Thy Doan Mai Le
Physics 311 - Mechanics
Modeling 10 - Report 10
April 20^{th}, 2018

Chaos in A Double Pendulum with Length Variation

I. Revisions

- Matthew C. Sullivan was correct. There was an error in the algebraic manipulation of my Lagrangian.

- The final Lagrangian of the double pendulum system comes down to be

$$\mathcal{L} = ml_1^2\dot{\theta_1}^2 + \frac{1}{2}l_2^2\dot{\theta_2}^2 + ml_1\dot{\theta_1}l_2\dot{\theta_2}\cos(\theta_1 - \theta_2) + 2mgl_1\cos(\theta_1) + mgl_1\cos(\theta_2) \tag{1}$$

- Using the Euler-Lagrange, the equations of motion in terms of θ_1 and θ_2 are found to be respectively:

$$-mgl_1\sin(\theta_1) - ml_1l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) = 2ml_1^2\ddot{\theta}_1^2 + ml_1l_2\ddot{\theta}_2\cos(\theta_1-\theta_2) \tag{2}$$

$$-mgl_2\sin(\theta_2) + ml_1\dot{\theta}_1^2l_2\sin(\theta_1-\theta_2) = ml_2^2\ddot{\theta}_2^2 + ml_1\ddot{\theta}_1l_2\cos(\theta_1-\theta_2) \tag{3}$$

- Dividing both equations by ml_1l_2, we have the following equations of motion for θ_1 and θ_2 respectively:

$$-2\frac{g}{l_2}\sin(\theta_1) - \dot{\theta}_2^2\sin(\theta_1-\theta_2) = 2\frac{l_1}{l_2}\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1-\theta_2) \tag{4}$$

$$-\frac{g}{l_1}\sin(\theta_2) + \dot{\theta}_1^2\sin(\theta_1-\theta_2) = \frac{l_2}{l_1}\ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_1-\theta_2) \tag{5}$$

- Let C = $\cos(\theta_1-\theta_2)$, S = $\sin(\theta_1-\theta_2)$, $\omega_1 = \sqrt{\frac{g}{l_1}}$, $\omega_2 = \sqrt{\frac{g}{l_2}}$, Q = $\frac{l_1}{l_2}$, we can rewrite the equations of motion for θ_1 and θ_2 respectively as follows:

$$-2\omega_2^2\sin(\theta_1) - S\dot{\theta}_2^2 = 2Q\ddot{\theta}_1 + C\ddot{\theta}_2 \tag{6}$$

$$-\omega_1^2\sin(\theta_2) + \dot{\theta}_1^2S = \frac{1}{Q}\ddot{\theta}_2 + C\ddot{\theta}_1 \tag{7}$$

- Using substitution, this system of coupled second-order differential equations can be de-coupled into two equations of motions as follows:

$$\ddot{\theta}_1 = \Big(-\frac{2}{Q}\omega_2^2\sin(\theta_1) - \frac{S}{Q}\dot{\theta}_2^2 + C\omega_1^2\sin(\theta_2) - CS\dot{\theta}_1^2\Big)/(2-C^2) \tag{8}$$

$$\ddot{\theta}_2 = \Big(-2Q\omega_1^2\sin(\theta_2) + 2QS\dot{\theta}_1^2 + 2C\omega_2^2\sin(\theta_1) + CS\dot{\theta}_2^2\Big)/(2-C^2) \tag{9}$$

- Let $y_1 = \theta_1$, $y_2 = \dot{\theta}_1$, $y_3 = \theta_2$, $y_4 = \dot{\theta}_2$ and $\ddot{\theta}_1 = \dot{y_2}$ and $\ddot{\theta}_2 = \dot{y_4}$, we have the following system of first-order differential equations that can be numerically integrated using Matlab.

$$\dot{y_2} = \Big(-\frac{2}{Q}\omega_2^2\sin(y_1) - \frac{S}{Q}y_4^2 + C\omega_1^2\sin(y_3) - CSy_2^2\Big)/(2-C^2) \tag{10}$$

