Nuclear Physics B83 (1974) 493–514. North-Holland Publishing Company

FIELD THEORY RENORMALIZATION USING THE CALLAN-SYMANZIK EQUATION

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Received 18 June 1974

Abstract: Quantum electrodynamics (QED) is renormalized by exploiting the Callan-Symanzik equations for broken scale invariance. The basic idea is to make enough mass insertions in a Green function G requiring renormalization so that the resulting Green function G' (a) is superficially convergent and (b) has a skeleton expansion. G' is trivially renormalized and G is recovered through the Callan-Symanzik equations. This offers a simple proof that QED can be multiplicatively renormalized. This technique avoids all divergent integrals, in particular the difficult overlapping divergences. Moreover, it is shown that the renormalized Green functions calculated according to the recipe given by Bogoliubov, Parasiuk and Hepp agree with the result of the present analysis. This offers an alternate and somewhat simpler proof that the BPH method (a) gives a finite answer and (b) corresponds to a multiplicative renormalization. The method used is generalizable to other field theories in an obvious manner.

1. Introduction

We consider here the problem of renormalization in field theory, with special attention towards the problem of overlapping divergences. To be specific, we shall study in detail spinor quantum electrodynamics (QED), though it will be clear that the techniques can be easily generalized. Various renormalization procedures have been given in the past; the scheme offered here is simple in its statement and its proof, and offers the advantage of explicitly incorporating the renormalization group [1]. We shall also demonstrate explicitly the equivalence of our procedure to the subtraction method of Bogoliubov, Parasiuk and Hepp [2] (BPH) and thus offer a simple, albeit indirect, proof of the BPH renormalization procedure.

It is well known [3] that a superficially convergent graph with a skeleton expansion can be renormalized by just substituting the (lower order) renormalization parts into the skeleton diagram (e.g. electron-electron scattering); while for a superficially divergent graph (with degree of divergence $d \geqslant 0$) with a skeleton expansion, $d + 1$ overall subtractions are required after substituting renormalization parts (e.g. the vertex in QED, with $d = 0$). The major complication arises in the case of overlapping divergences [4], i.e. for graphs which do not have skeleton expansions. In QED these include the electron self-energy and the vacuum polarization. From the integral equation formulation [3] applied to the 4th order vacuum polarization for example [5], it is already clear that a complicated subtraction procedure is involved, corresponding to subtracting "in all possible ways".

Part of the overlapping divergence problem in QED can be avoided by exploiting gauge invariance. The Ward identities [6] which express the constraint of gauge invariance relate Green functions differing from each other by the addition of an external photon, for example the electron self-energy and the vertex. Thus, instead of renormalizing directly the electron self-energy (which has an overlapping divergence) one could instead renormalize the vertex (which has a skeleton expansion) and use the Ward identity to recover the electron self-energy [7]. Note that the insertion of the extra photon, in addition to removing the overlapping divergence, also lowers the superficial degree of divergence by one. More generally, one can differentiate a Green function with respect to the external momenta in order to obtain a more tractable object. A recent work along this line is that of Poggio [8]. This method suffers an essential drawback. For example, in the original work of Ward, the extra photon is not inserted in all possible ways – indeed, if it were, this method would not be useful for vacuum polarization [9]. Instead, the insertion is performed only on lines carrying the external momentum, so that a careful specification of the routing of momentum becomes necessary [10]. The method given here will avoid the routing problem.

We shall present here a renormalization procedure which is a simple extension of the work of Callan [11] and relies on Ward identities for broken scale invariance to relate any Green function G to another G^θ with an extra zero momentum θ-insertion. (Here θ denotes the trace of the energy-momentum tensor; in the case of QED, a θ-insertion is simply a mass insertion on fermion lines.) The idea is to make enough insertions on a Green function requiring renormalization so that the resulting Green function (a) possesses a skeleton expansion and (b) is superficially convergent. This latter Green function is trivially renormalized and the original Green function is recovered through the Ward identities for broken scale invariance.

There is one essential difference between the gauge invariance Ward identities and the scaling Ward identities. Renormalization necessarily involves extra mass scales (e.g. regulator masses) which lead to violations of scale invariance not apparent from formal manipulations with the Lagrangian. Therefore the scaling Ward identities, better known as Callan-Symanzik (CS) equations [12] will contain anomalous terms. Indeed, these anomalous terms will correspond to various well-defined classes of subtractions in the BPH scheme, as we shall see later.

In sect. 2 we define the Green functions of interest and briefly derive CS equations relating them. These are of course widely known and we include the derivation only for the sake of completeness and to establish notations. In sect. 3, we show that the procedure introduced yields a finite result to any order in e. The analogous result for θ^4 has been proved by Callan. In ref. [9], it was shown that the Green functions constructed by exploiting the CS equations satisfy the requisite analyticity and unitarity conditions, and are thus respectable Green functions. We go slightly further by showing that such a construction corresponds to a multiplicative renormalization. In sect. 4, the BHP subtraction procedure is proved. We hope the present discussion will be of some interest on account of the difficulty of the direct proof of BPH. The present discussion should also shed some light on how the subtractions of BPH correspond to a multiplicative renormalization.

2. Definition of renormalization parts and mass insertions, and derivation of CS equations

2.1. The renormalization parts

In QED, there are three Green functions requiring non-trivial renormalization: the vertex Γ_μ, the electron propagator S and the photon propagator $D_{\mu\nu}$. These are defined by the usual Feynman rules * and are regarded as functions of a bare mass m_0, a bare charge e_0, a gauge parameter λ_0, and a set of cut-offs or regulator masses collectively denoted by Λ. The task of renormalization is to extract cut-off dependent factors Z_1 and Z_3 ** so that the renormalized Green functions, defined by

$$\tilde{\Gamma}_\mu = Z_1 \Gamma_\mu \,, \qquad \tilde{S} = Z_1^{-1} S \,, \qquad \tilde{D}_{\mu\nu} = Z_3^{-1} D_{\mu\nu} \,, \tag{1}$$

are finite (i.e. Λ-independent) in the limit of large Λ, when expressed in terms of some physical mass parameter m, the renormalized charge e and the renormalized gauge parameter λ, where

$$e = Z_3^{\frac{1}{2}} e_0 \,, \qquad \lambda = Z_3^{-1} \lambda_0 \,. \tag{2}$$

One would not only want to prove the above statement, but one should also like to provide a recipe for calculating the renormalized Green functions which bypasses the explicit and cumbersome extraction of the renormalization constants Z_i. This we shall do in sects. 3 and 4.

In order to avoid the infrared singularities associated with on-shell amplitudes, and in order to be able to invoke Weinberg's theorem [13], we shall renormalize at zero momentum. (All momenta are understood to be Euclidean and analytic continuation back to Minkowski momenta is implied in the whole discussion that follows.)

* To be precise, the Feynman rules are $-ie_0\gamma_\mu$ at each vertex, $iS = i/(\not{p} - m_0)$ for each fermion line, $-iD_{\mu\nu} = -i\,[g_{\mu\nu} - (1-\lambda_0) q_\mu q_\nu / q^2]/q^2$ for each photon line, with integrations rendered finite by regulators or cut-offs. We assume the cut-offs are implemented in a gauge-invariant way.

