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DIMENSIONAL REGULARISATION AND BROKEN CONFORMAL WARD IDENTITIES

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Abstract: Dimensional regularization is used to give a simple treatment of broken conformal Ward identities. The method reproduces the known answers for standard theories. In addition it permits a derivation of the relevant identities for non-Abelian gauge theories which have not been obtained by other means. For the latter class of theories asymptotic scale invariance is found not to imply asymptotic conformal invariance for gauge variant Green functions. This is due to two gauge dependent insertions occurring in the identities. One can be predicted classically, whereas the other contains a new term dependent on Faddeev-Popov ghosts.

1. Introduction

Intuitively we expect a theory to behave at large energies as if it were massless, i.e., scale invariant. In the context of quantum field theory it is by now well known that this is wrong in general [1]. We would like to ask the same question for special conformal transformations [1], which are just co-ordinate dependent scale transformations. We expect anomalies, but, we would like to know whether the anomalies are related to those in scale invariance. For non-gauge theories the answer is simple. The anomalies [2] in conformal invariant Ward identities are in one to one correspondence with those in scale invariance. For gauge theories, classically, we expect the gauge fixing term to break conformal invariance; so apart from the terms related to the Callan-Symanzik anomalies [1] and the gauge fixing term we might expect nothing. For Fermi type gauges this is true in Abelian [3] gauge theories but false in non-Abelian ones [4]. We want to derive these results in a new and simple way [4].

Whereas broken scale invariance Ward identities can be written down with ease for any renormalizable field theories [1], this is not the case with broken conformal invariance Ward identities. It just happens that the scale properties of a field theory are very closely related to its behaviour under changes of subtraction points. This, in turn, reduces to a simple expression of the renormalizability of the theory. The situation is much less simple for conformal symmetry. Until very recently there existed one treatment for a quartically self-interacting scalar theory [2], another rather different treatment for quantum electrodynamics [3], and nothing for non-Abelian gauge theories. This is clearly not satisfactory. We will give a method which seems to show up the relation between the conformal anomalies and the Callan-Symanzik ones, and works for both gauge and non-gauge field theories. The method will consist of writing down the conformal Ward identities for regulated, but unrenormalized, Green functions. This can be easily done when we regulate the theories dimensionally [5], since the divergences of Feyman graphs manifest themselves as poles at rational values of the space-time dimension n. Consequently, away from the poles, the unrenormalized Green functions are finite. Canonical Ward identities for non-rational n are true relations between finite quantities. After writing the Ward identities in n dimensions, all that then remains is to express the regulated, but unrenormalized, Green functions in terms of the renormalized ones. Furthermore, owing to the requirements of gauge invariance, it would seem to be necessary to use dimensional regularisation when discussing non-Abelian gauge theories. We will, in the main, concern ourselves with multiplicative renormalisability. However, subtractive renormalisability poses no special problems.

In sect. 2 we shall consider the problem of obtaining Ward identities in n dimensions. In sect. 3 we shall treat the familiar problem of a quartically self-interacting scalar field. In sect. 4 we shall consider a non-Abelian gauge theory which is general enough to show up any complications that may arise. The method used in sect. 2 is found to apply without modification. In an appendix we consider renormalization which is subtractive.

2. Ward identities

It is convenient to study Ward identities in the framework of generating functionals, Z, W and Γ [6]. All can be expressed in terms of Z which we shall represent as a Feynman path integral. For relativistic field theories, the latter is of course a badly defined quantity. Nevertheless, it has been shown that the path integral method yields the same identities as those obtained from combinatorics [7]. We regard the path integral as purely a device for performing combinatorics. It will be assumed throughout this section that we are working in a space-time of n dimensions at non-rational n.

In order to keep our notation compact we shall use a generalized summation convention first introduced by De Witt [8]. Every index introduced will have internal symmetry as well as space-time connotations. A contraction of indices implies a summation over internal symmetry indices as well as an integration over space-time. The path integral representation for Z is

$$Z[J_i, K_a, K_a^\dagger] = \int \prod_{i=1}^{s} \mathcal{D}\phi_i \prod_{a=1}^{t} \mathcal{D}c_a^\dagger \prod_{a'=1}^{t} \mathcal{D}c_{a'} \exp i\{c_a^\dagger M_{ab}(\phi)c_b + S[\phi] - \tfrac{1}{2}F_\alpha(\phi)F^\alpha(\phi)$$

$$+ \phi_i J^i + K_a^\dagger c^a + K^a c_a^\dagger\}, \tag{2.1}$$

where

$$\{\phi_i;\quad i=1,\ldots,s\}$$

is the set of fields in the theory,

$$\{c_a^\dagger, c_a;\quad a=1,\ldots,t\}$$

is the set of Faddeev-Popov ghosts [9], $S[\phi]$ is the action, $F_\alpha(\phi)$ is the gauge fixing function, J, K, and $K^\dagger$ are sources and $M_{ab}(\phi)$ is a functional matrix required for unitarity [10]. For a simple theory, such as ϕ^4, the matrix $M(\phi)$ and $F_\alpha(\phi)$ are zero. For Abelian gauge theories, if $F_\alpha(\phi)$ is linear in ϕ, $M(\phi)$ is zero. For non-Abelian gauge theories, even if $F_\alpha(\phi)$ is linear in ϕ and co-variant, $M(\phi)$ is non zero.

The Green functions $G(x_1,\ldots,x_l)$ of interest are given by

$$G(x_1,\ldots,x_l)\equiv\left(\frac{1}{i}\right)^l \frac{\delta^l}{\delta J(x_1)\ldots\delta J(x_l)} Z[J,K^\dagger,K]\,|_{J=K^\dagger=K=0}. \tag{2.2}$$

$G(x_1,\ldots,x_l)$ is finite but not connected. We are interested in connected, one particle irreducible Green functions. These latter Green functions are generated by [6]

$$\Gamma[\Phi_i,\mathcal{C}_a,\mathcal{C}_a^\dagger] = W[J,K,K^\dagger] - J_i\Phi^i - K_a\mathcal{C}^{\dagger a} - K_a^\dagger\mathcal{C}^a, \tag{2.3}$$

with

$$\exp(iW[J,K^\dagger,K]) = Z[J,K^\dagger,K],$$

$$\Phi_i = \frac{\delta W}{\delta J_i}[J,K^\dagger,K],$$

$$\mathcal{C}_a^\dagger = \frac{\delta W}{\delta K_a}[J,K^\dagger,K],$$

$$\mathcal{C}_a = \frac{\delta W}{\delta K_a^\dagger}[J,K^\dagger,K].$$

We have also the dual equations

$$\frac{\delta\Gamma}{\delta\Phi_i} = -J_i,\qquad \text{etc.} \tag{2.4}$$

It is simplest to find a functional Ward identity for Z and then, making use of (2.3), to write an equivalent statement for Γ [11]. For generality we shall consider an infinitesimal transformation on the fields of the form

$$\phi_i = \phi_i' + \left(f(x_i,\frac{\partial}{\partial x_i})\phi'\right)_i, \tag{2.5}$$

$$c_a = c_a' + \left(e(x_i,\frac{\partial}{\partial x_i})c'\right)_a,$$

where e and f are infinitesimal functions of space-time, e.g., for scale invariance

$$\phi_i(x) = \phi_i'(x) + \epsilon(x\cdot\partial + d_{(n)})\,\phi_i'(x),$$

$$c_a(x) = c_a'(x) + \epsilon(x\cdot\partial + \tfrac{1}{2}(n-2))\,c_a'(x),$$

where ϵ is an infinitesimal parameter, and $d_{(n)}$ is $\frac{1}{2}(n-2)$ for bosons, but $\frac{1}{2}(n-1)$ for fermions. The above assignments of $d_{(n)}$ are necessary for the action in n dimensions to be dimensionless.

The transformations (2.5) can be regarded as a change of variables in the path integral for Z.

