Univariate Time Series Analysis | **Summer 2020**
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More on Causality, Invertibility, Introduction to Forecasting and Estimation

Exercise 1 (Consequences of Causality)

Recall the AR(2) process from the last problem set.

$$y_t = 0.7y_{t-1} - 0.1y_{t-2} + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{GWN}(0, \sigma^2), \quad t \in \mathbb{Z}. \tag{1}$$

Since this process is weakly stationary and causal, derive the MA(∞) representation of (1).

Hint: *Derive the recursive relationship*

$$\phi_j = \phi_{j-1}\alpha_1 + \phi_{j-2}\alpha_2,$$

where the ϕ_i's correspond to the coefficients of the MA(∞)-polynomial

$$\phi(L) = \phi_0 + \phi_1 L + \phi_2 L^2 + \dots$$

such that $x_t = \phi(L)\varepsilon_t$. The recursive relation stated above can be transformed into a homogeneous 2^{nd} order difference equation

$$\phi_j - \phi_{j-1}\alpha_1 - \phi_{j-2}\alpha_2 = 0.$$

The solution (see Shumway & Stoffer 2011 section 3.3) can be written as

$$\phi_j = c_1 k_1^j + c_2 k_2^j,$$

where k_1, k_2 are the reciprocal values of the characteristic roots λ_1, λ_2 (with $\lambda_1 \neq \lambda_2$) of the AR polynomial $a(\lambda) = 1 - \alpha_1\lambda - \alpha_2\lambda^2 = 0$.

For causal and weakly stationary ARMA processes it holds

$$\begin{aligned} a(L)x_t &= b(L)\varepsilon_t \\ \Rightarrow x_t &= \frac{b(L)}{a(L)}\varepsilon_t = \phi(L)\varepsilon_t. \end{aligned}$$

We write the polynomial $\phi(L)$ as a function of $\lambda \in \mathbb{C}$,

$$\begin{aligned} \phi(\lambda) &= \sum_{j=0}^{\infty} \phi_j \lambda^j \left(= \frac{b(\lambda)}{a(\lambda)} \right) \\ \Leftrightarrow \phi(\lambda)a(\lambda) &= b(\lambda). \end{aligned}$$

For the given parameter combination, i.e., the stated AR(2)-process we get

$$
\begin{aligned}
&(\phi_0 + \phi_1\lambda + \phi_2\lambda^2 + \cdots)(1 - \alpha_1\lambda - \alpha_2\lambda^2) = 1 \\
\Leftrightarrow\ &\phi_0 - \phi_0\alpha_1\lambda - \phi_0\alpha_2\lambda^2 \\
&\quad +\phi_1\lambda - \phi_1\alpha_1\lambda^2 - \phi_1\alpha_2\lambda^3 \\
&\qquad + \phi_2\lambda^2 - \phi_2\alpha_1\lambda^3 - \ldots \\
&\qquad + \ldots \qquad = 1.
\end{aligned}
$$

Applying the method of matching coefficients gives

$$
\begin{aligned}
\phi_0 &= 1 \\
\phi_1 - \phi_0\alpha_1 &= 0 \ \Rightarrow\ \phi_1 = \alpha_1 \\
\phi_2 - \phi_1\alpha_1 - \phi_0\alpha_2 &= 0 \ \Rightarrow\ \phi_2 = \phi_1\alpha_1 + \phi_0\alpha_2 = \alpha_1^2 + \alpha_2 \\
&\vdots
\end{aligned}
$$

Altogether we obtain the recursive relationship

$$\phi_j = \phi_{j-1}\alpha_1 + \phi_{j-2}\alpha_2.$$

The solution is given by (hint)

$$\phi_j = c_1 k_1^j + c_2 k_2^j,$$

with

$$k_1 = \frac{1}{2} = 0.5 \text{ and } k_2 = \frac{1}{5} = 0.2.$$

Using the initial conditions gives

$$
\begin{aligned}
1 = \phi_0 &= c_1 0.5^0 + c_2 0.2^0 \\
\alpha_1 = 0.7 = \phi_1 &= c_1 0.5^1 + c_2 0.2^1;
\end{aligned}
$$

Solving the system of equations implies

$$
\begin{aligned}
-0.4 &= 0.6c_2 \\
c_2 &= -\frac{2}{3} \\
c_1 &= \frac{5}{3}.
\end{aligned}
$$

