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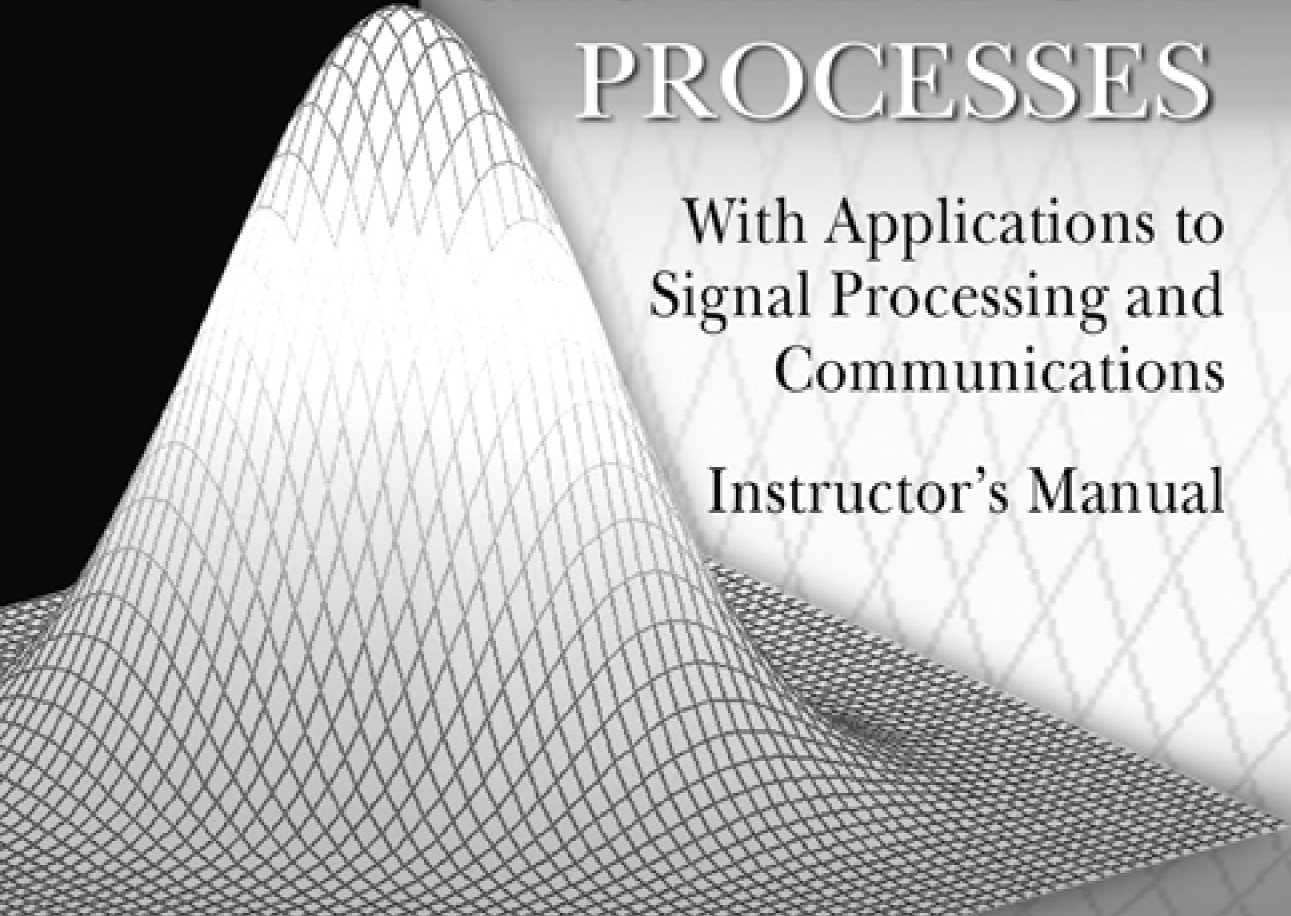
PROBABILITY and RANDOM PROCESSES

With Applications to Signal Processing and Communications

Instructor's Manual



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Probability and Random Processes

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With Applications to Signal Processing and Communications

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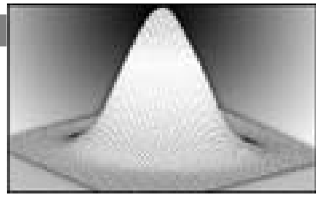

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Solutions to Chapter 2 Exercises

Problem 2.1

Given M events $A_1, A_2, \ldots, A_M$: $A_i \cap A_j = \phi \; \forall \; i \neq j$ prove that

$$Pr\left(\bigcup_{i=1}^{M} A_i\right) = \sum_{i=1}^{M} Pr\left(A_i\right).$$

We shall prove this using induction. For the case M=2

$$Pr\left(\bigcup_{i=1}^{2} A_i\right) = \sum_{i=1}^{2} Pr\left(A_i\right).$$

This we know to be true from the axioms of the probability. Let us assume that the proposition is true for M=k.

$$Pr\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} Pr\left(A_i\right)$$

We need to prove that this is true for M=k+1. Define an auxiliary event B :

$$B = \left(\bigcup_{i=1}^{k} A_i\right).$$

Then the event

$$\left(\bigcup_{i=1}^{k+1} A_i\right) = \left(B \cup A_{k+1}\right)$$

$$Pr\left(\bigcup_{i=1}^{k+1} A_i\right) = Pr\left(B \cup A_{k+1}\right).$$

Since the propostion is true for M=2 we can rewrite the above equation as

$$Pr\left(\bigcup_{i=1}^{k+1} A_i\right) = Pr\left(B\right) + Pr\left(A_{k+1}\right).$$

Since the proposition is true for M=k we can rewrite this as

$$\begin{aligned} &= \sum_{i=1}^{k} Pr\left(A_i\right) + Pr\left(A_{k+1}\right) \\ &= \sum_{i=1}^{k+1} Pr\left(A_i\right). \end{aligned}$$

Hence the propostion is true for M=k+1. And by induction the proposition is true for all M.

Problem 2.2

First, note that the sets $A \cup B$ and B can be rewritten as

$$A \cup B = \{A \cup B\} \cap \{\overline{B} \cup B\} = \{A \cap \overline{B}\} \cup B$$

$$B = B \cap \{A \cup \overline{A}\} = \{A \cap B\} \cup \{\overline{A} \cap B\}.$$

Hence, $A \cup B$ can be expressed as the union of three mutually exclusive sets.

$$A \cup B = \{\overline{B} \cap A\} \cup \{A \cap B\} \cup \{\overline{A} \cap B\}.$$

Next, rewrite $\overline{B} \cap A$ as

$$\overline{B} \cap A = \{\overline{B} \cap A\} \cup \{\overline{A} \cap A\} = A \cap \{\overline{A} \cup \overline{B}\} = A \cap \{\overline{A \cap B}\}.$$

Likewise, $\overline{A} \cap B = B \cap \{\overline{A \cap B}\}$. Therefore $A \cup B$ can be rewritten as the following union of three mutually exclusive sets:

$$A \cup B = \{A \cap (\overline{A \cap B})\} \cup \{A \cap B\} \cup \{B \cap (\overline{A \cap B})\}.$$

Hence

$$Pr\left(A \cup B\right) = Pr\left(A \cap (\overline{A \cap B})\right) + Pr\left(A \cap B\right) + Pr\left(B \cap (\overline{A \cap B})\right).$$

Next, write A as the union of the following two mutually exclusive sets

$$A = \{A \cap \{\overline{A \cap B}\}\} \cup \{A \cap B\}.$$

Hence,

$$Pr\left(A\right) = Pr\left(A \cap \{\overline{A \cap B}\}\right) + Pr\left(A \cap B\right)$$

and

$$Pr\left(A \cap \{\overline{A \cap B}\}\right) = Pr\left(A\right) - Pr\left(A \cap B\right).$$

Likewise,

$$Pr\left(B \cap \{\overline{A \cap B}\}\right) = Pr\left(B\right) - Pr\left(A \cap B\right).$$

Finally,

$$\begin{aligned}
Pr\left(A \cup B\right) &= Pr\left(A \cap (\overline{A \cap B})\right) + Pr\left(A \cap B\right) + Pr\left(B \cap (\overline{A \cap B})\right) \\
&= \left(Pr\left(A\right) - Pr\left(A \cap B\right)\right) + Pr\left(A \cap B\right) + Pr\left(B\right) - Pr\left(A \cap B\right) \\
&= Pr\left(A\right) + Pr\left(B\right) - Pr\left(A \cap B\right).
\end{aligned}$$

Problem 2.3

$$\begin{aligned}
&Pr(A \cup B \cup C) \\
&= Pr((A \cup B) \cup C) \\
&= Pr(A \cup B) + Pr(C) - Pr((A \cup B) \cap C) \\
&= Pr(A) + Pr(B) - Pr(A \cap B) + Pr(C) - Pr((A \cap C) \cup (B \cap C)) \\
&= Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) - Pr((A \cap C) \cup (B \cap C)) \\
&= Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) \\
&- (Pr((A \cap C) + Pr(B \cap C)) - Pr(A \cap C \cap B \cap C)) \\
&= Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) - Pr(A \cap C) - Pr(B \cap C) + Pr(A \cap B \cap C)
\end{aligned}$$

Problem 2.4

Since $A \subset B$ and $A \cap B = A$, $B = A \cup \{B \cap \overline{A}\}$. Since A and $B \cap \overline{A}$ are mutually exclusive, we have

$$Pr(B) = Pr(A) + Pr(B \cap \overline{A}) \ . \tag{1}$$

Hence, by (1) and considering $Pr(B \cap \overline{A}) \geq 0$, $Pr(A) \leq Pr(B)$.

Problem 2.5

$$Pr\left(\bigcup_{i=1}^{M}\right) \leq \sum_{i=1}^{M} Pr(A_i)$$

We shall prove this by induction. For $k = 1$ this reduces to

$$Pr(A_1) \leq Pr(A_1)$$

which is obviously true.For $k = 2$ we have

$$Pr(A_1 \cup A_2) = Pr(A_1) + Pr(A_2) - Pr(A_1 \cap A_2)$$

by the axioms of probability.

$$\Rightarrow Pr(A_1 \cup A_2) \leq Pr(A_1) + Pr(A_2)$$

The equality holding when the events are mutually exclusive. Assume that the propostion is true for $M = k$. Then we have

$$Pr\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} Pr(A_i)$$

We shall prove that this is true for $M = k + 1$. Let

$$B = \bigcup_{i=1}^{k} A_i$$

Then the following holds

$$Pr(B) \leq \sum_{i=1}^{k} Pr(A_i) \tag{2}$$

Since the proposition is true for $M = 2$

$$Pr(B \cup A_{k+1}) \leq Pr(B) + Pr(A_{k+1}) \tag{3}$$

Using (2) in (3) we have

$$Pr(B \cup A_{k+1}) \leq \sum_{i=1}^{k} Pr(A_i) + Pr(A_{k+1})$$

$$Pr(\bigcup_{i=1}^{k+1} A_i) \leq \sum_{i=1}^{k} Pr(A_i) + Pr(A_{k+1})$$

$$Pr(\bigcup_{i=1}^{k+1} A_i) \leq \sum_{i=1}^{k+1} Pr(A_i)$$

Thus the proposition is true for $M = k + 1$ and by the principle of induction it is true for all finite M.

Problem 2.6

Assume S is the sample space for a given experiment.
Axiom 2.1: For an event A in S,

$$P(A) = \lim_{n\to\infty} \frac{n_A}{n} \ .$$

Since $n_A \geq 0$, and $n > 0$, $P(A) \geq 0$.
Axiom 2.2: S is the sample space for the experiment. Since S must happen with each run of the experiment, $n_S = n$. Hence

$$P(S) = \lim_{n\to\infty} \frac{n_S}{n} = 1 \ .$$

Axiom 2.3a: Suppose $A \cap B = 0$. For an experiment that is run n times, assume the event $A \cup B$ occurs n' times, while A occurs n_A times and B occurs n_B times. Then we have $n' = n_A + n_B$. Hence

$$P(A \cup B) = \lim_{n\to\infty} \frac{n'}{n} = \lim_{n\to\infty} \frac{n_A + n_B}{n} = \lim_{n\to\infty} \frac{n_A}{n} + \lim_{n\to\infty} \frac{n_B}{n} = P(A) + P(B) \ .$$

Axiom 3.b: For an experiment that is run n times, assume the event A_i occurs n_{A_i} times, $i = 1, 2, \cdots$. Define event $C = A_1 \cup A_2 \cdots \cup A_i \cdots$. Since any two events are mutually exclusive, event C occurs $\sum_{i=1}^{\infty} n_{A_i}$ times. Hence,

$$P(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \frac{\sum_{i=1}^{\infty} n_{A_i}}{n} = \sum_{i=1}^{\infty} \lim_{n\to\infty} \frac{n_{A_i}}{n} = \sum_{i=1}^{\infty} P(A_i) \ .$$

Problem 2.7

Axiom 2.1:

$$Pr(A \mid B) = \frac{Pr(A,B)}{Pr(B)} \geq 0$$

since both $Pr(A,B)$ and $Pr(B)$ are greater than zero.
Axiom 2.2:

$$Pr(S \mid B) = \frac{Pr(S,B)}{Pr(B)} = \frac{Pr(B)}{Pr(B)} = 1$$

Axiom 2.3:

$$\begin{aligned} Pr(A \cup B \mid C) &= \frac{Pr((A \cup B) \cap C)}{Pr(C)} = \frac{Pr((A \cap C) \cup (B \cap C))}{Pr(C)} \\ &= \frac{Pr(A \cap C)}{Pr(C)} + \frac{Pr(B \cap C)}{Pr(C)} - \frac{Pr(A \cap B \cap C)}{Pr(C)} \\ &= Pr(A \mid C) + Pr(B \mid C) - Pr(A \cap B \mid C) \end{aligned}$$

Problem 2.8

Show that if $Pr(B \mid A) = Pr(B)$ then
(a) $Pr(A, B) = Pr(A)Pr(B)$

$$\begin{aligned} Pr(B \mid A) &= \frac{Pr(A, B)}{Pr(A)} && (4) \\ Pr(B \mid A) &= Pr(B) && (5) \end{aligned}$$

Equating (4) and (5) we get

$$\begin{aligned} Pr(B) &= \frac{Pr(A, B)}{Pr(A)} && (6) \\ Pr(A, B) &= Pr(A)Pr(B) && (7) \end{aligned}$$

(b) $Pr(A \mid B) = Pr(A)$

$$Pr(A \mid B) = \frac{Pr(A, B)}{Pr(B)} \qquad (8)$$

Since (5) $\Rightarrow$ (7) we get using (7)

$$\begin{aligned} Pr(A \mid B) &= \frac{Pr(A)Pr(B)}{Pr(B)} && (9) \\ Pr(A \mid B) &= Pr(A) && (10) \end{aligned}$$

Problem 2.9

Let $A_i = i$th dart thrown hits target.
Method 1:

$$\begin{aligned} Pr(\text{at least one hit out of 3 throws}) &= Pr(A_1 \cup A_2 \cup A_3) \\ &= Pr(A_1) + Pr(A_2) + Pr(A_3) - Pr(A_1 \cap A_2) \\ &- Pr(A_1 \cap A_3) - Pr(A_2 \cap A_3) \\ &+ Pr(A_1 \cap A_2 \cap A_3). \end{aligned}$$

$Pr(A_i) = 1/4$, $Pr(A_i \cap A_j) = 1/16$, and $Pr(A_i \cap A_j \cap A_k) = 1/64$.

$$Pr(\text{at least one hit}) = 3 \cdot \frac{1}{4} - 3 \cdot \frac{1}{16} + \frac{1}{64} = \frac{37}{64}.$$

Method 2:

$$\begin{aligned}
Pr(\text{at least one hit}) &= 1 - Pr(\text{no hits}) = 1 - Pr(\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}) \\
&= 1 - Pr(\overline{A_1}) \cdot Pr(\overline{A_2}) \cdot Pr(\overline{A_3}) = 1 - \frac{3}{4}^3 = \frac{37}{64}.
\end{aligned}$$

Note 1: To complete this problem, we must assume that the outcome of each throw is independent of all others.
Note 2: The sample space is not equally likely.

Problem 2.10

Let $D_i = i$th diode chosen is defective.

$$Pr(D_1 \cap D_2) = 1 - Pr(\overline{D_1} \cap \overline{D_2}) = 1 - Pr(\overline{D_1}) Pr(\overline{D_2} \mid \overline{D_1})$$

$$Pr(\overline{D_1}) = \frac{25}{30}$$

$$Pr(\overline{D_2} \mid \overline{D_1}) = \frac{24}{29}$$

(after the first diode is selected and found to be not defective, there are 29 diodes remaining of which 5 are defective and 24 are not)

$$Pr(D_1 \cap D_2) = 1 - \frac{25}{30} \cdot \frac{24}{29} = \frac{9}{29}$$

Problem 2.11

(a)

$$\begin{aligned}
&Pr(\text{1st} = \text{red}, \text{2nd} = \text{blue}) = Pr(\text{1st} = \text{red}) Pr(\text{2nd} = \text{blue} \mid \text{1st} = \text{red}) \\
&Pr(\text{1st} = \text{red}) = \frac{3}{12} \\
&Pr(\text{2nd} = \text{blue} \mid \text{1st} = \text{red}) = \frac{5}{11} \\
&Pr(\text{1st} = \text{red}, \text{2nd} = \text{blue}) = \frac{3}{12} \cdot \frac{5}{11} = \frac{5}{44}.
\end{aligned}$$

(b) $Pr(\text{2nd} = \text{white}) = \frac{4}{12} = \frac{1}{3}$.
(c) Same as part (b).