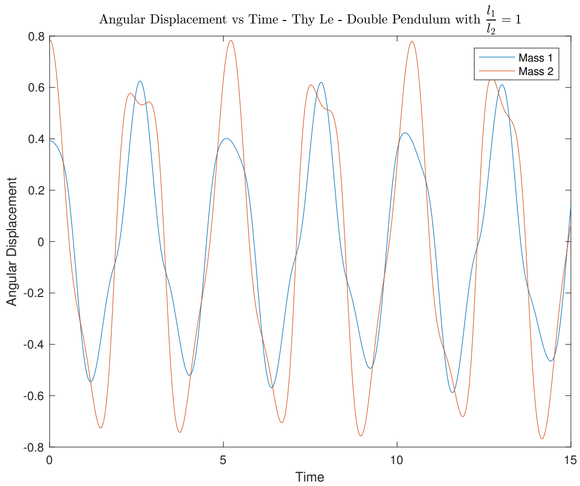
$$\ddot{y_4} = \Big(-2Q\omega_1^2\sin(y_3) + 2QSy_2^2 + 2C\omega_2^2\sin(y_1) + CSy_4^2\Big)/(2-C^2) \tag{11}$$

II. Investigation Procedure and Conclusion

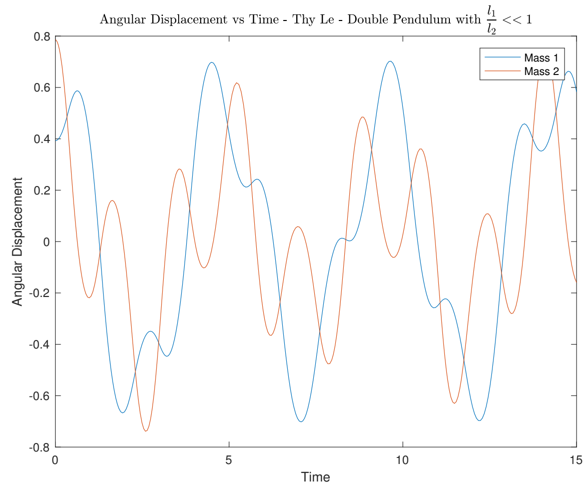
In order to investigate and answer the goal of this project, which was whether the double pendulum system exhibited chaotic behavior under different length ratios, the system was analyzed under three length ratios, specifically: $\frac{l_1}{l_2} = 1$; $\frac{l_1}{l_2} << 1$; $\frac{l_1}{l_2} >> 2$. The initial angles for mass 1 and mass 2 were set to be $\frac{\pi}{8}$ and $\frac{\pi}{4}$ respectively and both masses were given zero initial velocity. The small initial angle for mass 1 was primarily due to ODE45's limited abilities in numerical solution with any angle that was larger than $\frac{\pi}{8}$.

The behaviors shown on time plots, phase space plots and the Poincare maps of both masses exhibited chaotic behavior in all length ratios.

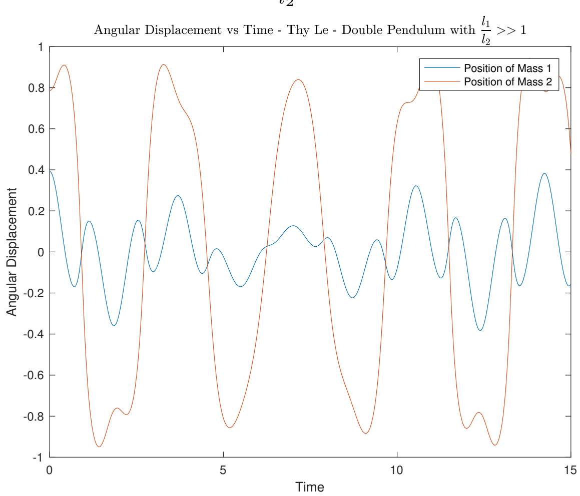
III. Time Plots



(a) Figure 3.1 Angular Position v. Time of Mass 1 & Mass 2 with Length Ratio $\frac{l_1}{l_2} = 1$