Exercise 2 (Uniqueness and invertibility of MA processes)

Recall: *A linear process is called **invertible**, if and only if there exists a polynomial*

$$\pi(L) = \pi_0 + \pi_1 L + \pi_2 L^2 + \dots$$

such that

$$\varepsilon_t = \pi(L) x_t.$$

(a) Show that the two MA(1) processes

$$\begin{aligned} x_t &= \varepsilon_t + \beta_1 \varepsilon_{t-1} \qquad && \{\varepsilon_t\} \sim \mathrm{WN}(0, \sigma^2) \\ y_t &= \epsilon_t + \frac{1}{\beta_1}\epsilon_{t-1} \qquad && \{\epsilon_t\} \sim \mathrm{WN}(0, \sigma^2 \beta_1^2) \end{aligned}$$

where $0 < |\beta_1| < 1$, have the same autocovariance functions.

In Problem Set 2 exercise 1, we derived the autocovariance function of the first MA(1) process.

$$\gamma_x(h) = \begin{cases} \sigma^2(1+\beta_1^2), & h = 0, \\ \beta_1 \sigma^2, & |h| = 1, \\ 0, & |h| > 1. \end{cases}$$

For y_t it then follows:

$$\gamma_y(h) = \begin{cases} \sigma^2 \beta_1^2 (1 + \frac{1}{\beta_1^2}) = \sigma^2(1+\beta_1^2), & h = 0, \\ \frac{1}{\beta_1}\sigma^2\beta_1^2 = \beta_1 \sigma^2, & |h| = 1, \\ 0, & |h| > 1. \end{cases}$$

Thus the equality follows.

(b) MA processes that can be represented by an infinite AR representation are called invertible. Write down the AR representations for both processes. Which one is invertible?

$$x_t = \varepsilon_t + \beta_1\varepsilon_{t-1} \quad \Longleftrightarrow \quad \varepsilon_t = x_t - \beta_1\varepsilon_{t-1}$$

Through substitution of all ε we get:

$$\begin{aligned}
\varepsilon_t &= x_t - \beta_1\varepsilon_{t-1} \\
&= -\beta(-\beta\varepsilon_{t-2} + x_{t-1}) + x_t \\
&= ... \\
&= \sum_{i=0}^{\infty}(-\beta_1)^i x_{t-j}
\end{aligned}$$

For y_t the procedure is the same

$$\begin{aligned}
\epsilon_t &= y_t - \frac{1}{\beta_1}\epsilon_{t-1} \\
&= -\frac{1}{\beta_1}(-\frac{1}{\beta_1}\epsilon_{t-2} + y_{t-1}) + y_t \\
&= ... \\
&= \sum_{i=0}^{\infty}(-\frac{1}{\beta_1})^i y_{t-j}
\end{aligned}$$

Since $|\beta_1| < 1$, it follows $|\frac{1}{\beta_1}| > 1$, therefore x_t is invertible and y_t is not because the sum is not well-defined.

Exercise 3 (Forecasting stationary processes)
Suppose we would like to predict a single stationary series $\{x_t\}_t$ with zero mean and autocovariance function $\gamma(h)$ at some time in the future, say, $t+\ell$, for $\ell > 0$.

(a) If we predict using x_t and some scale multiplier A only, show that the mean-square prediction error

$$MSE(A) = E[(x_{t+\ell} - Ax_t)^2], \tag{2}$$

is being minimized by

$$A = \rho(\ell). \tag{3}$$

Rewrite the MSE:

$$\begin{aligned} MSE(A) &= E(x_{t+\ell}^2 - 2Ax_t x_{t+\ell} + A^2 x_t^2) \\ &= E(x_{t+\ell}^2) - 2AE(x_t x_{t+\ell}) + A^2 E(x_t^2) \\ &= \gamma(0) - 2A\gamma(\ell) + A^2\gamma(0) \\ &= (1+A^2)\gamma(0) - 2A\gamma(\ell). \end{aligned}$$