Problem 2.12

(a) There are 2^n distinct words.
(b) Method 1:

$$Pr(2 \text{ ones}) = Pr(\{110\} \cup \{101\} \cup \{011\}) = Pr(110) + Pr(101) + Pr(011) = \frac{3}{8}.$$

Method 2:

$$Pr(2 \text{ ones}) = \binom{3}{2} \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{3-2} = \frac{3}{8}.$$

Problem 2.13

(a) There are m^n distinct words.
(b) For this case there are 4^3 distinct words.

$$Pr(2 \text{ pulses of level } 2) = \binom{4}{2} \cdot \left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right)^{4-2} = \frac{8}{27}.$$

Problem 2.14

(a)

$$Pr\,(3 \text{ heads}) = \binom{9}{3} (\frac{1}{2})^3 (\frac{1}{2})^{9-3} = \frac{21}{128} = 0.1641.$$

(b)

$$\begin{aligned} &Pr\,(\text{at least 3 heads}) \\ &\quad = \sum_{i=3}^{9} \binom{9}{i} (\frac{1}{2})^i (\frac{1}{2})^{9-i} \\ &\quad = 1 - \sum_{i=0}^{2} \binom{9}{i} (\frac{1}{2})^i (\frac{1}{2})^{9-i} = \frac{233}{256} = 0.9102 \ . \end{aligned}$$

(c)

$$\begin{aligned} &Pr\,(\text{at least 3 heads and at least 2 tails}) \\ &\quad = Pr\,(3 \leq \text{number of heads} \leq 7) \\ &\quad = \sum_{i=3}^{7} \binom{9}{i} (\frac{1}{2})^i (\frac{1}{2})^{9-i} = \frac{57}{64} = 0.8906 \ . \end{aligned}$$

Problem 2.15

(a)

$$\begin{aligned} Pr(5) &= \frac{1}{6} \\ Pr(\overline{5}) &= \frac{5}{6} \\ Pr(\overline{5},\overline{5}) &= \frac{5}{6}\cdot\frac{5}{6} = \frac{25}{36} \end{aligned}$$

(b) Pr($sum = 7$)
The sum of 7 can occur in the following 6 possible ways.
sum = {(1;6), (2;5) , (3;4) , (4;5) , (5;2) , (6;1) }. And there are a total of 36 outcomes in the sample space.

$$Pr(sum = 7) = \frac{6}{36} = \frac{1}{6}$$

(c) A= { (3;5) , (5;3) }

$$Pr(A) = \frac{2}{36} = \frac{1}{18}$$

(d)

$$\begin{aligned} Pr(A = 5) &= \frac{1}{6} \\ Pr(B = (5 \mid 4)) &= \frac{2}{6} \\ Pr(A, B) &= \frac{1}{6}\cdot\frac{2}{6} = \frac{1}{18} \end{aligned}$$

Alternatively the desired event is given by the following set of outcomes
X ={ (5;4) , (5,5) }

$$Pr(X) = \frac{2}{36} = \frac{1}{18}$$

(e)

$$\begin{aligned} Pr(5) &= \frac{1}{6} \\ Pr(5,5) &= \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36} \end{aligned}$$

(f)

$$\begin{aligned}
Pr(A=6) &= \frac{1}{6} \\
Pr(B=6) &= \frac{1}{6} \\
Pr(A \cup B) &= Pr(A) + Pr(B) - Pr(A \cap B) \\
&= \frac{1}{6} + \frac{1}{6} - \frac{1}{6} \cdot \frac{1}{6} \\
&= \frac{11}{36}
\end{aligned}$$

Problem 2.16

(a)

$$\begin{aligned}
& Pr\left(\{1st \in (2,3,4)\} \cap \{2nd \in (2,3,4)\}\right) \\
&= Pr\left(1st \in (2,3,4)\right) \times Pr\left(2nd \in (2,3,4)\right) \\
&= \frac{3}{6} \times \frac{3}{6} = \frac{1}{4}\,.
\end{aligned}$$

(b) The possible combinations of the two die can be $(6,4)$, $(5,3)$, $(4,2)$, $(3,1)$. Hence,

$$Pr\left(1st - 2nd = 2\right) = 4 \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{9}\,.$$

(c) Since one roll is 6, all possible combinations of two die are $(1,6)$, $(2,6)$, $(3,6)$, $(4,6)$, $(5,6)$, $(6,6)$, $(6,5)$, $(6,4)$, $(6,3)$, $(6,2)$, $(6,1)$. Since only two combinations satisfy the requirement, $(4,6)$ and $(6,4)$, we have

$$Pr\left(\text{sum} = 10 \mid \text{one roll} = 6\right) = \frac{2}{11}\,.$$

(d) Similarly, all possible combinations of two die are $(1,5)$, $(2,5)$, $(3,5)$, $(4,5)$, $(5,5)$, $(6,5)$, $(5,1)$, $(5,2)$, $(5,3)$, $(5,4)$, $(5,6)$. Since four combinations satisfy the requirement, $(5,2)$, $(5,3)$, $(2,5)$ and $(3,5)$, we have

$$Pr\left(\text{sum} \in (7{,}8) \mid \text{one roll} = 5\right) = \frac{4}{11}\,.$$

(e) Since the sum is 7, all possible combinations of two die are $(4,3)$, $(5,2)$, $(6,1)$, $(3,4)$, $(2,5)$, $(1,6)$. Hence, we have

$$Pr\left(\text{one roll} = 4 \mid \text{sum} = 7\right) = \frac{2}{6} = \frac{1}{3}\,.$$

Problem 2.17

$$\begin{aligned}
Pr(\text{defective}) = Pr(\text{defective} \mid A) \cdot Pr(A) + Pr(\text{defective} \mid B) \cdot Pr(B) \\
= (0.15) \cdot \frac{1}{1.15} + (0.05) \cdot \frac{0.15}{1.15} = 0.137.
\end{aligned}$$

Problem 2.18

(a)

$$\begin{aligned}
Pr\,(\text{two Aces}) &= \binom{3}{2} \cdot Pr\left(A, A, \overline{A}\right) = 3 \cdot \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{48}{50} = 0.0130. \\
Pr\,(\text{two of any kind}) &= 13 \cdot Pr\,(\text{two Aces}) = 0.1694.
\end{aligned}$$

(b)

$$\begin{aligned}
Pr\,(\text{three Aces}) &= Pr\,(A, A, A) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} = 0.000181. \\
Pr\,(\text{three of any kind}) &= 13 \cdot Pr\,(\text{three Aces}) = 0.00235.
\end{aligned}$$

(c)

$$\begin{aligned}
Pr\,(\text{three Hearts}) &= Pr\,(H, H, H) = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} = 0.0129. \\
Pr\,(\text{three of any suit}) &= 4 \cdot Pr\,(\text{three Hearts}) = 0.0518.
\end{aligned}$$

(d)

$$\begin{aligned}
Pr\,(\text{straight}) &= Pr\,(2, 3, 4) + Pr\,(3, 4, 5) + \ldots + Pr\,(Q, K, A) \\
&= 11 \cdot Pr\,(2, 3, 4) = 11 \cdot \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} = 0.003092.
\end{aligned}$$

Problem 2.19

(a)

$$\begin{aligned} Pr\,(\text{two Aces}) &= \binom{5}{2} \cdot Pr\left(A, A, \overline{A}, \overline{A}, \overline{A}\right) \\ &= \binom{5}{2} \cdot \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{48}{50} \cdot \frac{47}{49} \cdot \frac{46}{48} = 0.03993. \\ Pr\,(\text{two of any kind}) &= 13 \cdot Pr\,(\text{two Aces}) = 0.5191. \end{aligned}$$

Note that the above calculations also allow that the hand may have two pair or a full house. Hence to be completely accurate (in the poker sense) we must subtract these probabilities.

$$\begin{aligned} Pr\,(\text{two of a kind}) &= 0.5191 - Pr\,(\text{two pair}) - Pr\,(\text{full house}) \\ &= 0.5191 - 0.04754 - 0.001441 = 0.4701. \end{aligned}$$

Note that Pr(2 pair) is calculated in part (c) and Pr(full house) is calculated in part (e).

(b)

$$\begin{aligned} Pr\,(\text{three Aces}) &= \binom{5}{3} \cdot Pr\left(A, A, A, \overline{A}, \overline{A}\right) \\ &= \binom{5}{3} \cdot \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{48}{49} \cdot \frac{47}{48} = 0.001736. \\ Pr\,(\text{three of any kind}) &= 13 \cdot Pr\,(\text{three Aces}) = 0.02257. \end{aligned}$$

Note that the above calculations also allow that the hand may have a full house and hence this probability must be subtracted.

$$\begin{aligned} Pr\,(\text{three of a kind}) &= 0.02257 - Pr\,(\text{full house}) \\ &= 0.02257 - 0.001441 = 0.02113. \end{aligned}$$

(c)

$$\begin{aligned} &Pr\,(\text{two Aces, two Kings}) \\ &= \binom{5}{2} \cdot \binom{3}{2} \cdot Pr\left(A, A, K, K, \overline{A \cup K}\right) \end{aligned}$$

$$
\begin{aligned}
&= \binom{5}{2} \cdot \binom{3}{2} \cdot \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{4}{50} \cdot \frac{3}{49} \cdot \frac{44}{48} = 0.0006095. \\
Pr\,(\text{two pair}) &= \binom{13}{2} \cdot Pr\,(\text{two Aces, two Kings}) = 0.04754.
\end{aligned}
$$

(d)

$$
\begin{aligned}
Pr\,(\text{five Hearts}) &= \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48} = 0.0004952. \\
Pr\,(\text{flush}) &= 4 \cdot Pr\,(\text{five Hearts}) = 0.001981.
\end{aligned}
$$

Note that the above calculations also allow that the hand may have a straight-flush and hence this probability must be subtracted.

$$
\begin{aligned}
Pr\,(\text{straight-flush}) &= \frac{4 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}. \\
Pr\,(\text{flush}) &= \frac{4 \cdot 9 \cdot (13 \cdot 12 \cdot 11 \cdot 10 - 1)}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = 0.001981.
\end{aligned}
$$

(e)

$Pr\,(\text{three Aces, two Kings})$

$$
\begin{aligned}
&= \binom{5}{3} \cdot Pr\,(A, A, A, K, K) \\
&= 10 \cdot \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{4}{49} \cdot \frac{3}{48} = 9.235 \times 10^{-6}. \\
Pr\,(\text{full house}) &= 13 \cdot 12 \cdot Pr\,(\text{three Aces, two Kings}) = 0.001441.
\end{aligned}
$$

(f)

$$
\begin{aligned}
Pr\,(10,\text{J},\text{Q},\text{K},\text{A}) &= \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} \cdot \frac{4}{49} \cdot \frac{4}{48} = 3.283 \times 10^{-6}. \\
Pr\,(\text{straight}) &= 4 \cdot Pr\,(10,\text{J},\text{Q},\text{K},\text{A}) = 2.955 \times 10^{-5}.
\end{aligned}
$$

Problem 2.20

(a)

$$
\begin{aligned}
Pr\,(1\ \text{Heart}) &= 13 \cdot Pr\left(H, \overline{H}, \overline{H}, \ldots, \overline{H}\right) \\
&= 13 \cdot \frac{13}{52} \cdot \frac{39}{51} \cdot \frac{38}{50} \cdots \frac{28}{40} = \frac{\binom{13}{1} \cdot \binom{39}{12}}{\binom{52}{13}} = 0.08006
\end{aligned}
$$

(b)

We can choose anywhere between 7 to 13 cards of a given suit to satisfy the given condition. All these events are mutually exclusive. Let A_i denote the event of having i Hearts. Following a procedure similar to part (a), it is found that

$$Pr\left(A_i\right) = \frac{\binom{13}{i} \cdot \binom{39}{13-i}}{\binom{52}{13}}.$$

Therefore

$$\begin{aligned} Pr\left(\text{at least 7 Hearts}\right) = Pr\left(\bigcup_{i=7}^{13} A_i\right) &= \sum_{i=7}^{13} Pr(A_i) \\ = Pr\left(\bigcup_{i=7}^{13} A_i\right) &= \sum_{i=7}^{13} \frac{\binom{13}{i} \cdot \binom{39}{13-i}}{\binom{52}{13}}. \end{aligned}$$

The probability of at least 7 cards from any suit is simply 4 times the probability of at least 7 Hearts. Hence

$$Pr\left(\text{at least 7 cards from any suit}\right) = \frac{4}{\binom{52}{13}} \sum_{i=7}^{13} \binom{13}{i} \cdot \binom{39}{13-i} = 0.0403.$$

(c)

$$\begin{aligned} Pr\left(\text{no Hearts}\right) &= Pr\left(\overline{H}, \overline{H}, \overline{H}, \ldots, \overline{H}\right) \\ &= \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{37}{50} \cdots \frac{27}{40} = \frac{\binom{39}{13}}{\binom{52}{13}} = 0.01279 \end{aligned}$$

Problem 2.21

(a) Let $p = Pr\left(\text{Ace is drawn}\right) = 4/52 = 1/13$.

$$Pr\left(\text{1st Ace on 5th selection}\right) = (1-p)^4 p = \frac{12}{13}^4 \cdot \frac{1}{13} = 0.0558.$$

(b)

$$Pr\left(\text{at least 5 cards drawn before Ace}\right) = (1-p)^5 = \frac{12}{13}^5 = 0.6702.$$

(c)

$$Pr\,(\text{1st Ace drawn on 5th selection}) = \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} \cdot \frac{45}{49} \cdot \frac{4}{48} = 0.0599\ .$$

$$Pr\,(\text{at least 5 cards drawn before Ace}) = \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} \cdot \frac{45}{49} \cdot \frac{44}{48} = 0.6588\ .$$

Problem 2.22

(a)
If the cards are replaced the probability of drawing a club is $p = \frac{13}{52} = \frac{1}{4}$. We need to find the probability that we draw 2 clubs in 7 trials and a club on the 8th trial. This is given by $\begin{pmatrix} 7 \\ 2 \end{pmatrix} \cdot p^2(1-p)^5 \cdot p$.

$$Pr(\text{3rd club drawn on 8th selection}) = \begin{pmatrix} 7 \\ 2 \end{pmatrix} \cdot (\frac{1}{4})^3(\frac{3}{4})^5 = 0.0779.$$

(b)
We need to find the probability that we draw either $0, 1$ or 2 clubs in 8 trials. This follows the binomial distribution and this probability is given by

$$Pr(\text{at least 8 cards drawn before 3rd club}) = \sum_{i=0}^{2} \begin{pmatrix} 8 \\ i \end{pmatrix} \cdot (\frac{1}{4})^i(\frac{3}{4})^{8-i} = \frac{729}{1024} = 0.7119.$$

(c)

$$\begin{aligned}
&Pr(\text{3rd club drawn on 8th selection}) \\
&= Pr(\text{2 clubs drawn in 7 selections})Pr(\text{8th card} = \text{club} \mid \text{2 clubs in first 7 selections}). \\
&Pr(\text{2 clubs in 7 selections}) = \frac{\binom{13}{2}\binom{39}{5}}{\binom{52}{7}} = 0.3357. \\
&Pr(\text{8th card} = \text{club} \mid \text{2 clubs in first 7 selections}) = \frac{2}{45}. \\
&Pr(\text{3rd club drawn on 8th selection}) = \frac{2}{45} \cdot \frac{\binom{13}{2}\binom{39}{5}}{\binom{52}{7}} = 0.0149.
\end{aligned}$$

$$
\begin{aligned}
Pr(\text{at least 8 cards drawn before 3rd club}) &= Pr(\text{less than 3 clubs in 7 selections}) \\
&= \sum_{k=0}^{2} Pr(k \text{ clubs in 7 selections}) \\
&= \sum_{k=0}^{2} \frac{\binom{13}{k}\binom{39}{7-k}}{\binom{52}{7}} = 0.7677.
\end{aligned}
$$

Problem 2.23

$$
Pr(\text{all } n \text{ versions erased}) = \left(\frac{1}{2}\right)^n
$$

Problem 2.24

(i) (I, M) is not permissible. If A and B are independent, then $Pr(A, B) = Pr(A)Pr(B)$. By assumption, $Pr(A) \neq 0$ and $Pr(B) \neq 0$ and hence for independent events A and B it follows that $Pr(A, B) \neq 0$ and therefore they are not mutuallly exclusive.
(ii) (I, NM) is permissible. For example let $A = \{\text{throw a 1 on first roll of a die}\}$ and $B = \{\text{throw a 2 on second roll of a die}\}$. These two events are independent, but not mutually exclusive.
(iii) (NI, M) is permissible. For example let $A = \{\text{throw a 1 on first roll of a die}\}$ and $B = \{\text{sum of two rolls of a die is 8}\}$. These two events are not independent, but they are mutually exclusive.
(iv) (NI, NM) is permissible. This is the most general case. For example let $A = \{\text{throw a 2 on first roll of a die}\}$ and $B = \{\text{sum of two rolls of a die is 8}\}$. These two events are not independent, nor are they mutually exclusive.

Problem 2.25

(a) $Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$.
(b)

$$
\sum_{k=0}^{n} Pr(X = k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1^n = 1.
$$

(c) Binomial

Problem 2.26

(a) Since

$$\sum_{k=0}^{\infty} P_X(k) = 1 \ ,$$

i.e.,

$$c\sum_{k=0}^{\infty} 0.37^k = 1 \ .$$

Hence $\frac{c}{1-0.37} = 1$, which gives $c = 0.63$.
(b) Similarly,

$$c\sum_{k=1}^{\infty} 0.82^k = 1 \ .$$

Hence $\frac{c}{1-0.82} = 1 + c$, which gives $c = 0.2195$.
(c) Similarly,

$$c\sum_{k=0}^{24} 0.41^k = 1$$

which gives $c = 0.5900$.
(d) Similarly,

$$c\sum_{k=1}^{15} 0.91^k = 1$$

which gives $c = 0.1307$.
(e) Similarly,

$$c\sum_{k=0}^{6} 0.41^{2k} = 1$$

which gives $c = 0.8319$.

Problem 2.27

(a) The probability mass function for a binomial random variable is given by

$$Pr(X = k) = \left(\begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n-k}$$

Tabulating the probabilities for various values of k we get the following

k	Probability
0	0.10737418240000
1	0.26843545600000
2	0.30198988800000
3	0.20132659200000
4	0.08808038400000
5	0.02642411520000
6	0.00550502400000
7	0.00078643200000
8	0.00007372800000
9	0.00000409600000
10	0.00000010240000

Refer to Figure 1 for the plot.

(b) The probability mass function for a Poisson distribution is given by

$$Pr(X = k) = e^{-\alpha}\frac{\alpha^k}{k!}$$

Tabulating the probabilities for various values of k we get the following

k	Probability
0	0.13533528323661
1	0.27067056647323
2	0.27067056647323
3	0.18044704431548
4	0.09022352215774
5	0.03608940886310
6	0.01202980295437
7	0.00343708655839
8	0.00085927163960
9	0.00019094925324
10	0.00003818985065

Refer to Figure 1 for the plot.

(c) Binomial: $Pr(X \geq 5) = 1 - Pr(X < 5) = 1 - \sum_{k=0}^{4} \binom{10}{k} \frac{1}{5}^k \frac{4}{5}^{10-k} = 0.0328$

Poisson: $Pr(X \geq 5) = 1 - Pr(X < 5) = 1 - \sum_{k=0}^{4} \frac{2^k}{k!} e^{-2} = 0.0527$

The Poisson approximation is not particularly good for this example.

Problem 2.28

(a)

$$\lambda t = \left(\frac{10\text{calls}}{\text{minute}}\right) \cdot \left(\frac{1}{10}\text{minute}\right) = 1$$

$$Pr(X < 3) = \sum_{k=0}^{2} \frac{1}{k!} e^{-1} = \frac{1}{e}(1 + 1 + \frac{1}{2}) = \frac{5}{2e} = 0.9197.$$

(b)

$$\lambda t = \left(\frac{10\text{calls}}{\text{minute}}\right) \cdot (6 \text{ minutes}) = 60$$

$$Pr(X < 3) = \sum_{k=0}^{2} \frac{60^k}{k!} e^{-60} = e^{-60}(1+60+\frac{3600}{2}) = 1861 \cdot e^{-60} = 1.627 \times 10^{-23}.$$

Problem 2.29

(a)

$$Pr(\text{win with one ticket}) = p = \frac{1}{\binom{50}{6}} = 6.29 * 10^{-8} \ .$$

(b)

$$Pr(4 \text{ winners}) = \binom{6 * 10^6}{4} p^4 (1-p)^{6*10^6-4} = 5.8055 * 10^{-4} \ .$$

(c) The probability that n people win is

$$Pr(n \text{ winners}) = \binom{6 * 10^6}{n} p^n (1-p)^{6*10^6-n} \ .$$

Note that when $n = 0$, the above probability evaluates to 0.6855. Since this probability is greater than 1/2, it is impossible for any other number of winers to be more probable. Hence the most probable number of winning tickets is zero.
(d) Here, for the Possion approximation, $\alpha = np = \frac{1}{\binom{50}{6}} * 6 * 10^6 = 0.3774$. Hence,

$$P(4) = \frac{\alpha^4}{4!} e^{-\alpha} = \frac{0.3774^4}{4!} e^{-0.3774} = 5.7955 * 10^{-4} \ .$$

Compared with results in (b), the Possion distribution is an accurate approximation in this example.
For the case of (c),

$$P(n) = \frac{\alpha^n}{n!}e^{-\alpha} = \frac{0.3774^n}{n!}e^{-0.3774}\ .$$

When $n = 0$, $P(n)$ will be maximum, which is the same as the result in c).

