Geometric analysis of fast-slow PDEs with fold singularities via Galerkin discretisation

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Abstract. We study a singularly perturbed fast-slow system of two partial differential equations (PDEs) of reaction-diffusion type on a bounded domain via Galerkin discretisation. We assume that the reaction kinetics in the fast variable realise a generic fold singularity, whereas the slow variable takes the role of a dynamic bifurcation parameter, thus extending the classical analysis of the singularly perturbed fold. Our approach combines a spectral Galerkin discretisation with techniques from Geometric Singular Perturbation Theory (GSPT) which are applied to the resulting high-dimensional systems of ordinary differential equations (ODEs). In particular, we show the existence of invariant slow manifolds in the phase space of the original system of PDEs away from the fold singularity, while the passage past the singularity of the Galerkin manifolds obtained after discretisation is described by geometric desingularisation, or blow-up. Finally, we discuss the relation between the Galerkin manifolds and the underlying slow manifolds.

Keywords: geometric singular perturbation theory, fast-slow systems, fold singularities, reaction-diffusion equations, Galerkin discretisation

1. Introduction

Systems with multiple time scales have been established as a key mathematical tool across a broad number of applications [9, 27, 43]. At the centre of the theory of multiple-scale dynamics are so-called fast-slow systems, which are given in standard form by

$$\varepsilon \frac{\mathrm{d}u}{\mathrm{d}\tau} = \dot{u} = f(u, v, \varepsilon), \tag{1a}$$
$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = \dot{v} = g(u, v, \varepsilon), \tag{1b}$$

where $u = u(\tau) \in \mathbb{R}^m$ are the fast variables, $v = v(\tau) \in \mathbb{R}^n$ are the slow variables, $\varepsilon > 0$ is a small parameter, τ is the slow time, and f and g are sufficiently smooth functions of u, v, and ε. A wide variety of techniques have been developed for analysing ordinary differential equations (ODEs) of the form in (1), such as asymptotic analysis [5, 33, 34, 39], invariant manifold theory [13, 21, 38], nonstandard analysis [6, 7], geometric desingularisation [10, 24], and numerical methods [8, 14]. Appealingly, several of these techniques allow for a highly visual description of the geometry of trajectories, attractors, invariant sets, and sometimes even the entire phase space via a decomposition of the dynamics into its fast and slow components. This highly intuitive viewpoint is emphasised by reference to the corresponding techniques as geometric singular perturbation theory (GSPT). Indeed, in the singular limit as $\varepsilon \to 0$, we immediately identify the critical set

$$\mathcal{C}_0 := \{(u, v) \in \mathbb{R}^{m+n} : f(u, v, 0) = 0\}, \tag{2}$$

which is commonly referred to as the critical manifold for (1). The slow (or reduced) subsystem on that manifold is given by

$$0 = f(u, v, 0), \tag{3a}$$
$$\dot{v} = g(u, v, 0). \tag{3b}$$

The differential-algebraic Equation (3) has the geometric interpretation of a (generically) lower-dimensional dynamical system for the slow variables v. If $p = (u, v) \in \mathcal{C}_0$ is a normally hyperbolic point, i.e., if $\mathcal{C}_0$ is locally a sufficiently smooth manifold and the Jacobian matrix $\mathrm{D}_u f(p, 0)$ at $p \in \mathcal{C}_0$ has no spectrum on the imaginary axis, then Fenichel's Theorem [13, 20, 27] implies that the critical manifold $\mathcal{C}_0$ perturbs near p to a slow manifold $\mathcal{C}_\varepsilon$. The manifold $\mathcal{C}_\varepsilon$ is then $\mathcal{O}(\varepsilon)$-close, in the Hausdorff distance, to $\mathcal{C}_0$ as $\varepsilon \to 0$; moreover, the dynamics on $\mathcal{C}_\varepsilon$ is locally topologically conjugate to that on $\mathcal{C}_0$. Effectively, Fenichel's Theorem thus geometrically asserts that the normally hyperbolic regime can be viewed as a regular perturbation of its singular limit. However, it is relatively easy to prove that (1) also gives rise to singular perturbations, as non-hyperbolic points generically occur for $m, n \geq 1$, which can intuitively be understood by introducing the fast time scale $t := \tau/\varepsilon$ in (1) and by then taking again the singular limit of $\varepsilon \to 0$:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u' = f(u, v, 0), \tag{4a}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v' = 0. \tag{4b}$$

The fast subsystem, or layer problem, is a parametrised system of ODEs, and is hence not even structurally similar to the slow subsystem, or reduced problem, Equation (3). Important transitions between slow and fast dynamics occur at points where normal hyperbolicity is lost, which can also be interpreted as bifurcation points of the fast subsystem, Equation (4). The most important geometric technique for the analysis of such singularities is geometric desingularisation via "blow-up" [10, 24]; see also [19] for a recent review. A geometric blow-up of a point – or a more general submanifold – amounts to defining a vector field on a higher-dimensional manifold, such as a sphere, with the aim of regaining some hyperbolicity. That approach has been highly successful across a variety of classes of low-dimensional systems of ODEs, such as for the desingularisation of classical fold bifurcations [10, 24, 32], more degenerate folded singularities [26, 42], Hopf bifurcations [16], and transcritical or pitchfork bifurcations [25, 29].

However, for infinite-dimensional multiple-scale dynamical systems, there are significant conceptual and technical challenges to the generalisation of GSPT. Naively, one might anticipate that an extension of (1) to the partial differential equation (PDE)

$$u_t = u_{xx} + f(u, v, \varepsilon), \tag{5a}$$

$$v_t = \varepsilon \left(v_{xx} + g(u, v, \varepsilon) \right) \tag{5b}$$

with suitable boundary conditions, where $u = u(t, x) \in \mathbb{R}^m$, $v = v(t, x) \in \mathbb{R}^n$, and $x \in \Omega$, with Ω being a bounded interval, may yield a sufficiently basic reaction-diffusion system to which techniques from standard GSPT can be adapted. Naturally, on unbounded domains, an approach via spatial dynamics [22, 36] will allow one to apply finite-dimensional techniques directly. However, no ODE-based geometric approach is available for the study of bounded and ε-independent domains.

While some techniques from the theory of ODEs do translate well to an infinite-dimensional setting [17, 28] on such domains, the PDE in (1) presents challenges [40]. When there is only a bounded perturbation in the slow variables, i.e., when the term εv_{xx} is absent, the persistence of invariant manifolds in the normally hyperbolic regime was resolved in [3, 4]. In that case, the perturbation is, in essence, finite-dimensional, such that more classical invariant manifold techniques apply [1, 15, 37]. When the slow variables involve unbounded operators, however, the situation is far more complicated, as the εv_{xx}-term results in non-trivial interactions between fast and slow modes in the limit of $\varepsilon \to 0$. Therefore, there is a crucial need for developing an infinite-dimensional analogue of GSPT, which is the key motivation for this work. The normally hyperbolic regime in (5) was resolved only recently in [18], where an invariant manifold theory was developed for (5) on the basis of functional-analytic techniques.

An alternative approach via spectral Galerkin discretisation was proposed in [11], while a comparison of the two approaches can be found in [31]. Since Galerkin discretisation yields, upon truncation at a finite number of modes, large systems of singularly perturbed ODEs of fast-slow type, one may hope that even a loss of normal hyperbolicity at singular points can be treated by geometric desingularisation, or blow-up [10, 24], which was the focus in [11]. There, the blow-up technique was applied to a Galerkin truncation resulting from a transcritical singularity, i.e., for $f(u, v, \varepsilon) = u^2 - v^2 + \mu\varepsilon$ in (5), with μ a real parameter. As is well-known in the finite-dimensional context, transcritical (and pitchfork) singularities are slightly more straightforward to desingularise than fold singularities; cf. the analysis in [25] and [24], respectively. Hence, in this work we consider a generic fold singularity in (5) as a logical next step; specifically, we study the system

$$u_t = u_{xx} - v + u^2 + H^u(u, v, \varepsilon) \qquad \text{for } x \in (-a, a) \text{ and } t > 0, \tag{6a}$$

$$v_t = \varepsilon(v_{xx} - 1 + H^v(u, v, \varepsilon)) \qquad \text{for } x \in (-a, a) \text{ and } t > 0, \tag{6b}$$

$$u_x(t, x) = 0 = v_x(t, x) \qquad \text{for } x = \pm a \text{ and } t > 0, \tag{6c}$$

$$u(0, x) = u_0(x) \ \text{ and } \ v(0, x) = v_0(x) \qquad \text{for } x \in (-a, a) \tag{6d}$$

on bounded domains, where the domain length $a > 0$ is fixed, with zero Neumann boundary conditions. Here, H^u and H^v are higher-order terms which are specified below.

Remark 1.1. Note that locally well-defined (smooth) solutions for (6) can be obtained from classical theory on sectorial operators with reaction kinetics [17] and parabolic regularity [12].

This work is divided into two parts. In the first part, we apply results from [18] to obtain slow manifolds which drive the (semi)flow of (6) within a neighbourhood of the origin in an appropriately chosen phase space for suitable initial data. Using results from [31], we then approximate these manifolds by slow manifolds in a truncated – and thus *finite-dimensional* – Galerkin discretisation of (6). To avoid confusion, we henceforth refer to these finite-dimensional manifolds as *Galerkin manifolds*. The resulting approximation can be made arbitrarily precise provided an appropriate truncation level, denoted by $k_0 > 0$, is chosen; furthermore, we show that solutions of the Galerkin discretisation converge to those of Equation (6) under suitable assumptions, which allows us to interpret the corresponding Galerkin manifolds as "approximately invariant slow manifolds" for (6).

In the second part, which is the main result of this work, we apply the blow-up technique to extend these Galerkin manifolds around the singularity at the origin in the truncated, $2k_0$-dimensional Galerkin discretisation of (6). Under appropriate assumptions on initial conditions, e.g. by restricting to solutions of (6) that are close to spatially homogeneous ones in an appropriately chosen norm, we show that the dynamics of the Galerkin truncation in a neighbourhood of the origin can be reduced to that of the corresponding ODE for the singularly perturbed planar fold, a well-known prototypical fast-slow system that was studied via blow-up in [24].

There are evident similarities between our analysis and classical GSPT, where Fenichel's Theorem [13] is combined with geometric desingularisation in the form of blow-up; correspondingly, fast-slow systems of arbitrary dimension in both the fast and slow variables have been studied geometrically in previous work [30, 41].

However, the high dimensionality of our Galerkin discretisation, in combination with the inherent spatial dependence of Equation (6), poses both conceptual and technical challenges. Firstly, a preparatory rescaling of the domain length with (a fractional power of) ε is essential to our approach, and is required to obtain both well-defined and non-trivial dynamics in the singular limit after blow-up. A consequence of the rescaling is, however, that the approach in [41] does not apply, as the assumptions therein are not satisfied. Secondly, careful consideration of initial data, in tandem with precise estimates on the evolution of higher-order modes in the Galerkin discretisation, is required to ensure that solutions do not exhibit finite-time blowup before reaching the singularity at the origin; again, that blowup is inherently due to the Galerkin discretisation arising from a system of PDEs.

Our approach has a number of further advantages: it achieves an effective reduction to the singularly perturbed planar fold which can be studied via blow-up of the non-hyperbolic origin, rather than of a submanifold of singularities; furthermore, it allows us to account for the impact of data related to the original PDE, Equation (6), such as the domain length or the eigenvalues of the Laplacian, on the flow in its passage past the origin. Correspondingly, our approach yields explicit asymptotics, rather than merely an existence statement, and hence seems highly suited to the study of singular perturbation problems of fast-slow type obtained by Galerkin discretisation.

Our main results can hence be summarised as follows; precise statements will be given below.

- Equation (6) possesses a family of slow manifolds $S_{\varepsilon,\zeta}$ for small $\varepsilon > 0$ and an additional control parameter $\zeta > 0$. These can be approximated by Fenichel-type slow manifolds $\mathcal{C}_\varepsilon = \mathcal{C}_{\varepsilon,k_0}$ in the corresponding Galerkin discretisation truncated at $k_0 > 0$, provided k_0 is sufficiently large.
- For any $k_0 > 0$ fixed, the Galerkin manifolds $\mathcal{C}_{\varepsilon,k_0}$ are extended around the fold singularity at the origin in the Galerkin discretisation, which we show by combining the well-known fast-slow analysis of the singularly perturbed planar fold with *a priori* estimates that control higher-order modes.

In summary, this work is a stepping stone towards the development of a geometric approach for the study of singularities in multiple-scale (systems of) PDEs. However, it still remains to connect our two main results above: our work does not rigorously relate the extension of the Galerkin manifolds $\mathcal{C}_{\varepsilon,k_0}$ after passage past the fold singularity to corresponding manifolds for (6), which would require understanding of the limit as $k_0 \to \infty$ in our Galerkin truncation.

2. Galerkin discretisation

The starting point for our analysis is the singularly perturbed system of PDEs in (6). In analogy with the canonical form for the singularly perturbed planar fold studied in [24], we refer to u and v therein as the fast and slow variables, respectively. The functions H^u and H^v are assumed to be smooth and of the form

$$H^u(u, v, \varepsilon) = \mathcal{O}(\varepsilon, uv, v^2, u^3) \quad \text{and} \tag{7a}$$

$$H^v(u, v, \varepsilon) = \mathcal{O}(v^2). \tag{7b}$$

In addition, we assume that the higher-order terms H^v in (6d) are orthogonal in $L^2[-a, a]$ to the subspace of constant functions, which is not an essential restriction that is only imposed for technical reasons, as will become apparent in estimates for solutions of the system of ODEs resulting from a Galerkin discretisation of (6); see Lemma 5.9. In other words, we restrict H^v so that $H^v(u, v, \varepsilon)$ has zero mean over $[-a, a]$ for any $u, v \in L^2(-a, a)$. One specific example is given by $H^v(u, v, \varepsilon) = \tilde{H}^v(u, v, \varepsilon) - \frac{1}{2a}\int_{-a}^{a} \tilde{H}^v(u, v, \varepsilon)dx$, where $\tilde{H}^v : \mathbb{R}^3 \to \mathbb{R}$ is smooth. Note that we do not permit linear terms in H^v, since v_{xx} is a linear operator in (6d); however, we could consider more general H^v, such as $H^v(u, v, \varepsilon) = \mathcal{O}(u^2, uv, v^2, \varepsilon)$, with the caveat that we would have to impose further restrictions on the initial values for the higher-order modes u_k $(k \geq 2)$, in analogy to those imposed on v_k for $k \geq 2$.

We can write (6) as

$$w_t = Aw + F(w), \quad \text{with } w(0) = w_0,$$

where $w = (u, v)^T$, $w_0 = (u_0, v_0)^T$, $F(w) = (-v + u^2 + H^u(u, v, \varepsilon), -\varepsilon + \varepsilon H^v(u, v, \varepsilon))^T$, and

$$Aw = \begin{pmatrix} u_{xx} & 0 \\ 0 & \varepsilon v_{xx} \end{pmatrix}, \quad \text{with } \mathcal{D}(A) = \{w \in H^2(-a, a)^2 \ : \ u_x = 0 = v_x \ \text{ at } x = \pm a\}.$$

We have that $F(w)$ is locally Lipschitz continuous on $Z^\alpha = \mathcal{D}(A^\alpha)$ for $1/4 < \alpha < 1$. Moreover, the operator A is sectorial and a generator of an analytic semigroup on $Z = L^2(-a, a)^2$. Thus, for $w_0 \in Z^\alpha$, there exists a unique local-in-time solution $w \in C([0, t_*); Z^\alpha) \cap C^1((0, t_*); Z)$, with $w(t) \in \mathcal{D}(A)$, to (6) for some $t_* > 0$; see e.g. [17]. The quadratic nonlinearity in (6) implies a potential finite-time blowup of solutions to (6); cf. e.g. [2]. However, simple estimates show that, for initial values $u_0 < 0$ and $v_0 > 0$, a solution of (6) exists for $t > 0$ such that $u(t) \leq 0$ and $v(t) \geq 0$; see Appendix A for details.

Before giving a precise statement of our results, we introduce the Galerkin discretisation of the system of PDEs in (6) with respect to the eigenbasis $\{e_k(x) : k =$

$1, 2, \dots\}$ of the Laplacian on $L^2(-a, a)$ with Neumann boundary conditions. Specifically, the relevant orthonormal basis and the corresponding eigenvalues are given by

$$e_{k+1}(x) = \sqrt{\frac{1}{a}} \cos\left(\frac{k\pi(x+a)}{2a}\right) \quad \text{and} \quad \lambda_{k+1} = -\frac{k^2\pi^2}{4a^2} \quad \text{for } k = 1, 2, \dots, \tag{8}$$

with $e_1(x) = \frac{1}{\sqrt{2a}}$ and $\lambda_1 = 0$. We define

$$b_k := -(k-1)^2\pi^2, \tag{9}$$

so that $\lambda_{k+1} = \frac{b_{k+1}}{4a^2}$.

Then, solutions of (6) can be expanded as

$$u(x,t) = \sum_{k=1}^{\infty} e_k(x) u_k(t) \quad \text{and} \quad v(x,t) = \sum_{k=1}^{\infty} e_k(x) v_k(t). \tag{10}$$

Substitution of (10) into (6) results in the infinite system of ODEs

$$u_k' = \lambda_k u_k - \langle v, e_k \rangle + \langle u^2, e_k \rangle + \langle H^u, e_k \rangle, \tag{11a}$$

$$v_k' = \varepsilon \left(\lambda_k v_k - \langle 1, e_k \rangle + \langle H^v, e_k \rangle\right) \tag{11b}$$

for $k = 1, 2, \dots,$ where

$$\langle \phi, \psi \rangle = \int_{-a}^{a} \phi(x)\psi(x) dx \quad \text{for } \phi, \psi \in L^2(-a, a).$$

Using the formulae in (8), we can then derive the following explicit form of (11):

Proposition 2.1. *The system in* (11), *truncated at* $k_0 \in \mathbb{N}$, *reads*

$$u_1' = -v_1 + \frac{1}{\sqrt{2a}} \sum_{j=1}^{k_0} u_j^2 + H_1^u, \tag{12a}$$

$$v_1' = -\sqrt{2a}\varepsilon, \tag{12b}$$

$$u_k' = \frac{1}{4} a^{-2} b_k u_k - v_k + \frac{2}{\sqrt{2a}} u_1 u_k + \frac{1}{\sqrt{a}} \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_i u_j + H_k^u, \tag{12c}$$

$$v_k' = \varepsilon \frac{1}{4} a^{-2} b_k v_k + \varepsilon H_k^v \tag{12d}$$

for $2 \le k \le k_0$, *where* $0 \le \eta_{i,j}^k \le 1$ *is non-zero if and only if* $i + j - 2 = k - 1$ *or* $|i - j| = k - 1$, *and*

$$H_1^u = \mathcal{O}(\varepsilon, v_1^2, v_j^2, u_1 v_1, u_j v_j, u_1 u_j^2, u_i u_j u_l) \quad \textit{for } 2 \le i, j, l \le k_0, \tag{13}$$

$$H_k^u = \mathcal{O}(v_1 v_k, v_i v_j, u_1 v_k, u_k v_1, u_i v_j, u_1^2 u_k, u_1 u_i u_j, u_i u_j u_l) \quad \textit{for } 2 \le i, j, l \le k_0, \textit{ and} \tag{14}$$

$$H_k^v = \mathcal{O}(v_1 v_k, v_i v_j) \quad \textit{for } 2 \le i, j \le k_0. \tag{15}$$

Remark 2.2. Our assumption that the higher-order terms H^v are orthogonal to the subspace of constant functions ensures that $H_1^v = 0$ in (12).

Proof. Because the basis $\{e_j\}_{j\geq 1}$ is orthonormal in $L^2(-a,a)$, we have $\langle v, e_k\rangle = v_k$ for all $k \geq 1$. We observe that $\langle 1, e_1\rangle = \sqrt{2a}$ and $\langle 1, e_k\rangle = 0$ for any $k \geq 2$. Recalling that e_1 is a constant function, we find

$$\langle e_1 e_j, e_k\rangle = e_1\langle e_j, e_k\rangle = (2a)^{-1/2}\delta_{j,k}$$

for all $j, k \geq 1$, where $\delta_{j,k}$ denotes the standard Kronecker delta. In addition, simple calculations show that

$$\langle e_i e_j, e_k\rangle := a^{-1/2}\eta^k_{i,j},$$

where $\eta^k_{i,j}$ is independent of a and given by

$$\eta^k_{i,j} = \int_0^1 \cos((i+j-2)\pi x)\cos((k-1)\pi x)dx + \int_0^1 \cos((i-j)\pi x)\cos((k-1)\pi x)dx.$$

It follows that $0 \leq \eta^k_{i,j} \leq 1$ is non-zero if and only if $i+j = k+1$ or $|i-j| = k-1$. In particular, $\langle e_k^2, e_k\rangle = 0$ for $2 \leq k \leq k_0$. Equipped with the relations above, we can now calculate the term $\langle u^2, e_k\rangle$ in (11). For $k = 1$, we have

$$\left\langle \left(\sum_{j=1}^{k_0} u_j e_j\right)^2, e_1 \right\rangle = \sum_{i,j=1}^{k_0} u_i u_j \langle e_j e_i, e_1\rangle = \sum_{i,j=1}^{k_0} u_i u_j e_1 \langle e_j, e_i\rangle = (2a)^{-1/2}\sum_{j=1}^{k_0} u_j^2,$$

whereas for $2 \leq k \leq k_0$, it holds that

$$\begin{aligned}\left\langle \left(\sum_{j=1}^{k_0} u_j e_j\right)^2, e_k \right\rangle &= \sum_{i,j=1}^{k_0} u_i u_j \langle e_j e_i, e_k\rangle = 2u_1 \sum_{j=1}^{k_0} u_j \langle e_j e_1, e_k\rangle + \sum_{i,j=2}^{k_0} u_i u_j \langle e_j e_i, e_k\rangle \\ &= 2(2a)^{-1/2} u_1 u_k + \sum_{i,j=2}^{k_0} \eta^k_{i,j} u_i u_j,\end{aligned}$$

as in the first sum only the term with $j = k$ is non-zero. □

The relation between solutions of the Galerkin discretisation in (11) and those of Equation (11) is discussed briefly in Appendix A.

3. Slow and Galerkin manifolds

In analogy to standard procedure for fast-slow ODEs of singular perturbation type, the first step in our geometric analysis is to determine the critical manifold for (6). Considering the slow formulation of (6), obtained from the time rescaling $\tau = \varepsilon t$,

$$\varepsilon u_\tau = u_{xx} - v + u^2 + H^u(u, v, \varepsilon) \qquad \text{for } x \in (-a,a) \text{ and } \tau > 0, \tag{16a}$$

$$v_\tau = v_{xx} - 1 + H^v(u, v, \varepsilon) \qquad \text{for } x \in (-a,a) \text{ and } \tau > 0, \tag{16b}$$

$$u_x(\tau, x) = 0 = v_x(\tau, x) \qquad \text{for } x = \pm a \text{ and } \tau > 0, \tag{16c}$$

and setting $\varepsilon = 0$ therein, we find that the critical manifold is given by the set

$$\left\{(u,v) : 0 = u_{xx} - v + u^2 + H^u(u,v,0),\ u_x(\cdot,\pm a) = 0 = v_x(\cdot,\pm a)\right\}. \tag{17}$$

Restricting to spatially homogeneous solutions, we define the critical manifold S_0 as the set of functions

$$S_0 := \left\{(u,v) \in \mathbb{R}^2 : 0 = -v + u^2 + H^u(u,v,0)\right\}, \tag{18}$$

abusing notation and identifying constant functions $u : [-a,a] \to \mathbb{R}$ with the value u they take. Due to our assumptions on the form of H^u, near the origin $(u,v) = (0,0)$ in (u,v)-space the set S_0 is given as a graph

$$S_0 = \left\{(u,v) \in \mathbb{R}^2 : v = u^2 + \mathcal{O}(u^3)\right\}. \tag{19}$$

Proceeding again as in a finite-dimensional setting, the second step in our analysis concerns the persistence of the manifold S_0 for ε positive and sufficiently small. However, in an infinite-dimensional setting, the concept of "fast" and "slow" variables can be delicate, as for any $\varepsilon > 0$, there exists $k > 0$ such that $\varepsilon\lambda_k = O(1)$. One way to address this complication is to split the slow variables v into fast and slow parts, which we make precise in the following proof of Proposition 3.1. We refer to [18] for further discussion and details.

