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THE CALLAN–SYMANZIK EQUATION AND DIMENSIONAL REGULARIZATION

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Abstract: A study is made of the Callan–Symanzik equation using the dimensional regularization method. The final equation is far more general than in the usual derivation in that the effect of mass terms is explicitly included; moreover the momenta need not be restricted to the deep-Euclidean region when considering asymptotic limits.

1. Introduction

Recently, with the discovery in field theory of "asymptotic freedom" [1–4] and its possible relevance to explain the scaling behaviour of deep inelastic experiments [3], there has been a renewed interest in the renormalization group [5, 6] equations, which can be seen as asymptotic forms of the Callan–Symanzik (CS) equation [7–9].

In this paper, we study the Callan–Symanzik equation in renormalizable field theories regularized with the continuous dimension method [10, 11]. We find this method remarkably clear and powerful to study the scale transformations $p \to \lambda p$ of Green functions. This method introduces in general two different mass parameters; the first one is the usual "renormalized mass", which can of course be absent when dealing with massless theories, and the second one is a mass parameter, intrinsically connected with this regularization scheme, that "scales" the momentum dependence of Green functions and that will show up usually as the only remaining mass parameter in the region $p \to \infty$.

A beautiful aspect of the method is the role played in the equation by the renormalized mass parameter: in the original CS equation, one had to postulate the absence of mass terms in the asymptotic form of the equation, while in the form of the equation exhibited here, the role of this mass in the asymptotic domain can be explicitly studied. In this aspect, the method described here makes contact with the work of Weinberg [12].

Equally important is that the asymptotic domain is no longer restricted to the deep Euclidean region.

The results of this paper are closely connected with the work of 't Hooft [2]; they are in a sense a translation of his results in the framework of multiplicative renormalization of Green functions.

The paper is organized as follows: sect. 2 contains a short resumé of the basic results of the n-dimensional regularization method. Mostly, we will restrict ourselves to the $\lambda\phi^4$ interaction. In sect. 3, we derive the CS equation using this regularization method. In sect. 4, the solution of this equation is discussed and in 5 its asymptotic form. In sect. 6, we compare the CS equation obtained here with the usual derivation [7,8].

2. Some basic results of the n-dimensional regularization method

We will give here a formulation of the results of the n-dimensional regularization method suitable for the derivation of the CS equation.

Consider as an example the Lagrangian:

$$\mathcal{L}= \tfrac{1}{2}(\partial_\mu \Phi_u)(\partial^\mu \Phi_u) - \tfrac{1}{2} m_u^2 \Phi_u^2 - \frac{g_u}{4!} \Phi_u^4 , \tag{1}$$

where Φ_u, m_u, g_u are the unrenormalized field, mass and coupling constant.

This Lagrangian gives regularized Green functions when considered to be defined in n dimensions with $n \neq 4$ [10]. When $n \to 4$, the Green functions have poles of the type $(n-4)^{-k}$; as usual then, by redefining the parameters m_u, g_u and rescaling the fields, we introduce counter terms that cancel these poles.

There is always a considerable amount of freedom in choosing these counter terms; but each choice gives well-defined and finite Green functions. The subtraction procedure that is used in n-dimensional regularization is "to subtract only the poles" [10]: this guarantees that the renormalized theory will have the same symmetry properties as are formally present in the Lagrangian. Excluded are symmetries that involve the number of dimensions n, like properties connected with γ_5 invariance and dilations. In fact, the CS equation can be seen as the correct implementation of a dilation.

Theories which contain a γ_5 must partly be treated with another regularization method, as e.g. described by Bardeen [13].

The subtraction procedure in more detail, goes as follows (note that topologically, it is the same as in the BPH renormalization method). One formally starts with the Lagrangian

$$\mathcal{L}' = \tfrac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \tfrac{1}{2} m^2 \Phi^2 - \frac{g}{4!} \Phi^4 , \tag{2}$$

and calculates the two- and four-point Green functions in n dimensions.

At each order of perturbation theory, one subtracts the "poles" at $n = 4$, by introducing these as counter terms that are added to $\mathcal{L}'$. Higher order contributions are calculated each time by introducing the counter terms obtained in lower order. One generates then a new Lagrangian, which one identifies with $\mathcal{L}$.