Problem 2.30

$$\begin{aligned}
Pr(X = 0) &= \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} = \frac{1}{5}. \\
Pr(X = 1) &= 3 \cdot \frac{2}{6} \cdot \frac{4}{5} \cdot \frac{3}{4} = \frac{3}{5}. \\
Pr(X = 2) &= 3 \cdot \frac{4}{6} \cdot \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{5}.
\end{aligned}$$

Problem 2.31

Let $p = Pr(\text{success}) = \frac{1}{10}$.

$Pr(1 \text{ success}) = \binom{10}{1} \cdot p \cdot (1-p)^9 = 0.3874.$
$Pr(\geq 2 \text{ successes}) = 1 - Pr(\leq 1 \text{ success}) = 1 - Pr(0 \text{ successes}) - Pr(1 \text{ success}).$
$Pr(0 \text{ successes}) = (1-p)^1 0 = 0.3487.$
$Pr(\geq 2 \text{ successes}) = 1 - 0.3487 - 0.3874 = 0.2639.$

Problem 2.32

(a)

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\
\binom{n}{n-k} &= \frac{n!}{(n-k)!(n-(n-k))!}
\end{aligned}$$

$$\begin{aligned}
\binom{n}{n-k} &= \frac{n!}{(n-k)!(k)!} \\
\binom{n}{n-k} &= \binom{n}{k}
\end{aligned}$$

(b)

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k+1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} \\
&= \frac{n!}{k!(n-k-1)!}\left(\frac{1}{n-k} + \frac{1}{k+1}\right) \\
&= \frac{n!}{k!(n-k-1)!}\left(\frac{k+1+n-k}{(n-k)(k+1)}\right) \\
&= \frac{n!}{k!(n-k-1)!}\left(\frac{1+n}{(n-k)(k+1)}\right) \\
&= \frac{(n+1)!}{(k+1)!(n-k)!} \\
&= \binom{n+1}{k+1}
\end{aligned}$$

(c)

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Consider the binomial expansion of $(p+q)^n$. We have

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot p^k \cdot q^{(n-k)}$$

Put $p = q = 1$. Then we get

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

(d)

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k = 0$$

Consider the binomial expansion of $(p+q)^n$. We have

$$(p+q)^n = \sum_{k=0}^{n}\binom{n}{k}\cdot p^k\cdot q^{(n-k)}$$

Put $p=-1, q=1$. Then we get

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k = 0$$

(e)

$$\sum_{k=0}^{n}\binom{n}{k}k = n2^{n-1}$$

Once again consider the binomial expansion $(p+q)^n$.

$$(p+q)^n = \sum_{k=0}^{n}\binom{n}{k}\cdot p^k\cdot q^{(n-k)}$$

Differentiate both sides with respect to p.

$$\frac{d}{dp}(p+q)^n = \frac{d}{dp}\left(\sum_{k=0}^{n}\binom{n}{k}\cdot p^k\cdot q^{(n-k)}\right)$$

$$n(p+q)^{n-1} = \sum_{k=0}^{n}\binom{n}{k}\cdot k\cdot p^{k-1}\cdot q^{(n-k)} \tag{11}$$

Once again put $p=q=1$. We get the following relation.

$$n2^{n-1} = \sum_{k=0}^{n}\binom{n}{k}\cdot k$$

The above relation can be rewritten as follows.

$$n2^{n-1} = \sum_{k=1}^{n}\binom{n}{k}\cdot k + 0.\binom{n}{0}$$

$$n2^{n-1} = \sum_{k=1}^{n}\binom{n}{k}\cdot k$$

(f)

$$\sum_{k=0}^{n} \binom{n}{k} \cdot k \cdot (-1)^k = 0$$

For this we proceed in the same way as in (e) but we substitute $p = -1, q = 1$ in (11). This gives us

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \cdot k \cdot p^{k-1} \cdot q^{(n-k)} = 0$$

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \cdot k \cdot (-1)^{k-1} = 0$$

Multiplying by -1 through out gives the required identity.

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \cdot k \cdot (-1)^{k} = 0$$

Problem 2.33

(a) If more than 1 error occurs in a 7-bit data block, the decoder will be in error. Thus, the decoder error probability is

$$P_e = \sum_{i=2}^{7} \binom{7}{i} (0.03)^i (1 - 0.03)^{7-i} = 0.0171 \ .$$

(b) Similarly,

$$P_e = \sum_{i=3}^{15} \binom{15}{i} (0.03)^i (1 - 0.03)^{15-i} = 0.0094 \ .$$

(c) Similarly,

$$P_e = \sum_{i=4}^{31} \binom{31}{i} (0.03)^i (1 - 0.03)^{31-i} = 0.0133 \ .$$

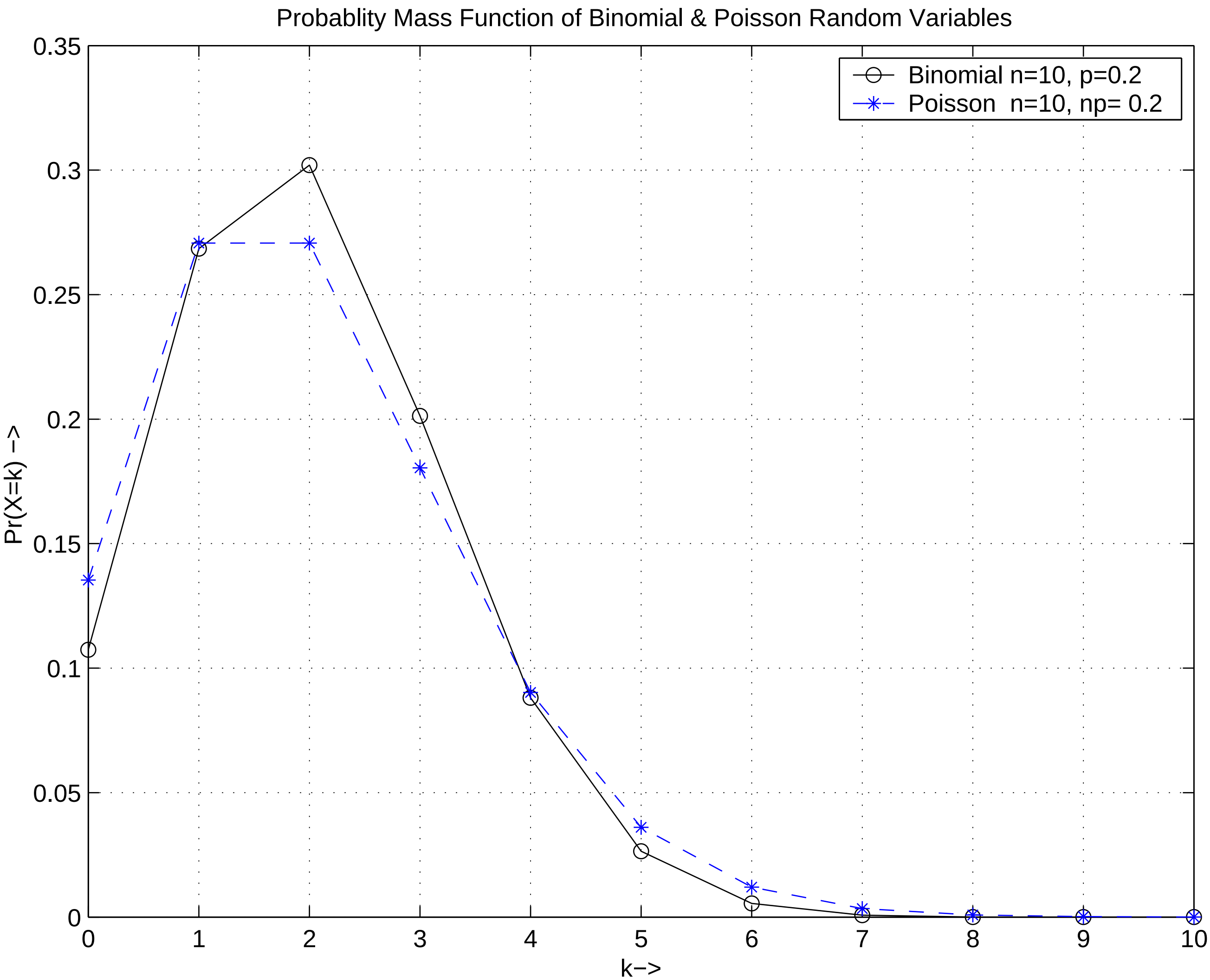

Figure 1: Probability Mass Function for Binomial and Poisson Dist.

Solutions to Chapter 3 Exercises

Problem 3.1

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

The CDF is given by

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$$

Integrating the above function for various regions of x we get the following

$$F_X(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x \geq b \end{cases}$$

A plot of this function is shown in Figure 1

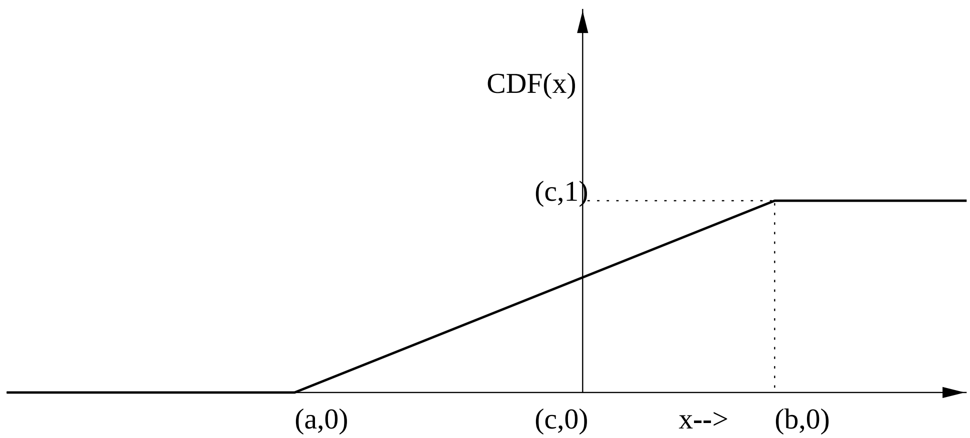

Figure 1: CDF for Problem 3.1

Problem 3.2

(a)

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases} .$$

(b)

$$\begin{aligned} F_X(x) = \textstyle\int_{-\infty}^{x} f_X(x)dx &= \begin{cases} 0 & x < 0 \\ \int_0^x x dx & 0 \le x < 1 \\ \int_0^1 x dx + \int_1^x (2-x)dx & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases} \\ &= \begin{cases} 0 & x < 0 \\ \frac{x^2}{2} & 0 \le x < 1 \\ -\frac{x^2}{2} + 2x - 1 & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases} . \end{aligned}$$

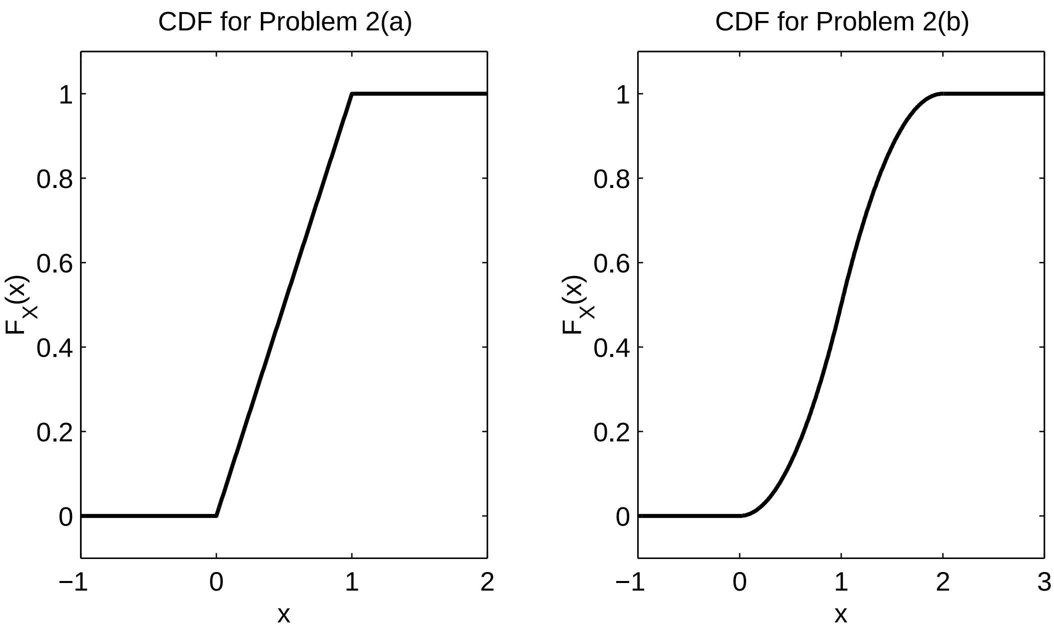

Figure 2: CDFs for Problem 3.2

Problem 3.3

$$f_X(x) = \begin{cases} a^{-bx} & x \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Since this is a pdf the following integral should evaluate to 1.

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\int_{-\infty}^{0} f_X(x)\,dx + \int_{0}^{\infty} f_X(x)\,dx = 1$$

$$\int_{-\infty}^{0} a^{-bx}\,dx + \int_{0}^{\infty} 0\,dx = 1$$

$$\left.\frac{-a^{-bx}}{b\ \ln(a)}\right|_{x=-\infty}^{x=0} = 1$$

$$\frac{-1}{b\ \ln(a)} + \lim_{x\to-\infty} \frac{-a^{-bx}}{b\ \ln(a)} = 1$$

Here two cases arise $a > 1$ and $0 < a < 1$. The case $a < 0$ is not possible because then $f_X(x)$ can become negative or imaginary and will no longer be a valid pdf. First considering the case $a > 1$ This evaluates to a finite value only if $b < 0$. And in the case $a < 1$ the limit will be finite only if $b > 0$. In both cases the limit evaluates to 0 and the above eqn reduces to

$$\frac{-1}{b\ \ln(a)} = 1$$

$$a = e^{\frac{-1}{b}}$$

We can see that this relation satisfies the requirements we laid on a and b earlier. Also using this relation the pdf can be written as

$$\begin{aligned} f_X(x) &= a^{-bx} \\ &= \left(e^{-\frac{1}{b}}\right)^{-bx} \\ f_X(x) &= e^{x} \end{aligned} \tag{1}$$

(b) The CDF is given by

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(x)\,dx \\ F_X(x) &= \int_{-\infty}^{x} e^{x}\,dx \end{aligned}$$

Integrating the above function for various regions of x we get the following

$$F_X(x) = \begin{cases} e^x & x \leq 0 \\ 1 & 0 \leq x \end{cases}$$

Problem 3.4

(a) Since

$$\int_{-\infty}^{\infty} f_X(x)dx = 1 \ ,$$

i.e.,

$$c * \int_{0}^{\infty} e^{-2x}dx = 1 \ ,$$

which gives c=2.
(b)

$$P_r(X > 2) = \int_{2}^{\infty} f_X(x)dx \ ,$$

Hence,

$$P_r(X > 2) = \int_{2}^{\infty} 2e^{-2x}dx = e^{-4} \ .$$

(c)

$$P_r(X < 3) = \int_{0}^{3} 2e^{-2x}dx = 1 - e^{-6} \ .$$

(d) Since

$$P_r(X < 3 \mid X > 2) = \frac{P_r(2 < X < 3)}{P_r(X > 2)} \ ,$$

and

$$P_r(2 < X < 3) = \int_{2}^{3} 2e^{-2x}dx = e^{-4} - e^{-6} \ ,$$

and from part (b)

$$P_r(X > 2) = e^{-4} ,$$

we have

$$P_r(X < 3 \mid X > 2) = \frac{e^{-4} - e^{-6}}{e^{-4}} = 1 - e^{-2} .$$

Problem 3.5

$$f_X(x) = \frac{c}{x^2 + 4}$$

(a) Since this is a probability function the following integral evaluates to 1.

$$\begin{aligned}
\int_{-\infty}^{\infty} f_X(x)\, dx &= 1 \\
\int_{-\infty}^{\infty} \frac{c}{x^2 + 4}\, dx &= 1 \\
\frac{c}{2} \arctan(\frac{x}{2})\Big|_{-\infty}^{\infty} &= 1 \\
\frac{c}{2}(\frac{\pi}{2} + \frac{\pi}{2}) &= 1 \\
c &= \frac{2}{\pi}
\end{aligned}$$

(b) $Pr(X > 2)$

$$\begin{aligned}
Pr(X > 2) &= \int_{2}^{\infty} \frac{2}{\pi(x^2 + 4)}\, dx \\
&= \frac{2}{2\pi} \cdot \arctan \frac{x}{2}\Big|_{x=2}^{x=\infty} \\
&= \frac{1}{\pi} \cdot (\frac{\pi}{2} - \frac{\pi}{4}) = \frac{1}{4}
\end{aligned}$$

(c) $Pr(X < 3)$

$$\begin{aligned}
Pr(X < 3) &= \int_{-\infty}^{3} \frac{2}{\pi(x^2 + 4)}\, dx \\
&= \frac{2}{2\pi} \cdot \arctan \frac{x}{2}\Big|_{x=-\infty}^{x=3}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\cdot(\arctan\frac{3}{2}+\frac{\pi}{2}) \\
&= \frac{1}{\pi}\cdot(\arctan\frac{3}{2}+\frac{\pi}{2}) = 0.8128.
\end{aligned}
$$

(d)$Pr(X<3|X>2)$

$$
\begin{aligned}
Pr(X<3|X>2) &= \frac{Pr(X>2, X<3)}{Pr(X>2)} = \frac{Pr(2<X<3)}{Pr(X>2)} \\
Pr(2<X<3) &= \int_2^3 \frac{2}{\pi(x^2+4)}\,dx \\
&= \frac{2}{2\pi}\cdot\arctan\frac{x}{2}\bigg|_{x=2}^{x=3} \\
&= \frac{1}{\pi}\cdot(\arctan\frac{3}{2}-\arctan\frac{2}{2}) \\
&= \frac{1}{\pi}\cdot(\arctan\frac{3}{2}-\frac{\pi}{4}) \\
Pr(X<3|X>2) &= \frac{\frac{1}{\pi}\cdot(\arctan\frac{3}{2}-\frac{\pi}{4})}{\frac{1}{4}} \\
&= \frac{4}{\pi}\cdot(\arctan\frac{3}{2}-\frac{\pi}{4}) = 0.2513.
\end{aligned}
$$