Proposition 3.1. *Let $(u,v) \in S_0$ with $u < 0$. Consider any small $\zeta > 0$ and $u \le \omega_A < 0$, $\omega_f \in \mathbb{R}$, and $L_f > 0$ such that $\omega_A + L_f < \omega_f < 0$. Then, there exist spaces $Y_S^\zeta \oplus Y_F^\zeta = L^2(-a,a)$ and a family of attracting slow manifolds around (u,v) that are given as graphs*

$$S_{\varepsilon,\zeta} := \left\{\left(h_X^{\varepsilon,\zeta}(v), h_{Y_F^\zeta}^{\varepsilon,\zeta}(v), v\right) : v \in Y_S^\zeta\right\} \tag{20}$$

for $0 < \varepsilon < C\frac{\omega_f}{\omega_A}\zeta$ and some fixed $C \in (0,1)$, where $\left(h_X^{\varepsilon,\zeta}(v), h_{Y_F^\zeta}^{\varepsilon,\zeta}(v)\right) : Y_S^\zeta \to H^2(-a,a) \times (Y_F^\zeta \cap H^2(-a,a))$.

Proof. We show that the assumptions of [31, Theorem 2.4] are satisfied, which will imply the existence of a family of slow manifolds stated in (20). Given a point $(u,v) = (c, c^2 + \mathcal{O}(c^3))$ on S_0, with $c < 0$ sufficiently small, we first translate that point to the origin in (6), which yields

$$u_t = u_{xx} - v + u^2 + 2cu + \widetilde{H}^u(u,v,\varepsilon) \qquad \text{for } x \in (-a,a) \text{ and } t > 0, \tag{21a}$$
$$v_t = \varepsilon\left(v_{xx} - 1 + H^v(u,v,\varepsilon)\right) \qquad \text{for } x \in (-a,a) \text{ and } t > 0, \tag{21b}$$
$$u_x(t,x) = 0 = v_x(t,x) \qquad \text{for } x = \pm a \text{ and } t > 0. \tag{21c}$$

Here, $\widetilde{H}^u$ are new higher-order terms that are obtained from H^u post-translation. We choose

$$X = L^2(-a,a) \quad \text{and} \quad Y = L^2(-a,a) \tag{22}$$

as the basis spaces for u and v, respectively, and consider $X_\alpha = H^{2\alpha}(-a,a)$ and $Y_\alpha = H^{2\alpha}(-a,a)$ for $\alpha \in [0,1)$. The linear operators L_1 and L_2 are defined as $L_1 u = u_{xx} + 2cu$ and $L_2 = v_{xx}$, respectively, with $\mathcal{D}(L_1) = \mathcal{D}(L_2) = \{\phi \in H^2(-a,a) : \phi_x(-a) = 0 = \phi_x(a)\}$.

Since we are interested in a neighbourhood of the origin in (21) and by rescaling $v = \kappa_v \tilde{v}$, for any $\kappa_v > 0$, we consider the modified nonlinear terms

$$f(u,v) = -\kappa_v v + \chi(u)u^2 + \chi(u)\chi(v)\widehat{H}^u \quad \text{and} \tag{23a}$$
$$g(u,v) = -1/\kappa_v + \chi(u)\chi(v)\widehat{H}^v, \tag{23b}$$

where $\chi : H^2(-a,a) \to [0,1]$ is such that

$$\chi(u) = 1 \;\text{ if } \|u\|_{H^2} \leq \sigma^2, \quad \chi(u) = 0 \;\text{ if } \;\|u\|_{H^2} \geq 2\sigma, \quad \text{and} \quad \|D\chi\|_{\mathcal{L}(H^2,\mathbb{R})} \leq \sigma$$

for some $0 < \sigma < 1$ and $\widehat{H}^u$ and $\widehat{H}^v$ denote the higher-order terms with rescaled $\tilde{v} = v/\kappa_v$, where the tilde is omitted. Then, these modified nonlinearities

$$f : H^2(-a,a) \times L^2(-a,a) \to L^2(-a,a) \quad \text{and} \quad g : H^2(-a,a) \times H^2(-a,a) \to H^2(-a,a) \tag{24}$$

satisfy

$$\begin{aligned} \|Df(u,v)\|_{\mathcal{L}(H^2\times L^2, L^2)} &\leq L_{f_1}, \\ \|Df(u,v)\|_{\mathcal{L}(H^2\times H^2, H^2)} &\leq L_{f_2}, \quad \text{and} \\ \|Dg(u,v)\|_{\mathcal{L}(H^2\times H^2, H^2)} &\leq L_g, \end{aligned} \tag{25}$$

where $\mathcal{L}(V,W)$ is the space of linear operators from V into W. Define $L_f := \min\{L_{f_1}, L_{f_2}\}$ and note that, by choosing $\sigma > 0$ small, the constants L_f and L_g can be made appropriately small.

Note also that, for any $\varepsilon > 0$, there exists $k > 0$ such that $\varepsilon\lambda_k = \mathcal{O}(1)$, where $\lambda_k = -\frac{k^2\pi^2}{4a^2}, k = 0, 1, \ldots$ are the eigenvalues of the operator L_2 with zero Neumann boundary conditions. Thus, to define fast and slow variables, we need to split the basic space $Y = L^2(-a,a)$ for v into $Y = Y_S^\zeta \oplus Y_F^\zeta$, where

$$Y_S^\zeta := \operatorname{span}\{e_k(x) : 0 \leq k \leq k_0\} \quad \text{and} \tag{26a}$$
$$Y_F^\zeta := \overline{\operatorname{span}\{e_k(x) : k > k_0\}}^{L^2}, \tag{26b}$$

with $\{e_k(x)\}_{k\in\mathbb{N}}$ being the eigenfunctions of the operator L_2 corresponding to the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ and $k_0 \in \mathbb{N}$ satisfying

$$-\frac{(k_0+1)^2\pi^2}{4a^2} < \zeta^{-1}\omega_A \leq -\frac{k_0^2\pi^2}{4a^2}, \tag{27}$$

for given $\zeta > 0$ and $\omega_A \in (2c, 0)$.

Then, for the semigroups generated by $-B_S$ and B_F, which are the realisations of the operator L_2 in $Y_S^\zeta \cap L^2(-a,a)$ and $Y_F^\zeta \cap L^2(-a,a)$, respectively, we have the following estimates:

$$\|e^{-tB_S}y_S\|_{H^2} \le e^{\frac{\pi^2 k_0^2}{4a^2}t}\|y_S\|_{H^2} \qquad \text{for } y_S \in Y_S^\zeta, \tag{28a}$$

$$\|e^{tB_F}y_F\|_{H^2} \le e^{-\frac{\pi^2(k_0+1)^2}{4a^2}t}\|y_F\|_{H^2} \qquad \text{for } y_F \in Y_F^\zeta \cap H^2(-a,a), \tag{28b}$$

see e.g. [17, p.20].

Now, using (26) and the estimates in (28) and following the proof of [31, Theorem 2.4] and [18], we obtain the stated results. □

Remark 3.2. Here, we have written Y_S^ζ instead of $Y_S^\zeta \cap H^2(-a,a)$, as Y_S^ζ is a finite-dimensional subspace of $H^2(-a,a)$.

Next, for given $\zeta > 0$, we also split the space $X = L^2(-a,a)$ into $X = X_S^\zeta \oplus X_F^\zeta$, where X_S^ζ and X_F^ζ are defined in the same manner as Y_S^ζ and Y_F^ζ, see (26).

Then, the truncation of the Galerkin system in (11) at k_0, which is related to ζ via (27), gives the projection of (21) onto $\left(X_S^\zeta, Y_S^\zeta\right)$. Thus, we obtain a family of so-called Galerkin manifolds

$$G_{\varepsilon,\zeta} := \left\{ \left(h_G^{\varepsilon,\zeta}(v), v\right) : v \in Y_S^\zeta \right\} \tag{29}$$

for a function $h_G^{\varepsilon,\zeta} : Y_S^\zeta \to X_S^\zeta$.

Proposition 3.3. *For $0 < \varepsilon < C\frac{\omega_f}{\omega_A}\zeta$ and some fixed $C \in (0,1)$, where ζ, ω_A and ω_f are as in Proposition 3.1, the following estimate holds:*

$$\left\|h_X^{\varepsilon,\zeta}(v) - h_G^{\varepsilon,\zeta}(v)\right\|_{H^2} + \left\|h_{Y_F^\zeta}^{\varepsilon,\zeta}(v)\right\|_{H^2} \le \tilde{C}\left(\frac{4a^2}{\pi^2(2k_0+1)} + \zeta\right)\|v\|_{H^2}. \tag{30}$$

In particular, using the relation between ζ and k_0 in (27), we have

$$\left\|h_X^{\varepsilon,\zeta}(v) - h_G^{\varepsilon,\zeta}(v)\right\|_{H^2} + \left\|h_{Y_F^\zeta}^{\varepsilon,\zeta}(v)\right\|_{H^2} \le \tilde{C}\frac{1}{k_0}\|v\|_{H^2}. \tag{31}$$

Proof. The proof follows the same steps as in [31]. □

Remark 3.4. Note that $k_0 \to \infty$ corresponds to $\zeta \to 0$ which, due to the relation $0 < \varepsilon < C\frac{\omega_f}{\omega_A}\zeta$, see Propositions 3.1 and 3.3, also implies $\varepsilon \to 0$ when $k_0 \to \infty$. Hence, the limit of the Galerkin manifolds $G_{\varepsilon,\zeta}$ as $k_0 \to \infty$ cannot, in general, be guaranteed for all $0 < \varepsilon < \varepsilon_0$, i.e., uniformly in ε. Thus, we perform the following analysis for $0 < \varepsilon < \varepsilon_0$, with ε_0 sufficiently small, and k_0 arbitrarily large, but fixed.

4. Fast-slow analysis

Consider an arbitrary, fixed $k_0 \in \mathbb{N}$ in Proposition 2.1. A rescaling of the variables in (12) via $\hat{u}_k = a^{-1/2}u_k$ and $\hat{v}_k = a^{-1/2}v_k$ gives the fast-slow system

$$u_1' = -v_1 + 2^{-1/2}u_1^2 + 2^{-1/2}\sum_{j=2}^{k_0} u_j^2 + H_1^u, \tag{32a}$$

$$v_1' = -2^{1/2}\varepsilon, \tag{32b}$$

$$u_k' = \frac{1}{4}a^{-2}b_k u_k - v_k + 2^{1/2}u_1 u_k + \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_i u_j + H_k^u, \tag{32c}$$

$$v_k' = \frac{1}{4}a^{-2}b_k \varepsilon v_k + \varepsilon H_k^v, \tag{32d}$$

for $2 \le k \le k_0$. The rescaled system in (32) is equivalent to the original one in (12), in that orbits of the latter are mapped to those of the former. Thus, without loss of generality, in our analysis, we will henceforth focus on (32). In the slow time variable $\tau = \varepsilon t$, Equation (32) becomes

$$\varepsilon\dot{u}_1 = -v_1 + 2^{-1/2}u_1^2 + 2^{-1/2}\sum_{j=2}^{k_0} u_j^2 + H_1^u, \tag{33a}$$

$$\dot{v}_1 = -2^{1/2}, \tag{33b}$$

$$\varepsilon\dot{u}_k = \frac{1}{4}a^{-2}b_k u_k - v_k + 2^{1/2}u_1 u_k + \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_i u_j + H_k^u, \tag{33c}$$

$$\dot{v}_k = \frac{1}{4}a^{-2}b_k v_k + H_k^v, \tag{33d}$$

with the overdot denoting differentiation with respect to τ.

Recalling the system of PDEs in (6), where the singularity is located at the origin of $L^2(-a, a)$, we will be considering initial data in a neighbourhood of the origin in the L^2-norm, with

$$\sum_{k=1}^{\infty} |u_k(0)|^2 \le \kappa \quad \text{and} \quad \sum_{k=1}^{\infty} |v_k(0)|^2 \le \kappa, \tag{34}$$

where $0 < \kappa < 1$. In addition, we impose the bounds

$$|u_k(0)| \le C_{k,u_0} \quad \text{and} \quad |v_k(0)| \le C_{k,v_0}\varepsilon^{4/3} \quad \text{for } k = 2, 3, \ldots, k_0, \tag{35}$$

where C_{k,u_0} and C_{k,v_0} are positive constants. The initial conditions for the first mode $\{u_1, v_1\}$ are taken as in the finite-dimensional (planar) case [24], and are specified in Equation (42) below.

The assumption in (35) implies that the higher-order modes $u_k(0)$ and $v_k(0)$, corresponding to non-constant eigenfunctions, are sufficiently small. The requirement that $v_k(0)$ is of the order $\mathcal{O}(\varepsilon^{4/3})$ is essential for ensuring that $v_k(t)$ does not exhibit finite-time blowup before transiting through a neighbourhood of the singularity at the origin; see Appendix B for an example and Lemma 5.9 for details.

4.1. Critical manifold

Clearly, the system in (32) is a fast-slow system in the standard form of GSPT, with ε the (small) singular perturbation parameter and $\{u_k\}$ and $\{v_k\}$, $k = 1, 2, \ldots, k_0$, the fast and slow variables, respectively. The critical manifold $\mathcal{C}$ for (32) is hence given, to leading order, as a graph over $(u_1, u_2, \ldots, u_{k_0})$, with

$$v_1 = f_1(u_1, u_2, \ldots, u_{k_0}) := 2^{-1/2} u_1^2 + 2^{-1/2} \sum_{j=2}^{k_0} u_j^2 \quad \text{and} \tag{36a}$$

$$v_k = f_k(u_1, u_2, \ldots, u_{k_0}) := \frac{1}{4} a^{-2} b_k u_k + 2^{1/2} u_1 u_k + \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_i u_j \tag{36b}$$

for $k = 2, \ldots, k_0$. Note that, in general, $\mathcal{C}$ is not normally hyperbolic: it contains attracting and saddle-type regions, as well as non-hyperbolic sets separating those regions; examples can be found in Appendix B. Of particular interest is the submanifold $\mathcal{C}_0 \subset \mathcal{C}$ of the critical manifold $\mathcal{C}$ which is defined as

$$\begin{aligned} \mathcal{C}_0 := \big\{ \, (u_1, \ldots, u_{k_0}, f_1(u_1, \ldots, u_{k_0}), \ldots, f_{k_0}(u_1, \ldots, u_{k_0}) \in \mathcal{C} \ : \\ u_1 < 0 \text{ and } u_k = 0 \text{ for } 2 \le k \le k_0 \big\}. \end{aligned} \tag{37}$$

In other words, $\mathcal{C}_0$ is obtained by setting $u_k = 0$ for $k = 2, \ldots, k_0$ in (36), and can hence be written as the curve

$$\begin{aligned} \mathcal{C}_0 = \big\{ (u_1, \ldots, u_{k_0}, v_1, \ldots, v_{k_0}) \in \mathbb{R}^{2k_0} : v_1 = 2^{-1/2} u_1^2 + \mathcal{O}\left(u_1^3\right), \\ \text{with } u_1 < 0 \text{ and } u_k = 0 = v_k \text{ for } 2 \le k \le k_0 \big\} \end{aligned} \tag{38}$$

that lies in the (u_1, v_1)-plane. The set $\mathcal{C}_0$ corresponds directly to the set of constant functions S_0, given by (18). We will denote the slow manifold that is obtained from $\mathcal{C}_0$ via GSPT by $\mathcal{C}_\varepsilon$ or by $\mathcal{C}_{\varepsilon,k_0}$, to emphasise the dependence thereof on k_0.

Remark 4.1. Note that in Section 2, both Y_S^ζ and X_S^ζ are finite-dimensional, and that $G_{\varepsilon,\zeta}$ can hence be viewed as the Fenichel slow manifold perturbing off the normally hyperbolic subset $\mathcal{C}_0$ of the critical manifold of the fast-slow system in (32), for k_0 defined by ζ through (27).

Lemma 4.2. *The subset $\mathcal{C}_0$ of the critical manifold $\mathcal{C}$ is normally hyperbolic and attracting under the layer flow that is obtained for $\varepsilon = 0$ in* (32)*.*

Proof. Linearising the fast flow of (32) about $\mathcal{C}_0$, we find the Jacobian matrix

$$\operatorname{diag}\left\{ 2^{1/2} u_1, 2^{1/2} u_1 + \frac{1}{4} a^{-2} b_2, \ldots, 2^{1/2} u_1 + \frac{1}{4} a^{-2} b_{k_0} \right\}, \tag{39}$$

which implies that $\mathcal{C}_0$ is normally hyperbolic and attracting for $u_1 < 0$ bounded away from zero. Recall that $b_k < 0$ for $k \in \mathbb{N}$ and $k \neq 1$. □

Since the eigenvalues of a matrix depend continuously on its entries, and since the eigenvalues of the linearisation about $\mathcal{C}_0$ are all strictly negative, there exists a full neighbourhood around $\mathcal{C}_0$ in $\mathcal{C}$, with $u_1 < 0$ bounded away from zero, which is normally hyperbolic and attracting under the layer flow of (32). The flow in that neighbourhood is directed towards the origin where, as can be seen from the above linearisation, normal hyperbolicity is lost and which is hence a partially degenerate steady state of (32). The description of the dynamics near the origin therefore requires the application of geometric desingularisation.

4.2. Statement of main result

We are now ready to formulate our main result, which concerns the transition between two appropriately defined sections Δ^{in} and Δ^{out} for the flow generated by (32). These sections of the phase space are defined as follows: consider the set

$$\{(u_1, v_1) \ : \ u_1 \in J \text{ and } v_1 = \rho^2\} \subset \mathbb{R}^{k_0} \times \mathbb{R}^{k_0} \tag{40}$$

for small $\rho > 0$ and a suitable interval J, and let Δ^{in} be a neighbourhood of that set in $\mathbb{R}^{k_0} \times \mathbb{R}^{k_0}$. Similarly, define Δ^{out} as a neighbourhood of the set

$$\{(u_1, v_1) \ : \ u_1 = \rho \text{ and } v_1 \in \mathbb{R}\} \subset \mathbb{R}^{k_0} \times \mathbb{R}^{k_0} \tag{41}$$

that is contained in the (u_1, v_1)-plane. More explicitly, let

$$\begin{aligned}\Delta^{\text{in}} = \Big\{(u_1, \ldots, u_{k_0}, v_1, \ldots, v_{k_0}) \in \mathbb{R}^{2k_0} : u_1 \in \left(-2^{1/4}\rho - C^{\text{in}}_{u_1}, -2^{1/4}\rho + C^{\text{in}}_{u_1}\right), \\ v_1 = \rho^2, |u_k| \le C^{\text{in}}_{u_k}, \text{ and } |v_k| \le C^{\text{in}}_{v_k} \text{ for } 2 \le k \le k_0\Big\}\end{aligned} \tag{42}$$

and

$$\begin{aligned}\Delta^{\text{out}} = \big\{(u_1, \ldots, u_{k_0}, v_1, \ldots, v_{k_0}) \in \mathbb{R}^{2k_0} : u_1 = \rho, \\ v_1 \in \mathbb{R}, |u_k| \le C^{\text{out}}_{u_k}, \text{ and } |v_k| \le C^{\text{out}}_{v_k} \text{ for } 2 \le k \le k_0\big\},\end{aligned} \tag{43}$$

where $C^{\text{in}}_{u_1}$, $C^{\text{in}}_{u_k}$, $C^{\text{in}}_{v_k}$, $C^{\text{out}}_{u_k}$, and $C^{\text{out}}_{v_k}$, for $2 \le k \le k_0$, are appropriately chosen small constants. Given these definitions, we have the following result on the transition map between the sections Δ^{in} and Δ^{out} that is induced by the flow of (32).

Theorem 4.3. *Fix $k_0 \in \mathbb{N}$, and consider the subset $R^{\text{in}} \subset \Delta^{\text{in}}$ defined by*

$$\begin{aligned}R^{\text{in}} = R^{\text{in}}(\varepsilon) := \Big\{(u_1, \ldots, u_{k_0}, v_1, \ldots, v_{k_0}) \in \mathbb{R}^{2k_0} : u_1 \in \left(-2^{1/4}\rho - C^{\text{in}}_{u_1}, -2^{1/4}\rho + C^{\text{in}}_{u_1}\right), \\ v_1 = \rho^2, |u_k| \le C^{\text{in}}_{u_k}, \ \text{ and } \ |v_k| \le C^{\text{in}}_{v_k}\varepsilon^{4/3} \text{ for } 2 \le k \le k_0\Big\}.\end{aligned} \tag{44}$$

Then, there exists $\varepsilon_0(k_0)$ such that for $0 < \varepsilon < \varepsilon_0$, the system in (32) admits a well-defined transition map

$$\Pi : R^{\text{in}} \to \Delta^{\text{out}}.$$

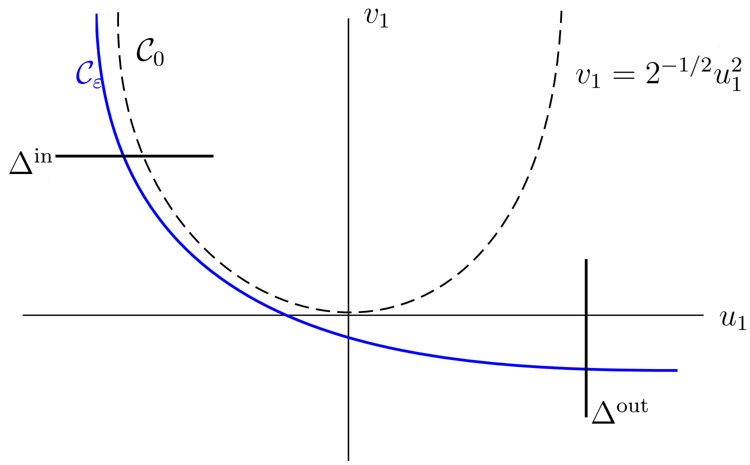


Figure 1: Illustration of the main result, Theorem 4.3, in its projection onto the (u_1, v_1)-plane. The sections $\Delta^{\rm in}$ and $\Delta^{\rm out}$ are, in fact, full neighbourhoods around the shown line intervals in u_1 and v_1. Given $k_0 \in \mathbb{N}$ fixed, trajectories of (32) that are initiated in $\Delta^{\rm in}$ will intersect $\Delta^{\rm out}$ transversely for ε sufficiently small.