Hence

$$Z[J,K^\dagger,K] = \int \prod_{i,a,a'} \mathcal{D}\phi_i'\mathcal{D}c_a'\mathcal{D}c_{a'}'^{\dagger}\, \det(1+f)\det(1+e)^2$$

$$\times \exp i\{(c_a'^{\dagger} + (ec'^{\dagger})_a)M_{ab}(\phi' + (f\phi'))(c_b' + (ec')_b) + S[\phi' + (f\phi)']$$

$$-\tfrac{1}{2}F_\alpha(\phi' + (f\phi'))F^\alpha(\phi' + f\phi') + J_i(\phi' + f\phi')^i\}. \tag{2.6}$$

Now the generating functional Z is defined up to a possibly infinite factor, which is independent of the sources and fields and represents vacuum graphs. In (2.6) the factor $\det(1+f)\det(1+e)^2$ contributes to this vacuum renormalization.

Equating (2.1) and (2.6) we have

$$0 = \int \prod_{i,a,a'} \mathcal{D}\phi_i\mathcal{D}c_a\mathcal{D}c_{a'}^{\dagger}\, \exp i(S[\phi] - \tfrac{1}{2}(F_\alpha(\phi))^2 + J_i\phi^i + c_a^\dagger M_{ab}c_b)$$

$$\times\{\delta(c_a^\dagger M_{ab}c_b) + \delta S[\phi] - \tfrac{1}{2}\delta(F_\alpha(\phi))^2 + J_i(f(x_i,\partial/\partial x_i)\phi)_i$$

$$+ K_a^\dagger(e(x_a,\partial/\partial x_a)c)_a + K_a(e(x_a,\partial/\partial x_a)c^\dagger)_a\}, \tag{2.7}$$

where δ is the symbol for infinitesimal transformations. For the situation of interest

$$M_{ab}(\phi) = m_{abc}\phi_c + p_{ab},$$

where m_{abc} and p_{ab} are independent of fields; we shall confine ourselves to this case from now on.

Eq. (2.7) contains the basis for the canonical Ward identity. Its functional form is

$$0 = \left\{\delta\left(\frac{\delta}{i\delta K_a}\left(m_{abc}\frac{\delta}{i\delta J_c} + p_{ab}\right)\frac{\delta}{i\delta K_b^\dagger}\right) + \delta S\left[\frac{1}{i}\frac{\delta}{\delta J}\right] - \tfrac{1}{2}\delta\left(F_\alpha\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right)^2\right.$$

$$\left. + K_a\left(e\left(x_a,\frac{\partial}{\partial x_a}\right)\frac{\delta}{i\delta K}\right)_a + J_i\left(f\left(x_i,\frac{\partial}{\partial x_i}\right)\frac{\delta}{i\delta J}\right)_i + K_a^\dagger\left(e\left(x_a,\frac{\partial}{\partial x_a}\right)\frac{\delta}{i\delta K^\dagger}\right)_a\right\} Z. \tag{2.8}$$

If we are interested in Green functions which are not necessarily connected (2.8) is adequate; however, we require the Green functions generated by Γ.

Translating (2.8) into an equation in terms of Γ we have

$$
\begin{aligned}
0 = &-\frac{\delta\Gamma}{\delta\Phi_i}\left(f\left(x_i, \frac{\partial}{\partial x_i}\right)\left(\frac{1}{i}\frac{\delta}{\delta J}+\Phi\right)\right)_i - \frac{\delta\Gamma}{\delta \mathcal{C}_a}\left(e\left(x_a, \frac{\partial}{\partial x_a}\right)\left(\frac{1}{i}\frac{\delta}{\delta K^\dagger}+\mathcal{C}\right)\right)_a \\
&-\frac{\delta\Gamma}{\delta \mathcal{C}^\dagger_a}\left(e\left(x_a, \frac{\partial}{\partial x_a}\right)\left(\frac{1}{i}\frac{\delta}{\delta K}+\mathcal{C}^\dagger\right)\right)_a - \tfrac{1}{2}\delta\left(F_\alpha\left[\frac{1}{i}\frac{\delta}{\delta J}+\Phi\right]\right)^2 \qquad (2.9) \\
&+\delta S\left[\frac{1}{i}\frac{\delta}{\delta J}+\Phi\right] + \delta\left(\left(\mathcal{C}^\dagger_a + \frac{1}{i}\frac{\delta}{\delta K_a}\right)\left(m_{abc}\left(\frac{1}{i}\frac{\delta}{\delta J_c}+\Phi_c\right)+p_{ab}\right)\left(\mathcal{C}_b + \frac{1}{i}\frac{\delta}{\delta K^\dagger_b}\right)\right),
\end{aligned}
$$

where

$$\frac{1}{i}\frac{\delta}{\delta K_a}$$

for example, is really to be thought of as

$$\frac{1}{i}\left(\frac{\delta\Phi^k}{\delta K_a}\frac{\delta}{\delta\Phi^k} + \frac{\delta\mathcal{C}^b}{\delta K_a}\frac{\delta}{\delta\mathcal{C}^b} + \frac{\delta\mathcal{C}^{\dagger b}}{\delta K_a}\frac{\delta}{\delta\mathcal{C}^{\dagger b}}\right).$$

Eq. (2.9) is the basis for Ward identities that we shall use. Provided that we are away from the poles in n, these identities are true and relate finite quantities.

Much tedious calculation can be avoided if we note that the infinitesimal scale and conformal variations [12], $\delta_s P$ and $\delta_c^\alpha P$, of a Poincare invariant functional $P(\phi)$ are given by

$$\delta_s P(\phi) = \partial_\nu(x^\nu P) + \left(\frac{\partial P}{\partial\phi_i}d_{(n)}\phi_i + \frac{\partial P}{\partial(\partial_\mu\phi_i)}(d_{(n)}+1)\partial_\mu\phi_i - nP\right), \qquad (2.10)$$

and

$$
\begin{aligned}
\delta_c^\alpha P(\phi) = &\ \partial_\nu((2x^\alpha x^\nu - g^{\alpha\nu}x^2)P(\phi)) \\
&+ 2x^\alpha\left(\frac{\partial P}{\partial\phi_i}d_{(n)}\phi_i + \frac{\partial P}{\partial(\partial_\mu\phi_i)}(d_{(n)}+1)\partial_\mu\phi_i - nP\right) \\
&+ 2\left(\frac{\partial P}{\partial(\partial_\alpha\phi_i)}d_{(n)}\phi_i - \frac{\partial P}{\partial(\partial_\nu\phi_i)}(\Sigma^{\nu\alpha})_{ij}\phi_j\right), \qquad (2.11)
\end{aligned}
$$

where $(\Sigma^{\nu\alpha})_{ij}$ is a direct product of the spin matrix and the unit matrices in internal symmetry space and in space-time. The proofs of (1.13) and (1.14) are very simple.

The family of Ward identities is found by functionally differentiating equation (2.9) with respect to ϕ and then setting all sources to zero [11]. For particular values of f and e we obtain the scale and conformal transformations.

3. The ϕ^4 case

We shall now consider the Lagrangian $\mathcal{L}$

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{3.1}$$

where ϕ is a scalar field.

For scale transformations

$$\delta\mathcal{L} = \partial_\nu(x^\nu \mathcal{L}) + m^2\phi^2 + \frac{\lambda}{4!}(4-n)\phi^4. \tag{3.2}$$

For conformal transformations

$$\delta\mathcal{L} = \partial_\nu((2x^\alpha x^\nu - g^{\alpha\nu}x^2)\mathcal{L}) + \tfrac{1}{2}(n-2)\partial^\alpha(\phi^2) + 2x^\alpha m^2\phi^2 + 2x^\alpha \frac{\lambda}{4!}(4-n)\phi^4. \tag{3.3}$$

It will be first useful to obtain the Callan-Symanzik equations in this approach in order to identify the anomalous terms, which, as we will see, are closely related to those in the conformal Ward identities. The scale invariance Ward identity is given by

$$\sum_{i=1}^{l}\left(-\frac{\partial}{\partial x_i}\left(x_i\Gamma\left(\prod_{j=1}^{l}\phi(x_j)\right)\right) + \tfrac{1}{2}(n-2)\Gamma\left(\prod_{j=1}^{l}\phi(x_j)\right)\right)$$
$$= \int d^n x\left(m^2\,\Gamma\left(\phi^2(x)\prod_{j=1}^{l}\phi(x_j)\right) + \frac{\lambda}{4!}(4-n)\Gamma\left(\phi^4(x)\prod_{j=1}^{l}\phi(x_j)\right)\right). \tag{3.4}$$

Here Γ denotes one particle irreducible Green functions.