Partial derivative:

$$\frac{\partial MSE(A)}{\partial A} = 2A\gamma(0) - 2\gamma(\ell).$$

Solve

$$2A\gamma(0) - 2\gamma(\ell) \stackrel{!}{=} 0,$$

which gives

$$A = \frac{\gamma(\ell)}{\gamma(0)} = \rho(\ell).$$

This solution yields a minimum because

$$\frac{\partial^2 MSE(A)}{\partial A^2} = 2\gamma(0) = 2E(x_t^2) > 0.$$

(b) Show that in this case the minimum mean-square prediction error is given by

$$MSE(A) = \gamma(0)[1 - \rho^2(\ell)]. \tag{4}$$

The minimum is given by

$$\begin{aligned} MSE(A_{min}) &= (1+\rho(\ell)^2)\gamma(0) - 2\rho(\ell)\gamma(\ell) \\ &= (1+\rho(\ell)^2)\gamma(0) - 2\rho(\ell)\rho(\ell)\gamma(0) \\ &= \gamma(0)[1-\rho(\ell)^2] \end{aligned}$$

Exercise 4 (Yule-Walker equations and a first look at estimation)
Consider the AR(2) process

$$y_t = \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \varepsilon_t, \quad \varepsilon_t \overset{iid}{\sim} GWN(0, \sigma^2), \quad t \in \mathbb{Z} \tag{5}$$

where GWN means *Gaussian white noise*. Furthermore, assume that this process is causal.

(a) Derive the Yule-Walker equations of the process, and determine its ACF for the first two lags, $\rho(1)$ and $\rho(2)$.

Multiply the base equation with y_{t-h} and take the expectation yields

$$E(y_t y_{t-h}) = \alpha_1 E(y_{t-1} y_{t-h}) + \alpha_2 E(y_{t-2} y_{t-h}) + E(\varepsilon_t y_{t-h}). \tag{6}$$

Note, that it holds

$$E(\varepsilon_t y_{t-h}) = \begin{cases} \sigma^2 & , h = 0, \\ 0 & , \text{otherwise}. \end{cases}$$

Altogether we get

$$\gamma(0) = \alpha_1 \gamma(1) + \alpha_2 \gamma(2) + \sigma^2 \quad \text{for } h = 0 \tag{7}$$

$$\gamma(h) = \alpha_1 \gamma(h-1) + \alpha_2 \gamma(h-2) \quad \text{for } h = \pm 1, \pm 2, \ldots, \tag{8}$$

which are the so-called **Yule-Walker equations**. For $h = 1, 2$ the ACF is given by

$$\begin{aligned}
\rho(1) &= \frac{\gamma(1)}{\gamma(0)} = \frac{\alpha_1 \gamma(0) + \alpha_2 \gamma(1)}{\gamma(0)} \\
&= \alpha_1 + \alpha_2 \rho(1) \\
\Rightarrow \rho(1) &= \frac{\alpha_1}{1 - \alpha_2}; \\
\rho(2) &= \frac{\gamma(2)}{\gamma(0)} = \frac{\alpha_1 \gamma(1) + \alpha_2 \gamma(0)}{\gamma(0)} \\
&= \alpha_1 \rho(1) + \alpha_2 \\
&= \frac{\alpha_1^2}{1 - \alpha_2} + \alpha_2 \\
\Rightarrow \rho(2) &= \frac{\alpha_1^2 + \alpha_2 - \alpha_2^2}{1 - \alpha_2}.
\end{aligned}$$

(b) Find the variance of $\{y_t\}_t$.

With (**??**) it follows

$$\begin{aligned}
\sigma^2 &= \gamma(0) - \alpha_1 \gamma(1) - \alpha_2 \gamma(2) \\
&= \gamma(0) - \alpha_1 \rho(1) \gamma(0) - \alpha_2 \rho(2) \gamma(0) \\
&= \gamma(0)(1 - \alpha_1 \rho(1) - \alpha_2 \rho(2))
\end{aligned}$$

implying

$$\begin{aligned}\gamma(0) &= \frac{\sigma^2}{1 - \frac{\alpha_1^2}{1-\alpha_2} - \alpha_2 \frac{\alpha_1^2+\alpha_2-\alpha_2^2}{1-\alpha_2}} \\ &= \frac{(1-\alpha_2)\sigma^2}{1-\alpha_2-\alpha_1^2-\alpha_2\alpha_1^2-\alpha_2^2+\alpha_2^3}.\end{aligned}$$