# Theoretical modeling of the collective tunneling of a Wigner necklace

Dominik szombathy

November 2, 2022

**Abstract**

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# A Mathematics



In this appendix I will write down some useful mathematical expressions, proofs or identities that are used in the discussion above

## A.1 Gaussian Integrals

## A.2 Cauchy's Resiude Theorem

## A.3 Besel Problem

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{A.3.1}$$

**Fourier Way** Let's consider first an arbitrary function $f(x)$. One could wirte the fourier series of this function as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi l}{L}\right) + b_n \sin\left(\frac{n\pi l}{L}\right) \right\}, \tag{A.3.2}$$

where $l \in [l_i, l_f]$ and $L = l_f - l_i$. Let $f(x) = x^2$ and $l_i = -l_f = -\pi$ so that $L = 2\pi$. The coefficients can be calculated as

$$a_0 = \frac{2}{L} \int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi} \int_0^{\pi} dx\, x^2 = \frac{2}{3}\pi^2 \tag{A.3.3}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_{-\pi}^{\pi} dx\, f(x) \cos\left(\frac{n\pi 2x}{L}\right) \\ &= \frac{2}{\pi} \left( \frac{x^2 \sin(nx)}{n} \right)\Bigg|_0^{\pi} - \frac{2}{\pi} \int_0^{\pi} dx\, \frac{2x \sin(nx)}{n} \\ &= \frac{4(-1)^n}{n^2}. \end{aligned} \tag{A.3.4}$$

Considering the problem at hand, consider the function as $x^2$, so $b_n = 0$ because $x^2$ is even. The Fourier series then takes the form

$$\begin{aligned} x^2 &= \frac{2\pi^2}{3 \cdot 2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos\left(\frac{2n\pi x}{L}\right) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} (-1)^n. \end{aligned} \tag{A.3.5}$$

Choose $x^2$ to be equal to 0,

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}. \tag{A.3.6}$$

Then subtract the $a_0$ term and devide by 4

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{A.3.7}$$

**Complex Integral Way** Consider the following integral:

$$I = \int_C \frac{1}{z^2} f(z) dz. \tag{A.3.8}$$

Let $f(z)$ be the Fermi function:

$$f(z) = \frac{1}{1 + e^{i\pi z}}, \tag{A.3.9}$$

then we have several first order poles at $z = \pm(2n+1) = p_n$ and a second order pole at $z = 0$.

The residue from the 1st order poles:

$$Res(f(z), p_n) = \lim_{z \to p_n} (z - p_n) f(z) = \frac{i}{\pi p_n^2}, \tag{A.3.10}$$

and from the 2nd order pole:

$$Res(f(z), 0) = \lim_{z \to 0} \frac{d}{dz} (z^2 f(z)) = \frac{-\pi i}{4}. \tag{A.3.11}$$

Writing now Cauchy's residue theorem A.2

$$\int_C g(z) dz = 2\pi i \sum Res(g(z)) \tag{A.3.12}$$

$$\int_C f(z) dz = 2\pi i \left( \frac{-\pi i}{4} \right) + 2\pi i \sum_{n=0}^{\infty} \frac{2i}{\pi (2n+1)^2} = 0 \tag{A.3.13}$$

$$\frac{\pi i}{4} = \sum_{n=0}^{\infty} \frac{2i}{\pi(2n+1)^2} \tag{A.3.14}$$

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \tag{A.3.15}$$

$$= \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{A.3.16}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{A.3.17}$$

# B Algorithms and Methods

## B.1 Monte Carlo and Simulated Annealing

## B.2 Minimalization by Nelder-Mead algorithm

# C Links, Data and other stuff

# D Research Diary

[1], [2]

## References

[1] Anna Anderegg, 2022, Example 1.

[2] Brenda Bradshaw, 2021, Example 2.