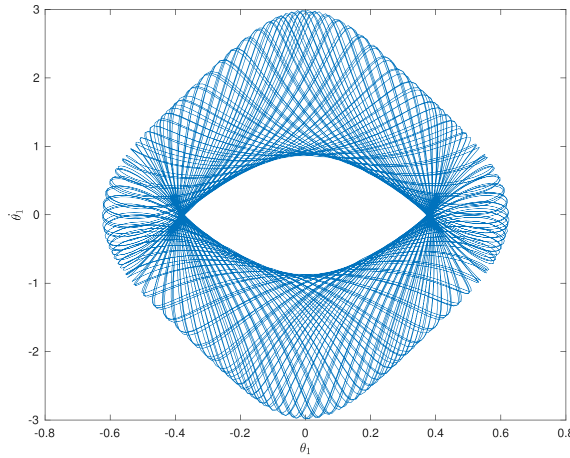


(b) Figure 3.2 Angular Position v. Time of Mass 1 & Mass 2 with Length Ratio $\frac{l_1}{l_2} << 1$

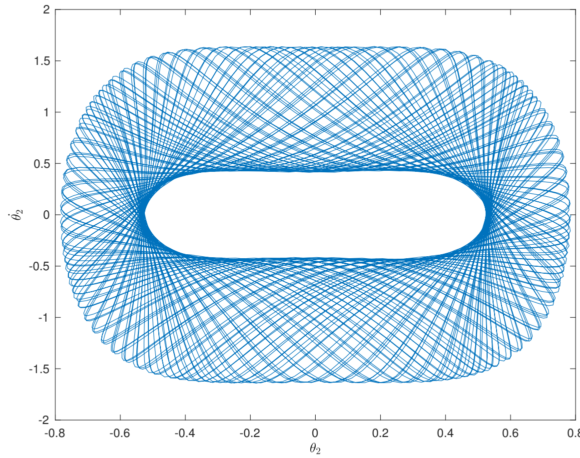


(c) Figure 3.3Angular Position v. Time of Mass 1 & Mass 2 with Length Ratio $\frac{l_1}{l_2} >> 1$

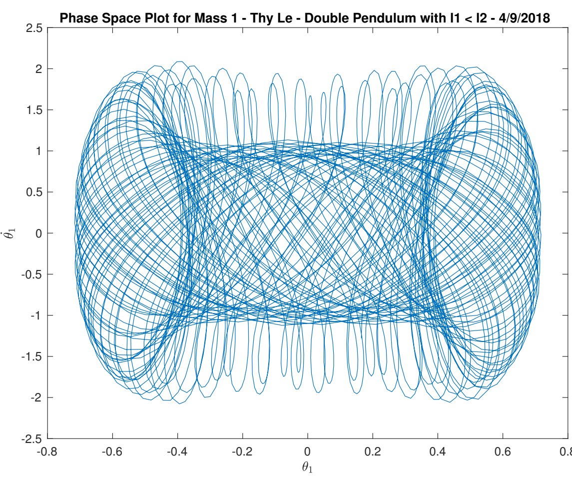
IV. Phase Space Plots



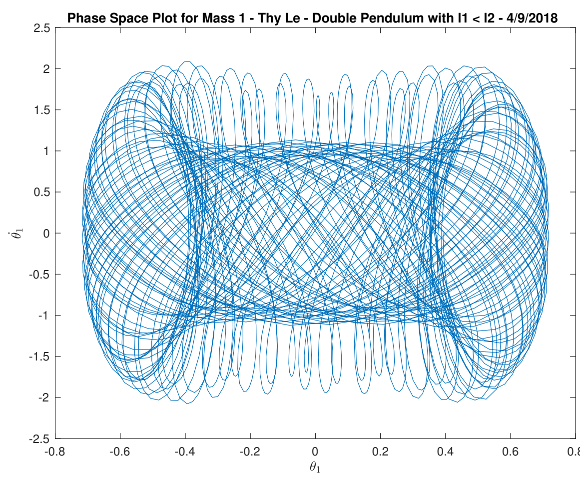
(a) Figure 4.1 Phase Space Plot of Mass 1 for Length Ratio $\frac{l_1}{l_2} = 1$