Problem 3.6

(a)

$$
F_X(x) = \int_{-\infty}^{x} f_X(x)dx = c*\int_{-\infty}^{x}\frac{1}{\sqrt{25-x^2}}dx = c*\arcsin(\frac{x}{5})+c*\frac{\pi}{2}\ ,
$$

since $F_X(\infty)$=1, we have

$$
c=\frac{1}{\pi}\ .
$$

and,

$$
F_X(x)=\frac{1}{\pi}\arcsin(\frac{x}{5})+\frac{1}{2}\ .
$$

(b)

$$
P_r(X>2) = 1-F_X(2) = 1-\frac{1}{\pi}\arcsin(\frac{2}{5})-\frac{1}{2} = 0.369\ .
$$

(c)

$$P_r(X < 3) = F_X(3) = \frac{1}{\pi}\arcsin(\frac{3}{5}) + \frac{1}{2} = 0.7048 \ .$$

(d) Since

$$P_r(X < 3 \mid X > 2) = \frac{P_r(2 < X < 3)}{P_r(X > 2)} \ ,$$

and

$$P_r(2 < X < 3) = F_X(3) - F_X(2) = 0.07384 \ ,$$

and from part (b)

$$P_r(X > 2) = 0.369 \ ,$$

we have

$$P_r(X < 3 \mid X > 2) = \frac{0.07384}{0.369} = 0.2001 \ .$$

Problem 3.7

$$\begin{aligned} Pr(|S-10| > 0.075) &= 2\int_{9.9}^{9.925} f_S(s)ds \\ &= 2\int_{9.9}^{9.925} 100\cdot(s-9.9)ds \\ &= 2\int_{0}^{0.025} 100\cdot u\cdot du \\ &= 100\cdot(0.025)^2 \\ &= 0.0625. \end{aligned}$$

Problem 3.8

$$\begin{aligned} I &= \int_{-\infty}^{\infty} exp(-\frac{x^2}{2})\,dx \\ I &= \int_{-\infty}^{\infty} exp(-\frac{y^2}{2})\,dy \\ I^2 &= \left(\int_{-\infty}^{\infty} exp(-\frac{x^2}{2})\,dx\right)\left(\int_{-\infty}^{\infty} exp(-\frac{y^2}{2})\,dy\right) \end{aligned}$$

Since these are totally independent and the operation is linear we can rewrite the above equation as

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(exp(-\frac{x^2}{2})\,dx \right) \left(exp(-\frac{y^2}{2})\,dy \right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} exp(-\frac{x^2+y^2}{2})\,dx\,dy \end{aligned}$$

Using polar coordinates to do this integral we make the substitution $x = \rho\cos\theta$ and $y = \rho\sin\theta$. Then the integral can be written as

$$\begin{aligned} &= \int_{0}^{2\pi}\int_{0}^{\infty} exp(-\frac{\rho^2}{2})\,\rho\,d\rho\,d\theta \\ &= \int_{0}^{2\pi} -\frac{exp(-\frac{\rho^2}{2})}{2}\Bigg|_0^{\infty} d\theta \\ &= \int_{0}^{2\pi} \frac{1}{2}\,d\theta \\ &= \frac{2\pi}{2} = \pi \\ \Rightarrow I &= \sqrt{\pi} \end{aligned}$$

Problem 3.9

Since

$$\exp(-(ax^2+bx+c)) = \exp\left(-\left(\frac{(x+\frac{b}{2a})^2+\frac{c}{a}-\frac{b^2}{4a^2}}{\frac{1}{a}}\right)\right) = \exp(\frac{b^2}{4a}-c)\exp\left(-\frac{(x+\frac{b}{2a})^2}{\frac{1}{a}}\right),$$

then we have

$$\begin{aligned} \int_{-\infty}^{\infty} \exp(-(ax^2+bx+c))dx &= \int_{-\infty}^{\infty} \exp(\frac{b^2}{4a}-c) * \exp\left(-\frac{(x+\frac{b}{2a})^2}{\frac{1}{a}}\right) dx \\ &= \exp(\frac{b^2}{4a}-c)\sqrt{\frac{\pi}{a}} * \frac{1}{\sqrt{\frac{\pi}{a}}}\int_{-\infty}^{\infty} \exp\left(-\frac{(x+\frac{b}{2a})^2}{\frac{1}{a}}dx\right). \end{aligned}$$

Since

$$\frac{1}{\sqrt{\frac{\pi}{a}}}\int_{-\infty}^{\infty} \exp\left(-\frac{(x+\frac{b}{2a})^2}{\frac{1}{a}}\right) dx = 1,$$

then

$$\int_{-\infty}^{\infty} \exp(-(ax^2+bx+c))dx = \sqrt{\frac{\pi}{a}}\exp(\frac{b^2}{4a}-c) \ .$$

Problem 3.10

$$f_X(x) = ce^{-2x^2-3x-1}$$

(a) As usual the integral of the pdf evaluates to 1.

$$\begin{aligned}
\int_{-\infty}^{\infty} f_X(x)\,dx &= 1 \\
c\int_{-\infty}^{\infty} \exp\{(-2x^2-3x-1)\}\,dx &= 1 \\
c\int_{-\infty}^{\infty} \exp\{(-2(x^2+\frac{3x}{2}+\frac{1}{2})\}\,dx &= 1 \\
c\int_{-\infty}^{\infty} \exp\{(-2(x+\frac{3}{4})^2-(\frac{3}{4})^2+\frac{1}{2})\}\,dx &= 1 \\
c\int_{-\infty}^{\infty} \exp\{(-2(x+\frac{3}{4})^2-(\frac{3}{4})^2+\frac{1}{2})\}\,dx &= 1 \\
c\int_{-\infty}^{\infty} \exp\{(-2(x+\frac{3}{4})^2-(\frac{1}{16})\}\,dx &= 1 \\
c\,e^{\frac{1}{8}}\int_{-\infty}^{\infty} \exp\{(-2(x+\frac{3}{4})^2\}\,dx &= 1 \\
c\,e^{\frac{1}{8}}\sqrt{\frac{\pi}{2}} &= 1 \\
c = \sqrt{\frac{2}{\pi}}e^{\frac{-1}{8}} &= 0.7041
\end{aligned}$$

(b) The previous pdf can be rewritten in the form of a standard gaussian pdf as follows

$$\begin{aligned}
f_X(x) &= c \quad \exp\{-(2x^2+3x+1)\} \\
&= \sqrt{\frac{2}{\pi}}e^{-\frac{1}{8}}\ \exp\{-(2x^2+3x+1)\} \\
&= \sqrt{\frac{2}{\pi}}e^{-\frac{1}{8}}\ \exp\{-2(x+\frac{3}{4})^2\}\ e^{\frac{1}{8}}
\end{aligned}$$

$$= \sqrt{\frac{2}{\pi}} \exp\{-2(x+\frac{3}{4})^2\}$$

This is in the form of standard gaussian pdf given by $\sqrt{\frac{1}{2\pi\sigma^2}} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}$ and we can easily identify the mean m and the standard deviation σ as

$$m = -\frac{3}{4}, 2\sigma^2 = \frac{1}{2} \Rightarrow \sigma = \frac{1}{2}$$

Problem 3.11

For the given Gaussian pdf, $m = 10$ and $\sigma = 5$. Hence,

(a)

$$Pr(X > 17) = Q(\frac{17-10}{5}) = Q(\frac{7}{5}) = 0.0808 \ .$$

(b)

$$Pr(X > 4) = Q(\frac{4-10}{5}) = Q(\frac{-6}{5}) = 1 - Q(\frac{6}{5}) = 0.8849 \ .$$

(c)

$$Pr(X < 15) = 1 - Q(\frac{15-10}{5}) = 1 - Q(1) = 0.8413 \ .$$

(d)

$$Pr(X < -2) = 1 - Q(\frac{-2-10}{5}) = 1 - Q(\frac{-12}{5}) = Q(\frac{12}{5}) = 0.0082 \ .$$

(e)

$$\begin{aligned} Pr(|X-10| > 7) &= Pr(X < 3) + Pr(X > 17) \\ &= 1 - Q(\frac{3-10}{5}) + Q(\frac{17-10}{5}) = 2Q(\frac{7}{5}) = 0.1615 \ . \end{aligned}$$

(f)

$$\begin{aligned} Pr(|X-10| < 3) &= Pr(7 < X < 13) \\ &= Q((\frac{7-10}{5}) - Q(\frac{13-10}{5}) = 1 - 2Q(\frac{3}{5}) = 0.4515 \ . \end{aligned}$$

(g)

$$
\begin{aligned}
Pr(|X-7|>5) &= Pr(X<2)+Pr(X>12) \\
&= 1-Q(\frac{2-10}{5})+Q(\frac{12-10}{5}) = Q(\frac{8}{5})+Q(\frac{2}{5}) = 0.3994 \ .
\end{aligned}
$$

(h)

$$
\begin{aligned}
Pr(|X-4|<7) &= Pr(-3<X<11) \\
&= Q((\frac{-3-10}{5})-Q(\frac{11-10}{5}) = 1-Q(\frac{13}{5})-Q(\frac{1}{5}) = 0.5746 \ .
\end{aligned}
$$

Problem 3.12

(a) $\Gamma(n) = (n-1)!$

$$
\begin{aligned}
\Gamma(x) &= \int_0^\infty e^{-t}t^{x-1}\,dt \\
\Gamma(x+1) &= \int_0^\infty e^{-t}t^{x}\,dt
\end{aligned}
$$

Integrating by parts we get

$$
\begin{aligned}
\Gamma(x+1) &= -e^{-t}t^x\big|_0^\infty - \int_0^\infty (-e^{-t})\,x\,t^{x-1}\,dt \\
&= \int_0^\infty e^{-t}\,x\,t^{x-1}\,dt \\
&= x\int_0^\infty e^{-t}\,t^{x-1}\,dt \\
&= x\,\Gamma(x) \\
\Gamma(1) &= \int_0^\infty e^{-t}t^{1-1}\,dt \\
&= \int_0^\infty e^{-t}\,dt \\
&= -e^{-t}\big|_0^\infty \\
&= 1 \\
\Gamma(2) &= (2-1)\,\Gamma(1) \\
\Gamma(2) &= 1 = 1!
\end{aligned}
$$

$$\begin{aligned}
\Gamma(3) &= 2\,\Gamma(2) = 2! \\
\Gamma(4) &= 3\,\Gamma(3) = 3! \cdots \\
\Gamma(n) &= (n-1)!
\end{aligned}$$

(b) $\Gamma(n) = n\Gamma(n-1)$
Refer to (a)

(c) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{-\frac{1}{2}}\, dt$$

Put $\sqrt{t} = y$. Then $dt = 2ydy$

$$\begin{aligned}
\Gamma(\frac{1}{2}) &= \int_0^\infty e^{-y^2} \frac{1}{y} 2y\, dy \\
&= 2\int_0^\infty e^{-y^2}\, dy \\
&= 2\frac{\sqrt{\pi}}{2} \\
&= \sqrt{\pi}
\end{aligned}$$

Problem 3.13

(a) $F_{X|A}(-\infty) = Pr(X < -\infty|A) = 0$ since it's impossible for anything to be less than $-\infty$.

$$F_{X|A}(\infty) = Pr(X < \infty|A) = \frac{Pr(X < \infty, A)}{Pr(A)}$$

Since $\{X < \infty\} \cap A = A$, then $Pr(X < \infty|A) = 1$.
(b)

$$F_{X|A}(x) = Pr(X < x|A) = \frac{Pr(X < x, A)}{Pr(A)}.$$

Since all probabilities are ≥ 0, the ratio is also ≥ 0.

$$\Rightarrow Pr(X < x|A) \geq 0.$$

Since $\{\{X \le x\} \cap A\} \subseteq A$, $Pr(X \le x, A) \le Pr(A)$.

$$\Rightarrow Pr(X < x|A) \le 1.$$

(c)

$$F_{X|A}(x) = \frac{Pr(X < x, A)}{Pr(A)}.$$

For $x_1 < x_2$, $\{X \le x_1\} \subseteq \{X \le x_2\}$.

$$\Rightarrow \{\{X \le x_1\} \cap A\} \subseteq \{\{X \le x_2\} \cap A\}.$$

$$\Rightarrow \mathrm{Pr}(\{X \le x_1\} \cap A) \le Pr\{X \le x_2\} \cap A).$$

$$\Rightarrow F_{X|A}(x_1) \le F_{X|A}(x_2.$$

(d)

$$\begin{aligned} Pr(x_1 < X \le x_2) &= \frac{Pr(\{x_1 < X \le x_2\} \cap A)}{Pr(A)}. \\ &= \frac{Pr(X \le x_2, A) - Pr(X \le x_1, A)}{Pr(A)} \\ &= F_{X|A}(x_2) - F_{X|A}(x_1). \end{aligned}$$

Problem 3.14

(a)

$$\begin{aligned} f_{X|X>0}(x) &= \begin{cases} \frac{f_X(x)}{P_r(X>0)} & x > 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{f_X(x)}{Q(0)} & x > 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 2f_X(x) & x > 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{2}{\sqrt{2\pi\sigma^2}} \exp(-\frac{x^2}{2\sigma^2}) & x > 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(b)

$$
\begin{aligned}
f_{X||X|<3}(x) &= \begin{cases} \frac{f_X(x)}{P_r(-3<X<3)} & -3 < x < 3 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{f_X(x)}{1-2Q(\frac{3}{\sigma})} & -3 < x < 3 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{1-2Q(\frac{3}{\sigma})}\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{x^2}{2\sigma^2}) & -3 < x < 3 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

(c)

$$
\begin{aligned}
f_{X||X|>3}(x) &= \begin{cases} \frac{f_X(x)}{P_r(|X|>3)} & |X| > 3 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{f_X(x)}{2Q(\frac{3}{\sigma})} & |X| > 3 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{2Q(\frac{3}{\sigma})}\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{x^2}{2\sigma^2}) & |X| > 3 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Problem 3.15

$$
\begin{aligned}
&r_1 = 1.5 \text{ ft} = \text{radius of target.} \\
&r_2 = 0.25 \text{ ft} = \text{radius of bulls-eye.} \\
&\sigma = 2 \text{ ft.} \\
&f_R(r) = \frac{r}{\sigma^2}\cdot\exp\left(-\frac{r^2}{2\sigma^2}\right)\cdot u(r)
\end{aligned}
$$

(a)

$$
\begin{aligned}
Pr(\text{hit target}) &= Pr(R \le r_1) = \int_0^{r_1} f_R(r)dr = 1 - \exp(-\frac{r_1^2}{2\sigma^2}) \\
&= 1 - \exp(-\frac{1}{2}\cdot\left(\frac{1.5}{2}\right)^2) = 0.2452.
\end{aligned}
$$

Maybe Mr. Hood isn't such a good archer after all!!

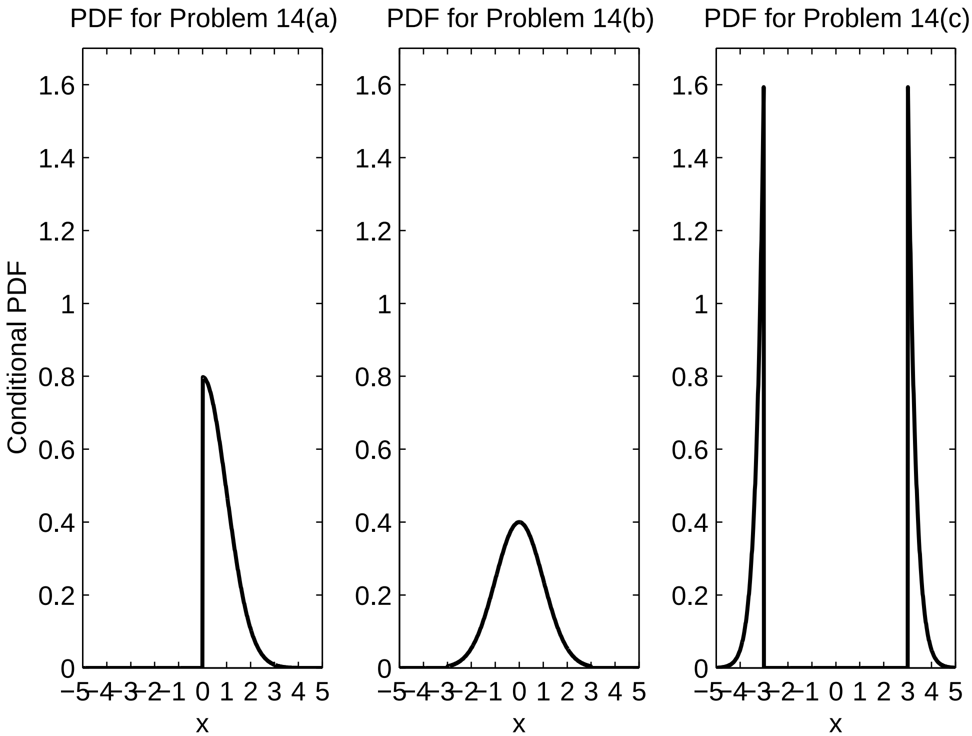

Figure 3: PDF plots for Problem 3.14; $(\sigma^2 = 1)$.