** We demand $Z_1 = Z_2$ from the start.

Since the renormalization is at zero momentum, it is convenient to choose the mass parameter as

$$m = -\tilde{S}^{-1}(0)\,. \tag{3}$$

We emphasize that this is *not* the physical electron mass. We also have to specify $\tilde{\Gamma}_\mu$ and $\tilde{D}_{\mu\nu}$ at one point in order to fix Z_1 and Z_3 uniquely. We do so by demanding

$$\tilde{\Gamma}_\mu(0) = \gamma_\mu\,.$$

$$\tilde{D}_{\mu\nu}(q) \to \frac{g_{\mu\nu}}{q^2} + (q_\mu q_\nu \text{ terms}) \quad \text{as } q \to 0\,. \tag{4}$$

It shall be convenient to work with the electron self-energy Σ and the vacuum polarization $\Pi_{\mu\nu}$, which are related to the propagators by

$$S = S_\mathrm{o} + S_\mathrm{o}\Sigma S_\mathrm{o} + \ldots = [S_\mathrm{o}^{-1} - \Sigma]^{-1}\,,$$

$$S^{-1} = S_\mathrm{o}^{-1} - \Sigma\,, \tag{5}$$

$$D_{\mu\nu} = D_{\mathrm{o}\mu\nu} + D_{\mathrm{o}\mu\alpha}(-e^2\Pi_{\alpha\beta})D_{\mathrm{o}\beta\nu} + \ldots\,, \tag{6}$$

where the subscript o denotes zeroth order in perturbation theory. As a consequence of gauge invariance, $\Pi_{\mu\nu}$ is transverse, so we may write

$$\Pi_{\mu\nu}(q) = (g_{\mu\nu}q^2 - q_\mu q_\nu)\Pi(q^2)\,, \tag{7}$$

so that (6) reduces to

$$D_{\mu\nu}(q) = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\frac{1}{q^2[1 + e_\mathrm{o}^2\Pi(q^2)]} + \lambda_\mathrm{o}\frac{q_\mu q_\nu}{q^2}\,,$$

$$\tilde{D}_{\mu\nu}(q) = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\frac{1}{q^2[1 + e^2\Pi_c(q^2)]} + \lambda\frac{q_\mu q_\nu}{q^2}\,, \tag{8}$$

where

$$\Pi_c(q^2) = \Pi(q^2) - \Pi(0)\,, \qquad Z_3^{-1} = 1 + e_\mathrm{o}^2\Pi(0)\,,$$

$$e^2 = Z_3 e_\mathrm{o}^2\,, \qquad \lambda = Z_3^{-1}\lambda_\mathrm{o}\,. \tag{9}$$

Likewise if we let

$$\Sigma_c(p) = \Sigma(p) - \Sigma(0) - p_\mu\frac{\partial}{\partial p_\mu}\Sigma(0)\,,$$

then

$$\tilde{S}^{-1}(p) = \tilde{S}^{-1}(0) + p_\mu \frac{\partial}{\partial p_\mu} \tilde{S}^{-1}(0) - \Sigma_c(p) . \tag{10}$$

The first term in (10) is by definition $-m$, while the second term, by the Ward identity is simply $p_\mu \tilde{\Gamma}^\mu(0) = \not{p}$, so (10) may be written as

$$\tilde{S}^{-1}(p) = -m + \not{p} - \Sigma_c(p) . \tag{11}$$

The material in this section so far is entirely conventional.

2.2. Mass insertions

Green functions with mass insertions can be introduced by defining

$$\Gamma^\theta_\mu = \frac{\partial}{\partial m_0} \Gamma_\mu ,$$

$$S^\theta = STS = \frac{\partial}{\partial m_0} S ,$$

i.e.

$$T = -\frac{\partial}{\partial m_0} S^{-1} ,$$

$$D^\theta_{\mu\nu} = D_{\mu\alpha} [-e^2 \Pi^\theta_{\alpha\beta}] D_{\beta\nu} = \frac{\partial}{\partial m_0} D_{\mu\nu} ,$$

i.e.

$$e^2 \Pi^\theta_{\mu\nu} = \frac{\partial}{\partial m_0} (D^{-1})_{\mu\nu} ,$$

$$T^\theta = \frac{\partial}{\partial m_0} T , \tag{12}$$

with $\partial/\partial m_0$ performed at fixed e_0, λ_0 and Λ. Note that we are considering two mass differentiations (T and T^θ) for the electron propagator. This is related to the fact that Σ requires two subtractions. It is clear that diagramatically $\partial/\partial m_0$ corresponds to mass insertions or all fermion lines. External propagators have been removed from T and $\Pi^\theta_{\mu\nu}$ so that they are proper three-point functions. Of course these Green functions are not expected to be finite, and we define renormalized counterparts by

$$\tilde{\Gamma}^\theta_\mu = Z_\theta^{-1} \Gamma^\theta_\mu , \qquad \tilde{\Pi}^\theta_{\mu\nu} = Z_\theta^{-1} Z_1^{-1} Z_3^{-1} \Pi^\theta_{\mu\nu} ,$$

$$\tilde{T} = Z_\theta^{-1} T , \qquad \tilde{T}^\theta = Z_\theta^{-1} Z_1^{-1} T^\theta , \tag{13}$$

The task of renormalization is again to prove that the left-hand side of (13), for a suitable choice of Z_θ, is finite for large Λ when expressed in terms of e, m and λ. We are obliged to fix the new renormalization constant Z_θ. One way of doing so is to specify $\widetilde{T}(0)$. We shall however not choose to do so, but instead make use of this freedom in Z_θ to simplify the CS equation later on. The reason for doing so is to follow the conventional form of the CS equation.

Again, gauge invariance can be invoked to write

$$\widetilde{\Pi}^\theta_{\mu\nu}(q) = (g_{\mu\nu} q^2 - q_\mu q_\nu)\widetilde{\Pi}^\theta(q^2) . \tag{14}$$

We also note that the Ward identity gives

$$p_\mu \frac{\partial}{\partial p_\mu} \widetilde{T}(p)\big|_0 = -p^\mu \widetilde{\Gamma}^\theta_\mu(0) ,$$

so that

$$\widetilde{T}(p) = \widetilde{T}(0) - p^\mu \widetilde{\Gamma}^\theta_\mu(0) + O(p^2) . \tag{15}$$

2.3. The Callan-Symanzik equations

The CS equations are relations among the renormalized Green functions $\widetilde{\Gamma}_\mu$, Σ_c, Π_c and $\widetilde{T}$ and their counterparts with a θ-insertion: $\widetilde{\Gamma}^\theta_\mu$, $\widetilde{T}$, Π^θ, $\widetilde{T}^\theta$. Given the definitions, the derivation of the CS equations is a trivial exercise in differentiation. We record here the equations involving the proper Green functions.

$$\widetilde{\Gamma}^\theta_\mu = \alpha \frac{\partial \widetilde{\Gamma}_\mu}{\partial m} + \frac{\beta}{m} \frac{\partial \widetilde{\Gamma}_\mu}{\partial e} - \frac{2\beta}{me} \lambda \frac{\partial \widetilde{\Gamma}_\mu}{\partial \lambda} + \frac{\gamma}{m} \widetilde{\Gamma}_\mu ,$$

$$\widetilde{T} = \alpha \frac{\partial \Sigma_c}{\partial m} + \frac{\beta}{m} \frac{\partial \Sigma_c}{\partial e} - \frac{2\beta}{me} \lambda \frac{\partial \Sigma_c}{\partial \lambda} - \frac{\gamma}{m} (\not p - m) + 1 ,$$

$$\widetilde{\Pi}^\theta = \alpha \frac{\partial \Pi_c}{\partial m} + \frac{\beta}{m} \frac{\partial \Pi_c}{\partial e} - \frac{2\beta}{me} \lambda \frac{\partial \Pi_c}{\partial \lambda} - \frac{2\beta}{me^3} ,$$

$$\widetilde{T}^\theta = \alpha \frac{\partial \widetilde{T}}{\partial m} + \frac{\beta}{m} \frac{\partial \widetilde{T}}{\partial e} - \frac{2\beta}{me} \lambda \frac{\partial \widetilde{T}}{\partial \lambda} + \frac{\delta}{m} \widetilde{T} , \tag{16}$$

where

$$\alpha = Z_\theta^{-1} \frac{\partial m}{\partial m_0} = O(1) , \qquad \frac{\gamma}{m} = Z_\theta^{-1} \frac{\partial \log Z_1}{\partial m_0} = O(e^2) ,$$

$$\frac{\beta}{m} = Z_\theta^{-1} \frac{\partial e}{\partial m_0} = O(e^3) , \qquad \frac{\delta}{m} = Z_\theta^{-1} \frac{\partial \log Z_\theta}{\partial m_0} = O(e^2) , \tag{17}$$

and these are functions of e, m/Λ and λ. Note that the coefficient of the $\partial/\partial\lambda$ term is expressible in terms of β on account of (2). It is also well to emphasize that at this point there is no guarantee that any of the quantities appearing in (16) is cut-off independent. That this is so will be proved inductively in sect. 3.