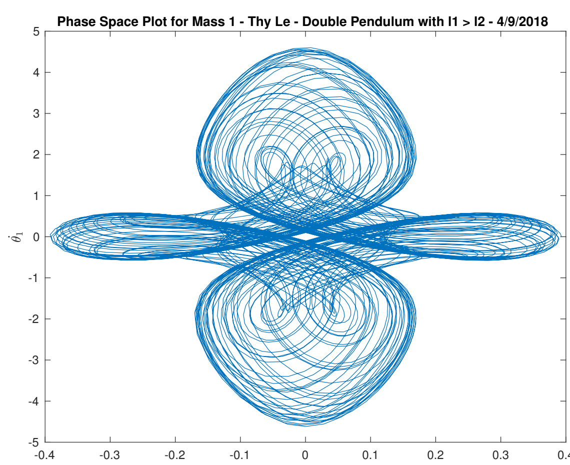
(b) Figure 4.2 Phase Space Plot of Mass 2 for Length Ratio $\frac{l_1}{l_2}$



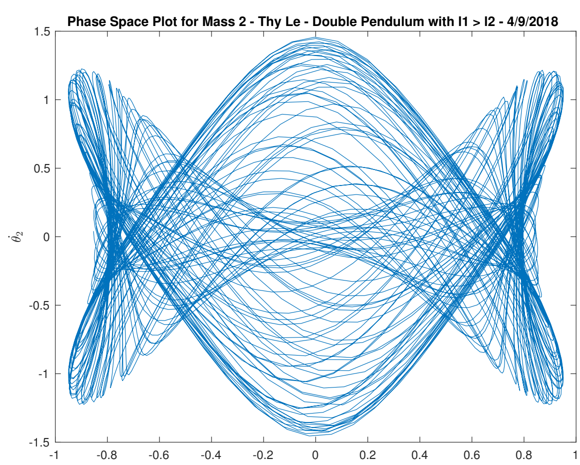
(c) Figure 4.3 Phase Space Plot of mass 1 for Length Ratio $\frac{l_1}{l_2} \ll 1$



(d) Figure 4.4 Phase Space Plot of Mass 2 for Length Ratio $\frac{l_1}{l_2} \ll 1$

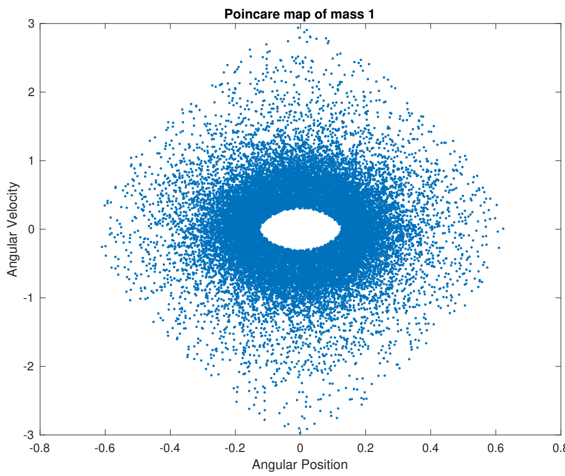


(e) Figure 4.5 Phase Space Plot of Mass 1 for Length Ratio $\frac{l_1}{l_2} \gg 1$

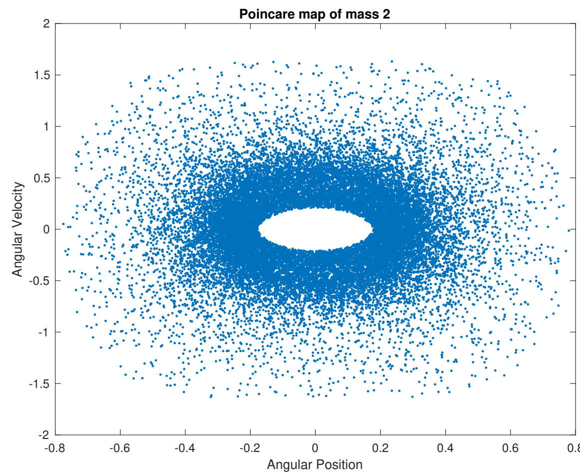


(f) Figure 4.6 Phase Space Plot of Mass 2 for Length Ratio $\frac{l_1}{l_2} \gg 1$