(b)

$$
\begin{aligned}
Pr(\text{hit bulls-eye}) &= Pr(R \leq r_2) = \int_0^{r_2} f_R(r)dr = 1 - \exp(-\frac{r_2^2}{2\sigma^2}) \\
&= 1 - \exp(-\frac{1}{2} \cdot \left(\frac{0.25}{2}\right)^2) = 0.0078.
\end{aligned}
$$

(c)

$$
\begin{aligned}
Pr(\text{hit bulls-eye}|\text{hit target}) &= \frac{Pr(\{\text{hit bulls-eye}\} \cap \{\text{hit target}\})}{Pr(\text{hit target})} = \frac{Pr(\text{hit bulls-eye})}{Pr(\text{hit target})} \\
&= \frac{1 - \exp(-\frac{1}{2} \cdot \left(\frac{0.25}{2}\right)^2)}{1 - \exp(-\frac{1}{2} \cdot \left(\frac{1.5}{2}\right)^2)} = \frac{0.0078}{0.2452} = 0.0317.
\end{aligned}
$$

Problem 3.16

Let $Pr(M=0)=p_0$ and $Pr(M=1)=p_1$

$$\begin{aligned}
f_{X|M=0}(x) &= \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{x^2}{2\sigma^2}\right) \\
f_{X|M=1}(x) &= \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(x-1)^2}{2\sigma^2}\right) \\
f_X(x) &= f_{X|M=0}(x).Pr(M=0) + f_{X|M=1}Pr(M=1) \\
&= \frac{p_0}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{x^2}{2\sigma^2}\right) + \frac{p_1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(x-1)^2}{2\sigma^2}\right) \\
Pr(M=0|X=x) &= \frac{f_{X|M=0}(x)}{f_X(x)} \\
&= \frac{p_0 exp\left(-\frac{x^2}{2\sigma^2}\right)}{p_0 exp\left(-\frac{x^2}{2\sigma^2}\right) + p_1 exp\left(-\frac{(x-1)^2}{2\sigma^2}\right)} \\
&= \frac{1}{1+\frac{p_1}{p_0} exp\left(-\frac{(x-1)^2}{2\sigma^2} + \frac{x^2}{2\sigma^2}\right)} \\
&= \frac{1}{1+\frac{p_1}{p_0} exp\left(\frac{x}{\sigma^2} - \frac{1}{2\sigma^2}\right)}
\end{aligned}$$

We have to plot this function for the following combinations

(a)
$p_0 = \frac{1}{2}$, $p_1 = \frac{1}{2}$, $\sigma^2 = 1$

$$Pr(M=0|X=x) = \frac{1}{1+exp\left(x-\frac{1}{2}\right)}$$

$p_0 = \frac{1}{2}$, $p_1 = \frac{1}{2}$, $\sigma^2 = 5$

$$Pr(M=0|X=x) = \frac{1}{1+exp\left(\frac{x}{5}-\frac{1}{10}\right)}$$

(b)
$p_0 = \frac{1}{4}$, $p_1 = \frac{3}{4}$, $\sigma^2 = 1$

$$Pr(M=0|X=x) = \frac{1}{1+3exp\left(x-\frac{1}{2}\right)}$$

$p_0 = \frac{1}{4}$, $p_1 = \frac{3}{4}$, $\sigma^2 = 5$

$$Pr(M = 0|X = x) \quad = \quad \frac{1}{1 + 3exp\left(\frac{x}{5} - \frac{1}{10}\right)}$$

Plot of all the distributions

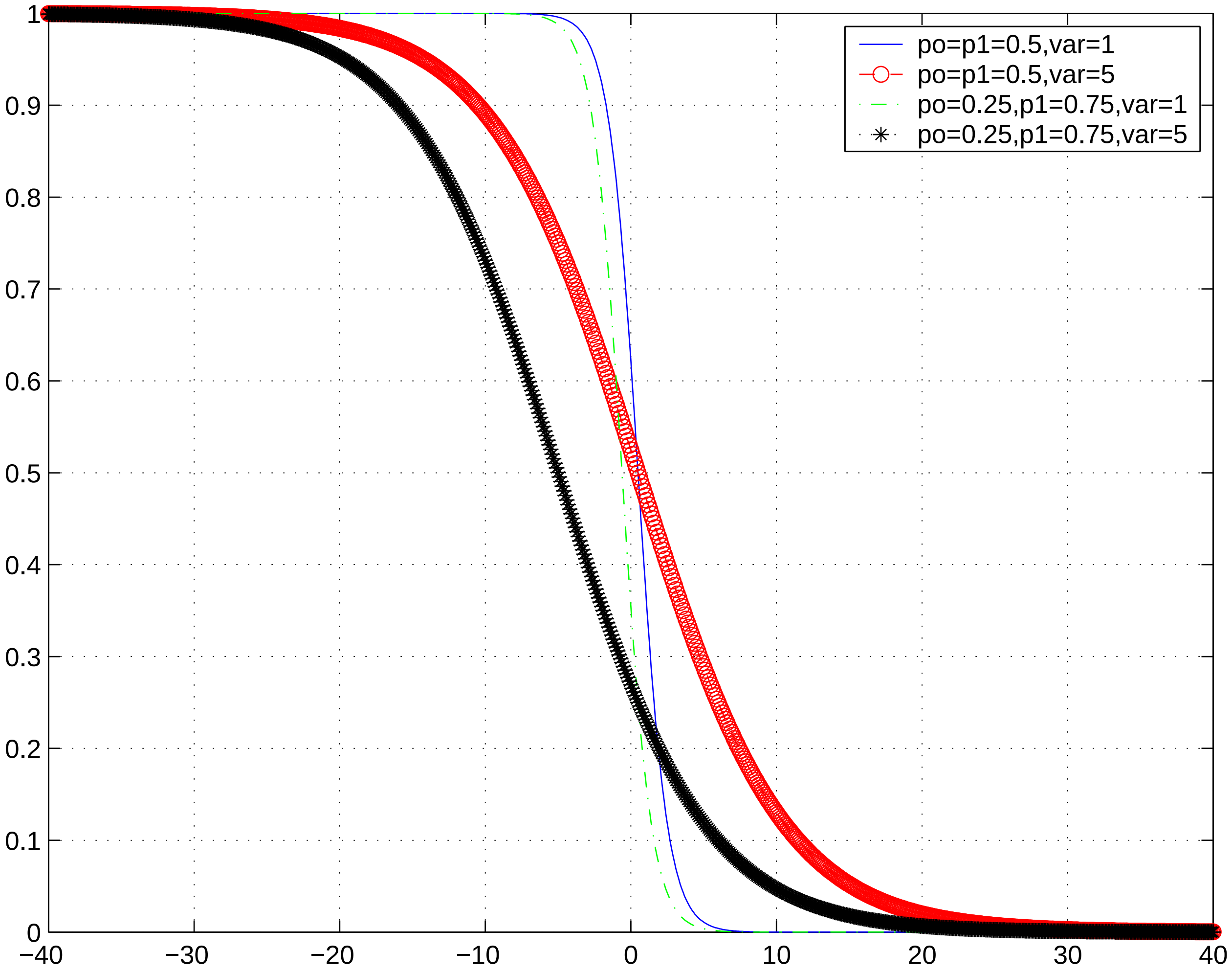

Figure 4: $Pr(M = 0|X = x)$ for various values of σ and p_0

Problem 3.17

(a) We decide a "0" was sent if

$$P_r(M = 0|X = x) \geq 0.9 \ .$$

Since

$$\begin{aligned} P_r(M = 0|X = x) &= \frac{f_{X|M=0}(x)P_r(M=0)}{f_{X|M=0}(x)P_r(M=0)+f_{X|M=1}(x)P_r(M=1)} \\ &= \frac{f_{X|M=0}(x)}{f_{X|M=0}(x)+f_{X|M=1}(x)} \\ &= \frac{e^{-\frac{x^2}{2\sigma^2}}}{e^{-\frac{x^2}{2\sigma^2}}+e^{-\frac{(x-1)^2}{2\sigma^2}}} \ , \end{aligned}$$

we have

$$x \leq \frac{1}{2} - \sigma^2 \ln 9 \ . \tag{2}$$

Similarly, we decide a "1" was sent if

$$x \geq \frac{1}{2} + \sigma^2 \ln 9 \ . \tag{3}$$

Finally, the symbol will be erased if neither (2) nor (3) hold, that is if

$$\frac{1}{2} - \sigma^2 \ln 9 \ < x < \ \frac{1}{2} + \sigma^2 \ln 9 \ . \tag{4}$$

Hence, we have the following decision rules

$$\begin{cases} 0 \ , & x \leq -1.6972 \ , \\ 1 \ , & x \geq 2.6972 \ , \\ \text{erased} \ , & -1.6972 < x < 2.6972 \ . \end{cases}$$

(b) The probability that the receiver erases a symbol is given by

$$\begin{aligned} P(\text{erased}) &= P(\text{erased}|M = 0)P(M = 0) + P(\text{erased}|M = 1)P(M = 1) \\ &= P(\text{erased}|M = 0)/2 + P(\text{erased}|M = 1)/2 \\ &= P(-1.6972 < x < 2.6972|M = 0)/2 + P(-1.6972 < x < 2.6972|M = 1)/2 \\ &= \left[Q(-1.6972) - Q(2.6972)\right]/2 + \left[Q(-1.6972 - 1) - Q(2.6972 - 1)\right]/2 \\ &= 1 - Q(1.6972) - Q(2.6972) \\ &= 0.95167 \ . \end{aligned}$$

(c) The probability that the receiver makes an error is given by

$$\begin{aligned}
P(\text{error}) &= P(\text{error}|M=0)P(M=0) + P(\text{error}|M=1)P(M=1) \\
&= P(\text{error}|M=0)/2 + P(\text{error}|M=1)/2 \\
&= P(x \geq 2.6972|M=0)/2 + P(x \leq -1.6972|M=1)/2 \\
&= Q(2.6972)/2 + [1 - Q(-2.6972)]/2 \\
&= Q(2.6972) \\
&= 0.0034963 \ .
\end{aligned}$$

Problem 3.18

$$r_N(t) = \lambda_n = \frac{1}{100}\text{days}^{-1}.$$

The overall failure rate for a serial connection is

$$r(t) = \sum_{n=1}^{10} r_n(t) = 10 \cdot \lambda_n = \frac{1}{10}\text{days}^{-1} \triangleq r.$$

(a) The reliability function is $R(t) = \exp(-rt)$ and the probability that the system functions longer than 10 days is

$$R(10\text{ days}) = \exp\left(-(10\text{ days})\left(\frac{1}{10}\text{days}^{-1}\right)\right) = e^{-1} = 0.3679.$$

(b)

$$R_n(t) = \exp(-r_n \cdot t)$$

$$R_n(10\text{ days}) = \exp\left(-(10\text{ days})\left(\frac{1}{100}\text{days}^{-1}\right)\right) = e^{-\frac{1}{10}} = 0.9048.$$

(c)

$$\begin{aligned}
R_n(t) &= \exp(-r_n t), \qquad r_n = \frac{1}{100}\text{days}^{-1} \text{ (component)} \\
R(t) &= \exp(-rt), \qquad r = \frac{1}{10}\text{days}^{-1} \text{ (system)}
\end{aligned}$$

Solutions to Chapter 4 Exercises

Problem 4.1

(a) Mean of the Distribution

$$\begin{aligned} P_X(k) &= \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k}, k = 0, 1, 2, \ldots n. \\ \mu &= \sum_{k=0}^{n} k \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} \end{aligned}$$

Consider the binomial expansion of $(p+q)^n$

$$(p+q)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} p^k q^{n-k} \tag{1}$$

Differentiating (1) wrt to p we get

$$n(p+q)^{n-1} = \sum_{k=0}^{n} k \binom{n}{k} p^{k-1} q^{n-k}.$$

Multiplying both sides by p we get

$$np(p+q)^{n-1} = \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k} \tag{2}$$

Substituing $q = 1 - p$ we get

$$np = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

The mean μ is therefore

$$\mu = np$$

Second moment of the Distribution

$$E[X^2] = \sum_{k=0}^{n} k^2 \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k}$$

Starting with (2) and differentiating it again wrt to p we get

$$np(n-1)(p+q)^{n-2} + n(p+q)^{n-1} = \sum_{k=0}^{n} k^2 \binom{n}{k} p^{k-1}\, q^{n-k}$$

Multipilying throughout by p we get

$$np^2(n-1)(p+q)^{n-2} + np(p+q)^{n-1} = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k q^{n-k}.$$

Substituting $q = 1-p$ in the above equation we get

$$np^2(n-1) + np = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k\,(1-p)^{n-k}$$

$$n^2p^2 + np(1-p) = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k\,(1-p)^{n-k}.$$

The Variance of the distribution is given by using the relation

$$\sigma^2 = E[X^2] - E[X]^2$$

$$\sigma^2 = n^2p^2 + np(1-p) - (np)^2$$

$$\sigma^2 = np(1-p)$$

(b) Poisson Distribution $P_X(k) = \frac{\alpha^k}{k!} e^{-\alpha}$

$$\begin{aligned} E[X] &= \sum_{k=0}^{\infty} k \frac{\alpha^k}{k!} e^{-\alpha} = e^{-\alpha} \sum_{k=1}^{\infty} k \frac{\alpha^k}{k!} = e^{-\alpha} \sum_{k=1}^{\infty} \frac{\alpha^k}{(k-1)!} \\ &= e^{-\alpha} \sum_{m=0}^{\infty} \frac{\alpha^{m+1}}{m!} = \alpha\, e^{-\alpha} \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} = \alpha\, e^{-\alpha} e^{\alpha} = \alpha \\ \mu &= \alpha \end{aligned}$$

The Second Moment of the Distribution

$$\begin{aligned} E[X^2] &= \sum_{k=0}^{\infty} k^2\, \frac{\alpha^k}{k!} e^{-\alpha} = e^{-\alpha} \sum_{k=1}^{\infty} k^2 \frac{\alpha^k}{k!} \\ &= e^{-\alpha} \sum_{k=1}^{\infty} k \frac{\alpha^k}{(k-1)!} = e^{-\alpha} \sum_{m=0}^{\infty} (m+1)\, \frac{\alpha^{m+1}}{m!} \\ &= e^{-\alpha} \left(\sum_{m=0}^{\infty} m\, \frac{\alpha^{m+1}}{m!} + \sum_{m=0}^{\infty} \frac{\alpha^{m+1}}{m!} \right) = \alpha\, e^{-\alpha} \left(\sum_{m=0}^{\infty} m\, \frac{\alpha^m}{m!} + \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} \right) \\ &= \alpha\, e^{-\alpha}\,(\alpha e^{\alpha} + e^{\alpha}) = \alpha\,(\alpha+1) = \alpha^2 + \alpha \\ \sigma^2 &= E[X^2] - E[X]^2 = \alpha^2 + \alpha - \alpha^2 \\ \sigma^2 &= \alpha \end{aligned}$$

(c) Laplace distribution

$$\begin{aligned} f_X(x) &= \frac{1}{2b} exp(-\frac{\mid x \mid}{b}) \\ E[X] &= \int_{-\infty}^{\infty} \frac{x}{2b} exp(-\frac{\mid x \mid}{b})\, dx \end{aligned}$$

This is an odd function therefore this integral will evaluate to zero.

$$\mu = 0$$

Second Moment

$$E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{2b} exp(-\frac{\mid x \mid}{b})\, dx = 2\int_{0}^{\infty} \frac{x^2}{2b} exp(-\frac{x}{b})\, dx$$

Putting $t = \frac{x}{b}$,

$$E[X^2] = 2\int_0^{\infty} \frac{(bt)^2}{2b} exp(-t)\, b\, dt = b^2 \int_0^{\infty} t^2 exp(-t)\, b\, dt = b^2\, \Gamma(3).$$

Hence,

$$\sigma^2 = E[X^2] - E[X]^2 = b^2\, \Gamma(3) - 0 = 2\, b^2.$$

(d) Gamma Distribution

$$\begin{aligned} f_X(x) &= \frac{(\frac{x}{b})^{c-1} exp(-\frac{x}{b})}{b\, \Gamma(c)} U(x) \\ E[X] &= \int_0^{\infty} x \frac{(\frac{x}{b})^{c-1} exp(-\frac{x}{b})}{b\, \Gamma(c)}\, dx \\ &= b\frac{\Gamma(c+1)}{\Gamma(c)} \int_0^{\infty} \frac{(\frac{x}{b})^{c} exp(-\frac{x}{b})}{b\, \Gamma(c+1)}\, dx \\ &= b\frac{\Gamma(c+1)}{\Gamma(c)} \\ \mu &= b\, c \end{aligned}$$

Second Moment

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{(\frac{x}{b})^{c-1} exp(-\frac{x}{b})}{b\, \Gamma(c)} U(x)\, dx$$

$$
\begin{aligned}
&= \int_0^\infty b^2 \frac{(\frac{x}{b})^{c+1} exp(-\frac{x}{b})}{b\,\Gamma(c)}\,dx \\
&= \frac{b^2\,\Gamma(c+2)}{\Gamma(c)} \int_0^\infty \frac{(\frac{x}{b})^{c+1} exp(-\frac{x}{b})}{b\,\Gamma(c+2)}\,dx \\
&= \frac{b^2\,\Gamma(c+2)}{\Gamma(c)} = c\,(c+1). \\
\sigma^2 &= E[X^2] - E[X]^2 \\
&= b^2 c\,(c+1) - b^2\,c^2 = b^2\,c.
\end{aligned}
$$

Problem 4.2

Let

$$
\begin{aligned}
N &= \text{number of trips until escape} \\
T &= \text{time to escaspe} = 2\,N\text{hours.}
\end{aligned}
$$

Then $E[T] = 2E[N]$ hours.

$$\Pr(N=n) = p(1-p)^{n-1}, \qquad n \geq 1 \qquad (p = \frac{1}{3}).$$

$$E[N] = \sum_{n=1}^{\infty} np(1-p)^{n-1} = -p\frac{d}{dp}\sum_{n=0}^{\infty}(1-p)^n = -p\frac{d}{dp}\left(\frac{1}{p}\right) = \frac{1}{p} = 3.$$

Therefore, $E[T] = 6$ hours.

Problem 4.3

Let N = number of packets transmitted until first success.

$$
\begin{aligned}
\Pr(N=n) &= q^{n-1}(1-q), \qquad n = 1, 2, 3, \ldots. \\
E[N] &= \sum_{n=1}^{\infty} nq^{n-1}(1-q) = (1-q)\sum_{n=1}^{\infty} nq^{n-1} \\
&= (1-q)\frac{d}{dq}\sum_{n=1}^{\infty} q^n = (1-q)\frac{d}{dq}\frac{1}{1-q} = \frac{1}{1-q}.
\end{aligned}
$$

Problem 4.4

$$\begin{aligned} T &= \text{total transmission time} \\ &= (N-1)T_i + NT_t = N(T_i+T_t) - T_i. \\ E[T] &= E[N](T_i+T_t) - T_i = \frac{T_i+T_t}{1-q} - T_i. \end{aligned}$$

Problem 4.5

Let $\frac{X-\mu_X}{\sigma_X} = Y$, then $Y \sim N(0,1)$

$$c_s = E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^3\right] = E\left[Y^3\right] \ . \tag{3}$$

$$c_k = E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)^4\right] = E\left[Y^4\right] \ . \tag{4}$$

The coefficient of skewness is calculated as follows:

$$E[Y^3] = \int_{-\infty}^{\infty} \frac{y^3}{\sqrt{2\pi}} \exp(-\frac{y^2}{2})dy = 0.$$

The integral is zero because the integrand is an odd function of y integrated over an interval syymetric about the origin.