The result is then as follows (introducing $\epsilon \equiv n - 4$):

$$\mathcal{L} = \tfrac{1}{2} C(m, g, \epsilon)(\partial_\mu \Phi)(\partial^\mu \Phi) - \tfrac{1}{2} m^2 B(m, g, \epsilon) \Phi^2 - \frac{g}{4!} A(m, g, \epsilon) \Phi^4 , \tag{3}$$

where e.g. $A(m, g, \epsilon)$ is of the form:

$$\begin{aligned} A(m, g, \epsilon) = 1 &+ A_{11}(m) \frac{g}{\epsilon} + A_{12}(m) \frac{g^2}{\epsilon^2} + \ldots \\ &+ A_{22}(m) \frac{g^2}{\epsilon^2} + A_{23}(m) \frac{g^3}{\epsilon^2} + \ldots \\ &+ \ldots . \end{aligned} \tag{4}$$

The result of multiplicative renormalization, is then, that any Green function, when calculated with (3) and for $\epsilon \neq 0$, wil have a finite limit when $\epsilon \to 0$, to any given order in g.

There is however a peculiarity with this regularization method; it is due to the fact that when (2) is considered in n dimensions, the fields and coupling constants change dimension, e.g.

$$\begin{aligned} &[\Phi] \sim M^{\frac{1}{2}(n-2)} , \\ &[g] \sim M^{4-n} , \\ &(m \text{ clearly keeps the dimension of mass}). \end{aligned} \tag{5}$$

This change of dimension does not influence the counter terms (4), since the coefficients of the powers ϵ^{-k} are residues and therefore calculated for $n = 4$. It implies then, that for dimensional reasons, the functions $A(m, g, \epsilon)$, $B(m, g, \epsilon)$ and $C(m, g, \epsilon)$ must be independent of m.

The same is not true for the renormalized Green fucntions: these get of course contributions that are ϵ dependent (i.e. contributions away from $\epsilon = 0$) and will therefore not have the "correct" dimensionality. To avoid this, one introduces into the theory an arbitrary mass scale M, such that all parameters, even when $n \neq 4$, will keep their physical dimension. Using (5), the new and more useful parametrization will then become:

$$\begin{aligned} \mathcal{L} = \tfrac{1}{2} C(g_R, \epsilon) M^{n-4} (\partial_\mu \Phi_R)(\partial^\mu \Phi_R) &- \tfrac{1}{2} M^2 m_R^2 B(g_R, \epsilon) M^{n-4} \Phi_R^2 \\ &- M^{4-n} \frac{g_R}{4!} A(g_R, \epsilon) M^{2(n-4)} \Phi_R^4 . \end{aligned} \tag{6}$$

We also made here m_R dimensionless: this is strictly speaking not necessary, but it gives some formulas a simpler outlook. Another way of stating this result is that in calculating the renormalized Green functions from (3), a unity of mass had been chosen equal to one.

The mass parameter M plays a basic role in our derivation of the CS equation, because it really scales the momentum dependence of renormalized Green functions.

By comparing the two expressions for $\mathcal{L}$, (1) and (6), one gets the usual identification:

$$\Phi_u = \sqrt{Z_3}\,\Phi_R \, ,$$

with

$$Z_3 = M^{n-4} C(g_R, \epsilon)$$

$$= M^{n-4}\left(1 + \frac{c_1(g_R)}{\epsilon} + \frac{c_2(g_R)}{\epsilon^2} + \dots\right), \tag{7a}$$

$$m_u = M m_R \left[\frac{B(g_R, \epsilon)}{C(g_R, \epsilon)}\right]^{\frac{1}{2}}$$

$$= M m_R \left(1 + \frac{b_1(g_R)}{\epsilon} + \frac{b_2(g_R)}{\epsilon^2} + \dots\right), \tag{7b}$$

$$g_u = M^{4-n} g_R \frac{A(g_R, \epsilon)}{C^2(g_R, \epsilon)}$$

$$= M^{4-n}\left(g_R + \frac{a_1(g_R)}{\epsilon} + \frac{a_2(g_R)}{\epsilon^2} + \dots\right). \tag{7c}$$

The procedure sketched for the ϕ^4 interaction can be taken over to any renormalizable theory even when gauge dependent terms are present.