Now

$$\frac{\partial}{\partial m^2}\Gamma\left(\prod_{j=1}^{l}\phi(x_j)\right)\Bigg|_\lambda = -\tfrac{1}{2}i\int d^n x\,\Gamma\left(\phi^2(x)\prod_{j=1}^{l}\phi(x_j)\right), \tag{3.5}$$

and

$$\frac{\partial}{\partial\lambda}\Gamma\left(\prod_{j=1}^{l}\phi(x_j)\right)\Bigg|_m = -\frac{1}{4!}i\int d^n x\,\Gamma\left(\phi^4(x)\prod_{j=1}^{l}\phi(x_j)\right). \tag{3.6}$$

Hence

$$\int d^n x\left(m^2\,\Gamma\left(\phi^2(x)\prod_{j=1}^{l}\phi(x_j)\right) + \frac{\lambda}{4!}(4-n)\Gamma\left(\phi^4(x)\prod_{j=1}^{l}\phi(x_j)\right)\right)$$
$$= \left(-2\frac{m^2}{i}\frac{\partial}{\partial m^2} - \frac{\lambda}{i}(4-n)\frac{\partial}{\partial\lambda}\right)\Gamma\left(\prod_{j=1}^{l}\phi(x_j)\right). \tag{3.7}$$

It is convenient at this stage to exploit the renormalizability, of $\mathcal{L}$. We will introduce renormalized quantities which will invariably be denoted by a suffix R.

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi_R)^2+\tfrac{1}{2}(Z_3-1)(\partial_\mu\phi_R)^2-\tfrac{1}{2}m_R^2\phi_R^2$$
$$-\tfrac{1}{2}m_R^2(Z_3-1)\phi_R^2-\frac{\lambda_R}{4!}\phi_R^4-\frac{\lambda_R}{4!}(Z_3^2-1)\phi_R^4. \tag{3.8}$$

It has been shown that in the dimensional regularization formulation [13] we have

$$Z_3=\mu^{n-4}\left(1+\sum_{\nu=1}^{\infty}c_\nu(\lambda_R)/(n-4)^\nu\right),$$

$$m=\mu\left(m_R+m_R\sum_{\nu=1}^{\infty}b_\nu(\lambda_R)/(n-4)^\nu\right),$$

$$\lambda=\mu^{4-n}\left(\lambda_R+\sum_{\nu=1}^{\infty}a_\nu(\lambda_R)/(n-4)^\nu\right), \tag{3.9}$$

where

$$\{a_\nu, b_\nu, c_\nu;\quad \nu=1,\dots,\infty\}$$

are smooth functions of λ_R, and μ is the unit of mass required in dimensional regularization.

The renormalized Green functions of the theory

$$\Gamma\left(\prod_{j=1}^{l}\phi_R(x_j)\right)$$

obey the following relation

$$\Gamma\left(\prod_{j=1}^{l}\phi(x_j)\right)=Z_3^{\frac{1}{2}l}\,\Gamma\left(\prod_{j=1}^{l}\phi_R(x_j)\right). \tag{3.10}$$

In the limit "$n\to 4$" $\Gamma[\Pi_{j=1}^{l}\varphi_R(x_j)]$ is finite; so the scale invariance Ward identity becomes

$$\sum_{i=1}^{l}\left(-\frac{\partial}{\partial x_i}\left(x_iZ_3^{\frac{1}{2}l}\Gamma\left(\prod_{j=1}^{l}\phi_R(x_j)\right)\right)+\tfrac{1}{2}(n-2)\,Z_3^{\frac{1}{2}l}\Gamma\left(\prod_{j=1}^{l}\phi_R(x_j)\right)\right)$$
$$=\left(-\frac{2}{i}m^2\frac{\partial}{\partial m^2}-\frac{\lambda}{i}(4-n)\frac{\partial}{\partial\lambda}\right)\left(Z_3^{\frac{1}{2}l}\Gamma\left(\prod_{j=1}^{l}\phi_R(x_j)\right)\right)$$
$$=Z_3^{\frac{1}{2}l}\left(-2\frac{m^2}{i}\frac{\partial}{\partial m^2}-\frac{\lambda}{i}(4-n)\frac{\partial}{\partial\lambda}\right)\Gamma\left(\prod_{j=1}^{l}\phi_R(x_j)\right)$$
$$+\left(\left(-2\frac{m^2}{i}\frac{\partial}{\partial m^2}-\frac{\lambda}{i}(4-n)\frac{\partial}{\partial\lambda}\right)Z_3^{\frac{1}{2}l}\right)\Gamma\left(\prod_{j=1}^{l}\phi_R(x_j)\right). \tag{3.11}$$

This simplifies to

$$i\sum_{i=1}^{l}\left(-\frac{\partial}{\partial x_i}\left(x_i\bar{\Gamma}\left(\prod_{j=1}^{l}\phi_R(x_j)\right)\right)+\bar{\Gamma}\left(\prod_{j=1}^{l}\phi_R(x_j)\right)\right)$$

$$+\left(\bar{\beta}\frac{\partial}{\partial\lambda_R}+\bar{\alpha}\frac{\partial}{\partial m_R^2}+\tfrac{1}{2}l\bar{\gamma}\right)\bar{\Gamma}\left(\prod_{j=1}^{l}\phi_R(x_j)\right)=0, \tag{3.12}$$

with

$$\beta=\left.\left((4-n)\lambda\frac{\partial\lambda_R}{\partial\lambda}\right)\right|_{m,\mu},$$

$$\gamma=\left.\left((4-n)\lambda\frac{\partial}{\partial\lambda}\log Z_3\right)\right|_{m,\mu},$$

$$\alpha=2m^2\left.\frac{\partial m_R^2}{\partial m^2}\right|_{\lambda,\mu}+(4-n)\lambda\left.\frac{\partial m_R^2}{\partial\lambda}\right|_{m,\mu},$$

$$\bar{\alpha}=\lim_{n\to4}\alpha,\qquad \bar{\beta}=\lim_{n\to4}\beta,\qquad \bar{\gamma}=\lim_{n\to4}\gamma,$$

and

$$\bar{\Gamma}\left(\prod_{j=1}^{l}\phi_R(x_j)\right)=\lim_{n\to4}\Gamma\left(\prod_{j=1}^{l}\phi_R(x_j)\right).$$

For later use we note that

$$\frac{\partial}{\partial m_R^2}\Gamma\left(\prod_{j=1}^{l}\phi_R(x_j)\right)=i\int d^nx\,\Gamma\left(\mathrm{O}(x)\prod_{j=1}^{l}\phi_R(x_j)\right), \tag{3.13}$$

where

$$\mathrm{O}(x)=-\tfrac{1}{2}Z_3\phi_R^2(x),$$

and

$$\frac{\partial}{\partial\lambda_R}\Gamma\left(\prod_{j=1}^{l}\phi_R(x_j)\right)=i\int d^nx\,\Gamma\left(\mathrm{O}'(x)\prod_{j=1}^{l}\phi_R(x_j)\right), \tag{3.14}$$

with

$$\mathrm{O}'(x)=-\frac{1}{4!}\left(1+2\lambda_R\frac{\partial Z_3}{\partial\lambda_R}\right)\phi_R^4(x)+\tfrac{1}{2}\frac{\partial Z_3}{\partial\lambda_R}(\partial_\mu\phi_R(x))^2-\tfrac{1}{2}m_R^2\frac{\partial Z_3}{\partial\lambda_R}\phi_R^2(x).$$

