Estimation and Forecasting

Exercise 1 (Yule-Walker estimation)
Suppose we observe $n = 250$ data points generated by an AR(2) process,

$$x_t = 1.5x_{t-1} - 0.75x_{t-2} + \varepsilon_t, \tag{1}$$

that is, the datapoints were simulated with this model. Now suppose that we are only given these data without knowing the true model specified above. For these data, we can compute the following quantities: $\hat{\gamma}(0) = 8.1281$, $\hat{\rho}(1) = 0.8645$, $\hat{\rho}(2) = 0.5769$.

(a) Compute the parameter estimates for α_1 and α_2 using the Yule-Walker method.

(b) What is the estimated variance of the innovation process $\{\varepsilon_t\}_t$?

(c) Compute the estimated variance-covariance matrix of the parameter vector $\hat{\alpha} = (\hat{\alpha}_1, \hat{\alpha}_2)'$

Hint: The asymptotic ($n \to \infty$) behavior of the Yule-Walker estimators in the case of causal AR(p) processes is as follows:

$$\sqrt{n}(\hat{\alpha} - \alpha) \xrightarrow{d} N(0, \sigma^2\Gamma_p^{-1}), \qquad \hat{\sigma}^2 \xrightarrow{p} \sigma^2. \tag{2}$$

Use this to compute the covariance matrix with the estimates you obtained from the previous tasks.

(d) Use the last result to construct an approximate 95% confidence interval for $\hat{\alpha}_2$.

Exercise 2 (ML estimation)
Consider the AR(1) process

$$x_t = \alpha_1 x_{t-1} + \varepsilon_t, \quad \varepsilon_t \overset{iid}{\sim} GWN(0, \sigma^2), \tag{3}$$

with $|\alpha_1| < 1$.

(a) State the unconditional density of x_1. *Hint: The process is causal.*

(b) Determine the conditional density $f_{x_t|x_{t-1}}(x_t|x_{t-1})$.

(c) The process exhibits the so-called *Markov property*. Explain.

(d) Suppose you observe realizations $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_T)'$ of the process. Write down the likelihood and log-likelihood (with respect to α_1, σ^2 and $\tilde{x}$).

Forecasting

For the following exercises, we have to make some assumptions and introduce some notation.

- Throughout, we assume x_t is a causal and invertible ARMA(p,q) process, that is $\alpha(L)x_t = \beta(L)\varepsilon_t$.
- The minimum mean square error predictor based on the given data $\{x_n, x_{n-1}, ..., x_1\}$, is denoted by

$$x^n_{n+m} = \mathbb{E}[x_{n+m}|x_n, ..., x_1]$$

- If we assume to have knowledge of the complete history of the process,$\{x_n, x_{n-1}, ...\}$, we denote the predictor based on the infinite past by

$$\tilde{x}^n_{n+m} = \mathbb{E}[x_{n+m}|x_n, x_{n-1}, ...]$$

- Henceforth we confine attention to *linear* prediction.
  Given data $x_1, ..., x_n$, the best linear predictor,

$$x^n_{n+m} = \alpha_0 + \sum_{k=1}^{n} \alpha_k x_k$$

  of x_{n+m}, for m ≥ 1, is found by solving

$$\mathbb{E}[(x_{n+m} - x^n_{n+m})x_k] = 0, \qquad k = 0, 1, 2, ...n$$

  where $x_0 = 1$

The above equations are called the *prediction equations.*

Exercise 3 (Forecasting MA(1) processes)
Consider the MA(1) process

$$x_t = \varepsilon_t + \beta_1\varepsilon_{t-1}$$

where $\{\varepsilon\}_t$ is white noise with variance σ^2.

(a) Derive the minimum mean square error one-step forecast based on the infinite past, and determine the mean square error of this forecast.

(b) Let $\tilde{x}^n_{n+1}$ be the truncated one-step-ahead forecast, i.e. the forecast based on the last n observations. (In principle the entire past of the process is known.) Show that

$$\mathbb{E}[(x_{n+1} - \tilde{x}^n_{n+1})^2] = \sigma^2(1 + \beta_1^{2+2n})$$

Compare the result with (a), and indicate how well the finite approximation works in this case.

Exercise 4 (Forecasting AR(1) processes)
The following AR(1) model is given:

$$x_t = \alpha_1 x_{t-1} + \varepsilon_t$$

similar to the previous exercise, determine the general form of the m-step ahead forecast x^t_{t+m} and show

$$\mathbb{E}[(x_{t+m} - x^t_{t+m})^2] = \sigma^2 \frac{1 - \alpha_1^{2m}}{1 - \alpha_1^2} \tag{4}$$