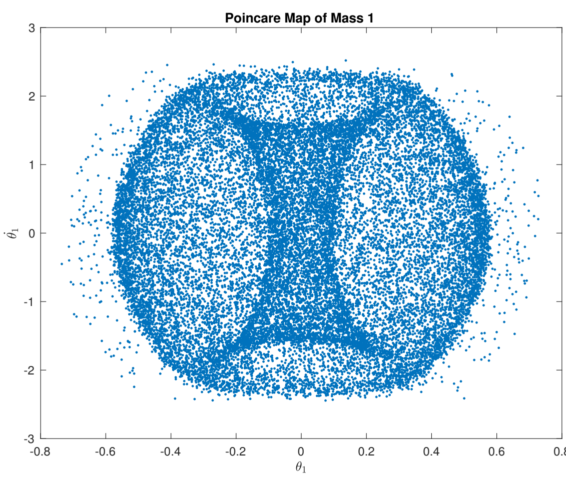
V. Poincare Maps



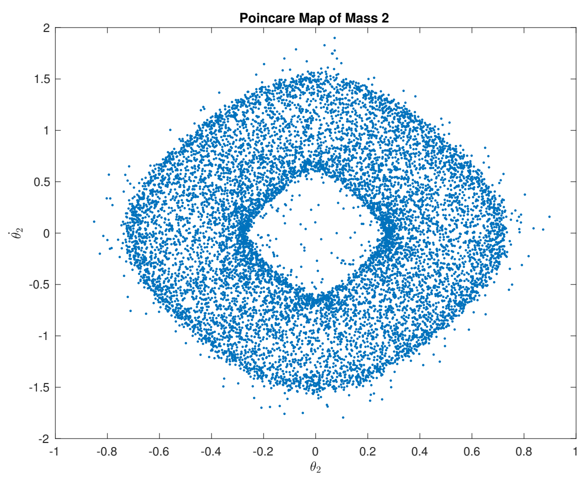
(a) Figure 5.1 Poincare Map of Mass 1 for Length Ratio $\frac{l_1}{l_2} = 1$



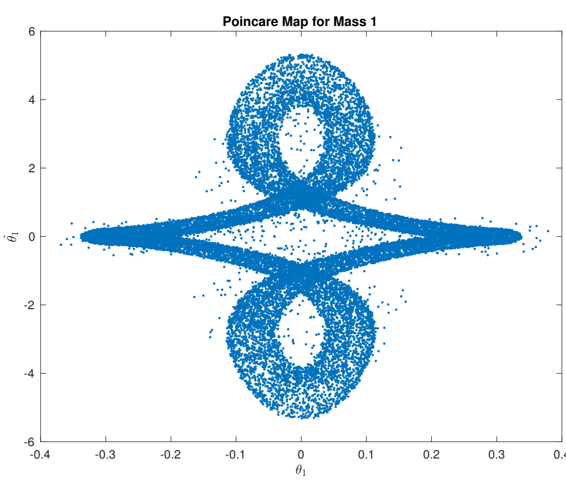
(b) Figure 5.2 Poincare Map of Mass 2 for Length Ratio $\frac{l_1}{l_2} = 1$



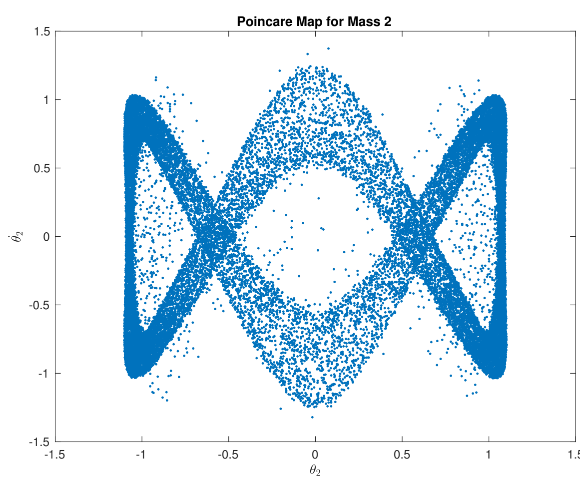
(c) Figure 5.3 Poincare Map of Mass 1 for Length Ratio $\frac{l_1}{l_2} \ll 1$