The freedom in Z_θ previously mentioned will now be exploited to set $\alpha = 1$ in (16). Since $\widetilde{\Gamma}_\mu(0)$, $\Sigma_c(0)$, $\partial\Sigma_c(0)/\partial p_\mu$, and $\Pi_c(0)$ are known, the choice $\alpha = 1$ lead immediately to the following "boundary conditions" obtainable by evaluating (16) at zero momentum:

$$\widetilde{\Gamma}^\theta_\mu(0) = \frac{\gamma}{m}\,\gamma_\mu\,, \qquad \widetilde{\Pi}^\theta(0) = -\frac{2\beta}{me^3}\,,$$
$$\widetilde{T}(p) = (1+\gamma) - \frac{\gamma}{m}\,p + O(p^2)\,, \qquad \widetilde{T}^\theta(0) = \frac{\beta}{m}\frac{\partial\gamma}{\partial e} - \frac{2\beta}{me}\,\lambda\frac{\partial\gamma}{\partial\lambda} + \frac{\delta}{m}(1+\gamma)\,. \tag{18}$$

Note that the first two of these equations are consistent with gauge invariance (15). It can be seen that this follows from the happy choice of $Z_1 = Z_2$.

The following notation will be useful in sect. 3:

$$\widetilde{\Gamma}_\mu = \sum_n e^{2n}\,\widetilde{\Gamma}_{\mu,2n}\,, \qquad \gamma = \gamma_2 e^2 + \gamma_4 e^4 + \dots\,, \quad \text{etc.}$$
$$\Sigma_c = \sum_n e^{2n}\,\Sigma_{c,2n}\,, \quad \text{etc.} \qquad \widetilde{\Gamma}_{\mu(n)} = \sum_{k=0}^{n} e^k\,\widetilde{\Gamma}_{\mu,k}\,,$$
$$\beta = \beta_3 e^3 + \beta_5 e^5 + \dots\,, \qquad \beta_{(n)} = \sum_{k=0}^{n} e^k\beta_k\,, \tag{19}$$

i.e. a bracketed index indicates up to order e^n. We shall refer to the superficially divergent Green functions $\widetilde{\Gamma}_\mu$, Σ_c, $\widetilde{T}$ and Π_c as renormalization parts.

3. Proof of renormalizability

In this section, we shall present an algorithm for calculating the renormalized Green functions as well as the coefficient functions β, γ, δ order by order in e, and prove that the result at each stage is cut-off independent ("finite"). The induction hypotheses are

(A) Assume that the following have already been calculated and are finite: $\widetilde{\Gamma}_{\mu(n)}$, $\widetilde{\Gamma}^\theta_{\mu(n)}$, $\Sigma_{c(n)}$, $\widetilde{T}_{(n)}$, $\widetilde{T}^\theta_{(n)}$, $\Pi_{c(n-2)}$, $\widetilde{\Pi}^\theta_{(n-2)}$, $\gamma_{(n)}$, $\beta_{(n+1)}$, $\delta_{(n)}$. These will be referred to as lower order Green functions.

(B) Assume the following asymptotic bounds (modulo powers of logarithms) for the renormalization parts:

$$\tilde{\Gamma}_{\mu(n)}(xp_1+k_1, xp_2+k_2) \sim \mathrm{O}(1)\,, \qquad \tilde{T}_{(n)}(xp+k) \sim \mathrm{O}(1)\,,$$

$$\Sigma_{c(n)}(xp+k) \sim \mathrm{O}(x)\,, \qquad \Pi_{c(n-2)}(xq+k) \sim \mathrm{O}(1) \tag{20}$$

as $x \to \infty$. (There are similar bounds for the other Green functions, but these will not be required below.) The essence of the above bounds is that up to $\mathrm{O}(e^n)$, $\tilde{\Gamma}_\mu$, $\tilde{S}$, $\tilde{T}$ and $\tilde{D}_{\mu\nu}$ have the same asymptotic power behaviour as in lowest order.

3.1. The vertex

We now proceed to construct the Green functions to the next order. First consider $\tilde{\Gamma}^\theta_{\mu(n+2)}$. Since $\tilde{\Gamma}^\theta_\mu$ possesses a skeleton expansion, we merely have to substitute the lower order renormalization parts into the skeleton. That this gives a multiplicative renormalization of $\tilde{\Gamma}^\theta_\mu$ is obvious and the proof will not be repeated here. By hypothesis (B), the integrand will have the same asymptotic behaviour as the skeleton graph itself. In a skeleton graph, the only suspect is the overall integration; but in the case of $\tilde{\Gamma}^\theta_\mu$, the superficial degree of divergence is $d = -1$ and thus ensures that all integration will be finite. (This is no different from the arguments which disposes of high point functions such as that for $\mathrm{e}^-\mathrm{e}^-$ scattering.) Moreover, Weinberg's theorem gives

$$\tilde{\Gamma}^\theta_\mu(xp_1+k_1, xp_2+k_2) \sim \mathrm{O}(x^{-1})\,, \qquad x \to \infty\,.$$

Next we consider the CS equation to $\mathrm{O}(e^{n+2})$:

$$\tilde{\Gamma}^\theta_{\mu(n+2)} = \frac{\partial}{\partial m}\tilde{\Gamma}_{\mu(n+2)} + \frac{\beta_{(n+1)}}{m}\frac{\partial}{\partial e}\tilde{\Gamma}_{\mu(n)}$$

$$-\frac{2\beta_{(n+1)}}{me}\lambda\frac{\partial}{\partial\lambda}\tilde{\Gamma}_{\mu(n)} + \frac{\gamma_{(n+2)}}{m}\tilde{\Gamma}_{\mu(n)} + \mathrm{O}(e^{n+4})\,. \tag{21}$$

Notice that in the last three terms only $\tilde{\Gamma}_{\mu(n)}$ is needed because $\gamma = \mathrm{O}(e^2)$, $\beta = \mathrm{O}(e^3)$. In this equation, $\beta_{(n+1)}$ and $\tilde{\Gamma}_{\mu(n)}$ are known by hypothesis (A) and $\tilde{\Gamma}^\theta_{\mu(n+2)}$ has just been calculated. Looking at (21) at zero momentum determines $\gamma_{(n+2)}$:

$$\gamma_{(n+2)}\gamma_\mu = m\tilde{\Gamma}^\theta_{\mu(n+2)}(0)\,.$$

We can now solve for $\tilde{\Gamma}_{\mu(n+2)}$:

$$\tilde{\Gamma}_{\mu(n+2)}(p_1, p_2) = \gamma_\mu + \int_0^1 \frac{\mathrm{d}\alpha}{\alpha}\,\Phi_{(n+2)}(\alpha p_1, \alpha p_2)\,, \tag{22}$$

where

$$\Phi_{(n+2)}(p_1, p_2) = -m\tilde{\Gamma}^{\theta}_{\mu(n+2)}(p_1, p_2) + \beta_{(n+1)} \frac{\partial}{\partial e} \tilde{\Gamma}_{\mu(n)}(p_1, p_2)$$

$$- \frac{2\beta_{(n+1)}}{e} \lambda \frac{\partial}{\partial \lambda} \tilde{\Gamma}_{\mu(n)}(p_1, p_2) + \gamma_{(n+2)} \tilde{\Gamma}_{\mu(n)}(p_1, p_2) . \tag{23}$$

In writing down (22), we have used the fact that $\tilde{\Gamma}_\mu$ is dimensionless to turn $m\partial/\partial m \to -\alpha\partial/\partial\alpha$, where the arguments of $\tilde{\Gamma}_\mu$ are αp_i. We have also invoked the boundary condition $\tilde{\Gamma}_\mu(0) = \gamma_\mu$. We shall *assume* that $\tilde{\Gamma}_\mu$ is continuous at zero momentum, i.e. †

$$\tilde{\Gamma}_\mu(\alpha p_1, \alpha p_2) \to \gamma_\mu \quad \text{as} \quad \alpha \to 0 \quad \text{for all } p_1, p_2 \,.$$

It is easy to see that this ensures the convergence of the integral in (22). Eq. (23) gives

$$\Phi_{(n+2)}(xp_1 + k_1, xp_2 + k_2) \sim \mathrm{O}(1) \quad \text{as} \quad x \to \infty$$

so that (22) yields

$$\tilde{\Gamma}_{\mu(n+2)}(xp_1 + k_1, xp_2 + k_2) \sim \mathrm{O}(1) \quad \text{as} \quad x \to \infty \,.$$

To recapitulate, we have recovered, to $\mathrm{O}(e^{n+2})$, the induction hypothesis for $\tilde{\Gamma}_\mu$, $\tilde{\Gamma}^\theta_\mu$ and γ.