Let $(u_1^{\rm in}, v_1^{\rm in}, u_k^{\rm in}, v_k^{\rm in}) \in R^{\rm in}$ *and*

$$(u_1^{\rm out}, v_1^{\rm out}, u_k^{\rm out}, v_k^{\rm out}) := \Pi(u_1^{\rm in}, v_1^{\rm in}, u_k^{\rm in}, v_k^{\rm in});$$

then,

$$\begin{aligned} |v_1^{\rm out}| &= \mathcal{O}\left(\varepsilon^{2/3}\right), \qquad & u_1^{\rm out} &= \rho, \\ |u_k^{\rm out}| &\le C|u_k^{\rm in}|, \quad \textit{and} & |v_k^{\rm out}| &\le C|v_k^{\rm in}| \end{aligned} \tag{45}$$

for $2 \le k \le k_0$ *and a positive generic constant* C *which may differ between estimates. In particular, the slow manifolds* $\mathcal{C}_\varepsilon$ *cross the section* $\Delta^{\rm out}$ *transversely. In addition, the restriction of* Π *to* $I := \{(u_1, u_k, v_1, v_k) \in R^{\rm in} : u_k, v_k \text{ fixed for } 2 \le k \le k_0\}$ *is a contraction with rate* $e^{-c/\varepsilon}$ *for any suitable choice of* $\{u_k, v_k\}$ *and some constant* $c > 0$.

Remark 4.4. In (32), the equations for (u_1, v_1) reduce to those for the classical singularly perturbed planar fold [24] if we set $u_k = 0$ for $2 \le k \le k_0$. Here, we perform a similar analysis as in [24] while controlling the higher-order modes $\{u_k, v_k\}$, $2 \le k \le k_0$. Note that we are restricting to initial data for the system of PDEs in (6) that are close to constant functions, which translates to small initial data $\{u_k(0), v_k(0)\}$ for the system of ODEs in (32). As mentioned already, the dependence on ε in the initial values for v_k, $2 \le k \le k_0$, is essential to ensure that trajectories of the Galerkin system in (32) do not exhibit finite-time blowup before reaching $\Delta^{\rm out}$; cf. again Appendix B for an illustrative example and Section 5 for the corresponding estimates.

5. Geometric desingularisation

To describe the dynamics of the system of equations in (32) near the origin, which is a partially degenerate steady state, we will apply the method of geometric desingularisation by considering ε as a variable in (32), which is included in the quasi-homogeneous spherical coordinate transformation

$$u_k = \bar{r}^{\alpha_k}\bar{u}_k, \quad v_k = \bar{r}^{\beta_k}\bar{v}_k, \quad \text{and} \quad \varepsilon = \bar{r}^{\gamma}\bar{\varepsilon}. \tag{46}$$

Here, $k = 1, 2, \dots, k_0$ and $(\bar{u}_1, \bar{v}_1, \dots, \bar{u}_{k_0}, \bar{v}_{k_0}, \bar{\varepsilon}) \in \mathbb{S}^{2k_0}$, with $\mathbb{S}^{2k_0}$ denoting the $2k_0$-sphere in $\mathbb{R}^{2k_0+1}$ and $r \in [0, r_0]$, for $r_0 > 0$ sufficiently small. The weights α_k, β_k, and γ in (46) will be determined by a rescaling argument below.

In analogy to the desingularisation of the well-known planar fold via blow-up, performed in [24], we shall introduce three coordinate charts K_1, K_2, and K_3, which are formally obtained by setting $\bar{v}_1 = 1$, $\bar{\varepsilon} = 1$, and $\bar{u}_1 = 1$, respectively, in (46). As is convention, we will denote the variables corresponding to u_k, v_k, and ε in chart K_i $(i = 1, 2, 3)$ by $u_{k,i}$, $v_{k,i}$, and ε_i, respectively.

In a nutshell, our strategy will be to retrace the analysis in [24] in each of these charts; crucially, we will need to control the higher-order modes in (32), i.e., the variables $\{u_k, v_k\}$ for $k = 2, \dots, k_0$, in the process. To be precise, we will verify that these additional variables will either remain uniformly bounded (in ε and k) or decay in the transition through the coordinate charts K_1, K_2, and K_3.

A significant challenge to our proposed strategy stems from the fact that, without taking into consideration the length of the spatial domain a, one cannot obtain non-trivial dynamics on the so-called blow-up locus that is given by $\{\bar{r} = 0\}$. To overcome that challenge, we could include a as an auxiliary variable in the quasi-homogeneous blow-up transformation in (46) by writing $a = \bar{r}^{\eta}\bar{a}$, which is the approach taken in [11]. That approach, however, has the disadvantage that the resulting vector fields are not even continuous for $a = 0$, as the exponent η is negative.

A key novelty here, in comparison to [11], is that we adopt an alternative approach by defining a new constant A via

$$a = A\varepsilon^{p}, \tag{47}$$

with $p \in \mathbb{R}$ to be determined, which allows us to obtain non-trivial dynamics for $\bar{r} = 0$ without the conceptual difficulties encountered in [11]. Regardless of the approach used, it appears that some rescaling of the domain in (6) is necessary to perform a successful geometric desingularisation, which is an intrinsic consequence of the Galerkin system in (32) originating from the discretisation of a system of PDEs. Substitution of (47) into (32) yields

$$u_1' = -v_1 + 2^{-1/2}u_1^2 + 2^{-1/2}\sum_{j=2}^{k_0} u_j^2 + H_1^u, \tag{48a}$$

$$v_1' = -2^{1/2}\varepsilon, \tag{48b}$$

$$u_k' = \frac{1}{4A^2} b_k \varepsilon^{-2p} u_k - v_k + 2^{1/2}u_1 u_k + \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_i u_j + H_k^u, \tag{48c}$$

$$v_k' = \frac{1}{4A^2} b_k \varepsilon^{-2p+1} v_k + \varepsilon H_k^v, \tag{48d}$$

$$\varepsilon' = 0. \tag{48e}$$

Remark 5.1. The ε-dependent rescaling of the domain for (6) through (47) changes the fast-slow structure of the original system in (32); in particular, the origin is now a fully degenerate steady state of (48). While singular objects such as steady states or manifolds for (48) in blow-up space no longer correspond directly to singular objects from the layer and reduced problems for (32), the two systems are equivalent for non-zero ε. Hence, our findings will equally apply to (32) in the original coordinates, i.e., after "blow-down".

A rescaling argument shows that the weights in (46), as well as the power p in (47), must satisfy the following relations:

$$\beta_1 = 2\alpha_1, \tag{49a}$$

$$\alpha_k = \alpha_1 \quad \text{for } 2 \leq k \leq k_0, \tag{49b}$$

$$\gamma - \beta_1 = \alpha_1, \tag{49c}$$

$$-2p\gamma \geq \alpha_1, \tag{49d}$$

$$\beta_j = 2\alpha_1 \quad \text{for } 2 \leq j \leq k_0, \tag{49e}$$

$$\gamma - 2p\gamma \geq \alpha_1. \tag{49f}$$

We see from the first three equations above that the consecutive ratios $\alpha_k : \beta_k : \gamma$ must be $1 : 2 : 3$, as in the finite-dimensional case, see e.g. [24]. The smallest integers and the resulting power p that satisfy these relations are

$$\alpha_k = 1, \quad \beta_k = 2, \quad \gamma = 3, \quad \text{and} \quad p = -\frac{1}{6}. \tag{50}$$

Remark 5.2. The choice $p = -\frac{1}{6}$ is the unique one that leaves no factor of r_i after desingularisation in the resulting equations for $u_{k,i}$ in chart K_i, with $i = 1, 2, 3$, where one also requires equality in (49d), making use of the relation $3\alpha_1 = \gamma$. Furthermore, note that the weights in (50) are consistent with the scaling obtained from a "desingularisation" of the system of PDEs in (6); see Section 6 for details.

For future reference, we also state the changes of coordinates between charts K_1, K_2, and K_3, as follows.

Lemma 5.3. *The change of coordinates κ_{12} between charts K_1 and K_2 and its inverse $\kappa_{21} = \kappa_{12}^{-1}$ are given by*

$$\kappa_{12}: \ u_{1,2} = \varepsilon_1^{-1/3} u_{1,1}, \ v_{1,2} = \varepsilon_1^{-2/3}, \ u_{k,2} = \varepsilon_1^{-1/3} u_{k,1}, \ v_{k,2} = \varepsilon_1^{-2/3} v_{k,1}, \ \text{and} \ r_2 = \varepsilon_1^{1/3} r_1; \tag{51}$$

$$\kappa_{21}: \ u_{1,1} = v_{1,2}^{-1/2} u_{1,2}, \ r_1 = v_{1,2}^{1/2} r_2, \ u_{k,1} = v_{1,2}^{-1/2} u_{k,2}, \ v_{k,1} = v_{1,2}^{-1} v_{k,2}, \ \text{and} \ \varepsilon_1 = v_{1,2}^{-3/2}. \tag{52}$$

Between charts K_2 and K_3, we have the following change of coordinates:

$$\kappa_{23}: \ r_3 = u_{1,2} r_2, \ v_{1,3} = u_{1,2}^{-2} v_{1,2}, \ u_{k,3} = u_{1,2}^{-1} u_{k,2}, \ v_{k,3} = u_{1,2}^{-2} v_{k,2}, \ \text{and} \ \varepsilon_3 = u_{1,2}^{-3}. \tag{53}$$

Proof. Direct calculation. □

5.1. Chart K_1

The coordinate chart K_1 is formally defined by $\bar{v}_1 = 1$. Expressed in the coordinates of that chart, the blow-up transformation in (46) reads

$$u_1 = r_1 u_{1,1}, \quad v_1 = r_1^2, \quad u_k = r_1 u_{k,1}, \quad v_k = r_1^2 v_{k,1}, \quad \text{and} \quad \varepsilon = r_1^3 \varepsilon_1.$$

With the above transformation and after desingularisation of the resulting vector field by a factor of r_1, the system in (48) becomes

$$u_{1,1}' = F_1 u_{1,1} - 1 + 2^{-1/2} u_{1,1}^2 + 2^{-1/2} \sum_{j=2}^{k_0} u_{j,1}^2 + H_{1,1}^u, \tag{54a}$$

$$r_1' = -r_1 F_1, \tag{54b}$$

$$u_{k,1}' = F_1 u_{k,1} + \frac{b_k}{4A^2} \varepsilon_1^{1/3} u_{k,1} - v_{k,1} + 2^{1/2} u_{1,1} u_{k,1} + \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_{i,1} u_{j,1} + H_{k,1}^u, \tag{54c}$$

$$v_{k,1}' = 2F_1 v_{k,1} + \frac{b_k}{4A^2} r_1^3 \varepsilon_1^{4/3} v_{k,1} + \varepsilon_1 H_{k,1}^v, \tag{54d}$$

$$\varepsilon_1' = 3F_1 \varepsilon_1, \tag{54e}$$

where

$$F_1 = F_1(\varepsilon_1) = 2^{-1/2} \varepsilon_1,$$

as well as

$$\begin{aligned}
H_{1,1}^u &= \mathcal{O}\left(r_1\varepsilon_1, r_1^2, r_1^2 v_{j,1}^2, r_1 u_{1,1}, r_1 u_{j,1} v_{j,1}, r_1 u_{1,1} u_{j,1}^2, r_1 u_{1,1} u_{j,1}^2, r_1 u_{j,1} u_{i,1} u_{l,1}\right), \\
H_{k,1}^u &= \mathcal{O}\left(r_1^2 v_{k,1}, r_1^2 v_{i,1} v_{j,1}, r_1 u_{1,1} v_{k,1},\right. \\
&\qquad\qquad \left. r_1 u_{k,1}, r_1 u_{i,1} v_{j,1}, r_1 u_{1,1}^2 u_{k,1}, r_1 u_{1,1} u_{i,1} u_{j,1}, r_1 u_{j,1} u_{i,1} u_{l,1}\right), \quad \text{and} \\
H_{k,1}^v &= \mathcal{O}\left(r_1^4 v_{k,1}, r_1^4 v_{i,1} v_{j,1}\right)
\end{aligned}$$

for $2 \leq i, j, l \leq k_0$ and $2 \leq k \leq k_0$. Due to the presence of fractional powers of ε_1 in Equations (54c) and (54d) for $u_{k,1}$ and $v_{k,1}$, respectively, the corresponding flow will not even be C^1 in ε_1. Hence, we rewrite (54) in terms of $\varepsilon_1^{1/3}$, which gives

$$
\begin{aligned}
u_{1,1}' &= F_1 u_{1,1} - 1 + 2^{-1/2} u_{1,1}^2 + 2^{-1/2} \sum_{j=2}^{k_0} u_{j,1}^2 + H_{1,1}^u, && \text{(55a)} \\
r_1' &= -r_1 F_1, && \text{(55b)} \\
u_{k,1}' &= F_1 u_{k,1} + \frac{b_k}{4A^2} \left(\varepsilon_1^{1/3}\right) u_{k,1} - v_{k,1} + 2^{-1/2} u_{1,1} u_{k,1} + \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_{i,1} u_{j,1} + H_{k,1}^u, && \text{(55c)} \\
v_{k,1}' &= 2F_1 v_{k,1} + \frac{b_k}{4A^2} r_1^3 \left(\varepsilon_1^{1/3}\right)^4 v_{k,1} + (\varepsilon_1^{1/3})^3 H_{k,1}^v, && \text{(55d)} \\
\left(\varepsilon_1^{1/3}\right)' &= F_1 \big(\varepsilon_1^{1/3}\big). && \text{(55e)}
\end{aligned}
$$

Clearly, the flow of Equation (55) will be smooth with respect to $\varepsilon_1^{1/3}$; in the following, we will hence refer to (55) when a higher degree of smoothness is required.

Equation (55) admits the two principal steady states

$$
p_a^{k_0} := (-2^{1/4}, 0, \mathbf{0}, \mathbf{0}, 0) \quad \text{and} \quad p_r^{k_0} := (2^{1/4}, 0, \mathbf{0}, \mathbf{0}, 0), \tag{56}
$$

where $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^{k_0-1}$.

Lemma 5.4. *The point $p_a^{k_0}$ is a partially hyperbolic steady state of Equation* (55)*, with the following eigenvalues and eigenvectors in the corresponding linearisation:*

- *the simple eigenvalue $-2^{3/4}$ with eigenvector $(1, 0, \ldots, 0)$, corresponding to $u_{1,1}$;*
- *the eigenvalue $-2^{3/4}$ with multiplicity $k_0 - 1$ and eigenvectors $(0, 0, 0, \ldots, 1, \ldots, 0)$, where non-zero entries appear at the $(k+2)$-th position, corresponding to $u_{k,1}$ $(2 \leq k \leq k_0)$; and*
- *the eigenvalue 0 with multiplicity $k_0 + 1$, corresponding to r_1, $v_{k,1}$ $(2 \leq k \leq k_0)$, and $\varepsilon_1^{1/3}$.*

Proof. Direct calculation. □

To describe the transition through chart K_1, i.e., to approximate the corresponding transition map, we define the following sections for the flow of (54):

$$
\begin{aligned}
\Sigma_{1,k_0}^{\text{in}} &:= \{(u_{1,1}, r_1, u_{k,1}, v_{k,1}, \varepsilon_1) \;:\; r_1 = \rho\} \quad \text{and} \\
\Sigma_{1,k_0}^{\text{out}} &:= \{(u_{1,1}, r_1, u_{k,1}, v_{k,1}, \varepsilon_1) \;:\; \varepsilon_1 = \delta\},
\end{aligned} \tag{57}
$$

for sufficiently small $\delta > 0$. Next, we need to determine the transition time between $\Sigma_{1,k_0}^{\text{in}}$ and $\Sigma_{1,k_0}^{\text{out}}$, which will allow us to derive estimates for the corresponding orbits as they pass through chart K_1.

Lemma 5.5. *The transition time between the sections* $\Sigma^{\mathrm{in}}_{1,k_0}$ *and* $\Sigma^{\mathrm{out}}_{1,k_0}$ *under the flow of* (54) *is given by*

$$T_1 = \frac{\sqrt{2}}{3}\left(\frac{1}{\varepsilon_1(0)} - \frac{1}{\delta}\right). \tag{58}$$

Proof. The explicit solution of Equation (54e) for ε_1 reads

$$\varepsilon_1(t) = \frac{2\varepsilon_1(0)}{2 - 3\sqrt{2}\varepsilon_1(0)t}, \tag{59}$$

where $\varepsilon_1(0)$ denotes an appropriately chosen initial value for ε_1 in $\Sigma^{\mathrm{in}}_{1,k_0}$. Solving the equation $\varepsilon_1(T_1) = \delta$ for T_1 results in (58), as stated. Note that the denominator in (59) remains strictly positive for all $t \in [0, T_1]$. □

Remark 5.6. We refer to the time variable by t throughout for simplicity of notation, even though we consider different systems in the three coordinate charts K_j, with $j = 1, 2, 3$, as well as multiple parametrisations of the same system in some cases.

To give a more complete description of the geometry and, in particular, of the steady state structure, we proceed as follows. Setting $r_1 = 0 = \varepsilon_1$ in (54), we find the singular system

$$u_{1,1}' = -1 + 2^{-1/2}u_{1,1}^2 + 2^{-1/2}\sum_{j=2}^{k_0} u_{j,1}^2, \tag{60a}$$

$$r_1' = 0, \tag{60b}$$

$$u_{k,1}' = -v_{k,1} + 2^{1/2}u_{1,1}u_{k,1} + \sum_{i,j=2}^{k_0} u_{i,1}u_{j,1}, \tag{60c}$$

$$v_{k,1}' = 0, \tag{60d}$$

$$\varepsilon_1' = 0, \tag{60e}$$

from which we see that the hyperplanes $\{r_1 = 0\}$ and $\{\varepsilon_1 = 0\}$ are invariant, as is their intersection. An application of the implicit function theorem shows that lines of steady states emanate from $p_a^{k_0}$ and $p_r^{k_0}$, respectively, for $u_{1,1}$ close to $\pm 2^{1/4}$ and $u_{k,1}$ and $v_{k,1}$ small, with $2 \le k \le k_0$. Locally, around $p_a^{k_0}$, these steady states will inherit the stability of $p_a^{k_0}$, which we will make use of in the estimates in the following subsection. For $k_0 = 2$, the geometry is exemplified in Figures 2a and 2b, in which case p_a^2 and p_r^2 are connected by curves of steady states which can be calculated explicitly from (60); see Figure 2a. The linearisation around those states has one zero eigenvalue and two non-trivial eigenvalues ℓ_1 and ℓ_2 which depend on the $u_{1,1}$-coordinate only; these eigenvalues are plotted in Figure 2b.

The geometry for general k_0 will be similar, in that $p_a^{k_0}$ and $p_r^{k_0}$ will not be isolated, with steady states lying in the plane $\{r_1 = 0 = \varepsilon_1\}$ that are neutral in the $v_{k,1}$-directions and of varying stability in $u_{1,1}$ and $u_{k,1}$, for $2 \le k \le k_0$. States that are close to the point $p_a^{k_0}$ will be stable in the latter directions, while those close to $p_r^{k_0}$ will be unstable in the same directions; in between, there will be steady states of saddle type. These statements are a direct consequence of the implicit function theorem, applied to the vector field in (60). It is unclear whether a curve of steady states that connects $p_a^{k_0}$ and $p_r^{k_0}$ will exist for general k_0, as is the case for $k_0 = 2$.

Furthermore, lines of equilibria are found emanating from each steady state in $\{r_1 = 0 = \varepsilon_1\}$, as can again be seen from the implicit function theorem. These lines locally inherit the stability of the corresponding steady state they are based on.

Remark 5.7. Typically, steady states in the subspace equivalent to $\{r_1 = 0 = \varepsilon_1\}$ after blow-up can be viewed as intersections of critical manifolds with the blow-up locus $\{\bar{r} = 0\}$ [24]. However, that is not the case here, as the rescaling of the spatial domain by ε in (47) alters the fast-slow structure of the original Equation (32). If the parameter a is blown up as in [11] instead, the correspondence with the flow pre-blow-up would be retained; however, the resulting dynamic boundary value problem poses different technical challenges, as detailed there.

The existence of non-hyperbolic steady states near $p_a^{k_0}$ that are attracting in the directions of $u_{1,1}$ and $u_{k,1}$ for $k = 2, \dots, k_0$ implies the following result.

Lemma 5.8. *For sufficiently small ρ, δ, $C^{\text{in}}_{u_{1,1}}$, $C^{\text{in}}_{u_{k,1}}$, and $C^{\text{in}}_{v_{k,1}}$, there exists an attracting, (k_0+2)-dimensional centre manifold $M_{k_0,1}$ at $p_a^{k_0}$ in* (55). *The manifold $M_{k_0,1}$ is given as a graph over $\left(u_{1,1}, r_1, v_{k,1}, \varepsilon_1^{1/3}\right)$, where $2 \le k \le k_0$. In particular, for initial conditions close to $p_a^{k_0}$, solutions of* (55) *satisfy $u_{1,1}(t) < 0$ for $t \in [0, T_1]$.*

Proof. The statements follow from centre manifold theory and Lemma 5.4. □

The centre manifold argument in Lemma 5.8 implies that if $u_{1,1}$ is close to $-2^{1/4}$ initially, then it will remain close throughout the transition through chart K_1; in particular, $u_{1,1}$ will remain negative. To obtain corresponding estimates for the remaining variables $u_{k,1}$ and $v_{k,1}$, with $k = 2, \dots, k_0$, we combine the classical variation of constants formula with a fixed point argument.

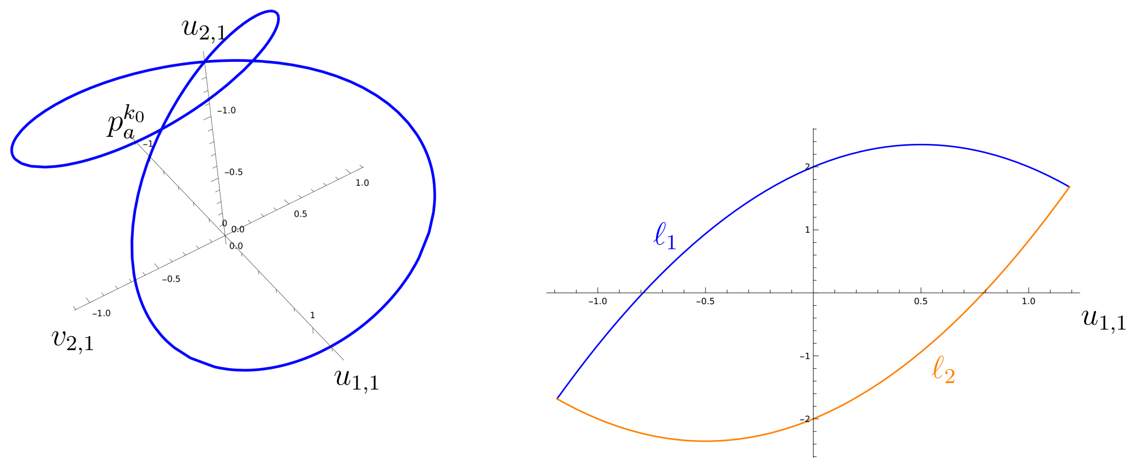
Lemma 5.9. *For $2 \le k \le k_0$ and $u_k(0)$ and $v_k(0)$ satisfying* (35), *solutions of Equation* (54) *satisfy the estimates*

$$|u_{k,1}(t)| \le \frac{1}{\rho}|u_k(0)| + \frac{8a^2\rho}{|b_k|}\left[\sigma_u + \sigma_v \varepsilon_1(0)^{2/3}\rho^2\delta^{2/3}\left(1 + \frac{8a^2}{|b_k|}\right)\right] \tag{61}$$

and

$$|v_{k,1}(t)| \le \frac{\delta^{2/3}}{\varepsilon_1(0)^{2/3}\rho^2}\left|v_k(0)\right| + \varepsilon_1(0)^{2/3}\delta^{2/3}\frac{8a^2\rho^2}{|b_k|}\sigma_v \le \varepsilon_1(0)^{2/3}\rho^2\delta^{2/3}\left(C_{k,v_0} + \frac{8a^2}{|b_k|}\sigma_v\right) \tag{62}$$

for all $t \in [0, T_1]$ and some $\kappa \le \sigma_u, \sigma_v < 1$, where T_1 is the transition time determined in (58) *and $\kappa > 0$ is as in* (34).



