Fixed Point Theorems and Evasiveness

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1 Introduction

In the previous paper, we have concluded that for any simplex associated with a graph property to be nonevasive it must be collapsible. It may still be the case that there is an evasive property which is also collapsible. What then does this property buy us?

We can ask the question: Are there any graph properties which are not collapsible? This is still very much an active field of research, we will focus on one theorem in particular in this paper:

> Any nontrivial monotone graph property on a graph with a prime power of vertices is evasive.

What is the general idea behind such a proof? Show that the associated simplicial complex is not collapsible!

2 Simplicial Complexes and Algebra

Before we begin to tackle decision trees we will have to make a detour and develop some tools for working with simplicial complexes. Here is a summary of what we will cover:

1. Definition of a Homology Group
2. Definition of F_p acyclic
3. Collapsible $\implies$ $F_p$acyclic

3 Collapsibility and Holes

We first should provide some intuition as to why collapsibility can (although not nearly as well as homology groups) detect holes in our structure. Imagine that the simplicial complex is a compressible fluid and that the space that is not an element of the complex is solid and rigid. Imagine taking your hand and squishing the fluid in order to reduce it to a single point. If there is a hole, it will feel like a rock in your hand. Otherwise, you can continue compressing everything down to a single point.

Of course, this is simply an analogy. Let's actually prove it.

3.1 Boundary Maps

Suppose we want to get the boundary of a simplicial complex. For example if we have a filled triangle Δ we may want some operator ∂ that $\partial\Delta$ gives us just the edges. In the case, a filled triangle should become an empty triangle.

We define the **Boundary Map** of a simplicial complex to be precisely:

$$d_n([v_0, v_1, \ldots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]$$

We now consider the $\ker(\partial)$. This ends up being the cycle space of a graph. They cycles are our tool for determining holes. But currently $\ker(\partial)$, by itself gives a few too many cycles needed to describe capture a hole. So we will define an equivalence class which does:

Example

3.2 Homology Groups

The nth homology group is, roughly speaking, a way of finding the number of n dimensional holes in some surface. It is also worth mentioning the 0th homology group corresponds to the number of connected componenets of a surface. Nevertheless, to say that it "counts" the number of holes is a bit misleading. A circle, for instance, has one 1-dimensional hole in the center and has one connected component. It is not the case that $H_0(S^1) = 1$ and $H_1(S^1) = 1$. Often one will see the following result instead:

$$H_k(S^1) = \begin{cases} \mathbb{Z}, & k = 0, 1 \\ \{0\}, & \text{otherwise} \end{cases}$$

Sometimes $\mathbb{Z}$ is replaced with $\mathbb{Z}_2$, and in our case we will be mostly concerning ourselves $\mathbb{F}_p$. Although it might not be clear where $\mathbb{Z}$ comes from, one should for now think of $\mathbb{Z}$ as the number 1, as this coincides with the informal defintion, namely, there is one connected componenet, and one 1-dimensional hole.

We define the n**th homology group** as:

$$H_n(\Delta, F_p) = \frac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})}$$

As the homology group is a definition there is no way to "prove" that the nth homology group gives us the n-dimensional holes. Nevertheless, let us consider an example to gain some for intuition for why this might be a good definition.

Example

3.3 Reduced Homology

We consider a variant of the homology just introduced known as the "Reduced Homology". The reduced homology makes working with certain spaces a bit easier. In particular, for a point the every homology group is 0 except for H_0. We want to make this nicer, so we modify the chain so that every homology group is 0.

To do this consider we have an arbitrary chain:

$$\cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

We then define the function r to be: (straight from wikipedia)

$$r\left(\sum_i n_i \sigma_i\right) = \sum_i n_i$$

And rework the chain to be:

$$\cdots \to C_2 \to C_1 \to C_0 \xrightarrow{r} F_p \to 0$$

3.3.1 Acyclic Simplicical Complexes

Intuitively speaking, a simplicial complex that is **acyclic** has no holes. More formally:

A simplicial complex is **acyclic** if the reduced homology is trivial.

4 Abstract Simplicial Complex versus the Geometric Realization

We discussed the differences between the algebraic interpretation and geometric realization of an abstract simplicial complex.

4.1 Exact Sequence

We now add an algebra tool to our disposal which will help us formally demonstrate that collapsible simplicial complex are acyclic. We introduce some notation to "visualize" the relationship between the boundary operator and homology groups. We consider any sequence of groups Y_n and a function that maps between them. We write this as:

$$\ldots \longrightarrow Y_2 \to Y_1 \to Y_0 \to 0$$

So in the case of homology groups:

$$\ldots \to H_2 \to H_1 \to H_0 \to 0$$

We can also arrange the lines vertically:

$$\begin{array}{c} \vdots \\ \downarrow \\ Y_2 \\ \downarrow \\ Y_1 \\ \downarrow \\ Y_0 \\ \downarrow \\ 0 \end{array}$$

We denote these **chain complexes** as $Y_\bullet$. Now we can introduce a useful structure know as a **short exact sequence**.