3.2. Electron self-energy

The renormalization of the other Green functions are sufficiently similar that we give only an outline. First we calculate $\tilde{T}^\theta_{(n+2)}$. It has a skeleton expansion. To see this, note that $\tilde{T}$ has the same topological structure as $\tilde{\Gamma}_\mu$ and thus has a skeleton expansion. An extra θ-insertion cannot create more renormalization parts and thus $\tilde{T}^\theta$ also has a skeleton expansion. The degree of divergence of $\tilde{T}^\theta$ is $d = -1$. Thus in complete analogy with $\tilde{\Gamma}^\theta_\mu$, $\tilde{T}^\theta_{(n+2)}$ can be renormalized by just substituting lower order renormalization parts into the skeleton expansion. We now use the CS equations twice. First (16d) is used to determine $\delta_{(n+2)}$ and to solve for $\tilde{T}_{(n+2)}$. Then (16b) is used to solve for $\Sigma_{c(n+2)}$. Again, finite results are assured at each stage. The only extra comment in order here is that the consistency of (16b) up to $\mathrm{O}(p^1)$ is guaranteed by (15).

Going back to $\tilde{T}^\theta_{(n+2)}$, we see that the integrand which determines it is asymptoti-

† Continuity at zero momentum would be no extra assumption in a massive theory. In a theory with massless particles such as QED, one has to be sure that discontinuities associated with multi-photon intermediate states vanish at zero external momentum. This could be a problem in a smaller number of dimensions. Discontinuity of the Green functions at zero momentum would ruin this and probably any other scheme which renormalizes at zero momentum.

cally similar to that of the corresponding skeleton graph. Weinberg's theorem thus gives $\widetilde{T}^{\theta}_{(n+2)}(xp+k) \sim O(x^{-1})$ and repeated use of the CS equations give $\widetilde{T}_{(n+2)}(xp+k) \sim O(1)$, $\Sigma_{c(n+2)}(xp+k) \sim O(x)$ as $x \to \infty$.

3.3. Vacuum polarization

Lastly we calculate $\widetilde{\Pi}^{\theta}_{(n)}$. It is easy to see that $\widetilde{\Pi}^{\theta}$ has a skeleton expansion. For a simple proof, see appendix A. Moreover, although the degree of divergence of $\widetilde{\Pi}^{\theta}_{\mu\nu}$ is $d = +1$, gauge invariance allows the extraction of a factor $(g_{\mu\nu}q^2 - q_\mu q_\nu)$ in forming $\widetilde{\Pi}^{\theta}$, thereby reducing the degree of divergence to $d = -1$. As in the vertex then, $\widetilde{\Pi}^{\theta}_{(n)}$ converges when we substitute renormalization parts into its skeleton. We then use the CS equation (16c) to obtain $\beta_{(n+3)}$ and hence $\Pi_{c(n)}$.

Weinberg's theorem also shows $\widetilde{\Pi}^{\theta}_{(n)}(xq+k) \sim O(x^{-1})$ and the use of the CS equation gives $\Pi_{c(n)}(xq+k) \sim O(1)$. This completes the inductive proof of renormalizability.

3.4. Gauge invariance

It can be seen from the foregoing that there was no need to discuss directly the renormalization of any Green function other than the superficially convergent ($d < 0$) ones. All divergences, in particular the troublesome overlapping divergences, have been avoided by the use of mass insertions. There is no need for cut-offs at any stage of the calculation. This renormalization scheme has the additional virtue of incorporating the renormalization group [1].

We close this section with a comment on gauge invariance. It is well known that if Pauli-Villars regulators are used [14], $\Pi(q, m_0, e_0, \Lambda)$ is gauge invariant, i.e. independent of the gauge parameter λ. The physical mass m_{ph} and the renormalized charge e are also gauge invariant:

$$\frac{m_{\mathrm{ph}}}{m_0} = f(e_0, m_0, \Lambda)\,, \tag{24}$$

$$e = e(e_0, m_0, \Lambda)\,. \tag{25}$$

So Π_c can be written

$$\begin{aligned}\Pi_c &= \Pi_c(q, m_{\mathrm{ph}}, e, \Lambda)\\ &= \Pi_c(q^2/m_{\mathrm{ph}}^2, e)\,,\end{aligned} \tag{26}$$

since we know Π_c to be cut-off independent. However, the mass parameter m, being essentially an off-shell amplitude, is gauge dependent:

$$\frac{m}{m_{\text{ph}}} = \phi(e, \lambda) . \tag{27}$$

The λ dependence can be verified in second order. Therefore if we express Π_c in terms of m, it will acquire a gauge dependence through ϕ:

$$\Pi_c = \Pi_c(q^2/m^2, e, \lambda) . \tag{28}$$

Thus the term $-(2\beta/me)\lambda\partial\Pi_c/\partial\lambda$ in the CS equation for Π_c is *not* zero, as might naively be thought; instead, it equals

$$\frac{2\beta}{e}\frac{\partial \log \phi}{\partial \log \lambda}\frac{\partial \Pi_c}{\partial m}$$

which enters in order e^4.

4. BPH renormalization

4.1. Graphs with skeleton expansion

Any renormalization program is faced with two tasks. First one has to show that the theory can be rendered finite and secondly one has to provide a recipe for calculating this finite answer. These have been accomplished in the preceding sections. It is nonetheless useful to compare the algorithm presented here with other recipes. We shall do this for the renormalization program of BPH [2], which is perhaps the most explicit formulation for complicated diagrams. This will provide an indirect but still simple proof that the BPH recipe gives a finite answer and this answer corresponds to a multiplicative renormalization. In order to first illustrate the technique without being bogged down by the cumbersome notation necessary in the case of the overlapping divergence, we begin by dealing with graphs with a skeleton expansion.

To be specific, we consider the vertex Γ (Lorentz indices will be suppressed in this section) to order e^{2n}. The BPH prescription is simple: substitute (lower order) re normalization parts into the skeleton, then make $d + 1$ overall subtractions, d being the degree of divergence, in this case 0. Explicitly let $J(p, k)$ be the Feynman integrand for Γ, to order e^{2n}, obtained by substituting renormalization parts into the skeleton graph. Here p denotes all external momenta and k all loop momenta. Define once subtracted objects

$$\tilde{\Gamma}_s(p) = \tilde{\Gamma}(p) - \tilde{\Gamma}(0) , \qquad J_s(p, k) = J(p, k) - J(0, k) . \tag{29}$$

Then the assertion of BPH is simply

$$\int J_s(p, k)\,\mathrm{d}k = \tilde{\Gamma}_s(p) . \tag{30}$$

We now give a proof of (30) by showing that $\int J_s(p, k)\,\mathrm{d}k$ satisfies the CS equation for $\tilde{\Gamma}_s(p)$.

