Math 251W: Foundations of Advanced Mathematics

Portfolio Assignment 3: §2.1-3

Name: Brendan Butler, Teddy Wachtler, Grace Yanzito

Problem 2.2.6

proposition: Let a, b, c, m, and n be integers. If $a|b$ and $a|c$ then $a|(bm + cn)$. [DISCUSS]

proof (Direct Proof)

■

Problem 2.2.8

proposition: If a and b are integers and $a|b$, then $a^n|b^n$. [DISCUSS]

proof ()

■

Problem 2.3.5

proposition: Let a, b, and c be integers. If there exists an integer d such that $d|a$ and $d|b$ but $d \nmid c$, then $ax + by = c$ has no integer solutions for x and y. [DISCUSS]

proof ()

■

Problem 2.3.6

proposition: If $c \geq 2$ is a composite integer, then there exists a positive integer $b \geq 2$ such that $b|c$ and $b \leq \sqrt{c}$. [DISCUSS, LATEX or 2.3.8]

proof ()

■

Problem 2.3.8

proposition: Let $q \geq 2$ be a positive integer. If for all integers a and b, whenever $q|ab$, $q|a$ or $q|b$, then q is prime. [DISCUSS, LATEX or 2.3.6]

proof ()

■

Problem 2.2.4

proposition: If n is an even integer, then n^2 is even. If n is odd, then n^2 is odd. [GIVEN]

$\boxed{\text{proof}}$ (Direct Proof)

Let n be an even integer. Then by definition, there exists some integer x such that $n = 2x$. Thus,

$$n^2 = (2x)^2 = (2x)(2x) = 2(2x^2)$$

by the associative and commutative properties of integer multiplication. Furthermore, since the integers are closed under multiplication, $2x^2$ is equal to some integer s. Thus, $n^2 = 2s$, which by definition of an even number, implies n^2 is even.

Let n be an odd integer. By definition of an odd integer, there exists some integer k such that $n = 2k + 1$. Thus,

$$n^2 = (2k+1)^2 = (2k+1)(2k+1) = 4k^2 + 4k + 1$$

by the associative, commutative, and distributive properties of the integers. Furthermore, by the distributive property, $4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since the integers are closed under multiplication and addition, $2k^2 + 2k = t$ for some integer t. Thus, $n^2 = 2t + 1$, which by definition of an odd integer, implies n^2 is odd. ■

Problem 2.2.7

<u>proposition:</u> Let a, b, c, and d be integers. If $a|b$ and $c|d$ then $ac|bd$. [GIVEN]

$\boxed{\text{proof}}$ (Direct Proof)

Suppose a, b, c, and d are integers and that $a|b$ and $c|d$. By definition of divisibility, there exist integers x and y such that $ax = b$ and $cy = d$. Thus, by the associative and commutative properties of integer multiplication, $bd = (ax)(cy) = ac(xy)$. Since the integers are closed under multiplication, $xy = s$ for some integer s. Thus, $bd = acs$, which by definition implies $ac|bd$. ■

Problem 2.3.2

<u>proposition:</u> If n is an integer and n^2 is even, then n is even. [GIVEN]

$\boxed{\text{proof}}$ (Contrapositive)

Suppose n is an odd integer. By definition, there exists an integer m such that $n = 2m + 1$. Thus, by the associative, commutative, and distributive properties, $n^2 = (2m+1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$. Furthermore, since the integers are closed under addition and multiplication, $2m^2 + 2m = s$ for some integer s. Thus, $n^2 = 2s + 1$. By definition of odd, this implies n^2 is odd. Thus, by the contrapositive, if n^2 is even, n must be even. ■

Problem 2.3.4

proposition: If y is a nonzero rational number and x is an irrational number, then xy is irrational. [GIVEN]

$\boxed{\text{proof}}$ (Contradiction)

Suppose y is a nonzero rational number, x is an irrational number, and xy is a rational number. We will show this leads to a contradiction. By definition of rational, since xy and y are rational numbers, there exist integers m, n, l and d such that $xy = \frac{m}{n}$ and $y = \frac{l}{d}$. Thus,

$$xy = x\frac{l}{d} = \frac{m}{n}.$$

Since y is nonzero, $l \neq 0$. Thus multiplication by $\frac{d}{l}$ is well defined. Multiplying both sides of $x\frac{l}{d} = \frac{m}{n}$ by $\frac{d}{l}$, we find

$$x = \frac{m}{n}\frac{d}{l} = \frac{s}{t},$$

where $s = md$ and $t = nl$. Since the integers are closed under multiplication, s and t are both integers. This implies x is a rational number, which contradicts our assumption that x is irrational. Thus, if y is a nonzero rational number and x is an irrational number, their product xy is irrational. $\blacksquare$

Problem 2.3.7

proposition: Let $q \geq 2$ be a positive integer. If for all integers a and b, whenever $q|ab$, $q|a$ or $q|b$, then $\sqrt{q}$ is irrational. [GIVEN]

$\boxed{\text{proof}}$ (Contradiction)

Let q be a positive integer greater than or equal to 2. Suppose that $\sqrt{q}$ is a rational number, and that for all integers a and b, whenever $q|ab$, $q|a$ or $q|b$. We will show this leads to a contradiction. By definition of a rational number, $\sqrt{q}$ can be expressed as $\frac{n}{m}$ where n and m are integers, and $m \neq 0$. Since $q \neq 1$, m and n can be chosen in such a way that they do not share any common factors. Given $\sqrt{q} = \frac{n}{m}$ it follows that $q = \frac{n^2}{m^2}$. Multiplying both sides of this equation by m^2, we find $n^2 = qm^2$. Since the integers are closed under multiplication, $m^2 = s$ for some integer s. Thus, $n^2 = qs$, which by definition, implies $q|n^2$. By our assumption, since $q|n^2$, q must divide n. Thus there exists some integer k such that $qk = n$. Substituting qk in for n in the equation $n^2 = qm^2$, and applying the associative and commutative properties of integer multiplication, we find $q^2k^2 = qm^2$. By the cancelation law, $qk^2 = m^2$. Again, by the closure of the integers under multiplication, there exists some integer $t = k^2$. Thus $qt = m^2$, which by definition, implies $q|m^2$. By our assumptions, q must also divide m. Thus, q is a common factor of m and n, which contradicts the fact that m and n were chosen to have no common factors. Therefore, if for all integers a and b, when $q|ab$, $q|a$ or $q|b$, then $\sqrt{q}$ must be irrational. $\blacksquare$