We see that $\mathrm{O}(x)$ and $\mathrm{O}'(x)$ contain all operators that can mix with ϕ^2 and ϕ^4 respectively, except, in the latter case, operators that differ by partial integrations from those in $\mathrm{O}'(x)$, i.e., $\phi(\partial^2\phi)$. Hence, from (3.12), (3.13) and (3.14)

$$Z_3^{-\frac{1}{2}l} \int \mathrm{d}^n x\, \Gamma\left(\left(m^2\phi^2(x) + \frac{\lambda}{4!}(4-n)\phi^4(x)\right) \prod_{i=1}^{l} \phi(x_i)\right)$$
$$= -\beta \int \mathrm{d}^n x \Gamma\left(\mathrm{O}'(x) \prod_{j=1}^{l} \phi_\mathrm{R}(x_j)\right) + \tfrac{1}{2} i\gamma \sum_{j=1}^{l} \int \mathrm{d}^n x\, \delta(x-x_j) \Gamma\left(\prod_{j=1}^{l} \phi_\mathrm{R}(x_j)\right)$$
$$- \alpha \int \mathrm{d}^n x\, \Gamma\left(\mathrm{O}(x) \prod_{j=1}^{l} \phi_\mathrm{R}(x_j)\right). \tag{3.15}$$

This has the consequence that

$$Z_3^{-\frac{1}{2}l} \Gamma\left(\left(m^2\phi^2(x) + \frac{\lambda}{4!}(4-n)\phi^4(x)\right) \prod_{i=1}^{l} \phi(x_i)\right)$$
$$= -\beta\Gamma\left(\mathrm{O}''(x) \prod_{j=1}^{l} \phi_\mathrm{R}(x_j)\right) + \tfrac{1}{2}\gamma i \sum_{j=1}^{l} \delta(x-x_j)\Gamma\left(\prod_{j=1}^{l} \phi_\mathrm{R}(x_j)\right)$$
$$- \alpha\Gamma\left(\mathrm{O}(x) \prod_{j=1}^{l} \phi_\mathrm{R}(x_j)\right), \tag{3.16}$$

where $\mathrm{O}''(x)$ differs from $\mathrm{O}'(x)$ by operators such as $\phi\partial^2\phi$, as indicated above. These extra operators just play the role of cancelling divergences proportional to the momentum inflow of the insertions.

Now following the method given in section one we can write down the broken conformal Ward identities for the regulated, but unrenormalized, Green functions, *viz.*,

$$\sum_{i=1}^{l} \left(-\frac{\partial}{\partial x_i^\nu} \left((2x_i^\alpha x_i^\nu - g^{\alpha\nu} x_i^2)\Gamma\left(\prod_{j=1}^{l} \phi(x_j)\right) + 2x_i^\alpha \tfrac{1}{2}(n-2)\Gamma\left(\prod_{j=1}^{l} \phi(x_j)\right)\right)\right.$$
$$\left. + \int \mathrm{d}^n x\; 2x^\alpha \Gamma\left(\left(m^2\phi^2(x) + \frac{\lambda}{4!}(4-n)\phi^4(x)\right) \prod_{j=1}^{l} \phi(x_j)\right) = 0. \tag{3.17}$$

The last term here is just the same as that appearing in (3.4) except for the $2x^\alpha$ factor Hence we have (on invoking multiplicative renormalizability)

$$\sum_{i=1}^{l} \left(-\frac{\partial}{\partial x_\nu^i} \left((2x_i^\alpha x_i^\nu - g^{\alpha\nu} x_i^2)\bar{\Gamma}\left(\prod_{j=1}^{l} \phi_\mathrm{R}(x_j)\right) + 2x_i^\alpha(-\tfrac{1}{2} i\bar{\gamma} + 1)\bar{\Gamma}\left(\prod_{j=1}^{l} \phi_\mathrm{R}(x_j)\right)\right)\right)$$
$$- \bar{\beta} \int \mathrm{d}^4 x\; 2x^\alpha \bar{\Gamma}\left(\mathrm{O}''(x) \prod_{j=1}^{l} \phi_\mathrm{R}(x_j)\right) - \bar{\alpha} \int \mathrm{d}^4 x\; 2x^\alpha \bar{\Gamma}\left(\mathrm{O}'(x) \prod_{j=1}^{l} \phi_\mathrm{R}(x_j)\right) = 0. \tag{3.18}$$

This is the broken conformal Ward identity of the theory and agrees with the equations obtained by other methods [2]. Since $\mathrm{O}'(x)$ is a soft operator, it is clear from (3.18) that at an eigenvalue of β we have conformal invariance asymptotically; so for ϕ^4 theory asymptotic scale invariance implies asymptotic conformal invariance.

4. The non-Abelian case

We will consider a gauge field theory for a compact semi-simple group. When the gauge group is non-Abelian, no derivations of conformal Ward identities exist using the usual methods.

The Lagrangian $\mathcal{L}$ to be studied involves Yang-Mills fields A^a_μ and fermions ψ transforming under a representation σ of the gauge group.

$$\mathcal{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \overline{\psi}i\gamma^\mu D_\mu\psi - \overline{\psi}m\psi,$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu, \tag{4.1}$$

$$D_\mu = \tfrac{1}{2}(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu) - ig\sigma^a A^a_\mu,$$

and m, for convenience, is proportional to the unit matrix. The constant g is the gauge coupling and $\{f^{abc}\}$ the set of structure constants of the group. We have not taken a Lagrangian with interactions involving γ_5, but there exist now methods [14] for incorporating this quantity naturally into the dimensional regularization scheme. The gauge fixing function $F_\alpha(A)$ will be taken to be

$$F_\alpha(A) = \xi^{-\frac{1}{2}}\,\partial\cdot A_\alpha, \tag{4.2}$$

ξ being some number which determines the gauge e.g., $\xi = 1$ gives the Feynman gauge, and $\xi = 0$ the Landau gauge.

The variations of $\mathcal{L}$ under infinitesimal scale and conformal transformations are denoted as in previous sections by $\delta_s\mathcal{L}$ and $\delta_c\mathcal{L}$:

$$\begin{aligned}\delta_s\mathcal{L} = {}& \partial_\nu(x^\nu\mathcal{L}) + m\overline{\psi}\psi + \tfrac{1}{2}g(n-4)\overline{\psi}\gamma^\mu\sigma^a A^a_\mu\psi \\ & + \tfrac{1}{4}g(n-4)f^{abc}A^b_\mu A^c_\nu F^a_{\mu\nu} \\ & - \tfrac{1}{2}g(n-4)f^{abc}\,\partial_\mu c^\dagger_a A^c_\mu c_b.\end{aligned} \tag{4.3}$$

Also

$$\begin{aligned}\delta^\alpha_c\mathcal{L} = {}& \partial_\nu((2x^\alpha x^\nu - g^{\alpha\nu}x^2)\mathcal{L}) + 2x^\alpha(m\overline{\psi}\psi + \tfrac{1}{2}(n-4)g\overline{\psi}\gamma^\mu\sigma_a A^a_\mu\psi \\ & + \tfrac{1}{4}g(n-4)f^{abc}A^b_\mu A^c_\nu F^a_{\mu\nu} - \tfrac{1}{2}g(n-4)f^{abc}\partial_\mu c^\dagger_a A^c_\mu c_b) \\ & + \tfrac{1}{2}(n-2)f^{abc}c^\dagger_a A^c_\alpha c_b - \tfrac{1}{4}(n-4)\partial_\alpha(A^a_\kappa)^2 + \tfrac{1}{2}(n-4)\partial_\kappa(A^a_\kappa A^a_\alpha) \\ & - \tfrac{1}{2}(n-4)(\partial_\kappa A^a_\kappa)A^a_\alpha + \frac{1}{2\xi}n(\partial\cdot A^a)A^a_\alpha.\end{aligned} \tag{4.4}$$

Both (4.3) and (4.4) are just particular applications of (2.10) and (2.11). Since we are interested in relating the anomalies in the conformal Ward identities to those in the Callan-Symanzik equations, we will first obtain the latter equations by our method.