(a) Steady states of (60) when $k_0 = 2$. (b) The two non-trivial eigenvalues ℓ_1 and ℓ_2.

Figure 2: Steady state structure of (60) for $k_0 = 2$. (a) The principal steady states p_a^2 and p_r^2 are connected by a pair of symmetric curves of steady states that are parametrised by $u_{1,1}$. (b) The two non-trivial eigenvalues ℓ_1 and ℓ_2 of the linearisation about these steady states are plotted against $u_{1,1}$.

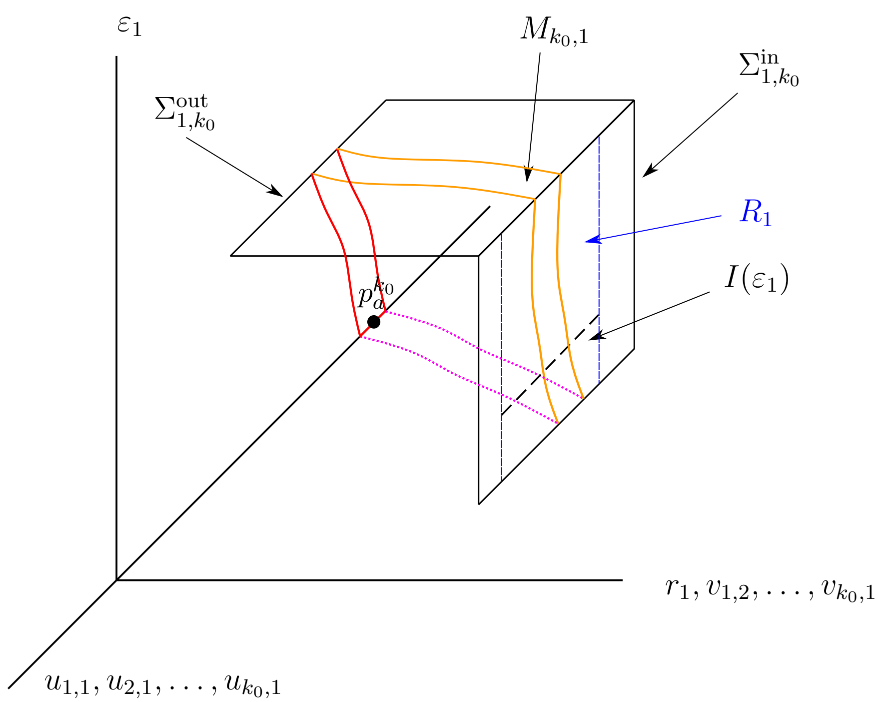


Figure 3: The dynamics in K_1 is organised around the attracting centre manifold $M_{k_0,1}$, which is anchored in a curve of steady states in the subspace $\{r_1 = 0 = \varepsilon_1\}$, one of which is $p_a^{k_0}$. The transition map Π_1 is defined on the subset $R_1 \subset \Sigma^{\text{in}}_{1,k_0}$ around the intersection $\Sigma^{\text{in}}_{1,k_0} \cap M_{k_0,1}$; slices of R_1 with ε_1 constant, denoted by $I(\varepsilon_1)$, will be mapped to slices with constant ε_3 in chart K_3. Since $M_{k_0,1}$ is a graph over $\left(u_{1,1}, r_1, v_{k,1}, \varepsilon_1^{1/3}\right)$, see Lemma 5.8, it is illustrated as having "thickness".

Proof. We first derive the estimates for $v_{k,1}$. Application of the variation of constants formula to (54d) yields

$$v_{k,1}(t) = \exp\Big(\int_0^t V_{k,1}(s)\,ds\Big)v_{k,1}(0) + \int_0^t \exp\Big(\int_s^t V_{k,1}(\tau)d\tau\Big)\varepsilon_1(s)H^v_{k,1}ds, \tag{63}$$

where $V_{k,1}(s) = 2^{1/2}\varepsilon_1(s) + \frac{b_k}{4A^2}r_1^3(s)\varepsilon_1^{4/3}(s)$. Equation (54b) can be solved explicitly for r_1 to give

$$r_1(t) = 2^{-1/3}\rho\big(2 - 3\sqrt{2}\varepsilon_1(0)t\big)^{1/3}, \tag{64}$$

where $\varepsilon_1(0)$ denotes the initial value for ε_1 and $r_1(0) = \rho$. Note that, due to $b_k < 0$, the second term in $V_{k,1}(s)$ is negative for all $s \in [0, T_1]$. Combination of the above expression with the explicit solutions for $\varepsilon_1(t)$ and $r_1(t)$ in (59) and (64), respectively, then implies

$$\begin{aligned}|v_{k,1}(t)| \le{} & \frac{2}{3}\exp\Big(\int_0^t \frac{\varepsilon_1(0)}{\phi_1(s)}\,ds\Big)|v_{k,1}(0)| \\ & + \Big(\frac{\sqrt{2}}{3}\Big)^{\frac{1}{3}}\rho^2\frac{e^{-\alpha\phi_1(t)^{2/3}}}{\phi_1(t)^{2/3}}\int_0^t \varepsilon_1(0)\phi_1(s)^{1/3}e^{\alpha\phi_1(s)^{2/3}}|\widetilde{H}^v_{k,1}(s)|ds,\end{aligned} \tag{65}$$

where $\phi_1(s) = \sqrt{2}/3 - \varepsilon_1(0)s$, $\alpha = (3/\sqrt{2})^{2/3}b_k\rho^2/(4a^2\sqrt{2})$, and $H^v_{k,1}(t) = r_1(t)^2\widetilde{H}^v_{k,1}(t)$. Evaluating the integrals in (65), we obtain

$$|v_{k,1}(t)| \le \frac{\delta^{2/3}}{\varepsilon_1(0)^{2/3}}\Big(|v_{k,1}(0)| + \frac{8a^2}{|b_k|}\sup_{[0,T_1]}|\widetilde{H}^v_{k,1}(t)|\Big) \qquad \text{for all } t \in [0, T_1]. \tag{66}$$

Recall that the term $\widetilde{H}^v_{k,1}$ is at least quadratic in $v_{k,1}$, with $k = 2, \ldots, k_0$. To estimate $u_{k,1}$, we first rewrite (54c) in the form

$$u'_{k,1} = \Big(F_1 + \frac{b_k}{4A^2}\varepsilon_1^{1/3} + 2^{1/2}u_{1,1}\Big)u_{k,1} - v_{k,1} + M_k(u_{2,1}, \ldots, u_{k_0,1}) + H^u_{k,1}$$

for $2 \le k \le k_0$, where

$$M_k(u_{2,1}, \ldots, u_{k_0,1}) := \sum_{i,j=2}^{k_0}\eta^k_{i,j}u_{i,1}u_{j,1}.$$

Application of the variation of constants formula yields

$$\begin{aligned}u_{k,1}(t) ={} & \exp\left(\int_0^t U_{k,1}(s)ds\right)u_{k,1}(0) - \int_0^t \exp\left(\int_s^t U_{k,1}(\tau)d\tau\right)v_{k,1}(s)ds \\ & + \int_0^t \exp\left(\int_s^t U_{k,1}(\tau)d\tau\right)\Big(M(u_{2,1}(s), \ldots, u_{k_0,1}(s)) + H^u_{k,1}\Big)ds,\end{aligned} \tag{67}$$

where $U_{k,1}(s) := 2^{-1/2}\varepsilon_1(s) + \frac{b_k}{4A^2}\varepsilon_1^{1/3}(s) + 2^{1/2}u_{1,1}(s)$.

Due to Lemma 5.8, we have $u_{1,1}(s) < 0$ for $s \in [0, T_1]$; hence, in the following estimates, we replace $U_{k,1}(s)$ with $\widetilde{U}_{k,1}(s) := 2^{-1/2}\varepsilon_1(s) + \frac{b_k}{4A^2}\varepsilon_1^{1/3}(s)$, as $\exp(\int_s^t U_{k,1}(\tau)d\tau) \le \exp(\int_s^t \widetilde{U}_{k,1}(\tau)d\tau)$ for all $0 \le s < t \le T_1$. Direct integration gives

$$\begin{aligned} 0 \le \mathcal{I}_1(t) &= \exp\left(\int_0^t \widetilde{U}_{k,1}(s)ds\right) \\ &= \frac{1}{(1-(3/\sqrt{2})\varepsilon_1(0)t)^{1/3}} \exp\Big(\frac{b_k}{4A^2\sqrt{2}}\Big[\Big(\frac{1}{\varepsilon_1(0)}\Big)^{\frac{2}{3}} - \Big(\frac{1}{\varepsilon_1(0)} - \frac{3}{\sqrt{2}}t\Big)^{\frac{2}{3}}\Big]\Big) \le 1, \end{aligned} \tag{68}$$

since $\mathcal{I}_1(t)$ is a non-increasing function for $\delta \le \pi^3 2^{-5/4}/(\rho^{3/2}\varepsilon_1(0)^{1/2}) = \pi^3 2^{-5/4}/\varepsilon^{1/2}$, and

$$\mathcal{I}_1(T_1) = \frac{\delta^{1/3}}{\varepsilon_1(0)^{1/3}} \exp\Big(\frac{b_k}{4A^2\sqrt{2}}\Big[\Big(\frac{1}{\varepsilon_1(0)}\Big)^{\frac{2}{3}} - \Big(\frac{1}{\delta}\Big)^{\frac{2}{3}}\Big]\Big) \le \exp\Big(\frac{b_k}{8A^2\sqrt{2}}\Big(1-\frac{1}{\alpha}\Big)\Big(\frac{1}{\varepsilon_1(0)}\Big)^{\frac{2}{3}}\Big)$$

for $\delta \ge \alpha\varepsilon_1(0) > 0$ with $\alpha \ge 1$. Next, we have that

$$\begin{aligned} 0 \le \mathcal{I}_2(t) &= \int_0^t \exp\Big(\int_s^t \widetilde{U}_{k,1}(\tau)d\tau\Big)ds = \varepsilon_1(t)^{\frac{1}{3}}\frac{4A^2}{|b_k|}\Big[\Big(\frac{1}{\varepsilon_1(0)} - \frac{3}{\sqrt{2}}t\Big)^{2/3} \\ &\quad - \Big(\frac{1}{\varepsilon_1(0)}\Big)^{\frac{2}{3}} \exp\Big(\frac{1}{\sqrt{2}}\frac{b_k}{4A^2}\Big[\Big(\frac{1}{\varepsilon_1(0)}\Big)^{\frac{2}{3}} - \Big(\frac{1}{\varepsilon_1(0)} - \frac{3}{\sqrt{2}}t\Big)^{\frac{2}{3}}\Big]\Big)\Big] \\ &\quad + \varepsilon_1(t)^{\frac{1}{3}}\frac{16A^4\sqrt{2}}{|b_k|^2}\left[1 - \exp\Big(\frac{1}{\sqrt{2}}\frac{b_k}{4A^2}\left[\Big(\frac{1}{\varepsilon_1(0)}\Big)^{\frac{2}{3}} - \Big(\frac{1}{\varepsilon_1(0)} - \frac{3}{\sqrt{2}}t\Big)^{\frac{2}{3}}\right]\Big)\right] \\ &\le \frac{8\rho a^2}{|b_k|}, \end{aligned}$$

where we have used $A^2 = \varepsilon^{1/3}a^2 = \rho\varepsilon_1(0)^{1/3}a^2$ and

$$\begin{aligned} \mathcal{I}_2(T_1) &= \delta^{-1/3}\frac{4A^2}{|b_k|}\left[1 - \Big(\frac{\delta}{\varepsilon_1(0)}\Big)^{2/3}\exp\Big(\frac{1}{\sqrt{2}}\frac{b_k}{4A^2}\Big[\Big(\frac{1}{\varepsilon_1(0)}\Big)^{2/3} - \frac{1}{\delta^{2/3}}\Big]\Big)\right] \\ &\quad + \delta^{1/3}\frac{16A^4\sqrt{2}}{|b_k|^2}\Big[1 - \exp\Big(\frac{1}{\sqrt{2}}\frac{b_k}{4A^2}\Big[\Big(\frac{1}{\varepsilon_1(0)}\Big)^{\frac{2}{3}} - \frac{1}{\delta^{2/3}}\Big]\Big)\Big] \le \frac{1}{\delta^{1/3}}\frac{4A^2}{|b_k|}\Big[1 + \frac{4A^2\sqrt{2}}{|b_k|}\Big]. \end{aligned}$$

Here, we again have $0 < \alpha\varepsilon_1(0) \le \delta < 1$, with $\alpha \ge 1$.

To complete the estimates, we shall use a fixed point argument and define the set

$$\begin{aligned} \mathcal{B}_1 = \Big\{&(\tilde{u}_{2,1}, \ldots, \tilde{u}_{k_0,1}, \tilde{v}_{2,1}, \ldots, \tilde{v}_{k_0,1}) : \tilde{u}_{k,1}, \tilde{v}_{k,1} \in C[0,T], 2 \le k \le k_0, \\ &\text{with } \sup_{[0,T_1]} |\tilde{u}_{k,1}(t)| \le C_{k,u}, \sup_{[0,T_1]} |\tilde{v}_{k,1}(t)| \le C_{k,v}, \\ &\sum_{k=2}^{k_0} C_{k,u}^2 \le \tilde{\sigma}_u, \text{ and } \sum_{k=2}^{k_0} C_{k,v}^2 \le \tilde{\sigma}_v\varepsilon_1(0)^{4/3}\Big\}, \end{aligned}$$

where $\tilde{\sigma}_u, \tilde{\sigma}_v \le 1$.

Considering $M_k(\tilde{u}_{2,1},\dots,\tilde{u}_{k_0,1})$ and

$$H^l_{k,1} = H^l_{k,1}(\tilde{u}_{2,1},\dots,\tilde{u}_{k_0,1},\tilde{v}_{2,1},\dots,\tilde{v}_{k_0,1}),$$

with $l=u,v$, in (63) and (67), for $(\tilde{u}_{2,1},\dots,\tilde{u}_{k_0,1},\tilde{v}_{k,1},\dots,\tilde{v}_{k_0,1}) \in \mathcal{B}_1$, we obtain a map $\mathcal{N}_1$ given by $\mathcal{N}_1(\tilde{u}_{2,1},\dots,\tilde{u}_{k_0,1},\tilde{v}_{2,1},\dots,\tilde{v}_{k_0,1}) = (u_{2,1},\dots,u_{k_0,1},v_{2,1},\dots,v_{k_0,1})$. Solutions of (63) and (67) correspond to the fixed points of $\mathcal{N}_1$.

We shall show that $\mathcal{N}_1 : \mathcal{B}_1 \to \mathcal{B}_1$. Our assumptions on the initial conditions, together with (66), yield

$$|v_{k,1}(t)| \le \delta^{2/3}\varepsilon_1(0)^{2/3}\Big(\rho^2 C_{k,v_0} + \frac{8a^2\rho^2}{|b_k|}\sigma_v\Big) \qquad \text{for all } t\in[0,T_1], \tag{69}$$

where we have used $|\widetilde{H}^v_{k,1}(t)| \le C_1\rho^2\sum_{k=2}^{k_0}|\tilde{v}_{k,1}|^2 \le \rho^2 C_2\varepsilon_1(0)^{4/3}\tilde{\sigma}_v \le \rho^2\varepsilon_1(0)^{4/3}\sigma_v$. Then,

$$|u_{k,1}(t)| \le |u_{k,1}(0)| + \frac{8a^2\rho}{|b_k|}\big(C_3\varepsilon_1(0)^{2/3} + \sigma_u\big) \qquad \text{for all } t\in[0,T_1],$$

where $|M_k + H^u_{k,1}| \le C_4\sum_{k=2}^{k_0}|\tilde{u}_k|^2 \le C_5\tilde{\sigma}_u = \sigma_u$.

Thus, for $0<\rho<1$ and $0<\sigma_u,\sigma_v<1$, we obtain that $\mathcal{N}_1 : \mathcal{B}_1 \to \mathcal{B}_1$, which implies the estimates in (61) and (62). $\square$

Remark 5.10. Note that if $H^v = 0$, then it is sufficient to consider $|v_k(0)| \le C_{k,v_0}\varepsilon^{1/2}$ and $|u_k(0)| \le C_{k,u_0}$. For more general higher-order terms of the form

$$H^v_k = \mathcal{O}(u_iu_j, v_iv_j, v_1v_k, v_iv_j),$$

with $i,j = 2,\dots,k_0$, we would have to assume that $|u_k(0)| \le C_{k,u_0}\varepsilon^{2/3}$. Then, in the definition of $\mathcal{B}_1$, we would consider $\sum_{k=2}^{k_0} C^2_{k,u} \le \varepsilon^{4/3}\tilde{\sigma}_u$, which would imply

$$|u_{k,1}(t)| \le \varepsilon_1(0)^{2/3}\rho^2 C_{k,u_0} + \varepsilon_1(0)^{2/3}\frac{8a^2\rho}{|b_k|}\big(\sigma_u+\sigma_v\big).$$

Given the above estimates, the transition map Π_1 in chart K_1 will be defined on the set $R_1 \subset \Sigma^{\text{in}}_{1,k_0}$, which is given by

$$\begin{aligned} R_1 := \Big\{ &(u_{1,1}, r_1, u_{k,1}, v_{k,1}, \varepsilon_1) : |u_{1,1} + 2^{1/4}| \le C^{\text{in}}_{u_{1,1}}, r_1 = \rho, \\ &|u_{k,1}| \le C^{\text{in}}_{u_{k,1}}, |v_{k,1}| \le C^{\text{in}}_{v_{k,1}}\varepsilon_1^{4/3} \text{ for } k = 2,\dots,k_0, \text{ and } \varepsilon_1 \in [0,\delta]\Big\}; \end{aligned} \tag{70}$$

see Figure 3. The set R_1 is precisely the set $R^{\text{in}} \subset \Delta^{\text{in}}$, transformed into the coordinates of chart K_1. For $\varepsilon_1 \in [0,\delta]$ fixed, we also define the slices $I(\varepsilon_1) \subset R_1$ as

$$I(\varepsilon_1) := \{(u_{1,1}, r_1, u_{k,1}, v_{k,1}, \varepsilon_1) \in R_1 \ : \ \varepsilon_1 \in [0,\delta] \ \text{ fixed}\}. \tag{71}$$

These slices will be useful when combining the transition through chart K_1 with those through charts K_2 and K_3, as $I(\varepsilon_1)$ will be mapped to sets with constant ε_3 in an appropriately defined section $\Sigma^{\text{out}}_{3,k_0}$.

We summarise our findings on the transition through chart K_1, and on the corresponding map Π_1.

Proposition 5.11. *The transition map $\Pi_1 : R_1 \to \Sigma^{\text{out}}_{1,k_0}$ is well-defined. For*

$$(u_{1,1}, \rho, u_{k,1}, v_{k,1}, \varepsilon_1) \in R_1, \quad \text{with } k = 2, \dots, k_0,$$

denote

$$\Pi_1(u_{1,1}, \rho, u_{k,1}, v_{k,1}, \varepsilon_1) = (u^{\text{out}}_{1,1}, r^{\text{out}}_1, u^{\text{out}}_{k,1}, v^{\text{out}}_{k,1}, \delta). \tag{72}$$

Then, the following estimates hold:

$$|u^{\text{out}}_{1,1} + 2^{1/4}| \leq C^{\text{out}}_{u_{1,1}}, \tag{73a}$$

$$r^{\text{out}}_1 \in [0, \rho], \tag{73b}$$

$$|u^{\text{out}}_{k,1}| \leq C^{\text{out}}_{u_{k,1}}, \quad \text{and} \tag{73c}$$

$$|v^{\text{out}}_{k,1}| \leq C^{\text{out}}_{v_{k,1}} \delta^{2/3}, \tag{73d}$$

where $C^{\text{out}}_{u_{k,1}}$, $C^{\text{out}}_{u_{k,1}}$, and $C^{\text{out}}_{v_{k,1}}$ are appropriately chosen constants. Furthermore, the restriction $\Pi_1|_{I(\varepsilon_1)}$ is a contraction, with rate bounded by $C \exp(cT_1)$, where $C > 0$ and $-2^{3/4} < c < 0$.

Proof. The estimates in (73c) and (73d) follow directly from the definition of R_1 in (70) and Lemma 5.9, while (73b) is immediate from the observation that $r_1(t)$ is decreasing, by (55b). Finally, (73a) and the stated contraction property are due to Lemma 5.8 and the existence of the attracting centre manifold $M_{k_0,1}$. $\square$

5.2. Chart K_2

As will become apparent, the dynamics of (48) in chart K_2 can be seen as a regular perturbation of the planar subsystem for the first two modes $\{u_1, v_1\}$, after transformation to K_2. In particular, for $r_2 = 0$, that subsystem reduces to the well-studied Riccati equation [33]. As the requisite analysis is similar to that in the corresponding rescaling chart for the singularly perturbed planar fold [24], we merely outline it here.

In chart K_2, the blow-up transformation in (46) reads

$$u_1 = r_2 u_{1,2}, \quad v_1 = r_2^2 v_{1,2}, \quad u_k = r_2 u_{k,2}, \quad v_k = r_2^2 v_{k,2}, \quad \text{and} \quad \varepsilon = r_2^3;$$

in particular, the variables u_k and v_k $(1 \leq k \leq k_0)$ are rescaled with powers of $r_2 = \varepsilon^{1/3}$, which justifies the terminology.

Substitution of the above transformation into (48) and desingularisation with a factor of r_2 gives

$$u_{1,2}' = -v_{1,2} + 2^{-1/2}u_{1,2}^2 + 2^{-1/2}\sum_{j=2}^{k_0} u_{j,2}^2 + H_{1,2}^u, \tag{74a}$$

$$v_{1,2}' = -2^{1/2}, \tag{74b}$$

$$u_{k,2}' = \frac{b_k}{4A^2}u_{k,2} - v_{k,2} + 2^{1/2}u_{1,2}u_{k,2} + \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_{j,2}u_{i,2} + H_{k,2}^u, \tag{74c}$$

$$v_{k,2}' = \frac{b_k}{4A^2}r_2^3 v_{k,2} + H_{k,2}^v, \tag{74d}$$

$$r_2' = 0 \tag{74e}$$

for $2 \leq k \leq k_0$, where

$$H_{1,2}^u = \mathcal{O}\left(r_2\right),$$
$$H_{k,2}^u = \mathcal{O}\left(r_2u_{1,2}v_{k,2}, r_2u_{k,2}v_{1,2}, r_2u_{i,2}u_{j,2}, r_2u_{1,2}^2u_{k,1}, r_2u_{1,2}u_{i,2}u_{j,2}, r_2u_{i,2}u_{j,2}u_{l,2}\right), \quad \text{and}$$
$$H_{k,2}^v = \mathcal{O}\left(r_2^4v_{1,2}v_{k,2}, r_2^4v_{i,2}v_{j,2}\right),$$

with $2 \leq i, j, l \leq k_0$.

