

Theoretical modeling of the collective tunneling of a Wigner necklace

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Abstract

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1 Wigner Crystals

2 Path Integral Formalism

2.1 Free Particle

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2.3 One particle tunneling

2.3.1 One Particle Hamiltonian

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2.3.3 Tunneling calculation in quartic potentials

3 Experimental Considerations

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5 Polarization

5.1 Classical Polarization Calculation

5.2 Polarization using ED

6 Open Questions

A Mathematics

In this appendix I will write down some useful mathematical expressions, proofs or identities that are used in the discussion above

A.1 Gaussian Integrals

A.2 Cauchy's Residue Theorem

A.3 Besel Problem

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad (\text{A.3.1})$$

Fourier Way Let's consider first an arbitrary function $f(x)$. One could write the fourier series of this function as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi l}{L}\right) + b_n \sin\left(\frac{n\pi l}{L}\right) \right\}, \quad (\text{A.3.2})$$

where $l \in [l_i, l_f]$ and $L = l_f - l_i$. Let $f(x) = x^2$ and $l_i = -l_f = -\pi$ so that $L = 2\pi$. The coefficients can be calculated as

$$a_0 = \frac{2}{L} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} dx x^2 = \frac{2}{3} \pi^2 \quad (\text{A.3.3})$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_{-\pi}^{\pi} dx f(x) \cos\left(\frac{n\pi 2x}{L}\right) \\ &= \frac{2}{\pi} \left(\frac{x^2 \sin(nx)}{n} \right) \Big|_0^{\pi} - \frac{2}{\pi} \int_0^{\pi} dx \frac{2x \sin(nx)}{n} \\ &= \frac{4(-1)^n}{n^2}. \end{aligned} \quad (\text{A.3.4})$$

Considering the problem at hand, consider the function as x^2 , so $b_n = 0$ because x^2 is even. The Fourier series then takes the form

$$\begin{aligned} x^2 &= \frac{2\pi^2}{3 \cdot 2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos\left(\frac{2n\pi x}{L}\right) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} (-1)^n. \end{aligned} \quad (\text{A.3.5})$$

Choose x^2 to be equal to 0,

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}. \quad (\text{A.3.6})$$

Then subtract the a_0 term and divide by 4

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad (\text{A.3.7})$$

Complex Integral Way Consider the following integral:

$$I = \int_C \frac{1}{z^2} f(z) dz. \quad (\text{A.3.8})$$

Let $f(z)$ be the Fermi function:

$$f(z) = \frac{1}{1 + e^{i\pi z}}, \quad (\text{A.3.9})$$

then we have several first order poles at $z = \pm(2n + 1) = p_n$ and a second order pole at $z = 0$.

The residue from the 1st order poles:

$$\text{Res}(f(z), p_n) = \lim_{z \rightarrow p_n} (z - p_n) f(z) = \frac{i}{\pi p_n^2}, \quad (\text{A.3.10})$$

and from the 2nd order pole:

$$\text{Res}(f(z), 0) = \lim_{z \rightarrow 0} \frac{d}{dz} (z^2 f(z)) = \frac{-\pi i}{4}. \quad (\text{A.3.11})$$

Writing now Cauchy's residue theorem A.2

$$\int_C g(z) dz = 2\pi i \sum \text{Res}(g(z)) \quad (\text{A.3.12})$$

$$\int_C f(z) dz = 2\pi i \left(\frac{-\pi i}{4} \right) + 2\pi i \sum_{n=0}^{\infty} \frac{2i}{\pi(2n+1)^2} = 0 \quad (\text{A.3.13})$$

$$\frac{\pi i}{4} = \sum_{n=0}^{\infty} \frac{2i}{\pi(2n+1)^2} \quad (\text{A.3.14})$$

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \quad (\text{A.3.15})$$

$$= \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} \quad (\text{A.3.16})$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} \quad (\text{A.3.17})$$

B Algorithms and Methods

B.1 Monte Carlo and Simulated Annealing

B.2 Minimalization by Nelder-Mead algorithm

C Links, Data and other stuff

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References

- [1] Anna Andereg, 2022, Example 1.
- [2] Brenda Bradshaw, 2021, Example 2.