First note that J has the form

$$J = (e^2\tilde{D})^n \tilde{\Gamma}^{2n+1} \tilde{S}^{2n} \tag{31}$$

($\tilde{D}$ = photon propagator, $\tilde{\Gamma}$ = vertex, $\tilde{S}$ = fermion propagator). Define a mass-insertion operator Δ acting on a product such as (31) by

$$\Delta(ABC\dots) = A^\theta BC_{\dots} + AB^\theta C_{\dots} + ABC^\theta_{\dots} + \dots \tag{32}$$

(Recall that $\tilde{D}^\theta$, $\tilde{S}^\theta$ are respectively $\tilde{\Pi}^\theta$ and $\tilde{T}$ with external legs restored.) If we operate on (31) with Δ,

$$\begin{aligned}\Delta J &= (e^2\tilde{D}^\theta)(e^2\tilde{D})^{n-1} \tilde{\Gamma}^{2n+1} \tilde{S}^{2n} + \dots \qquad (n \text{ terms})\\ &+ (e^2\tilde{D})^n \tilde{\Gamma}^\theta \tilde{\Gamma}^{2n} \tilde{S}^{2n} + \dots \qquad (2n+1 \text{ terms})\\ &+ (e^2\tilde{D})^n \tilde{\Gamma}^{2n+1} \tilde{S}^\theta \tilde{S}^{2n-1} + \dots \qquad (2n \text{ terms})\,.\end{aligned} \tag{33}$$

Next use the fact that the lower order Green functions on the right of (33) have been constructed to obey the CS equations

$$\begin{aligned}\Delta(e^2\tilde{D}) &= e^2\tilde{D}^\theta = [\mathcal{D}]\,\tilde{D}\,,\\ \Delta\tilde{\Gamma} &= \tilde{\Gamma}^\theta = [\mathcal{D} + \gamma/m]\tilde{\Gamma}\,,\\ \Delta\tilde{S} &= \tilde{S}^\theta = [\mathcal{D} - \gamma/m]\tilde{S}\,.\end{aligned} \tag{34}$$

(These are equivalent to (16)). The differential operator $\mathcal{D}$ is

$$\mathcal{D} = \frac{\partial}{\partial m} + \frac{\beta}{m}\frac{\partial}{\partial e} - \frac{2\beta}{me}\lambda\frac{\partial}{\partial\lambda}\,. \tag{35}$$

One can easily verify that (33) reduces to

$$\Delta J = [\mathcal{D} + \gamma/m]J\,. \tag{36}$$

In obtaining (36), it is easy to see that $\mathcal{D}$ distributes. The γ/m term works out because there is one more $\tilde{\Gamma}$ than $\tilde{S}$ in (31). This is trivially related to the fact that if we extract Z_1 from each $\tilde{\Gamma}$ and Z_1^{-1} from each $\tilde{S}$ in (31), the net result is to extract Z_1 from the whole expression, which is precisely the factor needed for renormalizing a vertex.

Eq. (36) still holds if we subtract at zero momentum as defined by (29):

$$\Delta J_s = [\mathcal{D} + \gamma/m]J_s = \left[\frac{\partial}{\partial m} + \frac{\beta}{m}\frac{\partial}{\partial e} - \frac{2\beta}{me}\lambda\frac{\partial}{\partial\lambda} + \frac{\gamma}{m}\right]J_s\,. \tag{37}$$

Now integrate over the loop momenta k, cutting off any possibly divergent integrals at some Λ:

$$\int^{\Lambda} \Delta J_s \, \mathrm{d}k = \frac{\partial}{\partial m} \int^{\Lambda} J_s \, \mathrm{d}k + \frac{\beta}{m} \frac{\partial}{\partial e} \widetilde{\Gamma}_s - \frac{2\beta}{me} \lambda \frac{\partial}{\partial \lambda} \widetilde{\Gamma}_s + \frac{\gamma}{m} \widetilde{\Gamma}_s \,. \tag{38}$$

In the last three terms, we have used the fact that the J_s involved is of lower order, for which (30) is assumed to hold.

Now ΔJ is obtained by making a θ-insertion on each of the parts of the graph for $\widetilde{\Gamma}$ in turn. This clearly yields the skeleton graph for $\widetilde{\Gamma}^{\theta}$ with renormalization parts put in. Since $\widetilde{\Gamma}^{\theta}$ has $d < 0$, from the discussion in the last section we know that

$$\int^{\Lambda} \Delta J_s \, \mathrm{d}k$$

converges and gives $\widetilde{\Gamma}_s^{\theta}$. So (38) now reads

$$\widetilde{\Gamma}_s^{\theta} = \frac{\partial}{\partial m} \int^{\Lambda} J_s \, \mathrm{d}k + \frac{\beta}{m} \frac{\partial}{\partial e} \widetilde{\Gamma}_s - \frac{2\beta}{me} \lambda \frac{\partial}{\partial \lambda} \widetilde{\Gamma}_s + \frac{\gamma}{m} \widetilde{\Gamma}_s \,.$$

The first term on the right side is now clearly Λ-independent, i.e. the integral converges. Moreover, by comparison with the CS equation (subtracted once at zero momentum), we see that

$$\frac{\partial}{\partial m} \int J_s \, \mathrm{d}k = \frac{\partial}{\partial m} \widetilde{\Gamma}_s$$

or

$$\frac{\partial}{\partial \alpha} \int J_s(\alpha p, k) \, \mathrm{d}k = \frac{\partial}{\partial \alpha} \widetilde{\Gamma}_s(\alpha p) \,. \tag{39}$$

(With the cut-off eliminated, $\widetilde{\Gamma}$ depends only on the ratios p_i/m.) But since $\int J_s \, \mathrm{d}k = \widetilde{\Gamma}_s = 0$ for $\alpha = 0$, integration of (39) with respect to α yields

$$\int J_s(p, k) \, \mathrm{d}k = \widetilde{\Gamma}_s(p)$$

proving (30). Note that convergence of the integral is part of the conclusion.

The spirit of the proof is extremely simple. This is hardly surprising, since (30) can be proved directly without great difficulty [3]. It is also evident that the above method applies to any other Green function with a skeleton expansion, e.g. $\widetilde{T}$.

The subtraction was necessary to avoid an integration constant in the last step. For a Green function with $d = 1$ say, two θ-insertions would have to be considered, and the CS equation invoked twice. At each stage, there will be an analog of the α integration; the two integration constants being zero if two (or in general $d + 1$) subtractions have been made.

4.2. Overlapping divergences

Again, to be specific, we shall consider the nth order vacuum polarization Π_n †, which, on account of gauge invariance, has an effective $d = 0$. The finite object to be calculated is Π_c to nth order, and the BPH recipe, in the form given by Zimmermann [15], is

$$e^n \Pi_{cn}(q) = \int e^n R(\Pi_{cn}, qk)\,\mathrm{d}k\,, \tag{40}$$

where k denotes all loop momenta. To give the integrand R, some preliminary definitions are in order. These follow Zimmermann [15], and the reader is referred there for details.

For a graph G, let $I(G, pk)$ be the integrand obtained by Feynman rules. (Note this differs from $J(G, pk)$ of subsect. 4.1, which is obtained by substituting renormalization parts into the skeleton of G.) Again p denotes external momenta and k loop momenta. We define the subtracted object

$$I_s(G, pk) = I(G, pk) - I(G, 0k)\,. \tag{41}$$

A forest F of G is a set (possibly null) of diagrams such that (a) each diagram is a renormalization part of G and (b) the diagrams do not overlap (but may be nested). Diagrammatically, each forest can be represented by the graph G together with a set (possibly null) of non-overlapping (but possible nested) boxes drawn around renormalization parts. Each box corresponds to an element $\gamma \in F$. A forest is normal if $G \notin F$. The class of all normal forests of G will be denoted by $\mathcal{F}_N(G)$. A Taylor operator t^γ means a Taylor series in the external momenta of the renormalization part γ up to order $d(\gamma)$, the degree of divergence of γ. Since γ are renormalization parts, $d(\gamma) \geqslant 0$. In fig. 1, we give one diagram G contributing to second order vacuum polarization, together with the normal forests of G.