The coefficient of kurtosis is calculated as follows:

$$E[Y^4] = \int_{-\infty}^{\infty} \frac{y^4}{\sqrt{2\pi}} \exp(-\frac{y^2}{2})dy.$$

Perform integration by parts once with $u = y^3$ and $dv = y \exp(-y^2/2)$ resulting in:

$$\int_{-\infty}^{\infty} \frac{y^4}{\sqrt{2\pi}} \exp(-\frac{y^2}{2})dy = 3\int_{-\infty}^{\infty} \frac{y^2}{\sqrt{2\pi}} \exp(-\frac{y^2}{2})dy.$$

The remaining integral represents the variance of a standard normal random variable and hence evaluates to 1. Therefore,

$$c_k = 3.$$

Problem 4.6

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

The central moments can be written as follows.

$$m_k = E[(X-\mu)^k] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-\mu)^k \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

Making a change of variable

$$\frac{x-\mu}{\sigma} = t$$

$$\Rightarrow m_k = \sigma^k \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^k \exp(-t^2/2)\, dt$$

When k is odd, the integrand is an odd function and the total area enclosed is zero. This implies the odd central moments are zero. For even $k \geq 2$, let

$$I_k = \int_{-\infty}^{\infty} t^k \exp(-t^2/2)\, dt.$$

Performing integration by parts with $u = t^{k-1}$ and $dv = t\exp(-t^2/2)dt$ produces

$$I_k = (k-1)\int_{-\infty}^{\infty} t^{k-2} \exp(-t^2/2)\, dt = (k-1)I_{k-2}.$$

With this recursion, it is seen that

$$I_k = (k-1)\cdot(k-3)\cdot(k-5)\ldots 3\cdot 1\cdot I_0.$$

Noting that $I_0 = \sqrt{2\pi}$ we get that for even $k \geq 2$

$$E[(X-\mu)^k] = \sigma^k (k-1)\cdot(k-3)\cdot(k-5)\ldots 3\cdot 1.$$

This can be written in a more compact form as

$$E[(X-\mu)^k] = \frac{\sigma^k\, k!}{(k/2)!\, 2^{k/2}}.$$

$$m_k = 2\sigma^k \sqrt{\frac{2^k}{\pi}} \int_0^\infty t^k \, e^{-t^2} \, dt$$

Substituting $t^2 = y$ we can rewrite this equation as

$$\begin{aligned} m_k &= 2\sigma^k \sqrt{\frac{2^k}{\pi}} \int_0^\infty y^{k/2} \, e^{-y} \, \frac{dy}{2\sqrt{y}} \\ &= \sigma^k \sqrt{\frac{2^k}{\pi}} \int_0^\infty y^{\frac{k-1}{2}} \, e^{-y} \, dy \\ &= \sigma^k \sqrt{\frac{2^k}{\pi}} \int_0^\infty y^{\frac{k+1}{2}-1} \, e^{-y} \, dy \\ &= \sigma^k \sqrt{\frac{2^k}{\pi}} \Gamma(\frac{k+1}{2}) \end{aligned}$$

Finally we can see that if we put $k = 2$ we should get the variance of the gaussian which is the same as σ^2

$$m_2 = \sigma^2 \sqrt{\frac{2^2}{\pi}} \frac{\sqrt{\pi}}{2} = \sigma^2$$

Problem 4.7

The Cauchy random variable has a PDF given by

$$f_X(x) = \frac{b/\pi}{b^2 + x^2} \ .$$

Since the PDF is symmetric about zero, the mean is zero. The second moment (and the variance) is

$$E[X^2] = \int_{-\infty}^{\infty} \frac{bx^2/\pi}{b^2 + x^2} dx.$$

As $x \to \infty$, the integrand approaches a constant (b/π) and hence this integral will diverge. Therefore the varinace of the Cauchy random variable is infinite (undefined).

Problem 4.8

$$\begin{aligned}
\mu_n &= \int_{-\infty}^{\infty} x^n f_X(x)\,dx \\
\mu = \mu_1 &= \int_{-\infty}^{\infty} x f_X(x)\,dx \\
c_n &= \int_{-\infty}^{\infty} (x-\mu)^n f_X(x)\,dx \\
&= \int_{-\infty}^{\infty} \sum_{k=0}^{n} \left\{ \left(\begin{array}{c} n \\ k \end{array} \right) x^k (-\mu)^{n-k} \right\} f_X(x)\,dx \\
&= \sum_{k=0}^{n} \left\{ \int_{-\infty}^{\infty} \left(\begin{array}{c} n \\ k \end{array} \right) x^k (-\mu)^{n-k} f_X(x)\,dx \right\} \\
&= \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \mu_k^k (-\mu)^{n-k}
\end{aligned}$$

Problem 4.9

(a)

$$E[2X-4] = 2E[X] - 4 = -2 \ .$$

(b)

$$E[X^2] = E[X]^2 + Var(X) = 5 \ .$$

(c)

$$\begin{aligned}
E[(2X-4)^2] &= E[(4X^2 + 16 - 16X)] \\
&= 4E[X^2] - 16E[X] + 16 = 20 \ .
\end{aligned}$$

Problem 4.10

$$\begin{aligned}
f_X(x) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \\
E[|X|] &= \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} |\mu + \sigma t| \, \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{t^2}{2}) \, dt \\
&= -\int_{-\infty}^{-\mu/\sigma} (\mu + \sigma t) \, \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{t^2}{2}) \, dt + \int_{-\mu/\sigma}^{\infty} (\mu + \sigma t) \, \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{t^2}{2}) \, dt \\
&= \frac{\sigma}{\sqrt{2\pi}} \left[\int_{-\mu/\sigma}^{\infty} t \exp(-\frac{t^2}{2}) \, dt - \int_{-\infty}^{-\mu/\sigma} t \exp(-\frac{t^2}{2}) \, dt \right] \\
&+ \mu \left[\int_{-\mu/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{t^2}{2}) \, dt - \int_{-\infty}^{-\mu/\sigma} \frac{1}{\sqrt{2\pi}} \exp(-\frac{t^2}{2}) \, dt \right] \\
&= \frac{\sigma}{\sqrt{2\pi}} \left[2 \exp\left(-\frac{\mu^2}{2\sigma^2} \right) \right] + \mu \left[Q(-\mu/\sigma) - (1 - Q(-\mu/\sigma)) \right] \\
E[|X|] &= \sqrt{\frac{2\sigma^2}{\pi}} \exp\left(-\frac{\mu^2}{2\sigma^2} \right) + \mu \left[1 - 2Q\left(\frac{\mu}{\sigma} \right) \right]
\end{aligned}
$$

Problem 4.11

$$
\begin{aligned}
E[X] &= \int_{-a/2}^{a/2} \frac{x}{a} dx = 0. \\
E[X^2] &= \int_{-a/2}^{a/2} \frac{x^2}{a} dx = \frac{a^2}{12} \Rightarrow \sigma^2 = \frac{a^2}{12}. \\
E[X^3] &= \int_{-a/2}^{a/2} \frac{x^3}{a} dx = 0. \\
E[X^4] &= \int_{-a/2}^{a/2} \frac{x^4}{a} dx = \frac{a^4}{80}.
\end{aligned}
$$

(a)

$$
c_s = E\left[\left(\frac{X - \mu}{\sigma} \right)^3 \right] = \frac{E[X^3]}{\sigma^3} = 0.
$$

(b)

$$
c_k = E\left[\left(\frac{X - \mu}{\sigma} \right)^4 \right] = \frac{E[X^4]}{\sigma^4} = \frac{a^4/80}{a^4/144} = \frac{144}{80} = \frac{9}{5}.
$$

(c) For a Gaussian random variable, $c_s = 0, c_k = 3$.

Problem 4.12

$$\int_0^\infty [1 - F_X(x)]dx = \int_0^\infty \int_y^\infty f_X(x)dxdy$$

By interchanging the two integrals, we have

$$\int_0^\infty [1 - F_X(x)]dx = \int_0^\infty f_X(x) \int_0^x dydx = \int_0^\infty x f_X(x)dx = E(x)$$

Problem 4.13

We know the pdf of a distribution can be written as sum of the conditional pdfs.

$$\begin{aligned}
f_X(x) &= \sum_{i=1}^{n} f_{X|A_i}(x)Pr(A_i) \\
E[X] &= \int_{-\infty}^{\infty} x f_X(x)\, dx \\
&= \int_{-\infty}^{\infty} x \sum_{i=1}^{n} f_{X|A_i}(x)Pr(A_i)\, dx
\end{aligned}$$

We can interchange the operations of the integration and summation as they are linear and rewrite the above equation as

$$\begin{aligned}
E[X] &= \sum_{i=1}^{n} \left(\int_{-\infty}^{\infty} x f_{X|A_i}(x)Pr(A_i)\, dx \right) \\
E[X] &= \sum_{i=1}^{n} \left(Pr(A_i) \int_{-\infty}^{\infty} x f_{X|A_i}(x)\, dx \right) \\
E[X] &= \sum_{i=1}^{n} Pr(A_i)E[X|A_i]
\end{aligned}$$

Problem 4.14

A convex function is one which has the property that for any α such that $0 \le \alpha \le 1$, and any $x_0 < x_1$

$$g(\alpha x_0 + (1 - \alpha)x_1) \le \alpha g(x_0) + (1 - \alpha)g(x_1).$$

Applying this property repeatedly, we can show that for any $x_0 < x_1 < x_2 < \ldots < x_n$ and any discrete distribution $\vec{p} = [p_0, p_1, p_2, \ldots, p_n]$ (i.e., $0 \le p_i \le 1$ and $\sum_{i=0}^{n} p_i = 1$) a convex function $g(x)$ will satisfy

$$g\left(\sum_{i=0}^{n} p_i x_i\right) \le \sum_{i=0}^{n} p_i g(x_i).$$

To start with, suppose X is a discrete random variable. Then we can choose p_i such that $p_i = Pr(X = x_i)$. In that case

$$\begin{aligned}\sum_{i=0}^{n} p_i x_i &= E[X] \\ \sum_{i=0}^{n} p_i g(x_i) &= E[g(X)]\end{aligned}$$

Hence, for discrete random variables

$$g(E[X]) \le E[g(X)].$$

Next, suppose X is a continuous random variable. In this case, let the x_i be a set of points evenly spaced by Δx and let $p_i = Pr(|X - x_i| < \Delta x/2)$. As $\Delta x \to 0$, $p_i \to f_X(x_i)\Delta x$. Therefore,

$$\begin{aligned}\lim_{\Delta x \to 0} g\left(\sum_{i=0}^{n} f_X(x_i) x_i \Delta x\right) &\le \lim_{\Delta x \to 0} \sum_{i=0}^{n} f_X(x_i) g(x_i) \Delta x \\ \Rightarrow g\left(\int x f_X(x) dx\right) &\le \int f_X(x) g(x) dx \\ \Rightarrow g\left(E[X]\right) &\le E[g(x)]dx\end{aligned}$$

Problem 4.15

$$f_\Theta(\theta) = \frac{1}{2\pi}$$

(a) $Y = \sin\theta$. This equation has two roots. At θ and $\pi - \theta$

$$f_Y(y) = \sum_{\theta_i} f_\Theta(\theta) \left|\frac{d\theta}{dy}\right|_{\theta=\theta_i}$$

$$\begin{aligned}
&= \left|\frac{1}{2\pi\cos\theta}\right|_{\theta=\theta} + \left|\frac{1}{2\pi\cos\theta}\right|_{\theta=\pi-\theta} \\
&= \frac{1}{\pi\cos\theta} \\
&= \frac{1}{\pi\sqrt{1-y^2}}
\end{aligned}$$

(b) $Z = \cos\theta$. This equation has two roots. At θ and $2\pi - \theta$

$$\begin{aligned}
f_Z(z) &= \sum_{\theta_i} f_\Theta(\theta)\left|\frac{d\theta}{dy}\right|_{\theta=\theta_i} \\
&= \left|\frac{1}{2\pi\sin\theta}\right|_{\theta=\theta} + \left|\frac{1}{2\pi\sin\theta}\right|_{\theta=2\pi-\theta} \\
&= \frac{1}{\pi\sin\theta} \\
&= \frac{1}{\pi\sqrt{1-y^2}}
\end{aligned}$$

(c) $W = \tan\theta$. This equation has two roots. At θ and $\pi + \theta$

$$\begin{aligned}
f_W(w) &= \sum_{\theta_i} f_\Theta(\theta)\left|\frac{d\theta}{dy}\right|_{\theta=\theta_i} \\
&= \left|\frac{1}{2\pi\sec^2\theta}\right|_{\theta=\theta} + \left|\frac{1}{2\pi\sec^2\theta}\right|_{\theta=\pi+\theta} \\
&= \frac{1}{\pi\sec^2\theta} \\
&= \frac{1}{\pi(\tan^2\theta+1)} \\
&= \frac{1}{\pi(w^2+1)}
\end{aligned}$$

Problem 4.16

For any value of $0 \le y \le a^2$, $y = x^2$ has two real roots, namely $x = \pm\sqrt{y}$ Then,

$$f_Y(y) = \left[\frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}\right]\left[U(y) - U(y-a^2)\right]$$

$$
\begin{aligned}
&= \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}} \left[U(y) - U(y-a^2)\right] \\
&= \frac{1}{2a\sqrt{y}} \left[U(y) - U(y-a^2)\right] \; .
\end{aligned}
$$

Problem 4.17

$$f_X(x) = 2\, e^{-2x}\, u(x).$$

(a)

$$X \geq 0, Y = 1 - X \Rightarrow Y \leq 1.$$

(b)

$$f_Y(y) = \left.\frac{2\, e^{-2x} u(x)}{|-1|}\right|_{x=1-y} = 2\, \exp(-2(1-y))\, u(1-y).$$

Problem 4.18

$$
\begin{aligned}
f_X(x) &= \frac{1}{\sqrt{2\pi}}\, exp(-\frac{x^2}{2}) \\
Y &= |X|
\end{aligned}
$$

The equation $Y = |X|$ has two roots at $\pm X$ at all x except at $x = 0$

$$
\begin{aligned}
f_Y(y) &= \sum_{x_i} f_X(x) \left|\frac{dx}{dy}\right|_{x=x_i} \\
&= \sum_{x_i} \frac{1}{\sqrt{2\pi}}\, exp(-\frac{x^2}{2}) \left|\frac{dx}{dy}\right|_{x=x_i} \\
&= \sqrt{\frac{1}{2\pi}}\, exp(-\frac{x^2}{2}) + \sqrt{\frac{1}{2\pi}}\, exp(-\frac{x^2}{2}) \\
&= \sqrt{\frac{2}{\pi}}\, exp(-\frac{x^2}{2}) \\
&= \sqrt{\frac{2}{\pi}}\, exp(-\frac{y^2}{2})
\end{aligned}
$$

Strictly speaking we cannot find the pdf for $f_Y(y = 0)$ using the above equation because $Y = |X|$ is not differentiable at $x = 0$. But we can see that $f_Y(y = 0) = f_X(x = 0) = \sqrt{\frac{1}{2\pi}}$

$$f_Y(y) = \begin{cases} \sqrt{\frac{2}{\pi}} exp(-\frac{y^2}{2}) & y \neq 0 \\ \sqrt{\frac{1}{2\pi}} & y = 0 \end{cases}$$

Problem 4.19

For $X > 0$ (and hence $Y > 0$), $Y = X$ and thus $f_Y(y) = f_X(y)$. For $X < 0$, $Y = 0$ and hence $\Pr(Y = 0) = \Pr(X < 0) = 1 - Q(0) = 1/2$. So the PDF of Y is of the form

$$f_Y(y) = \frac{1}{2}\delta(y) + \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) u(y).$$

Problem 4.20

$$\begin{aligned} f_Y(y) &= \Pr(X > 1)\delta(y-2) + \Pr(X < 1)\delta(y+2) + 1/2 f_X(y/2)u(y+2)u(2-y) \\ &= Q(1/\sigma_X)[\delta(y-2) + \delta(y+2)] + \frac{1}{2\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{y^2}{8\sigma_X^2}\right). \end{aligned}$$

Problem 4.21

$$f_Y(y) = \left.\frac{f_X(x)}{\left|\frac{dy}{dx}\right|}\right|_{x=y^{1/3}} = \left.\frac{\frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{x^2}{2\sigma_X^2}\right)}{|3x^2|}\right|_{x=y^{1/3}} = \frac{1}{3y^{2/3}\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{y^{2/3}}{2\sigma_X^2}\right).$$

Problem 4.22

(a) $X \in \left[-\frac{1}{2}, \frac{1}{2}\right)$.
(b) X is uniform over $\left[-\frac{1}{2}, \frac{1}{2}\right)$.
(c) $E[X^2] = \int_{-1/2}^{1/2} x^2 dx = \frac{1}{12}$.

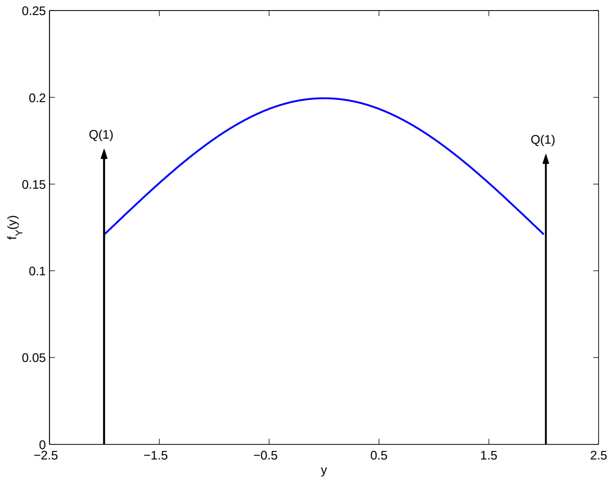

Figure 1: PDF for Problem 4.20; $\sigma_X = 1$.

Problem 4.23

(a)

$$\begin{aligned}\Pr(Y=0) &= \Pr(X<0) = \frac{1}{2}.\\ \Pr(Y=1) &= \Pr(X>0) = \frac{1}{2}.\end{aligned}$$

(b)

$$\begin{aligned}\Pr(Y=0) &= \Pr(X<0) = 1 - Q\left(-\frac{1}{2}\right) = Q\left(\frac{1}{2}\right) = 0.3085.\\ \Pr(Y=1) &= \Pr(X>0) = Q\left(-\frac{1}{2}\right) = 1 - Q\left(\frac{1}{2}\right) = 0.6915.\end{aligned}$$

Problem 4.24

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|\Bigg|_{x=g^{-1}(y)} = \frac{1}{y^2}f_X(x)\Bigg|_{x=\frac{1}{y}} = \frac{\frac{b/\pi}{b^2+1/y^2}}{y^2} = \frac{b/\pi}{b^2y^2+1}\ .$$

Problem 4.25

$$\begin{aligned} f_X(x) &= \frac{x^{c-1}exp(-\frac{x}{2})}{2^c\Gamma(c)} \\ Y &= \sqrt{X} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= f_X(x)\left|\frac{dx}{dy}\right| \\ &= \frac{x^{c-1}\exp(-\frac{x}{2})}{2^c\Gamma(c)}(2\sqrt{x}) \\ &= \frac{x^{c-1}\exp(-\frac{x}{2})}{2^{c-1}\Gamma(c)}\sqrt{x} \\ &= \frac{(y^2)^{c-1}\exp(-\frac{y^2}{2})}{2^{c-1}\Gamma(c)}y \\ f_Y(y) &= \frac{y^{2c-1}\exp(-\frac{y^2}{2})}{2^{c-1}\Gamma(c)}u(y) \end{aligned}$$

Problem 4.26

For the transformation from arbitrary to uniform, we want

$$f_Y(y) = \frac{f_X(x)}{\left|\frac{dg}{dx}\right|}\Bigg|_{x=g^{-1}(y)} = 1.$$

One obvious way to achieve this would be to select the transformation such that

$$\frac{dg}{dx} = f_X(x) \Rightarrow g(x) = F_X(x).$$

Hence $Y = F_X(x)$ transforms $X \sim f_X(x)$ to $Y \sim$ uniform$(0,1)$. For the transformation from uniform to abitrary, just use this result in reverse. $Y = F_Y^{-1}(X)$, will transform $X \sim$ uniform$(0,1)$ to $Y \sim f_Y(y)$.

