \sigma^2,\\
(iii) &= E(\varepsilon_t y_{t-1}) = 0,
\end{aligned}$$

yielding

$$\gamma(1) = \alpha_1\gamma(0) + \beta_1\sigma^2.$$

For $h \geq 2$:

$$\begin{aligned}
\gamma(2) &= \alpha_1\gamma(1) + \beta_1\underbrace{E(y_{t-2}\varepsilon_{t-1})}_{0} + \underbrace{E(\varepsilon_t y_{t-2})}_{0}\\
&\vdots\\
\gamma(h) &= \alpha_1\gamma(h-1) = \alpha_1^{h-1}\gamma(1). \quad \text{through backwards substitution}
\end{aligned}$$

In order to derive an explicit expression for $\gamma(0)$ and $\gamma(1)$, we have to solve a linear system given in the case of $h = 0$ and $h = 1$. Insert $\gamma(1)$ in $\gamma(0)$ yields:

$$\begin{aligned}
\gamma(0) &= \alpha_1[\alpha_1\gamma(0) + \beta_1\sigma^2] + \sigma^2[1 + \alpha_1\beta_1 + \beta_1^2]\\
&= \alpha_1^2\gamma(0) + \sigma^2[1 + 2\alpha_1\beta_1 + \beta_1^2],
\end{aligned}$$

and by rearranging

$$\gamma(0) = \frac{\sigma^2[1 + 2\alpha_1\beta_1 + \beta_1^2]}{1 - \alpha_1^2}.$$

Inserting this result back into $\gamma(0)$ yields

$$\begin{aligned}
\gamma(1) &= \alpha_1\gamma(0) + \beta_1\sigma^2\\
&= \sigma^2\frac{\alpha_1 + \beta_1 + \alpha_1^2\beta_1 + \alpha_1\beta_1^2}{1 - \alpha_1^2}\\
&= \sigma^2\frac{(1 + \alpha_1\beta_1)(\alpha_1 + \beta_1)}{1 - \alpha_1^2}.
\end{aligned}$$

Finally, altogether for $h \geq 1$:

$$\gamma(h) = \sigma^2\frac{(1 + \alpha_1\beta_1)(\alpha_1 + \beta_1)}{1 - \alpha_1^2}\alpha_1^{h-1}.$$

Dividing through $\gamma(0)$ yields the ACF

$$\rho(h) = \frac{(1+\alpha_1\beta_1)(\alpha_1+\beta_1)}{1+2\alpha_1\beta_1+\beta_1^2}\alpha_1^{h-1}, \quad h \geq 1.$$

Exercise 4 (Causality)

(a) Give the definition of causal ARMA process?

An ARMA process

$$\alpha(L)y_t = \beta(L)\varepsilon_t$$

is causal iff

$$\exists(\psi_j)_{j\in J} \quad \text{with} \quad \sum_{j=0}^{\infty}|\psi_j| < \infty : y_t = \sum_{j=0}^{\infty}\psi_j\epsilon_{t-j}$$

J is an index set. In other words, we can transform the ARMA process into a form in which it is represented by the infinite past of innovations. This form is helpful if we want to derive the stochastic properties of the time series process. Furthermore, the process is stationary if it is causal.

(b) Now, Consider the AR(2) process

$$y_t = 0.7y_{t-1} - 0.1y_{t-2} + \varepsilon_t, \quad \varepsilon_t \overset{iid}{\sim} GWN(0,\sigma^2), \quad t \in \mathbb{Z}. \tag{12}$$

Is it a causal process?

The process is causal iff the roots of the AR polynomial lie outside of the unit circle. Here:

$$\begin{aligned} & a(\lambda) = 1 - 0.7\lambda + 0.1\lambda^2 \overset{!}{=} 0 \\ \Leftrightarrow \qquad & \lambda_{1,2} = \frac{0.7 \pm \sqrt{0.7^2 - 4\cdot 0.1}}{2\cdot 0.1} \\ & \qquad = \frac{0.7\pm 0.3}{0.2} = 5 \vee 2. \end{aligned}$$

Thus, the process is causal and weakly stationary.

Alternative solution (cf. criterion from the lecture slide 161, equation (45)):

$$\alpha_1 + \alpha_2 = 0.6 < 1 \qquad \wedge \qquad \alpha_2 - \alpha_1 = -0.8 < 1 \qquad \wedge \qquad |\alpha_2| = 0.1 < 1, \tag{13}$$

which is also sufficient for the causality of the AR(2) process.