With these definitions, the integrand R in (40) is given by (we suppress momentum labels)

$$R(\Pi_{cn}) = \sum_{F \in \mathcal{F}_N(G)} \prod_{\gamma \in F} (-t^\gamma) I_s(\Pi_n)\,. \tag{42}$$

In words, R is formed as follows: Draw boxes around renormalization parts in all possible ways. Boxes are not to overlap, nor to contain the whole graph (i.e. normal forests only). Each distinct way of drawing boxes contributes one term, calculated by taking the content of each box γ to order $d(\gamma)$ in the external momenta of that box. This term contributes with a sign $(-1)^l$ where l is the number of boxes in that forest. Then perform the overall subtraction, indicated by the subscript s. (This is equivalent

† In our notation, the diagram with just one fermion loop is Π_0, etc. So Π_{2n} contains n internal photons.

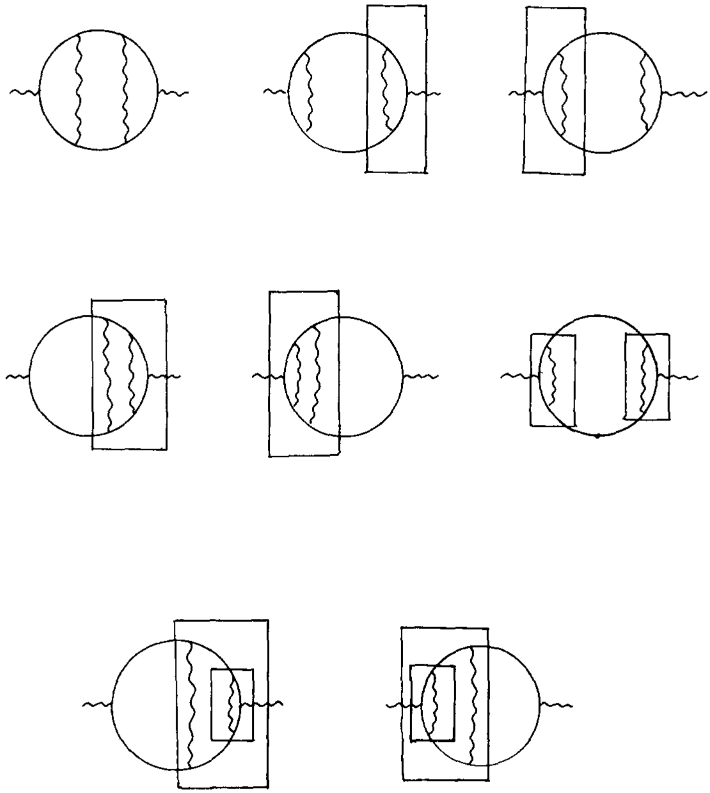

Fig. 1. The normal forests of one diagram G contributing to second order vacuum polarization.

to placing the operator $(1-t^G)$ in front of the whole expression.) Note that this last step commutes with the previous operations.

Eq. (40) is slightly different from (II.33) in Zimmermann in that we sum only over normal forests. This is compensated for by the subtraction indicated by s. To verify this, one merely have to note the one-to-two correspondence between normal forests and all forests given by $F \to F, F \cup \{G\}$.

Although we have written (40) only for Π, it is generally applicable. Indeed, in the case of a diagram with a skeleton expansion, it reduces to the more familiar prescription of subsect. 4.1.

The task at hand is to prove that $\int R(\Pi_{cn})$ indeed gives Π_{cn} as calculated by the technique of sect. 3. Evidently, we have to relate $\int R(\Pi_{cn})$ to Π^θ. We consider

$$e^n \frac{\partial}{\partial m} \int R(\Pi_{cn}) = e^n \sum_{F \in \mathcal{F}_N(G)} \int \prod_{\gamma \in F} (-t^\gamma) \frac{\partial}{\partial m} I_s(\Pi_n) , \tag{43}$$

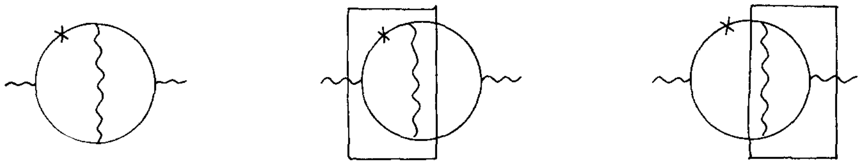

Fig. 2. Examples of diagrams contributing to $\partial R(\Pi_{c,2})/\partial m$. Mass insertion is indicated by x,

the integral being over all loop momenta. Cut-offs are implied wherever needed. The $\partial/\partial m$ on the right side of (43) is just a mass insertion into the diagram. Diagrammatically, the fermion line that is differentiated will be labelled by a cross, as illustrated in fig. 2. Now each term in (43) is not just a Feynman diagram, but a Feynman diagram adorned with a number of boxes. Therefore the mass insertion (the cross in the diagram) may either (a) lie outside all boxes or (b) lie inside a unique smallest box – the uniqueness being a consequence of the non-overlapping nature of the boxes.

We assign to each term a number n^* as follows. In case (a) $n^* = 0$. In case (b) n^* = the order $n(\gamma)$ of this unique smallest box γ, defined as follows. Since the box is a renormalization part of Π, it can be either Π, Σ or Γ. (We shall call these Π-box, Σ-box, Γ-box.) If γ is a self-energy (Π or Σ), then $n(\gamma)$ is the number of the vertices inside the box; if γ is the vertex Γ, then $n(\gamma)$ is the number of vertices inside the box minus one. Some forests with their n^* values are illustrated in fig. 3. We shall look at the various terms on the right-hand side of (43) according to their n^* value – this process will terminate because n^* cannot exceed n, the order of the graph itself. For simplicity, we shall work in Landau gauge ($\lambda = 0$); this eliminates the $\lambda\partial/\partial\lambda$ term from the CS equation, which obviously we are going to invoke. Of course we shall also assume that all lower order objects have been renormalized. The rest of the arguments to follow will be mostly combinatorial in nature.

n^* *= 0 terms.* These are insertions outside all boxes and are clearly forests for $\tilde{\Pi}^{\theta}_{sn}$. Since Π^{θ} has a skeleton expansion, by subsect. 4.1, the integral over momenta will converge. Call these terms σ_1.

n^* *= 2 terms.* There are three cases, depending on whether the insertion is in a second order Π-box, Σ-box or Γ-box. Let us examine these in turn.

(a) Π-box. The resulting diagram is not a forest for Π^{θ}_{sn} because one of the boxes (the one containing the mass insertion) is no longer a renormalization part. In fact, the mass insertion has rendered the integration over the box in question finite, and yields $\Pi^{\theta}_2(0)$. The rest of the diagram is a forest for the lower order diagram $\Pi_{c,n-2}$. Summing over this class of terms evidently gives †

$$e^n R(\Pi_{c,n-2})\Pi^{\theta}_2(0)N_1 \,. \tag{44}$$

Here N_1 is a combinatorial factor giving the number of ways of inserting a second or-

† The above arguments need slight modifications in any gauge other than Landau.