The plane $\{u_{k,2} = 0 = v_{k,2}, 2 \leq k \leq k_0\} \cap \{r_2 = 0\}$ is invariant under the flow of Equation (74); on that plane, (74) reduces to

$$u_{1,2}' = -v_{1,2} + 2^{-1/2}u_{1,2}^2, \tag{75a}$$

$$v_{1,2}' = -2^{1/2}, \tag{75b}$$

with $u_{1,2}, v_{1,2} \in \mathbb{R}$, which is a Riccati equation that corresponds to the one found in [24, Proposition 2.3], up to a rescaling. Correspondingly, we have the following result.

Proposition 5.12. *The Riccati equation in* (75) *has the following properties:*

(i) Every orbit has a horizontal asymptote $v_{1,2} = v_{1,2}^{+\infty}$, where $v_{1,2}^{+\infty}$ depends on the orbit, such that $u_{1,2} \to +\infty$ as $v_{1,2}$ approaches $v_{1,2}^{+\infty}$ from above.

(ii) There exists a unique orbit γ_2 which can be parametrised as $(u_{1,2}, s(u_{1,2}))$, with $u_{1,2} \in \mathbb{R}$, which is asymptotic to the left branch of the parabola $\{-v_{1,2} + 2^{-1/2}u_{1,2}^2 = 0\}$ for $u_{1,2} \to -\infty$. The orbit γ_2 has a horizontal asymptote $v_{1,2} = -\Omega_0 < 0$ such that $u_{1,2} \to \infty$ as $v_{1,2}$ approaches $-\Omega_0$ from above, where Ω_0 is a positive constant that is defined as in [24].

(iii) The function $s(u_{1,2})$ has the asymptotic expansions

$$s(u_{1,2}) = 2^{-1/2}u_{1,2}^2 + \frac{2^{-1/2}}{u_{1,2}} + \mathcal{O}\left(\frac{1}{u_{1,2}^4}\right) \quad \text{as } u_{1,2} \to -\infty \tag{76}$$

and

$$s(u_{1,2}) = -\Omega_0 + \frac{2^{1/2}}{u_{1,2}} + \mathcal{O}\left(\frac{1}{u_{1,2}^3}\right) \quad \text{as } u_{1,2} \to +\infty. \tag{77}$$

(iv) All orbits to the right of γ_2 are backward asymptotic to the right branch of the parabola $\{-v_{1,2} + 2^{-1/2}u_{1,2}^2 = 0\}$.

(v) All orbits to the left of γ_2 have a horizontal asymptote $v_{1,2} = v_{1,2}^{-\infty} > v_{1,2}^{+\infty}$, where $v_{1,2}^{-\infty}$ depends on the orbit, such that $u_{1,2} \to -\infty$ as $v_{1,2}$ approaches $v_{1,2}^{-\infty}$ from above.

If we transform the orbit γ_2 to chart K_1, we find that

$$\gamma_1 := \kappa_{12}^{-1}(\gamma_2) = \left\{ \left(u_{1,2} s(u_{1,2})^{-1/2}, 0, \mathbf{0}, \mathbf{0}, s(u_{1,2})^{-3/2} \right) \right\}, \tag{78}$$

where $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^{k_0-1}$.

In fact, expanding (78) in a power series as $u_{1,2} \to -\infty$, we obtain

$$\gamma_1 = \left\{ \left(-2^{1/4} + \frac{2^{-3/4}}{u_{1,2}^3} + \mathcal{O}\left(\frac{1}{u_{1,2}^6}\right), 0, \mathbf{0}, \mathbf{0}, -\frac{2^{-3/4}}{u_{1,2}^3} + \mathcal{O}\left(\frac{1}{u_{1,2}^6}\right) \right) \right\}, \tag{79}$$

which implies that γ_1 approaches the steady state $p_a^{k_0}$ in chart K_1, tangent to the vector $(-1, 0, \mathbf{0}, \mathbf{0}, 1)$.

Similarly, for $u_{1,2} > 0$, we can transform γ_2 to the coordinates in chart K_3 via

$$\begin{aligned} \gamma_3 := \kappa_{23}(\gamma_2) &= \left\{ \left(0, u_{1,2}^{-2} s(u_{1,2}), 0, 0, u_{1,2}^{-3}\right) \right\} \\ &= \left\{ \left(0, -\frac{\Omega_0}{u_{1,2}^2} + \frac{2^{1/2}}{u_{1,2}^3} + \mathcal{O}\left(\frac{1}{u_{1,2}^5}\right), \mathbf{0}, \mathbf{0}, \frac{1}{u_{1,2}^3} \right) \right\} \\ &= \left\{ \left(0, -\Omega_0 \varepsilon_3^{2/3} + 2^{1/2}\varepsilon_3 + \mathcal{O}\left(\varepsilon_3^{5/3}\right) \right), \mathbf{0}, \mathbf{0}, \varepsilon_3 \right\}, \end{aligned} \tag{80}$$

which shows that, as $u_{1,2} \to \infty$ or, equivalently, as $\varepsilon_3 \to 0$, γ_3 approaches the origin in chart K_3 tangent to the vector $(0, 1, \mathbf{0}, \mathbf{0}, 0)$.

To determine the transition map for chart K_2, we first transform the exit section $\Sigma_{1,k_0}^{\text{out}}$ from chart K_1 to the coordinates of chart K_2, applying the change of coordinates κ_{12} in (51), which will yield an entry section $\Sigma_{2,k_0}^{\text{in}}$ for the flow in K_2:

$$\Sigma_{2,k_0}^{\text{in}} := \left\{ (u_{1,2}, v_{1,2}, u_{k,2}, v_{k,2}, r_2) : v_{1,2} = \delta^{-2/3} \right\}.$$

In addition, the orbit γ_2 intersects that section in a single point q_0, so that

$$\gamma_2 \cap \Sigma_{2,k_0}^{\text{in}} = \{q_0\}. \tag{81}$$

The coordinates of q_0 satisfy $u_{k,2} = 0 = v_{k,2}$ and $r_2 = 0$. We also define the exit section

$$\Sigma_{2,k_0}^{\text{out}} := \left\{ (u_{1,2}, v_{1,2}, u_{k,2}, v_{k,2}, r_2) : u_{1,2} = \delta^{-1/3} \right\}. \tag{82}$$

The resulting geometry is illustrated in Figure 4. To define the transition map Π_2 in K_2, we consider initial conditions in a small neighbourhood R_2 around the point q_0.

Lemma 5.13. *The invariant set $\{u_{k,2} = 0 = v_{k,2},\ 2 \le k \le k_0\}$ is linearly stable under the flow of (74) if*

$$\frac{8^3 a^6}{\pi^6}\varepsilon_0 < \delta < \frac{\varepsilon_0}{\rho^3}. \tag{83}$$

Proof. Differentiation of (74c) with respect to $u_{k,2}$ shows that for linear stability, we require

$$\frac{b_k}{4A^2} + 2^{1/2} u_{1,2}(t) < 0 \quad \text{or, more strongly,} \quad u_{1,2}(t) < \frac{\pi^2}{8A^2} \tag{84}$$

for $2 \leq k \leq k_0$, as b_k is negative and decreasing with k; recall (9). Given that $u_{1,2}(T_2) = \delta^{-1/3}$ in $\Sigma_{2,k_0}^{\text{out}}$, where T_2 denotes the (finite) transition time of the orbit γ_2 between $\Sigma_{2,k_0}^{\text{in}}$ and $\Sigma_{2,k_0}^{\text{out}}$, it is sufficient to have

$$\delta^{-1/3} < \frac{\pi^2}{8a^2\varepsilon^{1/3}}, \tag{85}$$

where we have made use of $A = a\varepsilon^{1/6}$. We can simplify the last inequality to

$$\delta > \frac{8^3 a^6}{\pi^6}\varepsilon; \tag{86}$$

moreover, since $\varepsilon \in [0, \varepsilon_0)$, it is sufficient to assume

$$\delta > \frac{8^3 a^6}{\pi^6}\varepsilon_0, \tag{87}$$

which places a lower bound on δ. Finally, the upper bound in the statement of the lemma follows from the definition of δ in chart K_1. □

Remark 5.14. The linear stability condition in (83) can be satisfied by restricting ε_0 on the left-hand-side of the condition so that δ can be chosen sufficiently small for the analysis in charts K_1 and K_3 to hold, and by then choosing ρ small enough for the upper bound on the right-hand side to be satisfied.

Proposition 5.15. *The transition map* $\Pi_2 : \Sigma_{2,k_0}^{\text{in}} \to \Sigma_{2,k_0}^{\text{out}}$ *is well-defined in a neighbourhood of the point* q_0*, see Figure 4, which maps diffeomorphically to a neighbourhood of* $\Pi_2(q_0)$*, where*

$$\Pi_2(q_0) = \left(\delta^{-1/3}, -\Omega_0 + 2^{1/2}\delta^{1/3} + \mathcal{O}(\delta), \mathbf{0}, \mathbf{0}, 0\right).$$

Moreover, $|u_{k,2}|$ *and* $|v_{k,2}|$ *are non-increasing under* Π_2*.*

Proof. Given Lemma 5.13, the system in (74) can be considered as a regular perturbation of the Riccati equation (75) in a sufficiently small neighbourhood of q_0. Then, the assertions of the proposition follow from Proposition 5.12 and regular perturbation theory [39, 33]. □

We can also derive estimates on the higher-order modes $\{u_{k,2}, v_{k,2}\}$ during the transition from $\Sigma_{2,k_0}^{\text{in}}$ to $\Sigma_{2,k_0}^{\text{out}}$. Consider a point

$$q_1 = \left(u_{1,2}(0), \delta^{-2/3}, u_{k,2}(0), v_{k,2}(0), r_2(0)\right),$$

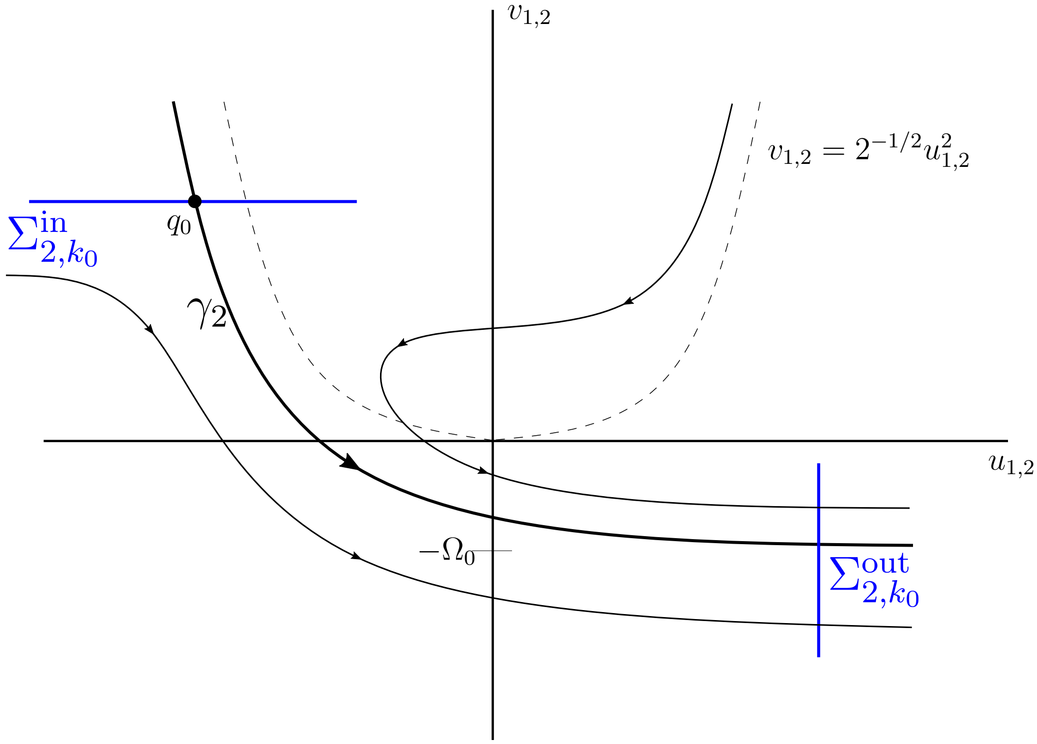


Figure 4: The dynamics in chart K_2 on the invariant plane $\{u_{k,2} = 0 = v_{k,2}\} \cap \{r_2 = 0\}$. For suitably chosen initial conditions and sufficiently small $r_2 = \varepsilon^{1/3}$, the general dynamics of (74) is a regular perturbation of the dynamics on that plane.

with $2 \le k \le k_0$, close to q_0. Since orbits of the full system, Equation (74), are regular perturbations of the orbit γ_2, the transition time $T_2(q_1)$ for the orbit initiated in q_1 will be equal, to leading order, to T_2, the transition time for γ_2,

$$T_2(q_1) = T_2 + \mathcal{O}\left(u_{k,2}(0), v_{k,2}(0), r_2(0)\right). \tag{88}$$

The lower bound on δ in (83) then yields the following estimates on $u_{k,2}$ and $v_{k,2}$.

Lemma 5.16. *For any $t \in [0, T_2(q_1)]$, the following estimates hold:*

$$|u_{k,2}(t)| \le \exp\left(\frac{b_k}{16a^2 r_2(0)} t\right)|u_{k,2}(0)| + \frac{16a^2 r_2(0)}{|b_k|}\Big[|v_{k,2}(0)| + \Big(1 + \frac{4a^2 r_2(0)^2}{|b_k|}\Big)\sigma\Big] \quad \textit{and} \tag{89}$$

$$|v_{k,2}(t)| \le \exp\left(\frac{b_k}{4a^2} r_2^2(0) t\right)|v_{k,2}(0)| + \frac{4a^2 r_2(0)^2 \sigma}{|b_k|}, \tag{90}$$

for some constant $C > 0$ and $0 < \kappa \le \sigma < 1$, where κ is as in (34).

Proof. For $(\tilde{u}_{1,2}, \dots, \tilde{u}_{k_0,2}, \tilde{v}_{1,2}, \dots \tilde{v}_{k_0,2})$ in

$$\mathcal{B}_2 = \Big\{(\tilde{u}_{1,2}, \dots, \tilde{u}_{k_0,2}, \tilde{v}_{1,2}, \dots, \tilde{v}_{k_0,2}) : \tilde{u}_{k,2}, \tilde{v}_{k,2} \in C[0, T_2(q_1)], \text{ with}$$

$$\sup_{[0,T_2(q_1)]} |\tilde{u}_{k,2}(t)| \le C_k, \sup_{[0,T_2(q_1)]} |\tilde{v}_{k,2}(t)| \le C_k \text{ for } 1 \le k \le k_0, \text{ and } \sum_{k=1}^{k_0} C_k^2 \le \tilde{\sigma}\Big\},$$

consider $M_k(\tilde{u}_{2,2},\dots\tilde{u}_{k_0,2}) = \sum_{i,j=2}^{k_0}\eta_{i,j}^k\tilde{u}_{j,2}\tilde{u}_{1,2}$ and the higher-order terms $H_{k,2}^u = H_{k,2}^u(\tilde{u}_{1,2},\dots,\tilde{u}_{k_0,2},\tilde{v}_{1,2},\dots\tilde{v}_{k_0,2})$ and $H_{k,2}^v = H_{k,2}^v(\tilde{u}_{1,2},\dots,\tilde{u}_{k_0,2},\tilde{v}_{1,2},\dots\tilde{v}_{k_0,2})$. Thus, we define a map $\mathcal{N}_2$ via $(\tilde{u}_{1,2},\dots,\tilde{u}_{k_0,2},\tilde{v}_{1,2},\dots\tilde{v}_{k_0,2}) \mapsto (u_{1,2},\dots,u_{k_0,2},v_{1,2},\dots v_{k_0,2})$, where $(u_{1,2},\dots,u_{k_0,2},v_{1,2},\dots v_{k_0,2})$ are solutions of (74). To obtain the estimates stated in the lemma, we shall show that $\mathcal{N}_2 : \mathcal{B}_2 \to \mathcal{B}_2$. As $r_2(t)$ is constant in chart K_2, from (74d) we conclude

$$\begin{aligned}|v_{k,2}(t)| &\le \exp(\frac{b_k}{4A^2}r_2^3(0)t)|v_{k,2}(0)| + \exp(\frac{b_k}{4A^2}r_2^3(0)t)\int_0^t \exp(-\frac{b_k}{4A^2}r_2^3(0)s)|H_{k,2}^v|ds\\ &\le \exp(\frac{b_k}{4A^2}r_2^3(0)t)|v_{k,2}(0)| + \frac{4A^2r_2(0)}{|b_k|}\sigma\end{aligned}$$

for all $t \in [0, T_2(q_1)]$, where $|H_{k,2}^v| \le Cr_2(0)^4\tilde{\sigma} \le r_2(0)^4\sigma$. Applying the variation of constants formula to (74c), we find

$$\begin{aligned}|u_{k,2}(t)| &\le \exp\left(\frac{(2-\sqrt{2})b_k}{8A^2}t\right)|u_{k,2}(0)| + \frac{8A^2}{(2-\sqrt{2})|b_k|}\left(\sup_{t\in[0,T_2(q_1)]}|v_{k,2}(t)| + C\tilde{\sigma}\right)\\ &\le \exp(\frac{b_k}{16A^2}t)|u_{k,2}(0)| \ + \frac{16A^2}{|b_k|}\left[|v_{k,2}(0)| + \left(1+\frac{4a^2r_2(0)^2}{|b_k|}\right)\sigma\right]\end{aligned}$$

for all $t \in [0, T_2(q_1)]$. Thus, for appropriately chosen $0 < r_2(0) < 1$ and $0 < \sigma < 1$, we obtain that $\mathcal{N}_2 : \mathcal{B}_2 \to \mathcal{B}_2$, as claimed, which implies (89) and (90). □

Remark 5.17. Since $A = a\varepsilon^{1/6} = a(r_2(0))^{1/2}$, the first term in (89) is equal to $\exp\left(-\frac{c}{r_2(0)}\right)|u_{k,2}|$ at $t = T_2(q_1)$, with $c > 0$ a constant, while the second term has the form of an $\mathcal{O}(r_2(0))$-correction.

The estimate for $v_{k,2}(t)$ in (90) implies the bound $\exp\left(-cr_2^2(0)\right)|v_{k,2}(0)| \approx |v_{k,2}(0)|$ at $t = T_2(q_1)$ for small $r_2(0)$, as considered here.

Taking more general higher-order terms of the form $H^v = H^v(u^2, uv, v^2)$ in (6), we would find $H_{k,2}^v$ to be of the order $\mathcal{O}(r_2(0)^2)$; then, the second term in (90) would read $4a^2\sigma/|b_k|$, which is uniformly bounded in $r_2(0)$ and k.

5.3. Chart K_3

In chart K_3, the blow-up transformation in (46) reads

$$u_1 = r_3, \quad v_1 = r_3^2 v_{1,3}, \quad u_k = r_3 u_{k,3}, \quad v_k = r_3^2 v_{k,3}, \quad \text{and} \quad \varepsilon = r_3^3\varepsilon_3.$$

After desingularising by dividing out a factor of r_3 from the resulting vector field, we obtain

$$
\begin{aligned}
r_3' &= r_3 F_3, && \text{(91a)}\\
v_{1,3}' &= -2F_3 v_{1,3} - 2^{1/2}\varepsilon_3, && \text{(91b)}\\
u_{k,3}' &= \left(-F_3 + \frac{b_k}{4A^2}\varepsilon_3^{1/3} + 2^{1/2}\right) u_{k,3} - v_{k,3} + \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_{i,3} u_{j,3} + H_{k,3}^u, && \text{(91c)}\\
v_{k,3}' &= \left(-2F_3 + \frac{b_k}{4A^2} r_3^3 \varepsilon_3^{4/3}\right) v_{k,3} + \varepsilon_3 H_{k,3}^v, && \text{(91d)}\\
\varepsilon_3' &= -3F_3\varepsilon_3, && \text{(91e)}
\end{aligned}
$$

where

$$F_3 = F_3(r_3, v_{1,3}, u_{k,3}, v_{k,3}, \varepsilon_3) = -v_{1,3} + 2^{-1/2} + 2^{-1/2}\sum_{j=2}^{k_0} u_{j,3}^2 + H_{1,3}^u,$$

with

$$
\begin{aligned}
H_{1,3}^u &= \mathcal{O}\left(r_3\varepsilon_3, r_3^2 v_{1,3}^2, r_3^2 v_{j,3}^2, r_3 v_{1,3}, r_3 u_{j,3} v_{j,3}, r_3 u_{j,3}^2, r_3 u_{i,3} u_{j,3} u_{l,3}\right), \\
H_{k,3}^u &= \mathcal{O}\left(r_3^2 v_{1,3} v_{k,3}, r_3^2 v_{i,3} v_{j,3}, r_3 v_{k,3},\right.\\
&\qquad \left. r_3 u_{k,3} v_{1,3}, r_3 u_{i,3} v_{j,3}, r_3 u_{k,3}, r_3 u_{i,3} u_{j,3}, r_3 u_{i,3} u_{j,3} u_{l,3}\right), \quad \text{and}\\
H_{k,3}^v &= \mathcal{O}\left(r_3^4 v_{1,3} v_{k,3}, r_3^4 v_{i,3} v_{j,3}\right)
\end{aligned}
$$

for $2 \leq i, j, l \leq k_0$.

As in K_1, we can rewrite (91) in the form

$$
\begin{aligned}
r_3' &= r_3 F_3, && \text{(92a)}\\
v_{1,3}' &= -2F_3 v_{1,3} - 2^{1/2}\varepsilon_3, && \text{(92b)}\\
u_{k,3}' &= \left(-F_3 + \frac{b_k}{4A^2}\varepsilon_3^{1/3} + 2^{1/2}\right) u_{k,3} - v_{k,3} + \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_{i,3} u_{j,3} + H_{k,3}^u, && \text{(92c)}\\
v_{k,3}' &= \left(-2F_3 + \frac{b_k}{4A^2} r_3^3 \left(\varepsilon_3^{1/3}\right)^4\right) v_{k,3} + \varepsilon_3 H_{k,3}^v, && \text{(92d)}\\
\left(\varepsilon_3^{1/3}\right)' &= -F_3 \varepsilon_3^{1/3} && \text{(92e)}
\end{aligned}
$$

for $2 \leq k \leq k_0$.

As mentioned already, the portion $\gamma_3 := \kappa_{23}(\gamma_2)$ of the orbit γ_2 from chart K_2 with $u_{1,2} > 0$, transformed to K_3, has the expansion

$$\gamma_3 = \left(0, -\Omega_0 \varepsilon_3^{2/3} + 2^{1/2}\varepsilon_3 + \mathcal{O}\big(\varepsilon_3^{5/3}\big), \mathbf{0}, \mathbf{0}, \varepsilon_3\right)$$

as $\varepsilon_3 \to 0$. Thus, we see that γ_3 approaches the origin in chart K_3. Hence, it follows that the centre manifold $M_{k_0,1}$ from chart K_1 passes through a neighbourhood of the origin, which is a hyperbolic steady state for (92).

Let $w_k := (0, 0, \ldots, 1, \ldots, 0)$, with $2 \leq k \leq k_0$, denote the vector with $k_0 - 1$ entries which are all equal to 0 except for the $(k-1)$-th entry, which equals 1. Using this notation, a direct calculation shows the following result.