In some cases, eq. (7a) will have to be modified: some fields are not multiplicatively renormalizable as is e.g. the case for the electromagnetic potential $A^\mu(x)$. Only the transverse component satisfies (7a). The renormalization of the other components depends on the chosen gauge. This is well-known and applies to any regularization method.

3. Derivation of the Callan–Symanzik equation

The CS equation has been developed to study the effect of the scale transformation $p_i \to \lambda p_i$ on Green functions.

For this one makes use of the fact that a renormalizable field theory is characterized by a few parameters with a dimension, whose role in the theory is *a priori* known. In the example considered, the only parameters of that type are m and M. For convenience, we can always reduce this number to one independent mass parameter with a dimension, which we choose to be M. Note that the parameter M, as a consequence of the renormalization procedure, is always present, even if one starts with a theory without a mass.

Consider now the renormalized Green functions $\Gamma_{\rm R}^{(k)}(p_i, M, m_{\rm R}, g_{\rm R})$ of the example considered. The $\Gamma_{\rm R}^{(k)}$ stands more precisely for the one-particle irreducible Green function of kth order.

For dimensional reasons, we get that

$$\Gamma_{\rm R}^{(k)}(\lambda p_i, \lambda M, m_{\rm R}, g_{\rm R}) = \lambda^{d_k}\, \Gamma_{\rm R}^{(k)}(p_i, M, m_{\rm R}, g_{\rm R}) \ , \tag{8}$$

where d_k is the dimension (in the usual sense) of $\Gamma_{\rm R}^{(k)}$. One can easily verify that $d_k = 4 - k$.

An immediate consequence of (8) is:

$$\lambda \frac{\partial}{\partial\lambda} \Gamma_{\rm R}^{(k)}(\lambda p_i, \lambda M, m_{\rm R}, g_{\rm R}) = d_k\, \Gamma_{\rm R}^{(k)}(\lambda p_i, \lambda M, m_{\rm R}, g_{\rm R}) \ , \tag{9}$$

or

$$\left(\sum_i p_i \frac{\partial}{\partial p_i} + M\frac{\partial}{\partial M}\right) \Gamma_{\rm R}^{(k)}(p_i, M, m_{\rm R}, g_{\rm R}) = d_k \Gamma_{\rm R}^{(k)}(p_i, M, m_{\rm R}, g_{\rm R}) \ , \tag{10}$$

which implies that the effect of a scale transformation is known, once we know the dependence on M. But this last dependence is in principle simply to get from the parametrization of the renormalized theory, expressed by (7).

For a given m_u, g_u and the given parametrization [eq. (7)], a change in M implies also a change in $g_{\rm R}$, $m_{\rm R}$ and Z_3. Then we use the fact that the parameter M has only been introduced at the level of the renormalized theory; the unrenormalized Green functions seen as functions of m_u and g_u are independent of M. Therefore we get:

$$M\frac{\partial}{\partial M} \Gamma_u^{(k)}(p_i, m_u, g_u, \epsilon)|_{m_u, g_u, \epsilon} = 0 \ . \tag{11}$$

Note that we could have chosen also $g_{\rm R}$ or $m_{\rm R}$ as the independent variable; e.g. for fixed m_u, g_u an arbitrary change in $g_{\rm R}$ would then imply a well determined change in M and $m_{\rm R}$. The final result would not change. It looks however more transparent to choose M as independent variable.

The point to note is that we have in the renormalized version three parameters $(M, m_{\rm R}, g_{\rm R})$, while the unrenormalized one involves only two parameters (m_u, g_u).