The scale invariance Ward identity for the regulated one particle irreducible Green functions Γ is given by

$$\int d^n x\, \Gamma\Big(m\overline{\psi}(x)\,\psi(x)\prod_{j=1}^{l} A_{\nu_j}^{a_j}(x_j)\Big)$$

$$+\int d^n x\, \tfrac{1}{2}g(n-4)\Gamma\Big((\overline{\psi}(x)\gamma^\mu\sigma^a A_\mu^a(x)\,\psi(x)+\tfrac{1}{2}f^{abc}A_\mu^b(x)A_\nu^c(x)F_{\mu\nu}^a(x)$$

$$-f^{abd}\,\partial_\mu c_a^\dagger(x)A_d^\mu(x)c_b(x)\Big)\prod_{j=1}^{l} A_{\nu_j}^{a_j}(x_j)\Big)$$

$$-\sum_{i=1}^{l}\Big((4-\tfrac{1}{2}(n-2))\,\Gamma\Big(\prod_{j=1}^{l} A_{\nu_j}^{a_j}(x_j)\Big)-\frac{\partial}{\partial x_i}\Big(x_i\Gamma\Big(\prod_{j=1}^{l} A_{\nu_j}^{a_j}(x_j)\Big)\Big)\Big)=0. \quad (4.5)$$

We can relate the Green function

$$\Gamma\Big(\overline{\psi}(x)\,\psi(x)\prod_{j=1}^{l} A_{\nu_j}^{a_j}(x_j)\Big),$$

and the other Green functions in (4.5) to simple derivatives with respect to g and m. In fact

$$\frac{\partial}{\partial m}\Gamma\Big(\prod_{j=1}^{l} A_{\nu_j}^{a_j}(x_j)\Big)=i\int d^n x\, \Gamma(\overline{\psi}(x)\,\psi(x)\prod_{j=1}^{l} A_{\nu_j}^{a_j}(x_j))\,, \quad (4.6)$$

$$\frac{\partial}{\partial g}\Gamma\Big(\prod_{j=1}^{l} A_{\nu_j}^{a_j}(x_j)\Big)=i\int d^n x\, \Gamma(\theta(x)\prod_{j=1}^{l} A_{\nu_j}^{a_j}(x_j))\,, \quad (4.7)$$

where

$$\theta(x)=\tfrac{1}{2}F_{\mu\nu}^a(x)A_b^\mu(x)A_c^\nu(x)f^{abc}-\overline{\psi}(x)\gamma^\mu\sigma_a A_\mu^a(x)\,\psi(x)-f_{abc}\partial^\mu c_a^\dagger(x)A_\mu^c(x)c_b(x).$$

Using (4.6) and (4.7) the identity (4.5) reduces to

$$\Big(-\frac{m}{i}\frac{\partial}{\partial m}+\tfrac{1}{2}g(n-4)\frac{1}{i}\frac{\partial}{\partial g}\Big)\,\Gamma\Big(\prod_{j=1}^{l} A_{\nu_j}^{a_j}(x_j)\Big)$$

$$-\sum_{i=1}^{l}\Big((4-\tfrac{1}{2}(n-2))\,\Gamma\Big(\prod_{j=1}^{l} A_{\nu_j}^{a_j}(x_j)\Big)-\frac{\partial}{\partial x_i}\Big(x_i\Gamma\Big(\prod_{j=1}^{l} A_{\nu_j}^{a_j}(x_j)\Big)\Big)\Big)=0. \quad (4.8)$$

The Lagrangian (4.1) is well known to be renormalizable [7, 15], i.e., the ultra-violet divergences of the Green functions can be absorbed by adding a finite set of counterterms to $\mathcal{L}$ without losing gauge invariance. In some cases the Green functions are infra-red infinite, but we shall ignore such problems. Indeed for many physical applications of gauge theories, such as in unified models of weak and electromagnetic interactions, the infra-red problem can be dealt with.

Multiplicative renormalizability can be expressed by

$$\Gamma\left(\prod_{i=1}^{l} A_{\mu_i}^{a_i}(x_i)\right) = Z_3^{\frac{1}{2}l}\,\Gamma\left(\prod_{i=1}^{l} A_{\mathrm{R}\mu_i}^{a_i}(x_i)\right), \tag{4.9}$$

where Z_3 is the wave function renormalization of A^a. Relations very similar to (3.9) hold.

Consequently

$$-\sum_{i=1}^{l} \left((4-\tfrac{1}{2}(n-2))\,\Gamma\left(\prod_{j=1}^{l} A_{\mathrm{R}\nu_j}^{a_j}(x_j)\right) - \frac{\partial}{\partial x_i}\left(x_i\Gamma\left(\prod_{j=1}^{l} A_{\nu_j}^{a_j}(x_j)\right)\right)\right)$$
$$+ Z_3^{-\frac{1}{2}l}\left(-\frac{m}{i}\frac{\partial}{\partial m} + \tfrac{1}{2}g(n-4)\frac{1}{i}\frac{\partial}{\partial g}\right)\left(Z_3^{\frac{1}{2}l}\Gamma\left(\prod_{j=1}^{l} A_{\mathrm{R}\nu_j}^{a_j}(x_j)\right)\right) = 0. \tag{4.10}$$

This is basically the Callan-Symanzik equation in a somewhat unusual form. If we express derivatives with respect to m and g in terms of m_R, ξ_R and g_R, we have

$$\tfrac{1}{2}l\gamma\Gamma\left(\prod_{j=1}^{l} A_{\mathrm{R}\nu_j}^{a_j}(x_j)\right) + \beta\frac{\partial}{\partial g_\mathrm{R}}\Gamma\left(\prod_{j=1}^{l} A_{\mathrm{R}\nu_j}^{a_j}(x_j)\right)$$
$$+\alpha\frac{\partial}{\partial m_\mathrm{R}}\Gamma\left(\prod_{j=1}^{l} A_{\mathrm{R}\nu_j}^{a_j}(x_j)\right) + \rho\frac{\partial}{\partial \xi_\mathrm{R}}\Gamma\left(\prod_{j=1}^{l} A_{\mathrm{R}\nu_j}^{a_j}(x_j)\right) \tag{4.11}$$
$$-i\sum_{i=1}^{l}\left((4-\tfrac{1}{2}(n-2))\Gamma\left(\prod_{j=1}^{l} A_{\mathrm{R}\nu_j}^{a_j}(x_j)\right) - \frac{\partial}{\partial x_i}\left(x^i\Gamma\left(\prod_{j=1}^{l} A_{\mathrm{R}\nu_j}^{a_j}(x_j)\right)\right)\right) = 0,$$

where

$$\beta = \tfrac{1}{2}g(n-4)\left.\frac{\partial g_\mathrm{R}}{\partial g}\right|_{m,\mu},$$
$$\gamma = \left(-m\frac{\partial}{\partial m}\log Z\bigg|_{g,\mu} + \tfrac{1}{2}g(n-4)\frac{\partial}{\partial g}\log Z\bigg|_{m,\mu}\right),$$
$$\rho = \tfrac{1}{2}(n-4)g\left.\frac{\partial \xi_\mathrm{R}}{\partial g}\right|_{m,\mu},$$
$$\alpha = \left(-m\frac{\partial m_\mathrm{R}}{\partial m}\bigg|_{g,\mu} + \tfrac{1}{2}g(n-4)\frac{\partial m_\mathrm{R}}{\partial g}\bigg|_{m,\mu}\right).$$

In the limit $n \to 4$ we have the conventional Callan-Symanzik equation. All the Green functions involved as well as the β, γ, ρ and α are finite (from standard arguments).