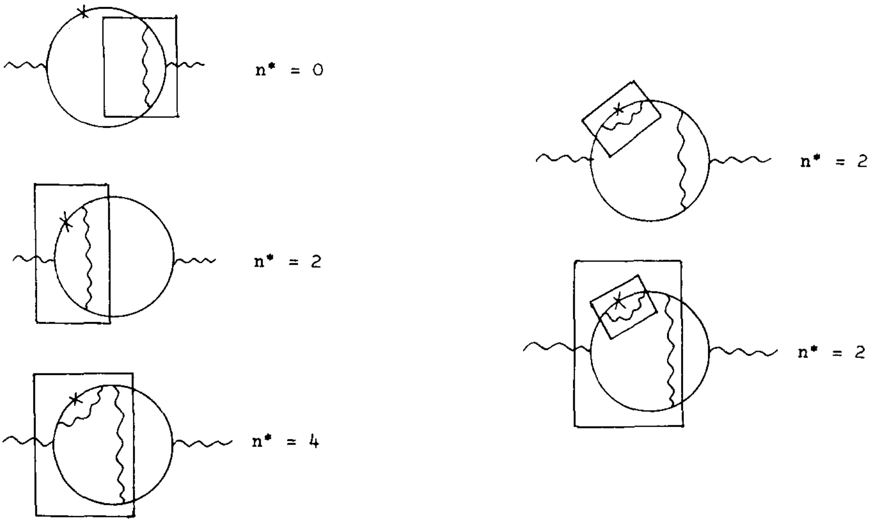

Fig. 3. Some diagrams with their n^* values.

der Π into Π_{n-2} to give a diagram for Π_n. Clearly it equals the number of internal photons in Π_{n-2}, i.e. $\frac{1}{2}(n-2)$. Also by the "boundary condition" (18), $\Pi_2^\theta(0)$ is nothing more than $-2\beta_3/m$. Moreover, the remaining momentum integrations in (44) may be carried out since by the assumption on lower order Green functions $\int R(\Pi_{c,n-2}) = \Pi_{c,n-2}$. Therefore (44) gives

$$e^n \Pi_{c,n-2} \left(-\frac{2\beta_3}{m}\right) \tfrac{1}{2}(n-2) = \frac{\beta_3 e^3}{m} \frac{\partial}{\partial e} \left[e^{n-2} \Pi_{c,n-2}\right] \equiv \sigma_2 \,. \tag{45}$$

(b) Γ-box. Again the resulting diagram is not a forest. The integration over the box is finite, giving $\Gamma_2^\theta(0)$, so the result, similar to (44) is

$$-e^n R(\Pi_{c,n-2}) \Gamma_2^\theta(0) N_2 \,. \tag{46}$$

N_2 is the number of ways of inserting a second order Γ into Π_{n-2} to give a diagram for Π_n, and equals the number of vertices in Π_{n-2}, i.e. n. (The two vertices coupling to external photon lines are of course included.) Using the "boundary condition" (18) $\Gamma_2^\theta(0) = \gamma_2/m$ and performing the remaining integrations over $R(\Pi_{c,n-2})$ as before yields

$$-e^n \Pi_{c,n-2} \left(\frac{\gamma_2}{m}\right) n = -(e^{n-2} \Pi_{c,n-2}) \left(\frac{e^2 \gamma_2}{m}\right) n \equiv \sigma_3 \,. \tag{47}$$

(c) Σ-box. This will be the most complicated because $d(\Sigma)$ is one rather than zero. The diagram in question contains a mass insertion inside a Σ-box. The mass insertion has of course turned the Σ-box into a T-box. The box around Σ indicates two terms in the Taylor expansion ($d(\Sigma) = 1$), whereas a box around T should mean only one term. ($d(T) = 0$). Therefore in addition to the "legitimate" zero order (in momentum) term contributing to $R(\Pi^{\theta}_{sn})$, which we denote by σ_4, there is an extra contribution corresponding to the second, linear term in the Taylor series, which gives

$$-e^n R(\Pi_{c,n-2})\left[p_\mu \frac{\partial}{\partial p_\mu} \tilde{T}_2(0)\right] \frac{1}{\not p - m} N_3 , \tag{48}$$

where p is the momentum of the fermion line on which the insertion is made. Briefly, $R(\Pi_{c,n-2})$ comes from the reduced graph, $p_\mu \partial \tilde{T}_2(0)/\partial p_\mu$ gives the extra linear term, and N_3 is the number of ways of inserting a second order Σ into Π_{n-2} to give a graph for Π_n, and equals the number of fermion lines in Π_{n-2}, i.e. n. Using the "boundary condition" $p_\mu \partial \tilde{T}_2(0)/\partial p_\mu = -\gamma_2 \not p/m$, we get

$$e^n R(\Pi_{c,n-2})\left(\frac{\gamma_2}{m}\not p\right)\frac{1}{\not p - m}\, n = e^n R(\Pi_{c,n-2})\,\frac{\gamma_2}{m}\left[1 + \frac{m}{\not p - m}\right] n$$

$$= e^n \,\frac{\gamma_2}{m}\, R(\Pi_{c,n-2})n + \gamma_2 e^n R(\Pi_{c,n-2})\frac{1}{\not p - m} .$$

Integrating over the remaining momenta yields

$$\frac{e^2\gamma_2}{m}\,(e^{n-2}\Pi_{c,n-2})n + \gamma_2 e^2 \int e^{n-2} R(\Pi_{c,n-2})\frac{1}{\not p - m} \equiv \sigma_5 + \sigma_6 ,$$

σ_5 cancels with σ_3. (This merely reflects the fact that equal numbers of Z_1 and Z_2 are involved in a vacuum polarization and we have chosen $Z_1 = Z_2$ to start with.)

The term σ_6 corresponds to an insertion (represented by the extra $1/(\not p - m)$) into $\Pi_{c,n-2}$. It differs from other contributions to $\Pi^{\theta}_{s,n-2}$ (e.g. terms like σ_1 and σ_4) however by the appearance of the factor $\gamma_2 e^2$. This is all to the good. Since $T(0) = 1 + \gamma$ rather than 1, the recipe for calculating $\tilde{\Pi}^{\theta}$ differs from the usual one by precisely this factor $1 + \gamma$. The term σ_6 will give the γ part of the $1 + \gamma$.

To summarize, the $n^* = 0$, 2 contributions add up to

$$(\text{Contributions to } \tilde{\Pi}^{\theta}_{s,n} \text{ given by } \sigma_1, \sigma_4) - \frac{\beta_3 e^3}{m}\frac{\partial}{\partial e}\,[e^{n-2}\Pi_{c,n-2}] .$$

It is evident what happens when higher n^* values are included. Without bothering with the formalities of an induction proof, the result is

$$\frac{\partial}{\partial m}\int R(\Pi_{cn}) = \Pi^{\theta}_{sn} - \left[\frac{\beta}{m}\frac{\partial}{\partial e}\Pi_c\right]_n . \tag{49}$$

Since all terms on the right are cut-off independent, we see the integral on the left on (49) converges. Moreover, by comparison with the CS equation (in Landau gauge and once subtracted at zero), we see that

$$\frac{\partial}{\partial m}\int R(\Pi_{cn}) = \frac{\partial}{\partial m}\Pi_{cn} ,$$

$$\frac{\partial}{\partial \alpha}\int R(\Pi_{cn}, \alpha q, k)\,\mathrm{d}k = \frac{\partial}{\partial \alpha}\Pi_{cn}(\alpha q) .$$

Integrating with respect to α and using the fact that both integrated terms vanish at $\alpha = 0$, we get

$$\int R(\Pi_{cn}, qk)\,\mathrm{d}k = \Pi_{cn}(q) ,$$

where the right-hand side is computed from $\widetilde{\Pi}_n^\theta$ using the CS equation as described in sect. 3. This proves the BPH method for vacuum polarization.

For best understanding of the above proof, which is necessarily sketchy in parts, the reader is invited to work out a low order example in detail. Π_4 will contain enough non-trivial features to be interesting.

We close this section by emphasizing that the technique presented above is generally applicable to other renormalization parts having an overlapping divergence, e.g. Σ and to other field theories.

5. Conclusion

We have presented a simple proof of the multiplicative renormalization of QED based on the CS equation. The finiteness of the calculation at each stage is evident and requires no elaborate proof. Overlapping divergences, which require considerable effort in conventional approaches, do not even have to be separately treated. We have also shown the equivalence of the present method to that of BPH. This provides a relatively easy proof of the BPH subtraction method. Such a proof is extendable to other field theories.

Appendix A

It was claimed in the text that Π^θ has no overlapping divergences. This will be proved here as a special case of the following lemma:

Lemma: In a renormalizable field theory, a graph C with superficial degree of divergence d_C will contain an overlapping divergence involving two m-particle irreducible parts A and B if and only if there exists a graph D, with at least $2(m+1)$ external lines and degree of divergence d_D, such that $d_\mathrm{C}+d_\mathrm{D} \geqslant 0$. Here m is any positive number and the dimension of space-time is arbitrary.