But,

$$\Gamma_u^{(k)}(m_u, g_u, \epsilon)$$
$$= [Z_3(M, g_u, m_u, \epsilon)]^{-\frac{1}{2}}\, \Gamma_{\rm R}^{(k)}(M, m_{\rm R}(M, g_u, m_u, \epsilon), g_{\rm R}(M, g_u, m_u, \epsilon)) \ . \tag{12}$$

(11) and (12) give:

$$\left[M\frac{\partial}{\partial M}+M\frac{\partial g_R}{\partial M}\frac{\partial}{\partial g_R}+M\frac{\partial m_R}{\partial M}\frac{\partial}{\partial m_R}-\frac{k}{2}M\frac{\partial \ln Z_3}{\partial M}\right]\Gamma_R^{(k)}(p_i,M,m_R,g_R)=0\ . \tag{13}$$

or

$$\left[\left(M\frac{\partial}{\partial M}+\beta(g_R)\frac{\partial}{\partial g_R}+\delta(g_R)m_R\frac{\partial}{\partial m_R}-\tfrac{1}{2}k\gamma(g_R)\right)\right]\Gamma_R^{(k)}(p_i,M,m_R,g_R)=0. \tag{14}$$

where

$$\beta(g_R)=\lim_{\epsilon\to 0}\left(M\frac{\partial g_R}{\partial M}\right)_{m_u,g_u,\epsilon}, \tag{15}$$

$$\delta(g_R)=\lim_{\epsilon\to 0}\left(\frac{M}{m_R}\frac{\partial m_R}{\partial M}\right)_{m_u,g_u,\epsilon}. \tag{16}$$

$$\gamma(g_R)=\lim_{\epsilon\to 0}\left(M\frac{\partial \ln Z_3}{\partial M}\right)_{m_u,g_u,\epsilon}. \tag{17}$$

The functions β, γ and δ must for dimensional reasons be independent of M; moreover, in the limit $\epsilon\to 0$, they clearly must be finite, since $\Gamma_R^{(k)}$ is independent of ϵ. The fact that they are independent of m_R is an immediate consequence of (7).

This last statement follows from the following manipulations (all differentiations with respect to M are meant to be for fixed g_u, m_u, ϵ):

$$\left(\frac{\partial g_u}{\partial M}\right)_{g_u,m_u,\epsilon}=0=-\epsilon g_u+\left(M\frac{\partial g_R}{\partial M}\right)\frac{\partial g_u}{\partial g_R}\ ,$$

or

$$\left(\epsilon g_R+\sum_{n=0}^{\infty}\frac{a_{n+1}(g_R)}{\epsilon^n}\right)=(\beta+O(\epsilon))\left(1+\sum_{n=1}^{\infty}\frac{a_{1,g_R}(g_R)}{\epsilon^n}\right).$$

This equation requires

$$\left(M\frac{\partial g_R}{\partial M}\right)_{g_u,m_u,\epsilon}=\beta+\epsilon g_R\ , \tag{19}$$

which gives then

$$\beta(g_R)=a_1(g_R)-g_R a_{1,g_R}(g_R)\ . \tag{20}$$

$$a_{n+1}-g_R a_{n+1,g_R}=\beta a_{n,g_R}\ . \tag{21}$$

Similar results can be derived form (16); one finds:

$$\delta=\ \ (1+g_R b_{1g_R}) \tag{22}$$

(note that δ does not depend on m_R),

$$g_{\mathrm{R}} b_{n+1 g_{\mathrm{R}}} + \beta b_{n, g_{\mathrm{R}}} = g_{\mathrm{R}}\, b_{1 g_{\mathrm{R}}} \,. \tag{23}$$

Eqs. (20)–(23) have also been obtained by 't Hooft [2] using a somewhat different method. They are a direct consequence of multiplicative renormalizability.

From (17), one derives directly:

$$\gamma(g_{\mathrm{R}}) = g_{\mathrm{R}} c_{1 g_{\mathrm{R}}} \,, \tag{24}$$

$$g_{\mathrm{R}} c_{n+1 g_{\mathrm{R}}} + \beta c_{n, g_{\mathrm{R}}} = c_n\, g_{\mathrm{R}}\, c_{1 g_{\mathrm{R}}} \,. \tag{25}$$

The net result of these calculations is that the theory is fully determined when the single pole terms in the counter terms have been claculated. Similar results – although not so directly obtainable – can also be derived when using other regularization methods.