The Lagrangian, in terms of renormalized quantities [15], is given by

$$\begin{aligned}
\mathcal{L} = & -\tfrac{1}{4}(\partial^\mu A^\nu_{\mathrm{R}a} - \partial^\nu A^\mu_{\mathrm{R}a} - g_{\mathrm{R}} f^{abc} A^\mu_{\mathrm{R}b} A^\nu_{\mathrm{R}c})^2 \\
& - \frac{1}{2\xi_{\mathrm{R}}}(\partial_\mu A^\mu_{\mathrm{R}a})^2 - \tfrac{1}{4}(Z_3 - 1)(\partial^\mu A^\nu_{\mathrm{R}a} - \partial^\nu A^\mu_{\mathrm{R}a})^2 \\
& + \tfrac{1}{2} g_{\mathrm{R}}(Z_1 - 1) f^{abc} A^\mu_{\mathrm{R}b} A^\nu_{\mathrm{R}c}(\partial_\mu A^a_{\mathrm{R}\nu} - \partial_\nu A^a_{\mathrm{R}\mu}) \\
& - \tfrac{1}{4} g^2_{\mathrm{R}}\left(\frac{Z_1^2}{Z_3} - 1\right)(f^{abc} A^\mu_{\mathrm{R}b} A^\nu_{\mathrm{R}c})^2 - \overline{\psi}_{\mathrm{R}} m_{\mathrm{R}} \psi_{\mathrm{R}} \\
& + \widetilde{Z}_3 c^\dagger_{\mathrm{R}a}\left(\partial^2 \delta_{ab} - g_{\mathrm{R}} \frac{Z_1}{Z_3} \overleftarrow{\partial}_\mu A^\mu_{\mathrm{R}c} f^{abc}\right) c_{\mathrm{R}b} \\
& + Z_{\mathrm{R}} \overline{\psi}_{\mathrm{R}}(i\gamma^\mu \tfrac{1}{2}(\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu) - (Z' - Z_2^{-1}) m_{\mathrm{R}})\psi_{\mathrm{R}} \\
& + g_{\mathrm{R}}(Z_1 Z_2 Z_3^{-1} - 1)\overline{\psi}_{\mathrm{R}} \gamma^\mu \sigma^a A^a_{\mathrm{R}\mu} \psi_{\mathrm{R}} + g_{\mathrm{R}} \overline{\psi}_{\mathrm{R}} \gamma^\mu \sigma^a A^a_{\mathrm{R}\mu} \psi_{\mathrm{R}},
\end{aligned} \tag{4.12}$$

where [15]

$$\begin{aligned}
A^\mu_a &= Z_3^{\frac{1}{2}} A^\mu_{\mathrm{R}a}, \\
g &= g_{\mathrm{R}} Z_1 Z_3^{-\frac{3}{2}}, \\
\xi &= \xi_{\mathrm{R}} Z_3, \\
\psi_j &= Z_2^{\frac{1}{2}} \psi_{\mathrm{R}j}, \\
m &= Z' m_{\mathrm{R}}.
\end{aligned}$$

From now on whenever we refer to $\mathcal{L}$ we shall mean the form given in (4.12).

Now equations (4.10) and (4.11) imply that

$$\begin{aligned}
& Z_3^{-\frac{1}{2}l} \int \mathrm{d}^n x\, \Gamma\Big((m\overline{\psi}(x)\psi(x) + \tfrac{1}{2} g(n-4)(\overline{\psi}(x)\gamma^\mu \sigma_a A^a_\mu(x)\psi(x) \\
& \quad + \tfrac{1}{2} f^{abc} A^\mu_b(x) A^\nu_c(x) F^a_{\mu\nu}(x) - f^{abd}\partial_\mu c^\dagger_a(x) A^\mu_d(x) c_b(x))) \prod_{j=1}^{l} A^{a_j}_{\nu_j}(x_j)\Big) \\
& = \tfrac{1}{2} i\gamma \sum_{j=1}^{l} \int \mathrm{d}^n x\, \delta(x - x_j)\Gamma\Big(\prod_{j=1}^{l} A^{a_j}_{\mathrm{R}\nu_j}(x_j)\Big) \\
& \quad - \alpha \int \mathrm{d}^n x\, \Gamma\Big(O_1(x) \prod_{j=1}^{l} A^{a_j}_{\mathrm{R}\nu_j}(x_j)\Big) \\
& \quad - \rho \int \mathrm{d}^n x\, \Gamma\Big(O_2(x) \prod_{j=1}^{l} A^{a_j}_{\mathrm{R}\nu_j}(x_j)\Big) \\
& \quad - \beta \int \mathrm{d}^n x\, \Gamma\Big(O_3(x) \prod_{j=1}^{l} A^{a_j}_{\mathrm{R}\nu_j}(x_j)\Big),
\end{aligned} \tag{4.13}$$

where

$$O_1(x) = \frac{\partial \mathcal{L}(x)}{\partial m_R}, \qquad O_2(x) = \frac{\partial \mathcal{L}(x)}{\partial \xi_R}, \qquad O_3(x) = \frac{\partial \mathcal{L}(x)}{\partial g_R}.$$

This is just the analogue of equation (3.15). $O_1(x)$ is simply bilinear in the ψ's whereas $O_3(x)$ contains $F^a_{\mu\nu}(x) f^{abc} A^b_\mu(x) A^c_\nu(x)$ and all the operators that mix with it (except for operators which differ by partial integrations). $O_2(x)$ has a similar structure to $O_3(x)$ which can be seen from evaluating $\partial\mathcal{L}(x)/\partial g_R$.

The unrenormalized Ward identity for broken conformal invariance is

$$\Bigg(\sum_{i=1}^{l}\Bigg(\partial_\nu\Bigg((2x_i^\alpha x_i^\nu - g^{\alpha\nu}x_i^2)\,\Gamma\Bigg(\prod_{j=1}^{l} A^{a_j}_{\nu_j}(x_j)\Bigg)\Bigg) - 2x_i^\delta \delta_{\alpha\nu_i}\Gamma\Bigg(\prod_{j\neq i} A^{a_j}_{\nu_j}(x_j) A^{a_i}_{\delta}(x_i)\Bigg)$$

$$+ 2x^i_{\nu_i}\Gamma\Bigg(\prod_{j\neq i} A^{a_j}_{\nu_j}(x_j) A^{a_i}_{\alpha}(x_i)\Bigg) - (n-2)x_i^\alpha\Gamma\Bigg(\prod_j A^{a_j}_{\nu_j}(x_j)\Bigg)\Bigg)\Bigg)$$

$$+ \int \mathrm{d}^n x\; 2x^\alpha\Gamma\Bigg((m\overline{\psi}(x)\psi(x) + \tfrac{1}{2}(n-4)g\overline{\psi}(x)\gamma^\mu\sigma^a A^a_\mu(x)\psi(x)$$

$$+ \tfrac{1}{4}g(n-4)f^{abc}A^\mu_b(x)A^\nu_c(x)F^a_{\mu\nu}(x) - \tfrac{1}{2}g(n-4)f^{abc}\partial_\mu c^\dagger_a(x)A^\mu_c(x)c_b(x))\prod_j A^{a_j}_{\nu_j}(x_j)\Bigg)$$

$$+ \int \mathrm{d}^n x\,\Gamma\;\;(-\tfrac{1}{2}(n-2)gf^{abc}c^\dagger_a(x)A^c_\alpha(x)c_b(x) + \tfrac{1}{2}\Bigg(\frac{n}{\xi} - (n-4)\Bigg)(\partial\cdot A^a_\alpha(x)A^a_\alpha(x)))$$

$$\times \prod_j A^{a_j}_{\nu_j}(x_j) = 0. \tag{4.14}$$

In (4.14) we have restricted ourselves to vector meson Green functions, purely for brevity. For the general case, when we have mixed fermion and vector meson Green functions, the conformal Ward identity differs from (4.14) merely by the addition of

$$\mathcal{C}^\alpha_{\text{fermion}}\Gamma,$$

where $\mathcal{C}^\alpha_{\text{fermion}}$ is the conformal operators for fermions *viz*.,

$$\mathcal{C}^\alpha_{\text{fermion}} = -\partial_\nu(2x^\alpha x^\nu - g^{\alpha\nu}x^2) + 2x_\nu(\tfrac{3}{2}g^{\alpha\nu} - \tfrac{1}{4}[\gamma^\nu,\gamma^\alpha])$$
$$- (2x^\alpha x^\nu - g^{\alpha\nu}x^2)\partial_\nu.$$