Proof: Suppose a graph C has an overlapping divergence. Then it is possible to draw two overlapping boxes A and B each of which contains a superficially divergent subgraph. Of course $C = A \cup B$. Let $D = A \cap B$ (the overlapping portion). Let $n(A)$ be the number of external lines for A etc. Then it is easy to see

$$n(A) + n(B) = n(C) + n(D) .$$

(Such an equation holds for lines of each type if both fermions and bosons are present.) Now in a renormalizable field theory, the degree of divergence of a graph takes the form $d_A = \alpha n(A) + \beta$, where α, β are constants. Hence

$$d_A + d_B = d_C + d_D .$$

But by assumption A and B are superficially divergent, so $d_A, d_B \geqslant 0$, hence $d_C + d_D \geqslant 0$.

It remains to show that D has at least $2(m+1)$ external lines. If A, B are both m-particle irreducible, there will be at least $m + 1$ lines leaving A but entirely in B and vice versa. All these will be external lines for D. This completes the proof.

In the Π^θ example, we choose $m = 1$ since we want to see if there are overlaps between renormalization parts, which are one-particle irreducible. Since $d_C = 0$, Π^θ will contain a overlapping divergence if and only if there exists a superficially divergent ($d_D \geqslant 0$) four-point function D. There is none (if gauge invariance is taken into account).

Appendix B

In the text we proved the BPH subtraction procedure in Landau gauge. Here we extend the proof to arbitrary gauge. Whereas in Landau gauge we showed that mass insertions on a photon line gave the extra contribution

$$-\frac{\beta}{m}\frac{\partial}{\partial e}\ldots, \tag{B.1}$$

we here show that in arbitrary gauge, in addition to (B.1) there is a contribution

$$\frac{2\beta}{me}\lambda\frac{\partial}{\partial\lambda}. \tag{B.2}$$

Let us decompose the bare propagator $D_{0\mu\nu}$ into transverse and longitudinal parts

$$D_{0\mu\nu} = D^{t}_{0\mu\nu} + \lambda D^{\ell}_{0\mu\nu} ,$$

where

$$D^{t}_{0\mu\nu} = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\frac{1}{q^2} \equiv t_{\mu\nu}/q^2 ,$$

$t_{\mu\nu}$ being the transverse projection operator, and

$$D^{\ell}_{o\mu\nu} = q_\mu q_\nu / q^4 \ .$$

The full propagator is

$$D_{\mu\nu} = D_{o\mu\nu} + D_{o\mu\alpha}(-e^2\Pi_{\alpha\beta})D_{o\beta\nu} + \ldots$$

$$= D^{t}_{o\mu\nu} + D^{t}_{o\mu\alpha}(-e^2\Pi_{\alpha\beta})D^{t}_{o\beta\nu} + \ldots + \lambda D^{\ell}_{o\mu\nu} \ . \qquad \text{(B.3)}$$

Since $\Pi_{\alpha\beta}$ is transverse:

$$\Pi_{\alpha\beta} = q^2 t_{\alpha\beta}\Pi$$

(B.3) may therefore be rewritten as

$$e^2 D_{\mu\nu} = e^2 \frac{t_{\mu\nu}}{q^2}\,[1 - e^2\Pi + (-e^2\Pi)^2 + \ldots] + e^2\lambda D^{\ell}_{o\mu\nu} \ .$$

Consider mass insertions on all internal photon lines:

$$\frac{\partial}{\partial m}(e^2 D_{\mu\nu}) = e^2 \frac{t_{\mu\nu}}{q^2}\left[1 + \frac{\partial}{\partial m}(-e^2\Pi) + \ldots\right] . \qquad \text{(B.4)}$$

Note that only the transverse part is involved. All diagrams in which the mass insertion $\partial/\partial m$ is made on a vacuum polarization Π not in a box will give contributions to $D^{\theta}_{\mu\nu}$ as before. Extra terms arise when the Π in question is boxed. This means $\partial\Pi/\partial m$ is to be replaced by $-\Pi^{\theta}(0)$, or

$$\frac{\partial}{\partial m}(e^2\Pi) \to -e^2\Pi^{\theta}(0) = \frac{2\beta}{me} \ .$$

Thus (B.4) yields, schematically,

$$e^2 \frac{t_{\mu\nu}}{q^2}\frac{2\beta}{me}\,[-1 + 2(e^2\Pi) - 3(e^2\Pi)^2 + \ldots]$$

$$= -\frac{\beta}{m}\frac{\partial}{\partial e}\left\{e^2\frac{t_{\mu\nu}}{q^2}\,[1 - e^2\Pi + (-e^2\Pi)^2 + \ldots]\right\}$$

$$= -\frac{\beta}{m}\frac{\partial}{\partial e}(e^2 D^{t}_{\mu\nu})$$

$$= -\frac{\beta}{m}\frac{\partial}{\partial e}(e^2 D_{\mu\nu} - e^2\lambda D^{\ell}_{o\mu\nu})$$

$$= -\frac{\beta}{m}\frac{\partial}{\partial e}(e^2 D_{\mu\nu}) + \frac{2\beta}{me}(e^2\lambda D^{\ell}_{o\mu\nu}) \ . \qquad \text{(B.5)}$$

But the longitudinal part can be projected out from the full propagator by $\lambda\partial/\partial\lambda$:

$$\lambda \frac{\partial}{\partial\lambda}(e^2 D_{\mu\nu}) = e^2 \lambda D^{\ell}_{o\mu\nu}$$

so (B.5) may be written as

$$-\frac{\beta}{m}\frac{\partial}{\partial e}(e^2 D_{\mu\nu}) + \frac{2\beta}{me}\lambda\frac{\partial}{\partial\lambda}(e^2 D_{\mu\nu}) .$$

Hence the net result of inserting on an internal photon is

$$\frac{\partial}{\partial m}(e^2 D_{\mu\nu}) = e^2 D^{\theta}_{\mu\nu} - \frac{\beta}{m}\frac{\partial}{\partial e}(e^2 D_{\mu\nu}) + \frac{2\beta}{me}\lambda\frac{\partial}{\partial\lambda}(e^2 D_{\mu\nu}) . \tag{B.6}$$

Since we are considering an internal photon line, all objects that occur above are of lower order and hence all manipulations are legitimate. The last term in (B.6) is the extra contribution needed when comparing with the CS equation in arbitrary gauge.

References

[1] M. Gell-Mann and F. Low, Phys. Rev. 95 (1954) 1300.
[2] N.N. Bogoliubov and O.S. Parasiuk, Acta Math. 97 (1957) 227;
K. Hepp, Comm. Math. Phys. 2 (1966) 301.
[3] F.J. Dyson, Phys. Rev. 75 (1949) 486, 1736.
[4] A. Salam, Phys. Rev. 82 (1951) 217; 84 (1951) 426.
[5] R. Jost and J.M. Luttinger, Helv. Phys. Acta 23 (1949) 201.
[6] J.C. Ward, Phys. Rev. 78 (1950) 182.
[7] J.C. Ward, Phys. Rev. 84 (1951) 897.
[8] E. Poggio, Ann. of Phys. 81 (1973) 481.
[9] W.H. Furry, Phys. Rev. 51 (1937) 125.
[10] T.T. Wu, Phys. Rev. 125 (1962) 1436;
R.L. Mills and C.N. Yang, cited in T.T. Wu, above.
[11] C.G. Callan, Phys. Rev. D2 (1970) 1541.
[12] C.G. Callan, see ref. [11];
K. Symanzik, Comm. Math. Phys. 18 (1970) 227.
[13] S. Weinberg, Phys. Rev. 118 (1960) 838.
[14] W. Pauli and F. Villars, Rev. Mod. Phys. 21 (1949) 434.
[15] W. Zimmermann, Lectures on elementary particles and quantum field theory, vol. I, ed. Desar, Grisaru and Pendleton (MIT Press, 1971).