4. Solution of the CS equation

In this section, we will discuss some general features of the solutions of (13). We also will use the notation g, m instead of g_{R} and m_{R}; we get then the equation:

$$\left[M \frac{\partial}{\partial M} + \beta(g) \frac{\partial}{\partial g} + \delta(g)\, m \frac{\partial}{\partial m} - \tfrac{1}{2} k\, \gamma(g)\right] \Gamma_{\mathrm{R}}^{(k)}(p_i, M, m, g) = 0 \,, \tag{26}$$

where

$$\beta(g) = a_1(g) - g\, a_{1 g}(g) \,,$$

$$\delta(g) = -(1 + g\, b_{1 g}(g)) \,,$$

$$\gamma(g) = g\, c_{1 g}(g) \,. \tag{27}$$

The solution of this equation goes as follows [9]:
We first make a change of variable:

$$\text{define } s = -\sum_i p_i^2 \,,$$

$$(\text{if } \textstyle\sum_i p_i^2 \text{ is positive, one omits the minus sign}) \tag{28}$$

and consider

$$\Gamma_{\mathrm{R}}^{(k)}(p_i, M, m, g) \to \Gamma_{\mathrm{R}}^{(k)}\left(\frac{p_i}{\sqrt{s}}, \tfrac{1}{2} \ln \frac{s}{M^2}, m, g, s\right)$$

$$= s^{\frac{1}{2}(4-k)}\, F^{(k)}\left(\frac{p_i}{\sqrt{s}}, \tfrac{1}{2} \ln \frac{s}{M^2}, m, g\right) . \tag{29}$$

The last expression follows from dimensional arguments. Introduce then the change of variable (for a given s)

$$M \to t = \tfrac{1}{2} \ln \frac{s}{M^2} \quad , \tag{30}$$

which gives:

$$-\frac{\partial}{\partial t} = M \frac{\partial}{\partial M} \quad ,$$

and (26) becomes:

$$\left[\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g} + \delta(g)\, m \frac{\partial}{\partial m} - \tfrac{1}{2} k\, \gamma(g)\right] \Gamma_{\mathrm{R}}^{(k)}(p_i/\sqrt{s},\ t, m, g, s) = 0 \, . \tag{31}$$

The solution of this equation is then obtained by, roughly speaking, writing it as a "total differential" in t. For this, one solves first:

$$\frac{\mathrm{d}g'(g, t)}{\mathrm{d}t} = \beta(g'(g, t)) \, , \tag{32}$$

with $g'(g, 0) = g$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln m'(m, t) = \delta(g'(g, t)) \quad , \tag{33}$$

with $m'(m, 0) = m$,

The solution of (33) is easy to give, using the important fact that $\delta(g)$ depends only on g:

$$m'(m, t) = m \exp \int_0^{\frac{1}{2}\ln(s/M^2)} \delta(g'(g, t))\, \mathrm{d}t \tag{34}$$

$$= m \exp \int_g^{g'(g,t)} \frac{\delta(g)}{\beta(g)}\, \mathrm{d}g \, . \tag{35}$$

The final result is then

$$\Gamma_{\mathrm{R}}^{(k)}(p_i, t, m, g, s) = \Gamma_{\mathrm{R}}^{(k)}(p_i, 0, m'(m, t), g'(g, t), s)$$

$$\times \exp\left[-\tfrac{1}{2} k \int_0^t \gamma(g(g, t))\, \mathrm{d}t\right] , \tag{36}$$

where the last factor can also be written as

$$\exp\left[-\tfrac{1}{2}k \int\limits_{g}^{g'(g,t)} \frac{\gamma(g)}{\beta(g)}\, dg\right] . \tag{37}$$

The main achievement of this solution is that the explicit t dependence of $\Gamma_R^{(k)}$ is transferred to an implicit t dependence through (32), (35) and (36), which are independent of the Green functions considered.

The most important fact of our solution is that (36) is valid for any choice of momenta p_i; we don't need to be in the deep Euclidean region.

The choice for s is not restricted to the form given in (28); any quadratic function of the momenta can be used.

These remarks will be especially relevant when we consider the asymptotic limit ($s \to \infty$) of (36).

Although we obtained the solution (36) by using arguments coming from perturbation theory, we will assume that the equation together with it's solution (36) is also valid for the summed up theory; moreover we assume that in the regions where $g'(g, t)$ gets small, we can again rely on perturbation theory.