The second set of terms in (4.14) is identical to those occurring in the Callan-Symanzik equation except for the factor $2x_\alpha$. By reasoning similar to that in the ϕ^4 case we have for the renormalized broken conformal Ward identity

$$\sum_{i=1}^{l}\Bigg\{\partial_\nu\Big((2x_i^\alpha x_i^\nu - g^{\alpha\nu}x_i^2)\bar\Gamma\Big(\prod_j A_{\mathrm{R}\nu_j}^{a_j}(x_j)\Big)\Big) - 2x_i^\delta\delta_{\alpha\nu_i}\bar\Gamma\Big(\Big(\prod_{j\neq i}A_{\mathrm{R}\nu_j}^{a_j}(x_j)\Big)A_\delta^{a_i}(x_i)\Big)$$
$$+ 2x_{\nu_i}^i\bar\Gamma\Big(\prod_{j\neq i}A_{\mathrm{R}\nu_j}^{a_j}(x_j)A_{\mathrm{R}\alpha}^{a_i}(x_i)\Big) - 2x_i^\alpha(1-\tfrac{1}{2}i\gamma)\bar\Gamma\Big(\prod_j A_{\nu_j}^{a_j}(x_j)\Big)\Bigg\} \tag{4.15}$$
$$+\int d^4x\, 2x^\alpha\beta\bar\Gamma\Big(O_3(x)\prod_{j=1}^{l}A_{\mathrm{R}\nu_j}^{a_j}(x_j)\Big) + \int d^4x\, 2x^\alpha\alpha\bar\Gamma\Big(O_1(x)\prod_{j=1}^{l}A_{\mathrm{R}\nu_j}^{a_j}(x_j)\Big)$$
$$+\int d^4x\, 2x^\alpha\rho\bar\Gamma\Big(O_2'(x)\prod_{j=1}^{l}A_{\mathrm{R}\nu_j}^{a_j}(x_j)\Big) + \lim_{n\to4}\int d^nx\,\Gamma\Big(O''(x)\prod_{j=1}^{l}A_{\mathrm{R}\nu_j}^{a_j}(x_j)\Big) = 0.$$

The operator $\hat O''(x)$ is defined as

$$\hat O''(x) = C_1(x) + C_2(x),$$
$$C_1(x) = \tfrac{1}{2}Z_3(-((n-4) - n/\xi)\,A_a^\alpha(x)\,\partial\cdot A^a(x),$$
$$C_2(x) = -\tfrac{1}{2}g(n-2)f^{abc}c_a^\dagger(x)A_c^\alpha(x)c_b(x)\,\widetilde Z_3 Z_3^{\frac{1}{2}}.$$

(Moreover, as in the ϕ^4 case, $O_2'(x)$ and $O_3'(x)$ are the same as the undashed operators except for terms which differ by partial integrations from those in $O_2(x)$ and $O_3(x)$.)

The main difference between (4.15) and (3.18) is the presence of the $O''(x)$ term. This term is finite because the scale invariance Ward identity allows us to conclude that all the other terms which appear are finite. The operator $C_1(x)$ is expected classically. For Abelian theories the $C_2(x)$ term is not present and so we recover the result found for quantum electrodynamics using normal product algorithms [3]. The C_2 term, however, is present in the general case of non-Abelian gauge theories. It serves to cancel some of the divergences associated with the insertion $A_a^\alpha\partial\cdot A^a$. From power counting we find that the Green functions with an $O''(x)$ insertion are not asymptotically negligible when compared to the other terms in (4.15). Hence, at an eigenvalue of ρ and a non-zero eigenvalue of β conformal invariance is broken (by the presence of $O''(x)$). (The case of the zero eigenvalue of β is trivial.) Our statements apply to gauge variant Green functions only, since we are arguing from (4.15). The gauge invariant operators are usually composite, and, although it is possible to obtain an analogue of (4.14), the renormalizability of the composite operators is in general not multiplicative. We expect still to have the insertions $O''(x)$ in conformal identities for gauge invariant Green functions; we might hope that such insertions, since they are essentially gauge dependent, may annihilate gauge invariant Green functions. At the level of perturbation theory, this is not true in general. It is possible to convince oneself of this by looking at the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a + i\bar\psi\gamma^\mu D_\mu\psi - \bar\psi m_1\psi$$
$$-\tfrac{1}{2}m_2^2\phi^2 - \frac{1}{4!}\lambda\phi^4 - \frac{1}{3!}\lambda'\phi^3 + h\bar\psi\psi\phi,$$

(where in addition to the gauge mesons, and fermions of mass m_1, we have a scalar gauge invariant field ϕ of mass m_2). For this Lagrangian

$$\Gamma(\hat{O}''(x)\phi_R(y)\phi_R(Z))$$

is non-zero in low orders of perturbation theory. It may be true that such Green functions are zero at a non-zero eigenvalue of β, but this is a non-perturbative statement and we cannot prove or disprove it.

Although we have concentrated on broken conformal Ward identities, the method employed should be useful for studying the response of Green functions to any infinitesimal transformations of the field operators in a renormalizable field theory; after all, the crucial ingredient in our approach was our ability to write down Ward identities in n dimensions for regulated but unrenormalized Green functions.

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Appendix

For the sake of completeness we will discuss the form of the conformal Ward identities for Green functions which are subtractively renormalisable. This is best illustrated by the vacuum polarisation tensor, $\Pi_{\mu\nu}(x)$, in quantum electrodynamics. The tensor is defined by

$$\begin{aligned}\Pi_{\mu\nu}(x) &= \Gamma(J_\mu(x)\, J_\nu(0)),\\ J_\mu(x) &= \overline{\psi}(x)\,\gamma^\mu\psi(x).\end{aligned} \tag{A.1}$$

Our approach and arguments will follow exactly those in the main text. The only point of difference is the relation between the renormalized and unrenormalised Green functions. Notation introduced in sec. 4 will be freely used.

The regularised, but unrenormalised, scale invariance identity is

$$\begin{aligned}&4n\,\Pi_{\mu\nu}(x) + x\cdot\frac{\partial}{\partial x}\,\Pi_{\mu\nu}(x) - 2(n-1)\,\Pi_{\mu\nu}(x)\\ &\quad+\int \mathrm{d}^n z\;\Gamma(m\overline{\psi}(z)\psi(z)\;\overline{\psi}(x)\gamma_\mu\psi(x)\overline{\psi}(0)\gamma_\nu\psi(0))\\ &\quad+\int \mathrm{d}^n z\,\tfrac{1}{2}g(n-4)\,\Gamma(\overline{\psi}(z)\gamma^\rho A_\rho(z)\,\psi(z)\overline{\psi}(x)\gamma_\mu\psi(x)\overline{\psi}(0)\gamma_\nu\psi(0)) = 0.\end{aligned} \tag{A.2}$$

This is just a straightforward rearrangement of the type of identities written in section three. Owing to current conservation $\Pi^{\mu\nu}(x)$ can be written as

$$\Pi^{\mu\nu}(x) = -\left(\frac{\partial}{\partial x^\mu}\,\frac{\partial}{\partial x^\nu} - g_{\mu\nu}\,\frac{\partial^2}{\partial x_\rho\,\partial x^\rho}\right)\Pi(x). \tag{A.3}$$