Lemma 5.18. *The origin is a hyperbolic steady state of Equation* (92)*, with the following eigenvalues and eigenvectors in the corresponding linearisation:*

- *the simple eigenvalue* $\frac{\sqrt{2}}{2}$ *with eigenvector* $(1, 0, \mathbf{0}, \mathbf{0}, 0)$*, corresponding to* r_3*;*
- *the simple eigenvalue* $-\sqrt{2}$ *with eigenvector* $(0, 1, \mathbf{0}, \mathbf{0}, 0)$*, corresponding to* $v_{1,3}$*;*
- *the eigenvalue* $\frac{\sqrt{2}}{2}$ *with multiplicity* $k_0 - 1$ *and eigenvectors* $(0, 0, w_k, \mathbf{0}, 0)$*, corresponding to* $u_{k,3}$ *(*$2 \leq k \leq k_0$*);*
- *the eigenvalue* $-\sqrt{2}$ *with multiplicity* $k_0 - 1$ *and eigenvectors* $(0, 0, \frac{\sqrt{2}}{3} w_k, w_k, 0)$*, corresponding to* $v_{k,3}$ *(*$2 \leq k \leq k_0$*); and*
- *the simple eigenvalue* $-\frac{\sqrt{2}}{2}$ *with eigenvector* $(0, 0, \mathbf{0}, \mathbf{0}, 1)$*, corresponding to* $\varepsilon_3^{1/3}$*.*

Remark 5.19. Since
$$-\frac{\sqrt{2}}{2} = -\sqrt{2} + \frac{\sqrt{2}}{2},$$
the eigenvalues of (92) are in resonance. Potential second-order resonant terms are $r_3 v_{1,3}$, $r_3 v_{k,3}$, and $u_{i,3} v_{j,3}$. While resonances are also observed in the singularly perturbed planar fold [24], the resonant terms differ, which is due to the formulation of the governing equations in chart K_3 in terms of $\varepsilon_3^{1/3}$. Furthermore, the higher dimensionality of (92) allows for a richer resonance structure which may be explored in future work.

The entry section $\Sigma^{\text{in}}_{3,k_0}$ in chart K_3, which is obtained by transformation of the exit section $\Sigma^{\text{out}}_{2,k_0}$ from K_2, is given by
$$\Sigma^{\text{in}}_{3,k_0} = \{(r_3, v_{1,3}, u_{k,3}, v_{k,3}, \mu_3) : \varepsilon_3 = \delta\}, \tag{93}$$
where we consider the set of initial conditions
$$\begin{aligned} R_3 = \{(r_3, v_{1,3}, u_{k,3}, v_{k,3}, \varepsilon_3) \mid r_3 \in [0, \rho], v_{1,3} \in [-\beta, \beta], \\ |u_{k,3}| \leq C_{u_{k,3}}, |v_{k,3}| \leq C_{v_{k,3}} \text{ for } 2 \leq k \leq k_0, \text{ and } \varepsilon_3 = \delta\} \subset \Sigma^{\text{in}}_{3,k_0}. \end{aligned} \tag{94}$$
Here, $\beta, C^{\text{in}}_{u_{k,3}}$, and $C^{\text{in}}_{v_{k,3}}$, for $2 \leq k \leq k_0$, are appropriately defined small constants. We also define the exit chart
$$\Sigma^{\text{out}}_{3,k_0} := \{(r_3, v_{1,3}, u_{k,3}, v_{k,3}, \varepsilon_3) : r_3 = \rho\}. \tag{95}$$
Our aim is to describe the transition map $\Pi_3 : R_3 \to \Sigma^{\text{out}}_{3,k_0}$. Therefore, since F_3 is bounded away from zero near the origin, we can divide the vector field in (92) by F_3,

which results in

$$r_3' = r_3, \tag{96a}$$

$$v_1' = -2v_1 - 2^{1/2}\frac{\left(\varepsilon_3^{1/3}\right)^3}{F_3}, \tag{96b}$$

$$u_k' = \left(-1 + \frac{b_k}{4A^2}\frac{\varepsilon_3^{1/3}}{F_3} + \frac{2^{1/2}}{F_3}\right)u_k - \frac{1}{F_3}v_k + \frac{1}{F_3}\sum_{i,j=2}^{k_0}\eta_{i,j}^k u_i u_j + \frac{1}{F_3}H_k^u, \tag{96c}$$

$$v_k' = \left(-2 + \frac{b_k}{4A^2}\frac{r_3^3\left(\varepsilon_3^{1/3}\right)^4}{F_3}\right)v_k + \frac{\left(\varepsilon_3^{1/3}\right)^3}{F_3}H_k^v, \tag{96d}$$

$$\left(\varepsilon_3^{1/3}\right)' = -\varepsilon_3^{1/3}, \tag{96e}$$

where the prime denotes differentiation with respect to the new, rescaled, time variable. Here, we have suppressed the subscript 3 in (96) for convenience of notation, and will do so for the remainder of the section.

The above rescaling of time by F_3 results in the eigenvalues of the linearisation about the origin being rescaled by a factor $2^{-1/2}$. Lemma 5.18 hence now implies the following:

Lemma 5.20. *The origin is a hyperbolic steady state of Equation* (96)*, with the following eigenvalues in the corresponding linearisation:*

- *the simple eigenvalue* 1*, corresponding to* r_3*;*
- *the simple eigenvalue* -2*, corresponding to* v_1*;*
- *the eigenvalue* 1 *with multiplicity* $k_0 - 1$*, corresponding to* u_k $(2 \le k \le k_0)$*;*
- *the eigenvalue* -2 *with multiplicity* $k_0 - 1$*, corresponding to* v_k $(2 \le k \le k_0)$*; and*
- *the simple eigenvalue* -1*, corresponding to* $\varepsilon_3^{1/3}$*.*

The associated eigenvectors are as in Lemma 5.18.

To obtain estimates for the transition map Π_3, we follow a procedure that is analogous to that in [24] for this chart. We begin by separating out terms containing r_3 in (96). To that end, we expand

$$\frac{1}{F_3(v_1, u_k, r_3)} = G_3(v_1, u_k) + r_3 J(v_1, u_k, r_3), \tag{97}$$

in a neighbourhood of the steady state at the origin, where

$$G_3(v_1, u_k) = \frac{1}{2^{-1/2} - v_1 + 2^{-1/2}\sum_{j=2}^{k_0} u_j^2} \tag{98}$$

and J is a smooth function of v_1, u_k, r_3 in the same neighborhood. With the above notation, we can rewrite Equation (96) as stated below.

Lemma 5.21. *For $r_3 \geq 0$ sufficiently small,* (96) *can be written as*

$$r_3' = r_3, \tag{99a}$$
$$v_1' = -2v_1 - 2^{1/2}\varepsilon_3 G_3 + \varepsilon_3 r_3 J_{v_1}, \tag{99b}$$
$$u_k' = \left(-1 + \frac{b_k}{4A^2}\varepsilon_3^{1/3} G_3 + 2^{1/2} G_3\right) u_k - G_3 v_k + G_3 \sum_{i,j=2}^{k_0} \eta_{i,j}^k u_i u_j + r_3 J_{u_k}, \tag{99c}$$
$$v_k' = \left(-2 + \frac{b_k}{4A^2}\left(\varepsilon_3^{1/3}\right)^4 r_3^3 G_3\right) v_k + \varepsilon_3 r_3 J_{v_k}, \tag{99d}$$
$$\left(\varepsilon_3^{1/3}\right)' = -\varepsilon_3^{1/3}, \tag{99e}$$

where $J_{v_1}(r_3, v_1, u_k, v_k, \varepsilon_3^{1/3})$, $J_{u_k}(r_3, v_1, u_k, v_k, \varepsilon_3^{1/3})$, *and* $J_{v_k}(r_3, v_1, u_k, v_k, \varepsilon_3^{1/3})$ *are smooth functions.*

Proof. Using (97) in (96) and collecting r_3-dependent terms, we obtain (99). The functions J_{v_1}, J_{u_k}, and J_{v_k} are defined by summation and multiplication between the variables in chart K_3 and the functions H_k^u, H_k^v, and J, and are hence smooth in their arguments in the neighbourhood of the origin we are considering. □

We now have the following result for the transition map Π_3.

Proposition 5.22. *The transition map $\Pi_3 : R_3 \to \Sigma_{3,k_0}^{\text{out}}$ is well-defined. Let $(r_3^{\text{in}}, v_1^{\text{in}}, u_k^{\text{in}}, v_k^{\text{in}}, \delta) \in R_3$, as defined in* (94)*, where $k = 2, \ldots, k_0$, and let T_3 be the corresponding transition time between $\Sigma_{3,k_0}^{\text{in}}$ and $\Sigma_{3,k_0}^{\text{out}}$ under the flow of* (99)*. Then, the map Π_3 is given by*

$$\Pi_3(r_3^{\text{in}}, v_1^{\text{in}}, u_k^{\text{in}}, v_k^{\text{in}}, \delta) = \left(\rho, \Pi_{3,v_1}^{k_0}, \Pi_{3,u_k}^{k_0}, \Pi_{3,v_k}^{k_0}, \delta^{1/3}\frac{r_3^{\text{in}}}{\rho}\right),$$

where

$$\left|\Pi_{3,v_1}(r_3^{\text{in}}, v_1^{\text{in}}, u_k^{\text{in}}, v_k^{\text{in}}, \delta)\right| \leq \left(\frac{r_3^{\text{in}}}{\rho}\right)^2 \left[|v_1^{\text{in}}| + C_{v_{1,3}}^{\text{out}}(1 + r_3^{\text{in}} \log r_3^{\text{in}})\right], \tag{100}$$
$$\left|\Pi_{3,u_k}(r_3^{\text{in}}, v_1^{\text{in}}, u_k^{\text{in}}, v_k^{\text{in}}, \delta)\right| \leq C_{u_{k,3}}^{\text{out}}, \quad \text{and} \tag{101}$$
$$\left|\Pi_{3,v_k}(r_3^{\text{in}}, v_1^{\text{in}}, u_k^{\text{in}}, v_k^{\text{in}}, \delta)\right| \leq C_{v_{k,3}}^{\text{out}}, \tag{102}$$

for positive constants $C_{v_{1,3}}^{\text{out}}$, $C_{u_{k,3}}^{\text{out}}$, and $C_{v_{k,3}}^{\text{out}}$.

Proof. From (99), we have that

$$r_3(t) = r_3^{\text{in}} e^t \quad \text{and} \quad \varepsilon_3(t) = \delta e^{-3t}, \tag{103}$$

which gives the transition time

$$T_3 = \log \frac{\rho}{r_3^{\text{in}}} \tag{104}$$

between $\Sigma_{3,k_0}^{\text{in}}$ and $\Sigma_{3,k_0}^{\text{out}}$.

For $\sum_{k=2}^{k_0} |\tilde{u}_k(t)|^2 \leq \sigma$ and $\sum_{k=2}^{k_0} |\tilde{v}_k(t)|^2 \leq \sigma$, with $0 < \sigma \leq 1$, and $|v_1(t)| \leq 1/(2\sqrt{2})$ for $t \in [0, T_3]$, consider $J_{v_1} = J_{v_1}(\tilde{v}_1, \tilde{u}_k, \tilde{v}_k)$ and $J_{l_k} = J_{l_k}(v_1, \tilde{u}_k, \tilde{v}_k)$, with $l = u, v$ and $k = 2, \ldots, k_0$, as well as $G_3 = G_3(v_1, \tilde{u}_k)$. We observe that

$$\begin{aligned} G_3(v_1(t), \tilde{u}_k(t)) &\leq \frac{1}{2^{-1/2} - |v_1(t)|} = 2^{1/2} + 2^{1/2} \sum_{n=1}^{\infty} 2^{\frac{n}{2}} |v_1(t)|^n \\ &\leq 2^{1/2} \left(1 + 2^{1/2} \frac{|v_1(t)|}{1 - \sqrt{2}|v_1(t)|}\right) \leq 2^{1/2}\big(1 + 2^{3/2}|v_1(t)|\big) \end{aligned} \tag{105}$$

for $|v_1(t)| \leq 1/(2\sqrt{2})$, and

$$G_3(v_1(t), \tilde{u}_k(t)) \geq \frac{1}{2^{-1/2} + |v_1(t)| + 2^{-1/2}\sigma} =: C_1 \geq 1. \tag{106}$$

Then, using the boundedness of G_3 and J_{v_k}, from Equation (99d) for v_k we obtain directly

$$|v_k(t)| \leq e^{-2t + b_3(t)} |v_k(0)| + C \int_0^t e^{-2(t-s) + B_3(t) - B_3(s)} \varepsilon_3(s) r_3(s) ds \leq e^{-2t} |v_k(0)| + \frac{C\rho a^2 \sigma}{|b_k|},$$

where $B_3(t) = r_3^{\text{in}} \delta^{4/3} b_k (1 - e^{-3t})/(12A^2)$. To determine the asymptotic behaviour of v_1, we define a new variable z by

$$v_1 = e^{-2t} \left(v_1^{\text{in}} + z\right), \tag{107}$$

where for $t = 0$ it follows that $z(0) = 0$. A direct calculation yields

$$\begin{aligned} v_1' &= e^{-2t} z' - 2e^{-2t} \left(v_1^{\text{in}} + z\right), \\ -2v_1 - \delta e^{-3t} G_3 + r_3^{\text{in}} \delta e^{-2t} J_{v_1} &= e^{-2t} z' - 2e^{-2t} \left(v_1^{\text{in}} + z\right), \quad \text{and} \\ -2e^{-2t} \left(v_1^{\text{in}} + z\right) + \delta e^{-3t} G_3 + r_3^{\text{in}} \delta e^{-2t} J_{v_1} &= e^{-2t} z' - 2e^{-2t} \left(v_1^{\text{in}} + z\right) \end{aligned}$$

or, equivalently,

$$z' = -2^{1/2} e^{-t} \delta G_3 + r_3^{\text{in}} \delta J_{v_1}. \tag{108}$$

Then, the boundedness of G_3 and J_{v_1} implies

$$|z(t)| \leq C\delta(1 - e^{-t} + r_3^{\text{in}} t).$$

Reverting to the original variable v_1 via (107), we find

$$|v_1(t)| \leq e^{-2t} \left[|v_1^{\text{in}}| + C\delta(1 - e^{-t} + r_3^{\text{in}} t)\right] \leq |v_1^{\text{in}}| + 2\delta C_2 \leq 1/(2\sqrt{2}) \quad \text{for all } t \in [0, T_3]$$

with sufficiently small $|v_1^{\text{in}}|$, and

$$|v_1(T_3)| \leq \left(\frac{r_3^{\text{in}}}{\rho}\right)^2 \left[|v_1^{\text{in}}| + C\delta \left(1 + r_3^{\text{in}} \log \frac{\rho}{r_3^{\text{in}}}\right)\right], \tag{109}$$

which proves the first estimate stated in the theorem.

Next, we show that u_k remains bounded throughout the transition through chart K_3. Once again, we perform an estimate using the variation of constants formula

$$u_k(t) = \exp\left(\int_0^t U_k(\tau)d\tau\right) u_k^{\text{in}}$$
$$+ \int_0^t \exp\left(\int_s^t U_k(\tau)d\tau\right)\left(-G_3(v_1,\tilde{u}_k)v_k + G_3(v_1,\tilde{u}_k)\sum_{i,j=2}^{k_0}\eta_{i,j}^k\tilde{u}_i\tilde{u}_j + r_3 J_{u_k}\right) ds,$$

where

$$U_k(\tau) = -1 + \frac{b_k}{4A^2}\delta^{1/3}e^{-\tau}G_3(v_1(\tau),\tilde{u}_k(\tau)) + 2^{1/2}G_3(v_1(\tau),\tilde{u}_k(\tau)).$$

Using the assumptions on $\tilde{v}_k$ and $\tilde{u}_k$, the estimate for $G_3(v_1,\tilde{u}_k)$ in (105), and the fact that $b_k < 0$, we find

$$\begin{aligned}
\mathcal{I}_1(s) &:= \int_s^t U_k(\tau)d\tau \\
&= s - t + \frac{b_k}{4A^2}\delta^{1/3}\int_s^t e^{-\tau}G_3(v_1(\tau),\tilde{u}_k(\tau))d\tau + 2^{1/2}\int_s^t G_3(v_1(\tau),\tilde{u}_k(\tau))d\tau \qquad (110)\\
&\le t - s + \frac{b_k}{4A^2}\delta^{1/3}C_1\left(e^{-s} - e^{-t}\right) + 4\int_s^t |v_1(\tau)|d\tau
\end{aligned}$$

for $0 \le s \le t \le T_3$. To estimate the integral in the last inequality, we observe that

$$|v_1(\tau)| \le e^{-2\tau}\left(|v_1^{\text{in}}| + C_2\delta(1+\tau)\right)$$

and write

$$\int_s^t |v_1(\tau)|d\tau \le \frac{1}{4}(2|v_1^{\text{in}}| + 3C_2\delta)(e^{-2s} - e^{-2t}) + C_2\frac{\delta}{2}(e^{-2s}s - e^{-2t}t).$$

The inequality in (110) then becomes

$$\mathcal{I}_1(s) \le t - s + \frac{b_k}{4A^2}\delta^{1/3}C_1\left(e^{-s} - e^{-t}\right) + (2|v_1^{\text{in}}| + 3C_2\delta)(e^{-2s} - e^{-2t}) + 2C_2\delta(e^{-2s}s - e^{-2t}t);$$

thus,

$$\exp(\mathcal{I}_1(0)) \le C\exp\left(t + \frac{b_k}{4A^2}\delta^{1/3}C_1\big(1 - e^{-t}\big)\right),$$

where $C = \exp(2|v_1^{\text{in}}| + 5C_2\delta)$. For the second term in $u_k(t)$, using that $t \le e^t$ and $1 \le e^t$ for $t \ge 0$, we have

$$\begin{aligned}
\int_0^t \exp(\mathcal{I}_1(s))ds &\le \mathcal{I}_2(t)\int_0^t \exp\Big(-s + \frac{b_k}{4A^2}\delta^{1/3}C_1e^{-s} + C_3e^{-2s} + 2C_2\delta e^{-2s}s\Big)ds \\
&\le \mathcal{I}_2(t)\int_0^t e^{-s}\exp\left(\Big[\frac{b_k}{4A^2}\delta^{1/3}C_1 + C_4\delta\Big]e^{-s}\right)ds \\
&= \mathcal{I}_2(t)\frac{4A^2}{C_1|b_k|\delta^{1/3} - 4A^2C_4}\left[\exp\left(\Big[\frac{b_k\delta^{1/3}}{4A^2}C_1 + C_4\delta\Big]e^{-t}\right) - \exp\left(\frac{b_k\delta^{1/3}}{4A^2}C_1 + C_4\right)\right] \\
&\le C_5\frac{4A^2}{C_1|b_k|\delta^{1/3} - 4A^2C_4}e^t \le C_6\frac{a^2}{|b_k|}\varepsilon^{1/3}\frac{\rho}{r_3^{\text{in}}} \le \frac{C_7\rho a^2}{|b_k|},
\end{aligned}$$

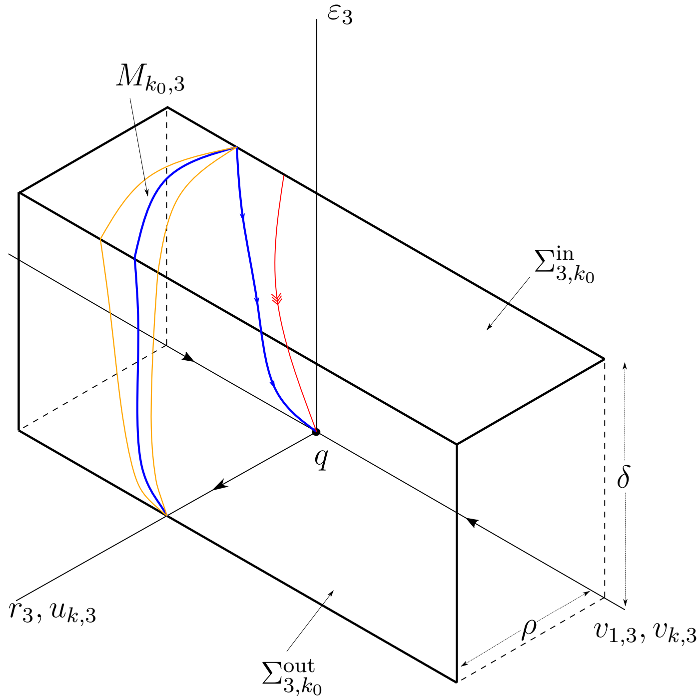


Figure 5: Dynamics in chart K_3. As the orbit γ_2 from chart K_2, after transformation to K_2 via $\gamma_3 := \kappa_{23}(\gamma_2)$ (in red), passes through the origin q, the invariant manifold $M_{k_0,3}$ contains q. The transition map Π_3 is defined in a neighbourhood of the intersection $M_{k_0,3} \cap \Sigma^{\text{in}}_{3,k_0}$.

where $C_3 = 2|v_1^{\text{in}}| + 3C_2\delta$ and

$$\mathcal{I}_2(t) = \exp\Big(t - \frac{b_k}{4A^2}\delta^{1/3}C_1e^{-t} - C_3e^{-2t} - 2C_2\delta e^{-2t}t\Big).$$

The estimates for v_1 and v_k then yield

$$|u_k(t)| \leq C_1|u_k(0)| + \frac{C_2\rho a^2}{|b_k|}\Big(|v_k(0)| + \frac{C\rho a^2\sigma}{|b_k|} + \sigma(1+\rho)\Big).$$

Hence, for sufficiently small $|u_k(0)|$, $|v_k(0)|$, and ρ, we find

$$\sum_{k=2}^{k_0}|u_k(t)|^2 \leq \sigma \quad \text{and} \quad \sum_{k=2}^{k_0}|v_k(t)|^2 \leq \sigma \qquad \text{for all } t \in [0, T_3].$$

Then, application of a fixed point argument as in chart K_1 yields the estimates stated in the theorem. □

5.4. Proof of main result

Let us now combine the analysis in the three charts K_1, K_2, and K_3 to give the proof of Theorem 4.3. For $2 \leq k \leq k_0$, the initial conditions $(u_{1,1}^{\text{in}}, r_1^{\text{in}}, u_{k,1}^{\text{in}}, v_{k,1}^{\text{in}}, \varepsilon_1^{\text{in}})$ in K_1 are assumed to lie in $R_1 \subset \Sigma^{\rm in}_{1,k_0}$, where R_1 is defined in (70). Applying the transition map Π_1, see Proposition 5.11, we obtain

$$\Pi_1(R_1) = \Big\{ |u^{\rm out}_{1,1} + 2^{1/4}| \le C^{\rm out}_{u_{1,1}},\ r^{\rm out}_1 \in [0,\rho],\ \varepsilon^{\rm out}_1 = \delta, \\ |u^{\rm out}_{k,1}| \le C^{\rm out}_{k,1},\ \text{and } |v^{\rm out}_{k,1}| \le C^{\rm out}_{v_{k,1}} \delta^{2/3} \Big\}.$$

Transformation of the above set to chart K_2 yields

$$\kappa_{12} \circ \Pi_1(R_1) = \Big\{ u^{\rm in}_{1,2} = \delta^{-1/3} u^{\rm out}_{1,1}, v^{\rm in}_{1,2} = \delta^{-2/3},\ |u^{\rm in}_{k,2}| \le \delta^{-1/3}|u^{\rm out}_{k,1}|, \\ |v^{\rm in}_{k,2}| \le \delta^{-2/3}|v^{\rm out}_{k,1}|,\ \text{and } r^{\rm in}_2 \in [0, \delta^{1/3}\rho] \Big\},$$

with $u_{k,1}$ and $v_{k,1}$ as above. Since the higher-order modes $\{u_{k,2}, v_{k,2}\}$ do not grow in K_2, we have

$$\Pi_2 \circ \kappa_{12} \circ \Pi_1(R_1) = \Big\{ u^{\rm out}_{1,2} = \delta^{-1/3},\ |v^{\rm out}_{1,2} + c_1| \le C^{\rm out}_{v_{1,2}}, \\ |u^{\rm out}_{k,2}| \le |u^{\rm in}_{k,2}|,\ |v^{\rm out}_{k,2}| \le |v^{\rm in}_{k,2}|,\ \text{and } r^{\rm out}_2 \in \left[0, \delta^{1/3}\rho\right] \Big\}. \tag{111}$$

Application of the change of coordinates κ_{23} yields

$$\kappa_{23} \circ \Pi_2 \circ \kappa_{12} \circ \Pi_1(R) = \Big\{ r^{\rm in}_3 \in [0,\varepsilon],\ v^{\rm in}_{1,3} \in [-\beta,\beta],\ \varepsilon^{\rm in}_3 = \delta, |u^{\rm in}_{k,3}| = \delta^{1/3}, \\ |u^{\rm out}_{k,2}| \le C^{\rm in}_{u_{k,3}},\ |v^{\rm in}_{k,3}| = \delta^{2/3},\ \text{and } |v^{\rm out}_{k,2}| \le C^{\rm in}_{v_{k,1}} \Big\},$$

where $\beta > 0$ is a small constant. Finally, we apply the map Π_3, see Proposition 5.22, to obtain

$$\Pi_3 \circ \kappa_{23} \circ \Pi_2 \circ \kappa_{12} \circ \Pi_1(R_1) \\ = \Big\{ r^{\rm out}_3 = \rho,\ v^{\rm out}_{1,3},\ \varepsilon^{\rm out}_3 \in [0,\delta],\ |u^{\rm out}_{k,3}| \le C^{\rm out}_{u_{k,3}},\ \text{and } |v^{\rm out}_{k,3}| \le C^{\rm out}_{v_{k,3}} \Big\},$$

where $v^{\rm out}_{1,3}$ is as in Proposition 5.22. The result then follows, since the sections $\Sigma^{\rm in}_{1,k_0}$ and $\Sigma^{\rm out}_{3,k_0}$ are equivalent to $\Delta^{\rm in}$ and $\Delta^{\rm out}$, respectively, transformed into the coordinates of charts K_1 and K_3, respectively, and since the systems in (32) and (48) are equivalent for $\varepsilon > 0$ sufficiently small.