5. Asymptotic forms of the equation

The asymptotic limit $t \to \infty$ of the solution (36) will mainly be governed by the limit of the function $g(g, t)$.

We discuss here two important cases:

(i) $g'(g, t) \to 0$ for $t \to \infty$, which gives a so-called asymptotic free theory.

Consider then (35): using perturbation theory to calculate $m'(m, t)$ in the limit $t \to \infty$, we will find:

$$m'(m, t) \to 0 \qquad \text{for } t \to \infty .$$

More explicitly, in most of these theories:

$$\delta(g) = \quad 1 + O(g^2) ,$$

$$\beta(g) = -c^2 g^3 + O(g^5) ,$$

and we then find:

$$g'(g, t) \to t^{-\frac{1}{2}}$$

$$\to \frac{1}{c[\ln(s/M^2)]^{\frac{1}{2}}} ,$$

i.e. the approach to "asymptotic freedom" is rather slow.

For $m'(t)$ we get:

$$m'(t) \to m e^{-t} t^{-\frac{1}{2}L} , \quad \text{(where } L \text{ is a constant)} ,$$

$$\to m \frac{M}{\sqrt{s}} , \quad \text{(up to logarithmic terms).}$$

Effectively, it amounts to the result that we could have started with (31) and omitting the mass term. In this case therefore, (31) is in appearence equivalent with the usual results obtained from the renormalization group.

The important distinction however, is that the asymptotic form that we obtain is valid not only in the deep-Euclidean region, but for any choice of the momenta p_i, only restricted by the condition that a quadratic function of these (which is then chosen to the s, instead of (28)) goes to infinity.

Assuming,

$$\gamma(g) = \mathrm{d}g^2 + O(g^3) ,$$

we get

$$\exp\left[-k \int_g^{g'(g,t)} \frac{\gamma(g)}{\beta(g)} \mathrm{d}g\right] \to \sim \left[\frac{1}{c \ln(s/M^2)}\right]^{kd/2c^2}$$

The final result can then easily be written down.

(ii) $g'(g, t) \to g_\infty$ for $t \to \infty$ where g_∞ can be finite (but $\neq 0$) or infinite.

We consider here only the case g_∞ finite: (35) gives then

$$m'(m, t) \to m \exp\left[\int_g^{g_\infty} \frac{\gamma(g)}{\beta(g)} \mathrm{d}g\right] .$$

Perturbation theory cannot tell us anything about this limit, we assume here that this limit is defined and finite:

$$m_\infty = \lim_{t \to \infty} m'(m, t) .$$

Note that except for the case that $m_\infty = 0$, one will not be allowed to neglect the mass term in (36). This means that one does not in general obtain the correct solution by starting with (26), omitting the mass term. Therefore the usual renormalization group equation will give in general an incorrect result.

6. Comparison with the usual formulation

The usual derivation [7,8] of the CS equation differs mainly from this method in that the former one does not distinguish between two mass parameters M and m, where m is the "mass" of the theory and M is the mass that scales the momentum dependence of the Green functions; they are chosen to be identical. It is also not clear how his differentiation could be done in the usual regularization procedure; for a possible approach, see a recent paper of Weinberg [12].

In our approach, a change in M implies a calculable change in both g and m, without changing g_u and m_u, i.e. without changing the Lagrangian; the equation that the Green functions satisfy is then homogeneous, without implying any restriction on the momenta.

In the approach of Callan and Symanzik, a change in m, implies a calculable change in g, but then also a change in m_u; the Lagrangian is then modified by an extra mass term and the corresponding equation for the Green functions becomes inhomogeneous.

To dispose of this term one has to invoke Weinberg's theorem [14] on the asymptotic behavior of Green functions, which requires that all momenta must be in the deep-Euclidean region. This gives a severe restriction on the range of applicability of the corresponding "asymptotic" CS equation. Moreover, it is not known whether this theorem, which has only been proven for a given order in perturbation theory, applies to the "summed up" theory.

In fact, sect. 5 shows that the theorem is only justified for an asymptotically free theory; again in this case perturbation theory makes sense and it is not surprising that one recovers the theorem.

Note that, even in this case of asymptotic freedom, our approach is superior, in that we need not restrict ourselves to the deep Euclidean region.

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