It is well known [16] that $\Pi(x)$ is subtractively renormalizable. This implies that

$$\Pi^{\mu\nu}_{\mathrm{R}}(x) = \Pi^{\mu\nu}(x) + (2\pi)^n \left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - g_{\mu\nu}\frac{\partial}{\partial x^\rho}\frac{\partial}{\partial x_\rho}\right)\delta^{(n)}(x)\int \Pi(y)\,\mathrm{d}^n y \tag{A.4}$$

has a finite limit $\overline{\Pi}^{\mu\nu}_{\mathrm{R}}$ as $n \to 4$. $\overline{\Pi}^{\mu\nu}_{\mathrm{R}}$ is the renormalised amplitude. In terms of this quantity we have

$$\left[i\left(10 + x^\rho\frac{\partial}{\partial x^\rho}\right) + \bar{\alpha}\,\frac{\partial}{\partial m_{\mathrm{R}}} + \bar{\beta}\,\frac{\partial}{\partial g_{\mathrm{R}}} + \bar{\rho}\,\frac{\partial}{\partial \xi_{\mathrm{R}}}\right]\overline{\Pi}^{\mathrm{R}}_{\mu\nu}(x) = C_{\mu\nu}(x), \tag{A.5}$$

where $C_{\mu\nu}(x)$ is

$$C_{\mu\nu}(x) = \lim_{n\to 4}\left[i\left(2(n+1) + x\cdot\frac{\partial}{\partial x}\right) + \alpha\,\frac{\partial}{\partial m_{\mathrm{R}}} + \beta\frac{\partial}{\partial g_{\mathrm{R}}} + \rho\frac{\partial}{\partial \xi_{\mathrm{R}}}\right] \times\left((2\pi)^n\left(g_{\mu\nu}\partial^2 - \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\right)\delta^{(n)}(x)\int \Pi(y)\,\mathrm{d}^n y\right), \tag{A.6}$$

and $\bar{\alpha}, \bar{\beta}, \bar{\rho}$ are as defined previously. $C_{\mu\nu}$ is finite since the other terms in the Ward identity are; we note also that there is no γ appearing among the anomalous terms and this is just a reflection of the fact that there are no Z factors required in the renormalisation. From (A.5) we learn that

$$\begin{aligned}&\int \mathrm{d}^n z\; \Gamma((m\overline{\psi}(z)\psi(z) + \tfrac{1}{2}g(n-4)\overline{\psi}(z)\gamma^\rho A_\rho(z)\psi(z))\overline{\psi}(x)\gamma^\mu\psi(x)\overline{\psi}(0)\gamma^\nu\psi(0))\\ &= \alpha\int \mathrm{d}^n z\;\Pi^{\mu\nu,\mathrm{R}}_{O_1(z)}(x) + \beta\int \mathrm{d}^n z\;\Pi^{\mu\nu,\mathrm{R}}_{O_i(z)}(x) + \rho\int \mathrm{d}^n z\;\Pi^{\mu\nu,\mathrm{R}}_{O_3(z)}(x)\\ &\quad - (2\pi)^n\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - g_{\mu\nu}\partial^2\right)\delta^{(n)}(x)\iint \mathrm{d}^n z\;\mathrm{d}^n y\;\Pi_{O(z)}(y)\end{aligned} \tag{A.7}$$

where Q is $\alpha O_1 + \beta O_2 + \rho O_3$, and $\Pi^{\mu\nu,\mathrm{R}}_{O_i(z)}$ is defined by an equation identical to (A.4) except that the Green functions on the right-hand side each contain an insertion of $O_i(z)$.

We will now turn to the regularised conformal Ward identity which is

$$\begin{aligned}&(2x^\alpha x^\kappa - g^{\alpha\kappa}x^2)\frac{\partial}{\partial x^\kappa}\,\Pi^{\mu\nu}(x) + 4n\,x^\alpha\,\Pi^{\mu\nu}(x) + 2g^{\alpha\mu}x_\kappa\,\Pi_{\kappa\nu}(x)\\ &\quad - 2x^\mu\,\Pi_{\alpha\nu}(x) - 2(n-1)x^\alpha\,\Pi_{\mu\nu}(x)\\ &\quad + \int \mathrm{d}^n z\; 2z^\alpha\,\Gamma((m\overline{\psi}(z)\psi(z) + \tfrac{1}{2}g(n-4)\overline{\psi}(z)\gamma^\rho A_\rho(z)\psi(z))\\ &\quad \times\overline{\psi}(x)\gamma^\mu\psi(x)\;\overline{\psi}(0)\;\gamma^\nu\psi(0))\\ &\quad + \tfrac{1}{2}\int \mathrm{d}^n z\left(\frac{n}{\xi} - (n-4)\right)\Gamma(\partial\cdot A(z)A^\alpha(z)\;\overline{\psi}(x)\gamma^\mu\psi(x)\;\overline{\psi}(0)\gamma^\nu\psi(0)) = 0.\end{aligned} \tag{A.8}$$

We see that apart from the spin and gauge terms, we just have the insertions appearing in the scale invariance case except for the $2z^\alpha$ factors. Consequently the renormalized conformal Ward identity is

$$
\begin{aligned}
&(2x^\alpha x^\kappa - g^{\alpha\kappa}x^2)\frac{\partial}{\partial x^\kappa}\overline{\Pi}^{\mathrm{R}}_{\mu\nu}(x) + 16x^\alpha\,\overline{\Pi}^{\mathrm{R}}_{\mu\nu}(x) + 2g^{\alpha\mu}x_\kappa\,\overline{\Pi}^{\mathrm{R}}_{\kappa\nu}(x)\\
&\quad - 2x_\mu\,\overline{\Pi}^{\mathrm{R}}_{\alpha\nu}(x) - 6x^\alpha\,\overline{\Pi}^{\mathrm{R}}_{\mu\nu}(x) + \bar{\alpha}\int \mathrm{d}^4z\; 2z^\alpha\,\overline{\Pi}^{\mu\nu,\mathrm{R}}_{O_1(z)}(x)\\
&\quad + \bar{\beta}\int \mathrm{d}^4z\; 2z^\alpha\,\overline{\Pi}^{\mu\nu,\mathrm{R}}_{O_2(z)}(x) + \bar{\rho}\int \mathrm{d}^4z\; 2z^\alpha\,\overline{\Pi}^{\mu\nu,\mathrm{R}}_{O_3(z)}(x) + \overline{D}^\alpha_{\mu\nu}(x) = 0,
\end{aligned}
\tag{A.9}
$$

where

$$
\begin{aligned}
D^\alpha_{\mu\nu}(x) = -(2\pi)^n \Bigg[&(2x^\alpha x^\kappa - g^{\alpha\kappa}x^2)\frac{\partial}{\partial x^\kappa} + 2(n+1)x^\alpha \left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - g_{\mu\nu}\partial^2\right)\delta^{(n)}(x)\int \Pi(y)\mathrm{d}^ny\\
&+ 2g^{\alpha\mu}x_\kappa\left(\frac{\partial}{\partial x^\kappa}\frac{\partial}{\partial x^\nu} - g_{\kappa\nu}\partial^2\right)\delta^{(n)}(x)\int \Pi(y)\,\mathrm{d}^ny\\
&- 2x^\mu\left(\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\nu} - g_{\alpha\nu}\partial^2\right)\delta^{(n)}(x)\int \Pi(y)\,\mathrm{d}^ny\\
&+ \left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - g_{\mu\nu}\partial^2\right)\int \mathrm{d}^nz\; 2z^\alpha\,\delta^{(n)}(x)\int \mathrm{d}^ny\;\Pi_{O(z)}(y)\Bigg]\\
&+ \tfrac{1}{2}\int \mathrm{d}^nz\left(\frac{n}{\xi} - (n-4)\right)\Pi^{\mu\nu}_{\partial\cdot A(z)A_\alpha(z)}(x)\,,
\end{aligned}
$$

and $\overline{D}^\alpha_{\mu\nu}(x) = \lim_{n\to 4} D^\alpha_{\mu\nu}(x)$. The limit exists and is finite since the remaining terms in the Ward identity are.

Hence, when we have subtractive renormalisation, the method goes through without modification yielding conformal identities similar to those obtained before.

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