6. Conclusions and outlook

In this work, we have studied, via discretisation, a fast-slow system of partial differential equations (PDEs) of reaction-diffusion type, Equation (6), under the assumption that a fold singularity is present at the origin in the fast kinetics. We have approximated a family $S_{\varepsilon,\zeta}$ of slow manifolds by their corresponding Galerkin manifolds $\mathcal{C}_{\varepsilon,k_0}$, which we have then extended past the singularity by applying the desingularisation technique known as blow-up [10]. Here, it is worth emphasising that the family $S_{\varepsilon,\zeta}$ is defined via a subsplitting ansatz for the slow variables; hence, it is not simply a "generic" perturbation of a classical critical manifold which is obtained in a finite-dimensional setting by a "quasi-steady state approximation".

As we have seen, our main result, Theorem 4.3, is analogous to what one would expect in the planar (finite-dimensional) setting [24]. While we have shown that the resulting Galerkin manifolds $\mathcal{C}_{\varepsilon,k_0}$ approximate the family $S_{\varepsilon,\zeta}$ away from the fold singularity, the crucial question of whether one can pass to the limit of $k_0 \to \infty$, which would require us to study a triple limit in k_0, ε, and ζ, and state that correspondence for the extension of $\mathcal{C}_{\varepsilon,k_0}$ via blow-up still remains open. Two conjectures seem plausible here: (i) an alternative approach may allow one to prove that, potentially under slightly stronger assumptions, a limiting invariant slow manifold exists as $k_0 \to \infty$ even near the singularity; or (ii) the limiting object must diverge due to the infinite-scale coupling between higher-order modes in the Galerkin discretisation, which would imply that no slow manifold can exist. Unfortunately, standard techniques [23, 35] for proving the non-existence of invariant manifolds do not seem to apply here.

Correspondingly, the presence of an additional $2k_0-2$ equations after discretisation, with k_0 arbitrarily large, causes several challenges. Thus, a preparatory rescaling of the domain length is introduced to allow for the application of the blow-up technique; an alternative approach in previous work on the transcritical and pitchfork singularities [11] results in a dynamic boundary value problem. Our rescaling appears natural, since it can be recovered directly from the original system of PDEs in (6). Specifically, taking $u = \varepsilon^{1/3}U$, $v = \varepsilon^{2/3}V$, $t = \varepsilon^{-1/3}\tau$, and $x = \varepsilon^{-1/6}X$, which is consistent with our scaling in (50), we obtain

$$\begin{aligned} \partial_\tau U &= \partial_X^2 U - V + U^2 + \varepsilon^p H_u(U,V) \quad &&\text{on } (-a\varepsilon^{1/6}, a\varepsilon^{1/6}),\\ \partial_\tau V &= \varepsilon\partial_X^2 V - 1 + \varepsilon^q H_v(U,V) &&\text{on } (-a\varepsilon^{1/6}, a\varepsilon^{1/6}), \end{aligned} \tag{112}$$

for some $p, q > 0$. Equation (112) defines a system of PDEs on a domain shrinking to the origin as $\varepsilon \to 0$, as is to be expected due to the singular nature of (6). Denoting by $(U_\varepsilon, V_\varepsilon)$ solutions of (112), and using the boundedness of higher-order terms and the non-positivity of U_ε or the boundedness of U_ε^2, which can be achieved by considering a cut-off function, we obtain the following estimates:

$$\begin{aligned} &\|V_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \varepsilon\|\partial_x V_\varepsilon\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} \le C\big(\|V_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{1/6}\big) \quad \text{and}\\ &\|U_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \|\partial_x U_\varepsilon\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} \le C\big(\|U_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)} + \|V_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{1/6}\big), \end{aligned}$$

where $\Omega_\varepsilon = (-a\varepsilon^{1/6}, a\varepsilon^{1/6})$ and C is some positive constant independent of ε. These estimates imply that $U_\varepsilon(\cdot, \varepsilon^{1/6}\cdot) \rightharpoonup U_0$ in $L^2(0,T;H^1(-a,a))$, which is independent of X, and $V_\varepsilon(\cdot, \varepsilon^{1/6}\cdot) \rightharpoonup V_0$, $\varepsilon^{1/2}\partial_X V_\varepsilon(\cdot, \varepsilon^{1/6}\cdot) \rightharpoonup W$ in $L^2((0,T)\times(-a,a))$ as $\varepsilon \to 0$, for some $W \in L^2((0,T)\times(-a,a))$. Thus, in the limit as $\varepsilon \to 0$, we see that (U_0, V_0) satisfies the system of ODEs

$$\begin{aligned} \frac{dU}{d\tau} &= -V + U^2,\\ \frac{dV}{d\tau} &= -1. \end{aligned} \tag{113}$$

Equation (113) is precisely the Riccati equation which lies at the heart of the dynamics in our rescaling chart K_2.

A consequence of our rescaling of the domain length is, however, that the original fast-slow structure which is present in the discretised system, Equation (32), does not translate to the blow-up analysis in the three coordinate charts. In particular, there is no longer a direct correspondence between singular objects in those charts and the layer and reduced problems pre-blow-up. Since the corresponding flows in the two scalings are equivalent after "blow-down", the loss of correspondence is merely of technical relevance: while it does entail that the approach in [41] does not apply to (48), we do not consider canard dynamics here, as is done in [41].

As elaborated in Appendix B, an additional challenge arises due to the finite-time blowup which can occur in (32) and which is due to the presence of additional slow variables v_k, $2 \leq k \leq k_0$, after Galerkin discretisation. To avoid solutions blowing up before they enter a neighbourhood of the singularity at the origin, we defined an ε-dependent set of initial values $R^{\text{in}}(\varepsilon) \subset \Delta^{\text{in}}$, which we combined with careful estimates for the higher-order modes $\{u_k, v_k\}$ resulting from the discretisation. We conjecture that this blowup is, in essence, caused by additional fold singularities that can be reached before the principal singularity at the origin which has been our focus here. In particular, a future direction of research would be the desingularisation of larger submanifolds where normal hyperbolicity is lost in the Galerkin discretisation; for example, one could blow up the blue curve in Figure B1 or the surface in Figure B2 in the cases where $k_0 = 2$ or $k_0 = 3$, respectively.

Finally, we briefly place our work into the broader context of singular perturbation problems arising in an infinite-dimensional context. Firstly, for fast-slow reaction-diffusion systems of the form in (5), we have recently gained a better understanding of transcritical points and generic fold points, including the results presented in this work. In finite dimensions, such non-hyperbolic points are known to generate only a dichotomy of either fast jumps of trajectories or an exchange of stability between slow manifolds. Yet, more degenerate fold points, such as folded nodes or folded saddle-nodes, may generate extremely complicated local dynamics, including oscillatory patterns, even in fast-slow systems of ODEs. That classification is likely to become even more complex in the infinite-dimensional setting of (systems of) PDEs. Secondly, systems of the form in (5) represent one class of interesting PDEs, where small perturbation parameters and singular limits occur. Other classes involve fast reaction terms, small diffusion problems, or heterogeneous media with highly oscillatory coefficients, which all commonly appear in the context of reaction-diffusion systems. Once one goes beyond reaction-diffusion systems, there are vast classes of PDE-type singular perturbation problems arising across the sciences. From a mathematical viewpoint, it is immediately clear that, in any parametrised PDE model, one anticipates possible distinctions between normally hyperbolic dynamics, where locally a good approximation is achieved by linearisation, and a loss of normal hyperbolicity along submanifolds in parameter space. Therefore, there is a need for developing techniques to tackle a loss of normal hyperbolicity in (systems of) PDEs. Our work is but one building block towards that general effort. Last, but not least, we have not yet related our theoretical approach via Galerkin discretisation with the performance of various numerical methods for PDEs. We conjecture that there is a link between (a loss of) performance and the presence of singularities, or non-hyperbolic points, in systems of nonlinear PDEs.

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Appendix A. Uniform boundedness and convergence

Under appropriate assumptions on initial data, it is possible to show the uniform boundedness of solutions to Equation (6) for sufficiently large times before those solutions reach the singularity at the origin. Uniform boundedness will then imply convergence of the Galerkin discretisation, as shown below. Hence, our finite-dimensional Galerkin manifolds can be interpreted as "approximately invariant slow manifolds"; the accuracy of the resulting approximation will improve with increasing k_0.

Appendix A.1. Uniform boundedness of solutions

For simplicity, we first consider the equations in (6) without higher-order terms H^u and H^v. The parabolic comparison principle for $\hat{v}_0 \geq v(0,x) \geq \tilde{v}_0 > 0$, with $x \in (-a,a)$, yields $\hat{v} \geq v(t,x) \geq \tilde{v}(t)$, where $\tilde{v}$ and $\hat{v}$ satisfy

$$\begin{aligned} \frac{d\tilde{v}}{dt} &= -\varepsilon, \qquad \text{with } \tilde{v}(0) = \tilde{v}_0, \\ \frac{d\hat{v}}{dt} &= -\varepsilon, \qquad \text{with } \hat{v}(0) = \hat{v}_0 \end{aligned}$$

and, hence,

$$\tilde{v}(t) = \tilde{v}_0 - \varepsilon t \quad \text{and} \quad \hat{v}(t) = \hat{v}_0 - \varepsilon t, \quad \text{with } \tilde{v}(t) \geq 0 \text{ for } t \leq \frac{\tilde{v}_0}{\varepsilon}.$$

Then, for $\tilde{u}_0 \le u(0,x) \le \hat{u}_0 < 0$, with $x \in (-a,a)$, we again apply the parabolic comparison principle to obtain that $\tilde{u}(t) \le u(t,x) \le \hat{u}(t)$, where

$$\begin{aligned} \frac{d\hat{u}}{dt} &= -\tilde{v} + \hat{u}^2, && \text{with } \hat{u}(0) = \hat{u}_0, \\ \frac{d\tilde{u}}{dt} &= -\hat{v} + \tilde{u}^2, && \text{with } \tilde{u}(0) = \tilde{u}_0. \end{aligned}$$

For $t \le \tilde{v}_0/(2\varepsilon)$, we have $\tilde{v} \ge \tilde{v}_0/2$ and can hence estimate $\hat{u}(t) \le \bar{u}(t)$, where

$$\frac{d\bar{u}}{dt} = -\frac{\tilde{v}_0}{2} + \bar{u}^2, \quad \text{with } \bar{u}(0) = \hat{u}_0,$$

and

$$\bar{u}(t) = \sqrt{\frac{\tilde{v}_0}{2}} \frac{\hat{u}_0(1 + e^{\sqrt{2\tilde{v}_0}t}) - \sqrt{\tilde{v}_0/2}(e^{\sqrt{2\tilde{v}_0}t} - 1)}{-\hat{u}_0(e^{\sqrt{2\tilde{v}_0}t} - 1) + \sqrt{\tilde{v}_0/2}(1 + e^{\sqrt{2\tilde{v}_0}t})},$$

which is bounded for all $\tilde{v}_0 > 0$ and $t \le \tilde{v}_0/(2\varepsilon)$. Similarly, we obtain that $\tilde{u}$ is also uniformly bounded for $\tilde{v}_0 > 0$ and $t \le \tilde{v}_0/(2\varepsilon)$. In sum, we hence have

$$\min\left\{\tilde{u}_0, -\sqrt{\hat{v}_0}\right\} \le u(t,x) \le \max_{t\in[0,\tilde{v}_0/(2\varepsilon)]} \bar{u}(t) \quad \text{for } 0 \le t \le \frac{\tilde{v}_0}{2\varepsilon} \text{ and } x \in (-a,a).$$

Thus, for all $v(0,x) \ge \tilde{v}_0 > 0$ and $u(0,x) \le \hat{u}_0 < 0$, we obtain that solutions of (6), without higher-order terms H^u and H^v, are uniformly bounded for $0 \le t \le \tilde{v}_0/(2\varepsilon)$.

When considering higher-order terms of the form $H^u(u,v,\varepsilon) = \mathcal{O}(\varepsilon, uv, v^2, u^3)$ and $H^v(u,v,\varepsilon) = \mathcal{O}(v^2)$ in (6), for $|u|,|v| \le 1$, we can assume

$$|H^u(u,v,\varepsilon)| \le \kappa_u(\varepsilon + |uv| + |v|^2 + |u|^3) \quad \text{and} \quad |H^v(u,v,\varepsilon)| \le \kappa_v|v|^2,$$

for some positive constants κ_u and κ_v. To derive estimates for the solutions of (6), we apply a fixed point argument: for given (u^*,v^*) with

$$|u^*| \le \min\left\{\frac{1}{4\kappa_u}, 1\right\} \quad \text{and} \quad 0 < v^* \le \min\left\{\frac{1}{2\sqrt{\kappa_v}}, \frac{1}{4\kappa_u}, \frac{1}{32\kappa_u^2}, 1\right\},$$

we consider $H^v(u^*,v^*), \varepsilon$ and $\widetilde{H}^u(u^*,u,v,\varepsilon)$, which is obtained from $H^u(u,v,\varepsilon)$ by replacing the terms of order vu and u^3 by u^*v and u^*u^2, respectively. The above assumptions on u^* and v^* yield

$$\begin{aligned} &|\widetilde{H}^u(u^*,u,v,\varepsilon)| \le \kappa_u(\varepsilon + |u^*v| + v^2 + |u^*|u^2) \le \kappa_u\varepsilon + \frac{|v|}{2} + \frac{u^2}{4} \quad \text{and} \\ &|H^v(u^*,v^*,\varepsilon)| \le \kappa_v|v^*|^2 \le \frac{1}{4}, \end{aligned}$$

which ensures

$$\begin{aligned} \frac{d\tilde{v}}{dt} &= \varepsilon(-1 + H^v(u^*,v^*,\varepsilon)) \ge -\frac{5}{4}\varepsilon \quad \text{and} \\ \frac{d\hat{v}}{dt} &= \varepsilon(-1 + H^v(u^*,v^*,\varepsilon)) \le -\frac{3\varepsilon}{4}. \end{aligned}$$

Then, for initial conditions satisfying

$$-\min\left\{\frac{1}{4\kappa_u},1\right\} \le \tilde{u}_0 < 0 \quad \text{and} \quad \frac{4}{5}\hat{v}_0 \le \tilde{v}_0 \le \hat{v}_0 \le \min\left\{\frac{1}{2\sqrt{\kappa_v}},\frac{1}{4\kappa_u},\frac{1}{32\kappa_u^2},1\right\} \tag{A.1}$$

and for $0 \le t \le 4\tilde{v}_0/(5\varepsilon)$, we have

$$0 \le \tilde{v}_0 - \frac{5\varepsilon}{4}t \le \tilde{v}(t) \le v(t,x) \le \hat{v}(t) \le \hat{v}_0 - \frac{3\varepsilon}{4}t \le \min\left\{\frac{1}{2\sqrt{\kappa_v}},\frac{1}{4\kappa_u},\frac{1}{32\kappa_u^2},1\right\}.$$

For $\hat{u}$, we obtain

$$\frac{d\hat{u}}{dt} = -\tilde{v} + \hat{u}^2 + |\widetilde{H}^u(u^*,\hat{u},\hat{v},\varepsilon)| \le -\tilde{v}_0 + \frac{5\varepsilon}{4}t + \frac{\hat{v}_0}{2} - \frac{3\varepsilon}{8}t + \kappa_u\varepsilon + \frac{3}{4}\hat{u}^2,$$

which, for $\varepsilon \le \tilde{v}_0/(16\kappa_u)$ and $t \le 2\tilde{v}_0/(7\varepsilon)$, implies

$$\frac{d\hat{u}}{dt} \le -\frac{\tilde{v}_0}{16} + \frac{5}{4}\hat{u}^2.$$

In combination with the previous estimates, the fixed point argument yields uniform boundedness of solutions of (6) for $t \le 2\tilde{v}_0/(7\varepsilon)$ and initial conditions satisfying (A.1).

Appendix A.2. Convergence of Galerkin discretisation

The Galerkin approximation (u_n, v_n) for solutions to (6), with $u_n(t,x) = \sum_{k=1}^n u_k(t)e_k(x)$ and $v_n(t,x) = \sum_{k=1}^n v_k(t)e_k(x)$, satisfies

$$\begin{aligned}
\partial_t u_n &= \partial_x^2 u_n - v_n + u_n^2 + H^u(u_n,v_n,\varepsilon) && \text{for } x \in (-a,a) \text{ and } t>0,\\
\partial_t v_n &= \varepsilon(\partial_x^2 v_n - 1 + H^v(u_n,v_n,\varepsilon)) && \text{for } x \in (-a,a) \text{ and } t>0,\\
&\partial_x u_n(t,x) = 0 = \partial_x v_n(t,x) && \text{for } x = \pm a \text{ and } t>0,\\
&u_n(0,x) = u_{n,0}(x) \quad \text{and} \quad v_n(0,x) = v_{n,0}(x) && \text{for } x \in (-a,a),
\end{aligned} \tag{A.2}$$

where $u_{n,0}$ and $v_{n,0}$ are projections of u_0 and v_0, respectively, onto the space $V = \operatorname{span}\{e_1(x),\dots,e_n(x)\}$. Using similar estimates as above and imposing the assumptions on initial conditions in (A.1), we obtain that $u_n(t,x)$ and $v_n(t,x)$ are uniformly bounded in $[0,T]\times[-a,a]$ for $\tilde{u}_0 \le u_{n,0}(x) \le \hat{u}_0 < 0$, $\hat{v}_0 \ge v_{n,0}(x) \ge \tilde{v}_0 > 0$, and $T \le 2\tilde{v}_0/(7\varepsilon)$. It follows that we have the a priori estimates

$$\begin{aligned}
&\|u_n\|^2_{L^\infty((0,T)\times(-a,a))} + \|u_n\|^2_{L^2(0,T;H^1(-a,a))} + \|\partial_t u_n\|^2_{L^2(0,T;H^1(-a,a)')} \le C \quad \text{and}\\
&\|v_n\|^2_{L^\infty((0,T)\times(-a,a))} + \varepsilon\|v_n\|^2_{L^2(0,T;H^1(-a,a))} + \|\partial_t v_n\|^2_{L^2(0,T;H^1(-a,a)')} \le C,
\end{aligned}$$

with a constant $C > 0$ that is independent of n, which ensures convergence of $u_n \to u$ weakly in $L^2(0,T;H^1(-a,a))$ and strongly in $L^2((0,T)\times(-a,a))$, as well as of $v_n \to v$ weakly-$*$ in $L^\infty(0,T;L^2(-a,a))$ and of $\sqrt{\varepsilon}v_n \to \sqrt{\varepsilon}v$ weakly in $L^2(0,T;H^1(-a,a))$ and strongly in $L^2((0,T)\times(-a,a))$; see e.g. [12]. Thus, we can pass to the limit as $n \to \infty$ in (A.2) to conclude that u and v are solutions to the original system (6).

Next, considering equations for the differences $u_n - u$ and $v_n - v$ and taking $u_n - u$ and $v_n - v$ as test functions, respectively, we obtain

$$\frac{1}{2}\partial_t\|u_n - u\|^2_{L^2(-a,a)} + \|\partial_x(u_n - u)\|^2_{L^2(-a,a)} \le \frac{1}{2}\|v_n - v\|^2_{L^2(-a,a)} + \frac{1}{2}\|u_n - u\|^2_{L^2(-a,a)}$$
$$+\|u_n + u\|_{L^\infty}\|u_n - u\|^2_{L^2(-a,a)} + h^u(\|u_n\|_{L^\infty}, \|u\|_{L^\infty})|u_n - u\|^2_{L^2(-a,a)} \quad \text{and}$$
$$\frac{1}{2}\partial_t\|v_n - v\|^2_{L^2(-a,a)} + \varepsilon\|\partial_x(v_n - v)\|^2_{L^2(-a,a)} \le \varepsilon h^v(\|u_n\|_{L^\infty}, \|u\|_{L^\infty})|v_n - v\|^2_{L^2(-a,a)},$$

with some smooth functions h^u and h^v representing contributions from the higher order terms H^u and H^v. Adding both inequalities, using the uniform boundedness of u_n, u, v_n, and v, and applying the Grønwall inequality, we obtain

$$\sup_{(0,T)}\|u_n - u\|^2_{L^2(-a,a)} + \|\partial_x(u_n - u)\|^2_{L^2((0,T)\times(-a,a))}$$
$$+ \sup_{(0,T)}\|v_n - v\|^2_{L^2(-a,a)} + \varepsilon\|\partial_x(v_n - v)\|^2_{L^2((0,T)\times(-a,a))}$$
$$\le C(T)\left[\|u_n(0) - u(0)\|^2_{L^2(-a,a)} + \|v_n(0) - v(0)\|^2_{L^2(-a,a)}\right],$$

which ensures the convergence of the Galerkin truncation to the solution of the original problem, Equation (6), as the approximation of the initial data converges strongly in $L^2(-a, a)$.