Problem 4.27

From the previous problem, the transformations should be chosen according to $Y = F_Y^{-1}(X)$.
(a) Exponential Distribution

$$f_Y(y) = b\,e^{-b\,y}\,u(y) \Rightarrow F_Y(y) = 1 - \exp(-b\,y).$$
$$X = 1 - \exp(-b\,Y) \Rightarrow Y = -\frac{1}{b}\ln(1-X).$$

Note that since X is uniform over $(0,1)$, $1-X$ will be as well. Hence $Y = -\frac{1}{b}\ln(X)$ will work also.
(b)Rayleigh Distribution

$$f_Y(y) = \frac{y}{\sigma^2}\exp\left(-\frac{y^2}{2\sigma^2}\right)u(y) \Rightarrow F_Y(y) = 1 - \exp\left(-\frac{y^2}{2\sigma^2}\right).$$
$$X = 1 - \exp\left(-\frac{Y^2}{2\sigma^2}\right) \Rightarrow Y = \sqrt{-2\sigma^2\ln(1-X)} \text{ or } \sqrt{-2\sigma^2\ln(X)}.$$

(c)Cauchy Distribution

$$f_Y(y) = \frac{b/\pi}{b^2+y^2} \Rightarrow F_Y(y) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{y}{b}\right).$$
$$X = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{Y}{b}\right). \Rightarrow Y = b\tan(\pi x - \pi/2) = -b\cot(\pi x).$$

(d)Geometric Distribution

$$\begin{aligned} f_Y(y) &= \sum_{k=0}^{\infty}(1-p)p^k\delta(y-k) \\ F_Y(y) &= \sum_{k=0}^{\infty}\beta_k u(y-k), \end{aligned}$$

where

$$\beta_k = \sum_{m=0}^{k}\Pr(X=m) = \sum_{m=0}^{k}(1-p)\,p^k = 1 - p^{k+1}.$$

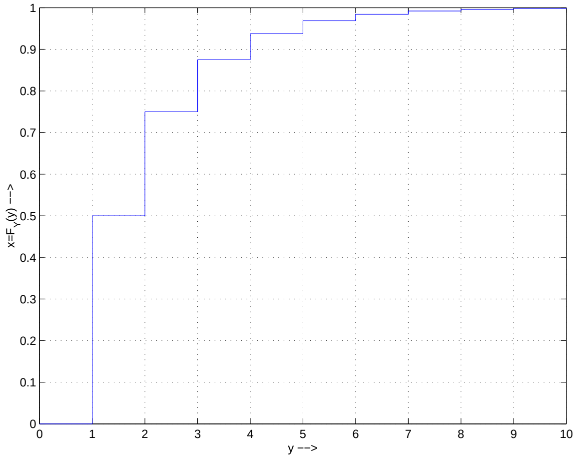

Figure 2: X vs Y for the Geometric Dist.

This is a staircase function and its inverse is also a staircase function as shown in Figure 2. The inverse function can be expressed in terms of step functions as:

$$F_Y^{-1}(x) = \sum_{k=0}^{\infty} u(y - \beta_k).$$

Y can be written in a little more convenient form as follows:

$$Y = \sum_{k=0}^{\infty} u(X - (1 - p^k)).$$

Note: This transformation works for any integer valued discrete rand variable. The only change would be the values of the β_k.
(e) Poisson Distribution - Using the results of part (d):

$$Y = F_Y^{-1}(x) = \sum_{k=0}^{\infty} u(y - \beta_k),$$

where in this case,

$$\beta_k = \sum_{m=0}^{k} \Pr(X = m) = \sum_{m=0}^{k} \frac{b^m}{m!} e^{-b}.$$

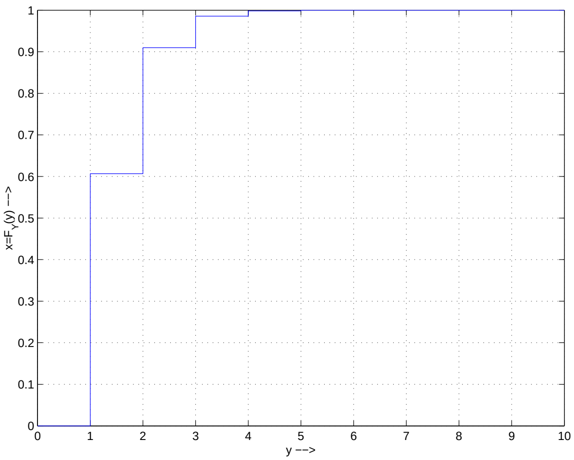

Figure 3: X vs Y for the Poisson Dist.

Problem 4.28

$$\Phi_Y(\omega) = E\left[e^{j\omega Y}\right] = E\left[e^{j\omega(aX+b)}\right] = e^{j\omega b} E\left[e^{j\omega a X}\right] = e^{j\omega b}\Phi_X(a\omega).$$

Problem 4.29

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx$$

(a)

$$\Phi_X^*(\omega) = \int_{-\infty}^{\infty} f_X^*(x)\, e^{-j\omega x}\, dx$$

Since the pdf is a real function its complex conjuate will be the same as itself. Using this we can rewrite the above equation as

$$\begin{aligned}\Phi_X^*(\omega) &= \int_{-\infty}^{\infty} f_X(x)\, e^{-j\omega x}\, dx \\ \Phi_X^*(\omega) &= \Phi_X(-\omega)\end{aligned}$$

(b)

$$\begin{aligned}\Phi_X(\omega) &= \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx \\ \Phi_X(0) &= \int_{-\infty}^{\infty} f_X(x)\, dx \\ \Phi_X(0) &= 1\end{aligned}$$

The integral on the RHS evaluating to 1 as it is a valid pdf.
(c)

$$
\begin{aligned}
\Phi_X(\omega) &= \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx \\
|\Phi_X(\omega)| &= \left| \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx \right| \\
&\leq \int_{-\infty}^{\infty} \left| f_X(x)\, e^{j\omega x} \right|\, dx \\
&\leq \int_{-\infty}^{\infty} f_X(x)\, dx \text{ because } |e^{j\omega x}| \leq 1 \\
&\leq 1 \text{ because } \textstyle\int_{-\infty}^{\infty} f_X(x) = 1 \text{ as } f_X(x) \text{ is a pdf}
\end{aligned}
$$

(d) If $f_X(x)$ is even,

$$
\begin{aligned}
f_X(-x) &= f_X(x) \\
\Phi_X(\omega) &= \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx \\
\Phi_X^*(\omega) &= \int_{-\infty}^{\infty} f_X(x)\, e^{-j\omega x}\, dx
\end{aligned}
$$

If we make a change of variable and put $t = -x$ we get

$$
\begin{aligned}
\Phi_X^*(\omega) &= -\int_{\infty}^{-\infty} f_X(-t)\, e^{j\omega t}\, dt \\
\Phi_X^*(\omega) &= \int_{-\infty}^{\infty} f_X(t)\, e^{j\omega t}\, dt \\
\Phi_X^*(\omega) &= \Phi_X(\omega)
\end{aligned}
$$

Since both the $\Phi_X(\omega)$ and its complex conjugate are equal it must be real.

(e) Let us assume that the $\Phi_X(\omega)$ is purely imaginary. Then $\Phi_X(0)$ is also purely imaginary. But we know that $\Phi_X(0) = 1$. This cannot be possible if $\Phi_X(\omega)$ were purely imaginary. So our assumption about $\Phi_X(\omega)$ being purely imaginary must be false and $\Phi_X(\omega)$ cannot be purely imaginary.

Problem 4.30

For an integer valued discrete random variable X,

$$\phi_X(2\pi n) = E\left[e^{j2\pi nX}\right] = \sum_k e^{j2\pi nk}\,\Pr(X=k).$$

Since $e^{j2\pi nk} = 1$ for any integers n and k, we have

$$\phi_X(2\pi n) = \sum_k \Pr(X=k) = 1.$$

To prove the reverse, note that since $|e^{j2\pi nX}| \leq 1$, the only way we can get $E[e^{j2\pi nX}] = 1$ would be if $e^{j2\pi nX} = 1$ with probability 1. This means that $2\pi nX$ must always be some multiple of 2π or equivalently nX must be an integer for any integer n. This will only happen if X takes on only integer values.

Problem 4.31

(a) Characteristic Function

$$\begin{aligned}
f_X(x) &= \frac{1}{2b}exp(-\frac{|x|}{b}) \\
\Phi_X(\omega) &= \int_{-\infty}^{\infty} \frac{1}{2b}e^{-\frac{|x|}{b}}\,e^{j\omega x} \\
&= \int_{-\infty}^{\infty} \frac{1}{2b}e^{-\frac{|x|}{b}}\,e^{j\omega x} \\
&= \int_{-\infty}^{0} \frac{1}{2b}e^{\frac{x}{b}}\,e^{j\omega x} + \int_{0}^{\infty} \frac{1}{2b}e^{-\frac{x}{b}}\,e^{j\omega x} \\
&= \frac{1}{2b}\left(\left[\frac{e^{\frac{x}{b}+j\omega x}}{\frac{1}{b}+j\omega}\right]\Bigg|_{x=-\infty}^{x=0} + \left[\frac{e^{-\frac{x}{b}+j\omega x}}{-\frac{1}{b}+j\omega}\right]\Bigg|_{x=0}^{x=\infty}\right) \\
&= \frac{1}{2b}\left(\frac{1}{\frac{1}{b}+j\omega} - \frac{1}{-\frac{1}{b}+j\omega}\right) \\
&= \frac{1}{2b}\left(\frac{1}{\frac{1}{b}+j\omega} + \frac{1}{\frac{1}{b}-j\omega}\right) \\
&= \frac{1}{1+b^2\omega^2}
\end{aligned}$$

(b) Taylor Series Expansion of $\Phi_X(\omega)$.

$$\begin{aligned}\Phi_X(\omega) &= \frac{1}{1+b^2\omega^2} \\ &= \sum_{k=0}^{\infty}(-1)^k(b\omega)^{2k}\end{aligned}$$

(c) k^{th} Moment of X.

$$E[X^k] = (-j)^k \frac{d^k\Phi_X(\omega)}{d\omega^k}\bigg|_{\omega=0}$$

$$\Phi_X(\omega) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{d^k\Phi_X(\omega)}{d\omega^k}\bigg|_{\omega=0}\right)\omega^k = \sum_{m=0}^{\infty}(-1)^m b^{2m}\omega^{2m}$$

Since there are no odd powers in the Taylor series expansion of $\Phi_X(\omega)$, all odd moments of X are zero. For even values of k, we note from the above expressions that

$$\begin{aligned}\frac{1}{(2k)!}\left(\frac{d^{2k}\Phi_X(\omega)}{d\omega^{2k}}\bigg|_{\omega=0}\right) &= (-1)^k b^{2k} \\ \Rightarrow \frac{d^k\Phi_X(\omega)}{d\omega^k}\bigg|_{\omega=0} &= (k!)j^k b^k \\ \Rightarrow E[X^k] &= (k!)b^k.\end{aligned}$$

Problem 4.32

$$h_2 = E[X(X-1)] = E[X^2] - E[X] = E[X^2] - h_1.$$

Then we have $E[X^2] = h_1 + h_2$. Hence the variance is

$$\sigma_X^2 = E[X^2] - \mu_X^2 = h_1 + h_2 - h_1^2\ .$$

Problem 4.33

$$H_X(z) = \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{d^k H_X(z)}{dz^k}\bigg|_{z=1}\right)(z-1)^k = \sum_{k=0}^{\infty}\frac{1}{k!}h_k(z-1)^k.$$

If the Taylor Series coefficients are η_k, that is

$$H_X(z) = \sum_{k=0}^{\infty} \eta_k (z-1)^k,$$

then $h_k = (k!)\eta_k$.

Problem 4.34

Poisson:

$$H_X(z) = \sum_{k=0}^{\infty} \frac{\mu^k}{k!} e^{-\mu} z^k = e^{-\mu} \sum_{k=0}^{\infty} \frac{(\mu z)^k}{k!} = e^{-\mu} e^{\mu z} = e^{\mu(z-1)}$$

.

Binomial:

$$\begin{aligned} H_X(z) &= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} z^k = \sum_{k=0}^{n} \binom{n}{k} (pz)^k (1-p)^{n-k} \\ &= (pz + 1 - p)^n = (1 + p(z-1))^n. \end{aligned}$$

Let $\mu = np$.

$$\Rightarrow H_X(z) = \left(1 + \frac{\mu(z-1)}{n}\right)^n$$

As $n \to \infty$

$$\lim_{n\to\infty} H_X(z) = \lim_{n\to\infty} \left(1 + \frac{\mu(z-1)}{n}\right)^n = \exp(\mu(z-1))$$

Problem 4.35

(a)

$$\begin{aligned} H_X(z) &= \sum_{k=0}^{\infty} P_X(k) z^k \\ &= \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} e^{-\alpha} z^k \\ &= e^{-\alpha} \sum_{k=0}^{\infty} \frac{(\alpha z)^k}{k!} \\ &= e^{\alpha(z-1)} . \end{aligned}$$

(b)

$$e^{\alpha(z-1)} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}(z-1)^k \ .$$

(c) Noting that

$$H_X(z) = \sum_{k=0}^{\infty} \frac{h_k}{k!}(z-1)^k,$$

we get $h_k = \alpha^k$.

Problem 4.36

$$\begin{aligned} H_X(z) &= \frac{1}{n}\frac{1-z^n}{1-z} \\ &= \frac{1}{n}\sum_{k=0}^{n-1} z^k \end{aligned}$$

We know that

$$\begin{aligned} H_X(z) &= \sum_{k=0}^{\infty} P_X(X=k) z^k \\ &= \frac{1}{n}\sum_{k=0}^{n-1} z^k = \sum_{k=0}^{n-1} \frac{1}{n}\, z^k \end{aligned}$$

Recognizing that the coeffecient of z^k in the above equation is the $P_X(X=k)$ we get the PMF of the distribution as

$$P_X(X=k) = \begin{cases} \frac{1}{n} & k = 0, 1, 2, \ldots n-1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 4.37

$$\begin{aligned} M_X(u) &= E[e^{uX}] = \int_0^{\infty} f_X(x) e^{ux} dx \\ &= \int_0^{\infty} x e^{-\frac{x^2}{2}} e^{ux} dx \end{aligned}$$

$$= \int_0^\infty x e^{-\frac{1}{2}(x-u)^2} e^{\frac{1}{2}u^2} dx \tag{5}$$

Let $t = x - u$, we have

$$\begin{aligned}
M_X(u) &= e^{\frac{1}{2}u^2}\left(\int_{-u}^\infty t e^{-\frac{1}{2}t^2} dt + u\int_{-u}^\infty e^{-\frac{1}{2}t^2} dt\right) \\
&= e^{\frac{1}{2}u^2}\left(e^{-\frac{1}{2}u^2} + u\int_{-u}^\infty e^{-\frac{1}{2}t^2} dt\right) \\
&= 1 + e^{\frac{1}{2}u^2} u\sqrt{2\pi} Q(-u) \\
&= 1 + e^{\frac{1}{2}u^2} u\sqrt{2\pi}[1 - Q(u)] \qquad (6)
\end{aligned}$$

Problem 4.38

$$\begin{aligned}
f_X(x) &= x\exp(-\frac{x^2+a^2}{2})\, Io(ax)\, U(x) \\
E[e^{uX^2}] &= \int_0^\infty e^{ux^2}\, x\, \exp(-\frac{x^2+a^2}{2})\, Io(ax)\ dx \\
&= \int_0^\infty x\ \exp(-\frac{x^2 - 2ux^2 + a^2}{2})\, Io(ax)\, dx
\end{aligned}$$

Put $t^2 = x^2(1-2u)$.

$$\begin{aligned}
E[e^{uX^2}] &= \int_0^\infty \frac{t}{\sqrt{1-2u}}\ \exp(-\frac{t^2+a^2}{2})\, Io(\frac{at}{\sqrt{1-2u}})\, \frac{dt}{\sqrt{1-2u}} \\
&= \frac{1}{1-2u}\ \exp(-\frac{a^2 - \frac{a^2}{(1-2u)}}{2}) \int_0^\infty t\ \exp(-\frac{t^2 + \frac{a^2}{(1-2u)}}{2})\, Io(\frac{at}{\sqrt{1-2u}})\, dt \\
&= \frac{1}{1-2u}\ \exp(\frac{ua^2}{1-2u}) \int_0^\infty t\ \exp(-\frac{t^2 + \frac{a^2}{(1-2u)}}{2})\, Io(\frac{at}{\sqrt{1-2u}})\, dt \\
&= \frac{1}{1-2u}\ \exp(\frac{ua^2}{1-2u})
\end{aligned}$$

The last step is accomplished by noting that the remaining integrand is a properly normalized Rician PDF and hence must integrate to one.