4.2 Useful tools for Homological Algebra

A **short exact sequence** is a structure of the form:

$$0 \longrightarrow X_\bullet \longrightarrow Y_\bullet \longrightarrow Z_\bullet \longrightarrow 0$$

Why is this useful? For our purposes, short exact sequences are useful because of the following lemma:

Lemma A For a short exact sequence:

$$0 \longrightarrow X_\bullet \longrightarrow Y_\bullet \longrightarrow Z_\bullet \longrightarrow 0$$

if $X_\bullet$ and $Z_\bullet$ are acyclic, then $Y_\bullet$ is acyclic.

This is a well known fact in homological algebra, that we will not prove this fact here. We will, however, use this fact to provide a reasonably rigorous proof that collapsible simplicial complexes are acyclic.

Another useful tool we will not prove:

Lemma B For a short exact sequence:

$$0 \longrightarrow X_\bullet \longrightarrow Y_\bullet \longrightarrow Z_\bullet \longrightarrow 0$$

4.2.1 Collapsible Simplicial Complexes are Acyclic

5 Fixed Points

At this point we have two major tools at our disposal:

1. All nonevasive properties are collapsible.
2. All collapsible simplexes are F_p acyclic.

5.1 Lefschetz Fixed Point Theorem

Now that we have introduced quite a bit of machine to prove a relatively intuitive fact, we can now use this machinery to prove a less obvious fact:

Theorem (Lefschetz) Every simplicial automorphism fixes some subcomplex.

To put this another way suppose we have some function $f : \Delta \to \Delta$ such that f takes any subcomplex of Δ to another subcomplex of Δ there is at least one such subcomplex δ such that:

$$f(\delta) = \delta$$

Hence δ is a "fixed point".

5.1.1 Proof

The strategy we will use to show F has a fixed point will go as follows:

1. Construct a linear transformation associated with f.

2. Compute the trace of this function two different ways.
3. Show the trace is different and arrive at a contradiciton.

We will now define the linear function we need:

Let F be the **associated chain map** with f:

$$F_n = \mathrm{sgn}(f(\Delta_n))(f(\Delta_n)$$

where Δ_n is the set of all faces in Δ of dimension n.

As F_n is a linear transformation, we can imagine stacking these transformations to get a matrix. From this we can consturct a matrix. From this matrix we can take the trace which we will denote: Tr_F. We then observe that:

$$\mathrm{Tr}_F(H_0) = 1$$
$$\mathrm{Tr}_F(H_n) = 0 \text{ if } n > 0$$

And observe that:

$$\mathrm{Tr}_F(K_n) = 0$$

Thus we see:

$$\begin{aligned}
0 &= \sum(-1)^n \mathrm{Tr}_F(K_n) && (1)\\
&= \mathrm{Tr}_F + \sum(-1)^n[\mathrm{Tr}_F(\mathrm{im}\partial) + \mathrm{Tr}_F(\ker\partial)] && (2)\\
&= \mathrm{Tr}_F - \mathrm{Tr}_F(\mathrm{im}\partial_1) && (3)\\
&= \mathrm{Tr}_F - \dots \qquad = 1 && (4)
\end{aligned}$$

So there must be a fixed point.

6 Abstract Simplicial Complex versus the Geometric Realization

We discussed the differences between the algebraic interpretation and geometric realization of an abstract simplicial complex. At this point we want make some things clear.

A **simplicial map** is a map $f : \Delta \to \Delta'$ where Δ and Δ' prime are *abstract* simplicial complexes and for any subsimplex δ:

$$f(\delta) \in \Delta'$$

We make the following observation that there is a geometric realization of f. Call it $\overline{f}$. Observe that $\overline{f}$ is a continuous map from the geometric realization of Δ (we will refer to this as $\overline{\Delta}$ to the geometric realization of Δ' ($\overline{\Delta}$).

If $\overline{f}$ maps from $\overline{\Delta}$ to $\overline{\Delta}$, it follows that we can apply Lefschetz Fixed Point theorem and notice that there is some fixed point $x \in \overline{\Delta}$ such that:

$$\overline{f}(x) = x$$

However, what f does on the simplex is much more limiting than any arbitrary conitinuous function so we can get an even greater set of fixed points!

If $x \in \overline{\Delta}$ then x is the sum of some linear combination of basis vectors. Define the **support simplex of** x Δ_x to be the collection of non-zero support vectors.

Because of the way f acts on a simplicial complex, for any the support of any fixed point:

$$f(\Delta_x) = \Delta_x$$

This intuitively says:

If x is a fixed point, then the face containing x is fixed by the permutation.

Furthermore, suppose $\overline{f}$ has some face. Then it may be the case the face was "rotated" from the original $\overline{\Delta}$, however it should be intuitive that this rotation does not affect the center point and so the center point is a fixed point for f.

6.1 Orbits

The rotations of these faces correspond to the "orbits" of f. Observing this fact we can almost conclude the following:

If P is a non-evasive monotone property that invariant under cycles it must be evasive.

Proof: Suppose otherwise, then P is non-evasive $\implies$ P is collapsible $\implies$ for any function f, f has a fixed point. But the orbit of such a function must contain the largest face which corresponds to the largest graph.

6.2 Proof of Theorem