Appendix B. Illustrative example: $k_0 = 2$

In order to develop intuition for the singular geometry and resulting dynamics of (32), it is instructive to examine the simple case where $k_0 = 2$. For simplicity, let $a = \frac{1}{2}$, and assume that the higher-order terms H_i^u and H_i^v for $i = 1, 2$ are identically zero. In that case, the system in (32) reads

$$u_1' = -v_1 + 2^{-1/2}u_1^2 + 2^{-1/2}u_2^2, \tag{B.1a}$$
$$v_1' = -2^{-1/2}\varepsilon, \tag{B.1b}$$
$$u_2' = -\pi^2 u_2 - v_2 + 2^{1/2}u_1u_2, \tag{B.1c}$$
$$v_2' = -\pi^2\varepsilon v_2, \tag{B.1d}$$

where the critical manifold $\mathcal{C}$ is given by the graph

$$v_1 = f_1(u_1, u_2) := 2^{-1/2}u_1^2 + 2^{-1/2}u_2^2 \quad \text{and} \tag{B.2a}$$
$$v_2 = f_2(u_1, u_2) := -\pi^2 u_2 + 2^{1/2}u_1u_2. \tag{B.2b}$$

Linearisation of the layer problem induced by (B.2) for $\varepsilon = 0$ about $\mathcal{C}$ reveals that one eigenvalue is always negative for any choice of (u_1, u_2), whereas the sign of the other eigenvalue depends on (u_1, u_2), as shown in Figure B1. The set $\mathcal{C}_0$, as defined in (37), is denoted in red there. To the left of the curve $u_1 = g(u_2) := \frac{1}{2}\left(\pi^2 - \sqrt{\pi^2 + 4u_2^2}\right)$ (illustrated in blue), the second eigenvalue is negative, whereas it is positive to the right of the curve. Normal hyperbolicity is lost on the curve itself.

Remark Appendix B.1. Similarly, one can visualise the stability properties of the critical manifold $\mathcal{C}$ in the case where $k_0 = 3$, which will be given as a graph over (u_1, u_2, u_3); see Figure B2. Specifically, the manifold $\mathcal{C}$ is then attracting inside the funnel-like region of (u_1, u_2, u_3)-space shown in the figure and of saddle type outside that region. In analogy to the case of $k_0 = 2$, normal hyperbolicity is lost on the surface separating those two regions which is now given by an implicit polynomial expression that can be obtained by application of the Routh-Hurwitz stability criterion. The set $\mathcal{C}_0$ is again drawn in red.

For the particular case when $k_0 = 2$, it is possible to find explicit formulae for the initial conditions which will allow us to reach the section Δ^{in} under the flow of (B.1). Firstly, (u_1, u_2) must be in the region of the (u_1, u_2)-plane that corresponds to the normally hyperbolic attracting part of the critical manifold $\mathcal{C}$, see Figure B1. Secondly, by GSPT, we have to be sufficiently close to the corresponding slow manifold $\mathcal{C}_\varepsilon$ for ε sufficiently small, which amounts to a condition of the form

$$\max\{|v_1 - f_1(u_1, u_2)|, |v_2 - f_2(u_1, u_2)|\} < C, \tag{B.3}$$

where $C > 0$ is some suitably chosen constant. Thirdly, we also need to impose corresponding restrictions on (v_1, v_2) to ensure that we will not reach an unstable part of the critical manifold $\mathcal{C}$ under the slow flow induced by (B.1). To that end, we first need to invert the line $u_1 = g(u_2)$ which separates the attracting and saddle-like parts of $\mathcal{C}$ in the (u_1, u_2)-plane by substituting into (B.2) and solving for v_1 and v_2. The result is now a curve in the (v_1, v_2)-plane of the form $v_1 = h(v_2)$, where the function h is a quadratic polynomial in v_2, as shown in blue in Figure B3. A further, fourth, restriction is given by solving explicitly the (v_1, v_2)-subsystem in (B.1), rewritten in terms of the slow time, and by then determining a relation between v_1 and v_2 so that the flow reaches the section Δ^{in}:

$$|v_2| \le \xi(v_1) := C^{\text{in}}_{v_2} e^{\frac{\pi^2}{\sqrt{2}}(v_1 - \rho^2)}; \tag{B.4}$$

see Figure B3, in purple, for an illustration. Initial values for (B.1) satisfying these four conditions will flow into the section Δ^{in}.

However, as the flow of (B.1) approaches the singularity at the origin, for some initial conditions u_2 may blow up before v_1 becomes negative. The flow will hence have reached an unstable part of the critical manifold $\mathcal{C}$. In the next section, we provide an explicit explanation for this blowup in finite time.

Appendix C. Finite-time blowup of solutions

To motivate the importance of restrictions on the initial data for the Galerkin truncation in (12), we prove that for some choices of initial conditions, a blowup in u_1 can occur before v_1 becomes negative already for $k_0 = 2$. Setting $a = \frac{1}{2}$ and rescaling u_1 and u_2

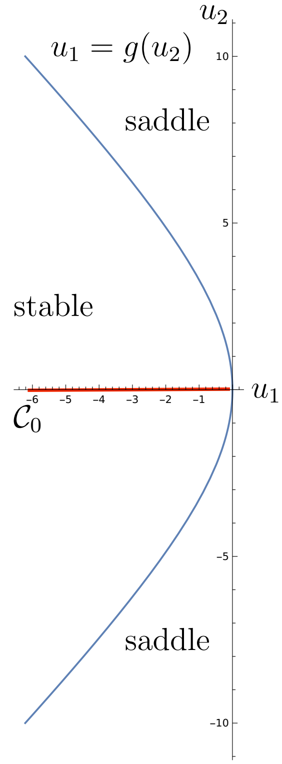

Figure B1: Stability properties of the critical manifold $\mathcal{C}$ which, for $k_0 = 2$, can be written as a graph over (u_1, u_2). A loss of normal hyperbolicity occurs along the curve $u_1 = g(u_2)$ (in blue) where one of the two eigenvalues of the linearisation about $\mathcal{C}$ changes sign. The manifold $\mathcal{C}_0$ is shown in red.

by a factor of $2^{-1/2}$, we obtain the two-dimensional system

$$\begin{aligned}
u_1' &= -v_1 + u_1^2 + u_2^2, && \text{with } u_1(0) = u_1^0, \\
v_1' &= -\varepsilon, && \text{with } v_1(0) = v_1^0, \\
u_2' &= -v_2 + u_2(2u_1 - \pi^2), && \text{with } u_2(0) = u_2^0, \\
v_2' &= -\varepsilon\pi^2 v_2, && \text{with } v_2(0) = v_2^0.
\end{aligned} \tag{C.1}$$

It is assumed that $v_1^0 > 0$. We will show that, for $v_2^0 \neq 0$ and $\varepsilon > 0$ sufficiently small, a finite-time blowup will occur in (C.1) before v_1 changes sign. For the sake of simplicity and without loss of generality, we may assume that $u_2^0 < 0$ and $v_2^0 > 0$; see also Remark Appendix C.6.

Appendix C.1. Main observation

Firstly, we establish our main observation on finite-time blowup for solutions of (C.1) when $k_0 = 2$. Various auxiliary results which are used in the proof are collated in Appendix C.2.

Proposition Appendix C.1. *Let $u_1^0 \in \mathbb{R}$, $u_2^0 < 0$, and $v_1^0, v_2^0 > 0$. Then, there exists $\varepsilon > 0$ such that the solution of* (C.1) *blows up before $t_0 = \frac{v_1^0}{\varepsilon}$, i.e., before v_1 changes*

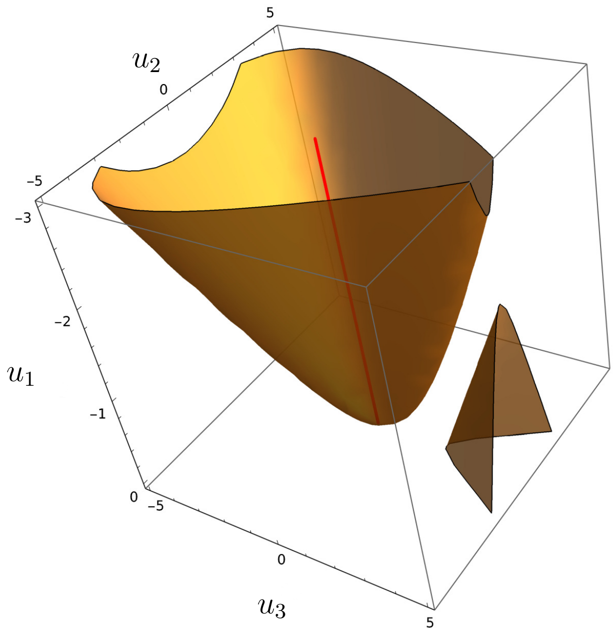

Figure B2: The fold surface for $k_0 = 3$. The critical manifold $\mathcal{C}$ can be written as a graph over (u_1, u_2, u_3) and is stable inside the funnel-like region, the boundary of which is a surface that is implicitly defined by a polynomial equation in u_1, u_2, and u_3. One of the three eigenvalues of the linearisation about $\mathcal{C}$ changes sign across the surface.

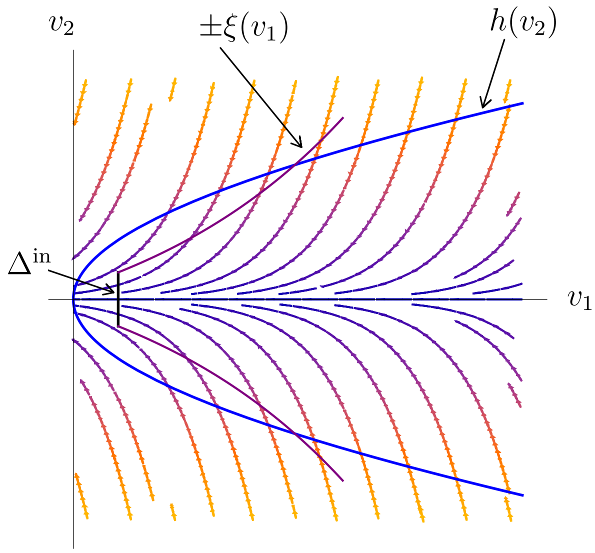

Figure B3: The reduced flow of (B.1). The region inside the curve $h(v_2)$ (in blue) corresponds to the stable part of the critical manifold $\mathcal{C}$; across that curve, one of the eigenvalues of the linearisation about $\mathcal{C}$ changes sign. Also illustrated are Δ^{in} (in black) and $\pm\xi(v_1)$ (in purple); recall (B.4). The set of initial conditions in the (v_1, v_2)-plane that reach Δ^{in} is found in the intersection of the regions to the right of $h(v_2)$ and $\pm\xi(v_1)$.

sign.

Proof. As observed in Proposition Appendix C.3 and Remark Appendix C.6 below, without loss of generality, we may assume that

$$-\pi/2 < u_1^0 \leq \pi/4 \quad \text{and} \quad v_1^0 < \min\left\{\tfrac{\pi^2}{16}, \left(\tfrac{e^{-\pi^4/32} v_2^0}{2(\pi+\pi^2)}\right)^2\right\}.$$

By Propositions Appendix C.4 and Appendix C.5, it follows that $-\pi/2 < u_1(t) \leq \pi/4$ for all $t \geq 0$ unless there is blowup in finite time independent of $\varepsilon > 0$. We consider the time interval $[0, \frac{v_1^0}{2\varepsilon}]$ in which v_1 remains positive. Moreover, we have $v_2(t) \in \left[\exp(-\frac{\pi^2 v_1^0}{2})v_2^0, v_2^0\right]$ for all $[0, \frac{v_1^0}{2\varepsilon}]$. Since $u_2(t) \leq 0$, by Remark Appendix C.6, and since $-\pi/2 < u_1(t) \leq \pi/4$ for all $t \geq 0$, it follows

$$-2v_2^0 - (\pi^2 - \tfrac{\pi}{2})u_2 < -v_2 + u_2(2u_1 - \pi^2) = \partial_t u_2 < -\exp(-\frac{\pi^2 v_1^0}{2})v_2^0 - (\pi + \pi^2)u_2$$

in $[0, \frac{v_1^0}{2\varepsilon}]$. Let now w_u and w_o be the solutions of

$$\begin{aligned} w_u' &= -2v_2^0 - (\pi^2 - \tfrac{\pi}{2})w_u, \\ w_o' &= -\exp\left(-\frac{\pi^2 v_1^0}{2}\right)v_2^0 - (\pi + \pi^2)w_o, \end{aligned}$$

in $[0, \frac{v_1^0}{2\varepsilon}]$, with $w_u(0) = u_2^0 = w_o(0)$. Lemma Appendix C.2 ensures $w_u \leq u_2 \leq w_o$. Thus, in $[\frac{v_1^0}{4\varepsilon}, \frac{v_1^0}{2\varepsilon}]$ we have

$$\begin{aligned} u_2(t) \leq w_o(t) &= \exp\left(-(\pi+\pi^2)t\right)\left[u_2^0 + \frac{1}{(\pi+\pi^2)}\exp\left(-\frac{\pi^2 v_1^0}{2}\right)v_2^0\right] \\ &\quad - \frac{1}{(\pi+\pi^2)}\exp\left(-\frac{\pi^2 v_1^0}{2}\right)v_2^0 \\ &\leq -\frac{1}{2(\pi+\pi^2)}\exp\left(-\frac{\pi^2 v_1^0}{2}\right)v_2^0, \end{aligned}$$

provided $\varepsilon > 0$ is sufficiently small. Correspondingly, in $\left[\frac{v_1^0}{4\varepsilon}, \frac{v_1^0}{2\varepsilon}\right]$ we obtain

$$\begin{aligned} u_1' = -v_1 + u_1^2 + u_2^2 &\geq -v_1^0 + \frac{e^{-\pi^2 v_1^0}}{4(\pi+\pi^2)^2}\left(v_2^0\right)^2 + u_1^2 \\ &\geq -v_1^0 + \frac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}\left(v_2^0\right)^2 + u_1^2 > c + u_1^2 \end{aligned} \tag{C.2}$$

for some $c > 0$ due to $v_1^0 < \min\left\{\frac{\pi^2}{16}, \left(\frac{e^{-\pi^4/32} v_2^0}{2(\pi+\pi^2)}\right)^2\right\}$. The equation

$$w' = \mu + w^2,$$

with $\mu > 0$ constant, experiences blowup for all initial conditions at a time t_0 that depends on the initial condition and on μ, but that is independent of ε. If $\varepsilon > 0$ is small enough, then the blowup occurs in $\left[0, \frac{v_1^0}{4\varepsilon}\right]$. Thus, Lemma Appendix C.2 implies that u_1 blows up before time $\frac{v_1^0}{2\varepsilon}$; in particular, it blows up before v_1 changes sign. $\square$

Appendix C.2. Proof of Proposition Appendix C.1

The following comparison principle is standard; however, we include it for completeness.

Lemma Appendix C.2. *Let $f, g\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be such that $f(t,x) > g(t,x)$ for all $(t,x) \in [0,\infty) \times \mathbb{R}$, and suppose that f and g are locally Lipschitz continuous. Furthermore, let $x_0 \in \mathbb{R}$, and let y_f and y_g be the solutions of*

$$y_f'(t) = f(t, y_f(t)) \quad \text{and} \quad y_g'(t) = g(t, y_g(t)), \quad \text{with } y_f(0) = y_g(0).$$

Then, $y_f(t) \geq y_g(t)$ for all t in the intersection of the maximal existence intervals of y_f and y_g.

Proposition Appendix C.3. *If the solution of* (C.1) *exists for a sufficiently long time, then there exists $\eta > 0$ independent of ε, but dependent on v_1^0 and v_2^0, such that*

$$0 < v_1\big(\tfrac{v_1^0-\eta}{\varepsilon}\big) < \min\left\{\tfrac{\pi^2}{16}, \tfrac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}\left[v_2\big(\tfrac{v_1^0-\eta}{\varepsilon}\big)\right]^2\right\}.$$

Proof. Solving explicitly, we can write $v_1(t) = v_1^0 - \varepsilon t$ and $v_2(t) = \exp(-\varepsilon\pi^2 t)v_2^0$. Hence,

$$v_1\big(\tfrac{v_1^0-\eta}{\varepsilon}\big) = \eta > 0.$$

On the other hand, for $\eta > 0$ sufficiently small, we have

$$v_1\big(\tfrac{v_1^0-\eta}{\varepsilon}\big) = \eta < \tfrac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}\exp\big(-2\pi^2(v_1^0-\eta)\big)(v_2^0)^2 = \tfrac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}\left[v_2\big(\tfrac{v_1^0-\eta}{\varepsilon}\big)\right]^2.$$

Obviously, if $\eta > 0$ is small enough, it also holds that

$$v_1\big(\tfrac{v_1^0-\eta}{\varepsilon}\big) = \eta < \tfrac{\pi^2}{16},$$

which shows the assertion. □

Given Proposition Appendix C.3, blowup in (C.1) can still occur in a time interval of length η/ε. Since η can be chosen independent of ε, that interval can be made arbitrarily large for ε sufficiently small. In particular, if we can now show that solutions of (C.1) blow up after a time which is independent of ε, then blowup will occur before v_1 changes sign if $\varepsilon > 0$ is small enough. By Proposition Appendix C.3, we may assume that $v_1^0 < \min\left\{\tfrac{\pi^2}{16}, \left[\tfrac{e^{-\pi^4/32}v_2^0}{2(\pi+\pi^2)}\right]^2\right\}$.

Proposition Appendix C.4. *If the solutions of* (C.1) *exist for a sufficiently long time, then there exists a time $t_0 \geq 0$, independent of ε, such that $u_1(t_0) > -\pi/2$.*

Proof. Since we can assume $v_1^0 < \frac{\pi^2}{16}$, it holds that

$$u_1' = -v_1 + u_1^2 + u_2^2 > -\tfrac{\pi^2}{16} + u_1^2.$$

As long as $u_1^2 \leq -\pi/2$, we also have $-\frac{\pi^2}{16} + u_1^2 > \frac{3\pi^2}{16}$ and, hence, $u_1' > \frac{3\pi^2}{16}$, which proves the assertion. □

Proposition Appendix C.5. *If $u_1^0 > \pi/4$, then solutions of* (C.1) *blow up after a finite time which is independent of ε.*

Proof. Since we may assume $v_1^0 < \frac{\pi^2}{16}$, it holds that

$$u_1' = -v_1 + u_1^2 + u_2^2 > -\tfrac{\pi^2}{16} + u_1^2.$$

If $u_1^0 > \pi/4$, then the right-hand side in the above expression is positive. It follows from Lemma Appendix C.2 that blowup occurs after a finite time which is independent of ε, as that is the case for the solution of

$$w' = -\tfrac{\pi^2}{16} + w^2, \quad \text{with } w(0) = u_1^0 > \pi/4.$$

□

Remark Appendix C.6. (i) Propositions Appendix C.4 and Appendix C.5 show that we may assume $-\pi/2 < u_1^0 \leq \pi/4$.

(ii) One can also show the following: if solutions to (C.1) exist long enough and if $\varepsilon > 0$ is sufficiently small, then there exists a time $t_0 \geq 0$, independent of ε, such that $u_2(t) \leq 0$ for all $t \geq t_0$. We simply take $u_2^0 < 0$ and observe that, hence, $u_2(t) \leq 0$ for all $t \geq 0$. Note, however, that one has to exchange signs here if $v_2^0 < 0$.

We now derive an estimate for how small ε has to be such that we observe blowup before v_1 changes sign. In a first step, we give an explicit expression for η – dependent on v_1^0 and v_2^0, but independent of ε – that satisfies the estimate in Proposition Appendix C.3. For the sake of simplicity, we will assume that $v_1^0 \in (0, \pi^2/16)$.

Lemma Appendix C.7. *If η is chosen as*

$$\eta = \frac{(v_2^0)^2}{4(\pi+\pi^2)^2} \exp(-\frac{\pi^4}{16} - 2\pi^2 v_1^0),$$

then the estimate in Proposition Appendix C.3 is satisfied.

Proof. The estimate in Proposition Appendix C.3 holds true if and only if

$$\begin{aligned} \eta = v_1^0 - \varepsilon\frac{v_1^0 - \eta}{\varepsilon} = v_1\Big(\frac{v_1^0 - \eta}{\varepsilon}\Big) &< \tfrac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}\Big[v_2\big(\tfrac{v_1^0-\eta}{\varepsilon}\big)\Big]^2 \\ = \tfrac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}(v_2^0)^2 \exp\Big(-2\varepsilon\pi^2\frac{v_1^0-\eta}{\varepsilon}\Big) &= \tfrac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}(v_2^0)^2 \exp\big[-2\pi^2(v_1^0-\eta)\big]. \end{aligned}$$

Multiplication by $e^{-2\pi^2\eta}$ yields

$$\eta\exp(-2\pi^2\eta) < \tfrac{e^{-\pi^4/16}}{4(\pi+\pi^2)^2}(v_2^0)^2\exp(-2\pi^2 v_1^0),$$

which is satisfied if

$$\eta = \frac{(v_2^0)^2}{4(\pi+\pi^2)^2}\exp(-\tfrac{\pi^4}{16} - 2\pi^2 v_1^0),$$

as stated in the assertion. □

Remark Appendix C.8. The main argument in the proof of Proposition Appendix C.1 was that solutions of $w' = \mu + w^2$ blow up in finite time if $\mu > 0$. The explicit solution is given by

$$w(t) = \sqrt{\mu} \tan \Big(\arctan(\tfrac{w(0)}{\sqrt{\mu}}) + \sqrt{\mu} t \Big),$$

and hence exists until time

$$t = \frac{\pi/2 - \arctan(\frac{w(0)}{\sqrt{\mu}})}{\sqrt{\mu}}.$$

In particular, blowup occurs before time $t = \pi/\sqrt{\mu}$. To determine how to choose μ in Proposition Appendix C.1, we recall Equation (C.2), which allows for

$$\mu = -v_1^0 + \frac{e^{-\pi^4/16}}{4(\pi + \pi^2)^2}(v_2^0)^2 = (e^{2\pi^2\eta} - 1)\eta;$$

here, we have used Lemma Appendix C.7.

Proposition Appendix C.9. *If $\varepsilon < \frac{\eta^2}{2\sqrt{2}}$, then the solution of* (C.1) *blows up before v_1 changes sign.*

Proof. In the proof of Proposition Appendix C.1, blowup is generated in the time interval $[0, \frac{v_1^0}{4\varepsilon}] = [0, \frac{\eta}{4\varepsilon}]$. In combination with Remark Appendix C.8, it follows that it suffices to take ε small enough such that $\varepsilon < \frac{\eta\sqrt{\mu}}{4\pi}$. To prove the assertion, we rewrite the right-hand side of that inequality as

$$\frac{\eta\sqrt{\mu}}{4\pi} = \frac{\eta\sqrt{\eta(e^{2\pi^2\eta} - 1)}}{4\pi},$$

which is, in fact, sharper than the right-hand side in the assertion; for conciseness, we observe that $e^{2\pi^2\eta} - 1 > 2\pi^2\eta$ and, hence, that

$$\frac{\eta\sqrt{\mu}}{4\pi} > \frac{\eta^2}{2\sqrt{2}},$$

whence the assertion follows. □

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