(d) Figure 5.4 Poincare Map of Mass 2 for Length Ratio $\frac{l_1}{l_2} \ll 1$

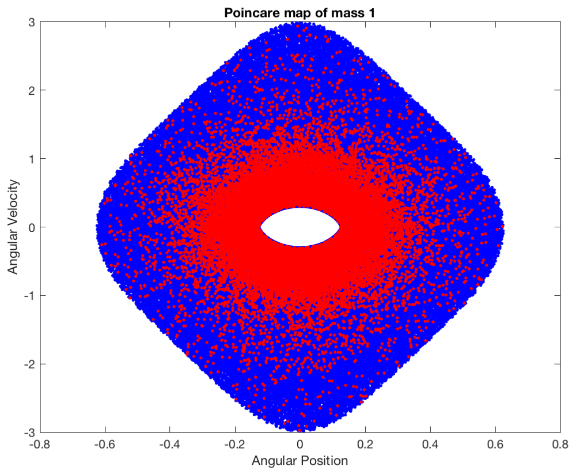


(e) Figure 5.5 Poincare Map of Mass 1 for Length Ratio $\frac{l_1}{l_2} \gg 1$

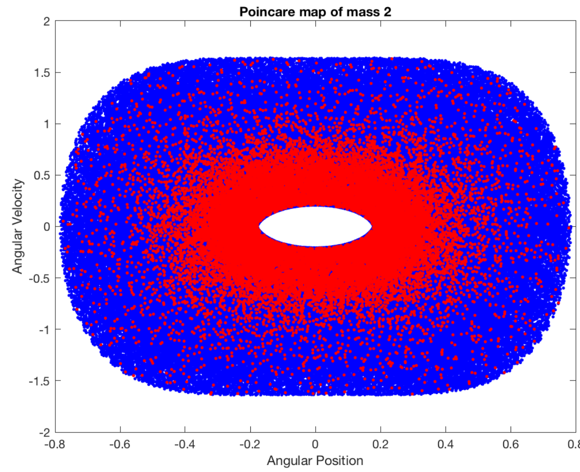


(f) Figure 5.6 Poincare Map of Mass 2 for Length Ratio $\frac{l_1}{l_2} \gg 1$

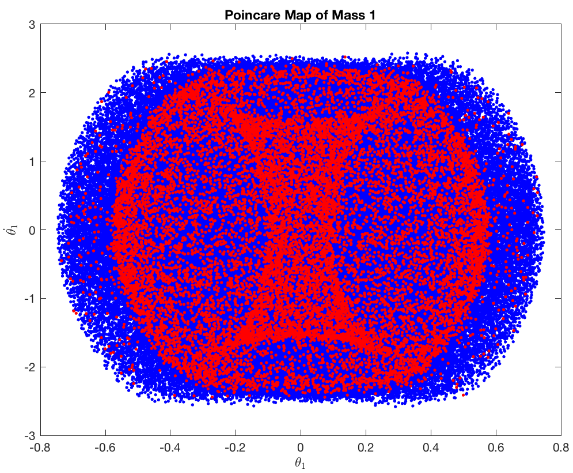
VI. Final Plots of Poincare sections with Phase Space



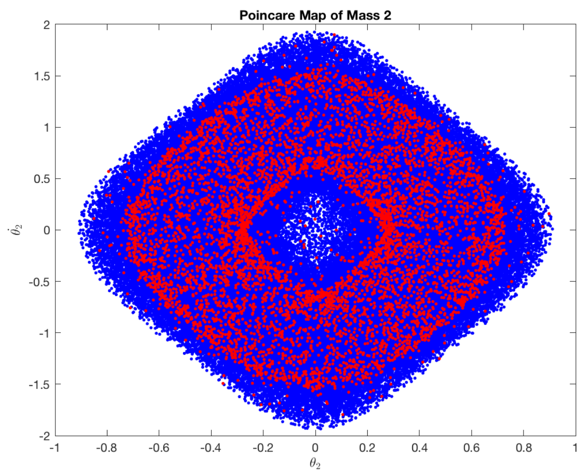
(a) Figure 6.1 Final Plot of Mass 1 for Length Ratio $\frac{l_1}{l_2} = 1$



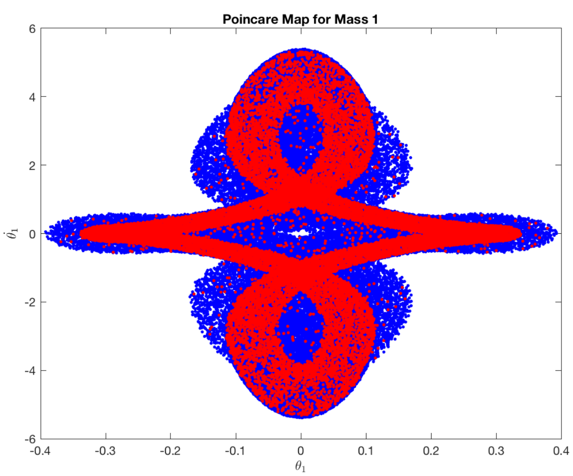
(b) Figure 6.2 Final Plot of Mass 2 for Length Ratio $\frac{l_1}{l_2} = 1$



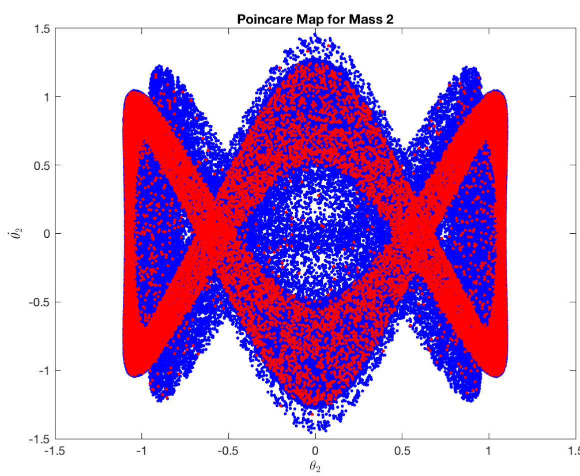
(c) Figure 6.3 Final Plot of Mass 1 for Length Ratio $\frac{l_1}{l_2} \ll 1$



(d) Figure 6.4 Final Plot of Mass 2 for Length Ratio $\frac{l_1}{l_2} \ll 1$



(e) Figure 6.5 Final Plot of Mass 1 for Length Ratio $\frac{l_1}{l_2} \gg 1$



(f) Figure 6.6 Final Plot of Mass 2 for Length Ratio $\frac{l_1}{l_2} \gg 1$

VII. Conclusion

Under the initial conditions previously determined, the double pendulum system demonstrates chaotic behavior in all cases of length variations: $\frac{l_1}{l_2} = 1$, $\frac{l_1}{l_2} \ll 1$ and $\frac{l_1}{l_2} \gg 1$. This determination came from the shape of the Final plots which are combinations of the Phase Space Plot and the Poincare section of each individual mass. The Phase Space plot showed that the masses traverse a larger set of data points of $(x, \dot{x})$ while the Poincare section showed that the masses do not necessarily travel to all of the data points from the Phase Space plot. A harmonic, nonchaotic system would show that the system comes back to a certain point at the end of each period and a system that deviates from this routine is to be considered chaotic. he Poincare section showed that the position and velocity of the masses are always at a different point after each period, which is a hallmark for chaotic behavior. Thus, the project comes to the conclusion that the double pendulum with the length variations and initial conditions listed in this investigation exhibits chaotic behavior throughout all length variations.