Problem 4.39

Note that $|X-\mu| \geq \epsilon$ is equivalent to $|X-\mu|^n \geq \epsilon^n$. Applying Markov's inequality results in

$$\Pr\left(|X-\mu|^n \geq \epsilon^n\right) \leq \frac{E[|X-\mu|^n]}{\epsilon^n}.$$

Problem 4.40

$$\begin{aligned}
\Pr(X \leq x_0) &= \int_{-\infty}^{x_0} f_X(x)dx = \int_{-\infty}^{\infty} f_X(x)u(x_0-x)dx \\
&\leq \int_{-\infty}^{\infty} f_X(x)\exp(u(x-x_0))dx \qquad (u \leq 0) \\
&= \exp(-ux_o)\int_{-\infty}^{\infty} f_X(x)\exp(ux)dx \\
&= \exp(-ux_o)M_X(u).
\end{aligned}$$

The bound is tightened by finding the value of u which makes the right hand side as small as possible. Therefore,

$$Pr(X \leq x_0) \leq \min_{u\leq 0} \exp(-ux_o)M_X(u).$$

Problem 4.41

$$M_X(s) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}e^{-\alpha}e^{sk} = \sum_{k=0}^{\infty} \frac{(\alpha e^s)^k}{k!} = \exp(\alpha e^s)\exp(-\alpha) = \exp(\alpha(e^s-1)).$$

$$\begin{aligned}
\Pr(X \geq n_0) &\leq \min_{s\geq 0} e^{-sn_0}M_X(s) \\
e^{-sn_0}M_X(s) &= \exp(-sn_0+\alpha(e^s-1)) \\
\frac{d}{ds}e^{-sn_0}M_X(s) &= \exp(-sn_0+\alpha(e^s-1))[-n_0+\alpha e^s] = 0 \\
\Rightarrow n_0 &= \alpha e^s \\
s &= \ln(n_0/\alpha).
\end{aligned}$$

This will be the minimizing value of s provided that $n_0 \geq \alpha$. In which case,

$$\begin{aligned}
\Pr(X \geq n_0) &\leq \exp(-n_0\ln(n_0/\alpha)+n_0-\alpha) \\
&= \left(\frac{\alpha}{n_0}\right)^{n_0}\exp(n_0-\alpha) \qquad (\text{for } n_0 \geq \alpha).
\end{aligned}$$

Problem 4.42

$$M_X(u) = \frac{1}{b\Gamma(c)}\int_0^\infty \left(\frac{x}{b}\right)^{c-1} e^{-x/b}e^{ux}dx = \frac{1}{b\Gamma(c)}\int_0^\infty \left(\frac{x}{b}\right)^{c-1}\exp(-x(1/b-u))dx.$$

Let $y = x(1/b - u) \Rightarrow dy = dx(1/b - u)$. Then the moment generating function is given by

$$M_X(u) = \frac{1}{(1-bu)^c\Gamma(c)}\int_0^\infty y^{c-1}\exp(-y)dy = \frac{1}{(1-bu)^c}.$$

The Chernoff bound is then computed as follows:

$$\begin{aligned}
\Pr(X \le x_o) &\le \min_{s\ge 0} e^{-sx_o}M_X(s) \\
e^{-sx_o}M_X(s) &= \frac{e^{-sx_o}}{(1-bu)^c} \\
\frac{d}{ds}e^{-sx_o}M_X(s) &= \frac{e^{-sx_o}}{(1-bu)^c}\left[-x_0 + \frac{bc}{1-bs}\right] = 0
\end{aligned}$$

Hence, the minimizing value of s is given by

$$s = \frac{x_0 - bc}{x_0 b}$$

provided that $(x_0 > bc)$. Plugging in this value produces

$$\begin{aligned}
e^{-sx_o} &= \exp\left(-\frac{x_0}{b} + c\right) \\
M_X(s) &= \frac{1}{\left(1 - \frac{x_0 - bc}{x_0}\right)^c} = \left(\frac{x_0}{bc}\right)^c \\
\Pr(X \le x_o) &\le \left(\frac{x_0}{bc}\right)^c \exp\left(-\frac{x_0}{b} + c\right) \qquad (x_0 > bc).
\end{aligned}$$

Problem 4.43

$$\begin{aligned}
M_X(u) &= \frac{1}{(1-u)^n} \qquad \text{(from results of the last exercise).} \\
\Psi(u) &= \ln(M_X(u)) = -n\ln(1-u)
\end{aligned}$$

$$\begin{aligned}
\lambda(u) &= \Psi(u) - ux_0 - \ln(u) = -n\,\ln(1-u) - ux_0 - \ln(u) \\
\lambda'(u) &= \frac{n}{1-u} - x_0 - \frac{1}{u} \\
\lambda''(u) &= \frac{n}{(1-u)^2} + \frac{1}{u^2}
\end{aligned}$$

The saddlepoint is given by

$$\lambda'(u) = 0 \Rightarrow \frac{n}{1-u} - x_0 - \frac{1}{u} = 0$$

This is a quadratic equation in u whose roots are given by

$$u_0 = \frac{x_0 - 1 - n \pm \sqrt{(n+1-x_0)^2 + 4x_0}}{2x_0}.$$

Note we take the negative square root so that the saddlepoint satisfies $u_0 < 0$. In summary, the saddlepoint approximation is

$$\begin{aligned}
\Pr(X \le x_0) &\approx -\frac{M_X(u_0)\exp(-u_0 x_0)}{u_0\sqrt{2\pi\lambda''(u_0)}} \\
u_0 &= \frac{n + x_0 - 1 \pm \sqrt{(n+x_0-1)^2 + 4x_0}}{2x_0} \\
M_X(u) &= \frac{1}{(1-u)^n} \\
\lambda''(u) &= \frac{n}{(1-u)^2} + \frac{1}{u^2}
\end{aligned}$$

The exact value of the tail probability is given by

$$\Pr(X \le x_0) = 1 - e^{-x_0}\sum_{k=0}^{n-1}\frac{x_0^k}{k!}.$$

Figure 4 shows a comparison of the exact probability and the saddlepoint approximation.

Problem 4.44

The tail probability can be evaluated according to

$$\Pr(X > x_0) = \Pr(X^2 > x_0^2) \approx \frac{M_{X^2}(u_0)\exp(-ux_0^2)}{u_0\sqrt{2\pi\lambda''(u_0)}}.$$

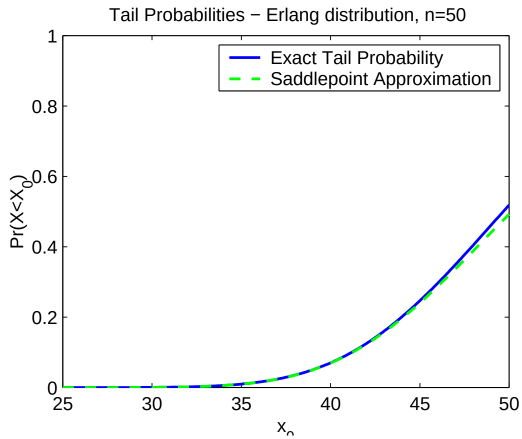

Figure 4: Tail Probabilities for Problem 4.43; $n = 50$.

The necessary functions are calculated as follows:

$$
\begin{aligned}
M_{X^2}(u) &= \frac{\exp\left(\frac{ua^2}{1-2u}\right)}{1-2u} \qquad \text{(from results of Problem 4.38)} \\
\lambda(u) &= -\ln(1-2u) + \frac{ua^2}{1-2u} - ux_0^2 - \ln(u) \\
\lambda'(u) &= \frac{2}{1-2u} + \frac{a^2}{(1-2u)^2} - x_0^2 - \frac{1}{u} \\
\lambda''(u) &= \frac{4}{(1-2u)^2} + \frac{4a^2}{(1-2u)^3} + \frac{1}{u^2}
\end{aligned}
$$

The saddlepoint is the solution to

$$\lambda'(u) = \frac{2}{1-2u} + \frac{a^2}{(1-2u)^2} - x_0^2 - \frac{1}{u} = 0$$

$$\Rightarrow u[2(1-2u) + a^2 - x_0^2(1-2u)^2] - (1-2u)^2 = 0.$$

Since this equation is cubic in u, we would need to solve it numerically. Once the necessary root is found (we must take care to find a root which is positive), the saddlepoint approximation is then found according to:

$$\Pr(X > x_0) \approx \frac{\exp\left(\frac{ua^2}{1-2u} - ux_0^2\right)}{u(1-2u)\sqrt{2\pi\left(\frac{4}{(1-2u)^2} + \frac{4a^2}{(1-2u)^3} + \frac{1}{u^2}\right)}}$$

$$= \frac{\exp\left(\frac{ua^2}{1-2u} - ux_0^2\right)}{\sqrt{2\pi\left(4u^2 + \frac{4a^2u^2}{(1-2u)} + (1-2u)^2\right)}}.$$

Problem 4.45

(a)

$$H(X) = \frac{1}{2}\log_2(2) + 2 \cdot \frac{1}{4}\log_2(4) = \frac{3}{2} = 1.5\text{bits/symbol}.$$

(b)

Symbol	Probability	Codeword	Length
1	1/2	(1)	1
2	1/4	(01)	2
3	1/4	(00)	2

The average codeword length is

$$\overline{L} = \frac{1}{2} \cdot 1 + 2 \cdot \frac{1}{4} \cdot 2 = 1.5\text{bits/symbol}.$$

Solutions to Chapter 5 Exercises

Problem 5.1

$$
\begin{aligned}
f_{X,Y}(x,y) &= ab\, e^{-(ax+by)} u(x)\, u(y) \\
F_{X,Y}(x,y) &= \int_0^x \int_0^y ab\, e^{-(ax+by)} dx\, dy
\end{aligned}
$$

We can write this integral as follows

$$
\begin{aligned}
F_{X,Y}(x,y) &= ab \int_0^x e^{-ax} dx \int_0^y e^{-by}\, dy \\
&= ab \left[-\frac{e^{-ax}}{a} \right] \Bigg|_{x=0}^{x=x} \left[-\frac{e^{-by}}{b} \right] \Bigg|_{y=0}^{y=y} \\
&= (1 - e^{-ax})(1 - e^{-by})
\end{aligned}
$$

(b) Marginal PDFs

$$
\begin{aligned}
f_X(x) &= \int_0^\infty f_{X,Y}(x,y)\, dy \\
&= \int_0^\infty ab\, e^{-(ax+by)}\, dy \\
&= ab\, e^{-ax} \left[-\frac{e^{-by}}{b} \right] \Bigg|_{y=0}^{y=\infty} \\
&= a\, e^{-ax}
\end{aligned}
$$

Because of the symmetry we see in the equation we can say that the marginal pdf of Y would be

$$
f_Y(y) = b\, e^{-by}
$$

(c) $Pr(X > Y)$

$$
Pr(X > Y) = \int_0^\infty \int_y^\infty ab\, e^{-(ax+by)} dx\, dy
$$

$$
\begin{aligned}
&= \int_0^\infty ab\, e^{-by}\, dy \left[-\frac{e^{-ax}}{a} \right] \Big|_{x=y}^{x=\infty} \\
&= \int_0^\infty b\, e^{-by}\, dy \left[-e^{-ax} \right] \Big|_{x=y}^{x=\infty} \\
&= \int_0^\infty b\, e^{-by}\, dy e^{-ay} \\
&= b \left[-\frac{e^{-(by+ay)}}{a+b} \right] \Big|_{y=0}^{y=\infty} \\
&= \frac{b}{a+b}
\end{aligned}
$$

(d) $Pr(X > Y^2)$

$$
\begin{aligned}
Pr(X > Y) &= \int_0^\infty \int_{y^2}^\infty ab\, e^{-(ax+by)} dx\, dy \\
&= \int_0^\infty ab\, e^{-by}\, dy \left[-\frac{e^{-ax}}{a} \right] \Big|_{x=y^2}^{x=\infty} \\
&= \int_0^\infty b\, e^{-by}\, dy \left[-e^{-ax} \right] \Big|_{x=y^2}^{x=\infty} \\
&= \int_0^\infty b\, e^{-by}\, dy\, e^{-ay^2} \\
&= \int_0^\infty b\, e^{-by}\, dy\, e^{-ay^2} \\
&= b \int_0^\infty e^{-(by+ay^2)}\, dy \\
&= be^{\frac{b^2}{4a}} \int_0^\infty e^{-a(y+\frac{b}{2a})^2}\, dy \\
&= be^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}} Q(\frac{0+b/2a}{\sqrt{1/2a}}) \\
&= be^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}} Q(\frac{b}{\sqrt{2a}})
\end{aligned}
$$

Problem 5.2

(a) Since

$$
\int_0^\infty \int_0^\infty \frac{d}{(ax+by+c)^n} u(x)u(y) dx dy = 1 \ , \tag{1}
$$

we should consider the cases for different n. When $n = 1$ or $n = 2$, the integration in (1) can not be 1. Hence we only consider when $n \geq 3$. Since

$$\int_0^\infty \int_0^\infty \frac{d}{(ax+by+c)^n} u(x)u(y)dxdy = \frac{d}{(n-1)(n-2)abc^{n-2}},$$

we have $d = (n-1)(n-2)abc^{n-2}$, $n \geq 3$.

(b)

$$\begin{aligned} f_X(x) &= \int_0^\infty \frac{d}{(ax+by+c)^n} u(y)dy \\ &= \frac{d}{b(n-1)(ax+c)^{n-1}} \\ &= \frac{(n-2)ac^{n-2}}{(ax+c)^{n-1}} \ . \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_0^\infty \frac{d}{(ax+by+c)^n} u(x)dx \\ &= \frac{(n-2)bc^{n-2}}{(by+c)^{n-1}} \ . \end{aligned}$$

(c)

$$\begin{aligned} Pr(X > Y) &= \int_0^\infty \int_y^\infty \frac{d}{(ax+by+c)^n} u(x)u(y)dxdy \\ &= \int_0^\infty \frac{d}{a(n-1)(ay+by+c)^{n-1}} dy \\ &= \frac{b}{(a+b)} \ . \end{aligned}$$

Problem 5.3

$$f_{X,Y}(x,y) = d \exp(-(ax^2 + bxy + cy^2))$$

We note immediately, that we must have $a > 0$ and $c > 0$ for this joint PDF to go to zero as $x, y \to \pm\infty$.

(a) Relation between a, b, c, d
To be a valid pdf, the following integral must evaluate to 1:

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx\,dy &= 1 \\
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\,\exp(-(ax^2+bxy+cy^2))\,dx\,dy &= 1 \\
d\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-a(x+\frac{by}{2a})^2\right)\,dx\;\exp(\frac{b^2y^2}{4a}-cy^2)\,dy &= 1 \\
d\int_{-\infty}^{\infty}\sqrt{\frac{\pi}{a}}\exp(\frac{b^2y^2}{4a}-cy^2)\,dy &= 1 \qquad (\text{for } a>0) \\
d\sqrt{\frac{\pi}{a}}\int_{-\infty}^{\infty}\exp\left(-(c-\frac{b^2}{4a})y^2\right)\,dy &= 1 \qquad (\text{for } 4ac>b^2) \\
d\sqrt{\frac{\pi}{a}}\sqrt{\frac{\pi}{(c-\frac{b^2}{4a})}} &= 1 \\
2\pi d\sqrt{\frac{1}{4ac-b^2}} &= 1 \\
\Rightarrow d &= \frac{\sqrt{4ac-b^2}}{2\pi}
\end{aligned}
$$

The restriction on a, b, c are that $a > 0$, $c > 0$ and $4ac > b^2$.
(b) Marginal pdfs of X, Y

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \\
&= d\int_{-\infty}^{\infty}\exp(-(ax^2+bxy+cy^2))\,dx \\
&= d\,\exp(\frac{b^2y^2}{4a}-cy^2)\int_{-\infty}^{\infty}\exp\left(-a(x+\frac{by}{2a})^2\right)\,dx \\
&= d\,\exp(\frac{b^2y^2}{4a}-cy^2)\sqrt{\frac{\pi}{a}} \\
&= \frac{\sqrt{4ac-b^2}}{2\pi}\sqrt{\frac{\pi}{a}}\exp\left(-y^2(c-\frac{b^2}{4a})\right)
\end{aligned}
$$

$$
f_Y(y) = \sqrt{\frac{4ac-b^2}{4\pi a}}\exp\left(-y^2(c-\frac{b^2}{4a})\right)
$$

Because of the symmetry in the equation in X, Y we can write the pdf of $f_X(x)$ as follows

$$f_X(x) = \sqrt{\frac{4ac-b^2}{4\pi c}} \exp\left(-y^2(a - \frac{b^2}{4c})\right)$$

(c)

$$\begin{aligned}
Pr(X > Y) &= \int_{-\infty}^{\infty}\int_{y}^{\infty} f_{X,Y}(x,y)\,dx\,dy \\
&= d\sqrt{\frac{\pi}{a}}\int_{-\infty}^{\infty} \exp\left(-y^2\left(\frac{4ac-b^2}{4a}\right)\right)\left[\int_y^\infty \sqrt{\frac{a}{\pi}} \exp\left(-a\left(x+\frac{by}{2a}\right)^2\right) dx\right] dy \\
&= d\sqrt{\frac{\pi}{a}}\int_{-\infty}^{\infty} \exp\left(-y^2\left(\frac{4ac-b^2}{4a}\right)\right) Q\left(\sqrt{2a}\left(1+\frac{b}{2a}\right)y\right) dy
\end{aligned}$$

In order to evaluate this last integral we rewrite the Q-function by defining $P(x) = Q(x) - \frac{1}{2}$. Then

$$\begin{aligned}
Pr(X > Y) &= d\sqrt{\frac{\pi}{a}}\int_{-\infty}^{\infty} \exp\left(-y^2\left(\frac{4ac-b^2}{4a}\right)\right)\left[\frac{1}{2} + P\left(\sqrt{2a}\left(1+\frac{b}{2a}\right)y\right)\right] dy \\
&= \frac{d}{2}\sqrt{\frac{\pi}{a}}\int_{-\infty}^{\infty} \exp\left(-y^2\left(\frac{4ac-b^2}{4a}\right)\right) dy \\
&+ d\sqrt{\frac{\pi}{a}}\int_{-\infty}^{\infty} \exp\left(-y^2\left(\frac{4ac-b^2}{4a}\right)\right) P\left(\sqrt{2a}\left(1+\frac{b}{2a}\right)y\right) dy
\end{aligned}$$

The second integral in this last expression is zero because the integrand is odd. The first integral is evaluated using the Gaussian normalization integral. Hence,

$$Pr(X > Y) = \frac{d}{2}\sqrt{\frac{\pi}{a}}\sqrt{\frac{4\pi a}{4ac-b^2}} = \frac{1}{2}.$$

Problem 5.4

(a) Let $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then we have

$$\int\int f_{X,y}(x,y)dxdy = c\int_0^{2\pi}\int_0^1 r\sqrt{1-r^2}drd\theta$$

$$
\begin{aligned}
&= 2\pi c \int_0^1 r\sqrt{1-r^2}dr \\
&= \frac{2\pi c}{3} \\
\Rightarrow c &= \frac{3}{2\pi}.
\end{aligned}
$$

(b)

$$
\begin{aligned}
Pr(X^2+Y^2 > 1/4) &= Pr(r > 1/2) \\
&= \tfrac{3}{2\pi} \int_0^{2\pi}\int_{1/2}^1 r\sqrt{1-r^2}drd\theta \\
&= 3\int_{1/2}^1 r\sqrt{1-r^2}dr \\
&= \frac{3\sqrt{3}}{8}.
\end{aligned}
$$

(c) Since the joint PDF is symmetric with respect to the line $y = x$,

$$
\Pr(X > Y) = \frac{1}{2}.
$$

Problem 5.5

$$
f_{X,Y}(x,y) = \frac{1}{8\pi}\exp\left(-\frac{(x-1)^2+(y+1)^2}{8}\right)
$$

(a)$Pr(X > 2, Y < 0)$

$$
\begin{aligned}
Pr(X > 2, Y < 0) &= \int_2^\infty \int_{-\infty}^0 \frac{1}{8\pi}\exp\left(-\frac{(x-1)^2+(y+1)^2}{8}\right) \\
&= \int_2^\infty \frac{1}{\sqrt{8\pi}}\exp(-\frac{(x-1)^2}{8})\,dx \int_{-\infty}^0 \frac{1}{\sqrt{8\pi}}\exp(-\frac{(y+1)^2}{8})\,dy \\
&= Q\left(\frac{2-1}{2}\right)\Phi\left(\frac{0+1}{2}\right) = Q\left(\frac{1}{2}\right)\Phi\left(\frac{1}{2}\right) \\
&= Q(1/2)\,(1-Q(1/2))
\end{aligned}
$$

(b)$Pr(0 < X < 2, |Y+1| > 2)$

$$
\begin{aligned}
&= Pr(\{0 < X < 2, Y > 1\} \cup \{0 < X < 2, Y < -3\}) \\
&= \int_0^2 \int_{-\infty}^{-3} f_{X,Y}(x,y) dx\, dy + \int_0^2 \int_1^{\infty} f_{X,Y}(x,y) dx\, dy \\
&= \int_0^2 \frac{1}{\sqrt{8\pi}} exp(-\frac{(x-1)^2}{8})\, dx \left(\int_{-\infty}^{-3} \frac{1}{\sqrt{8\pi}} \exp(-\frac{(y+1)^2}{8})\, dy + \int_1^{\infty} \frac{1}{\sqrt{8\pi}} \exp(-\frac{(y+1)^2}{8})\, dy \right) \\
&= \left(Q\left(\frac{0-1}{2}\right) - Q\left(\frac{2-1}{2}\right) \right) \left(\Phi\left(\frac{-3+1}{2}\right) + Q\left(\frac{1+1}{2}\right) \right) \\
&= \left(Q\left(-\frac{1}{2}\right) - Q\left(\frac{1}{2}\right) \right) \left(\Phi\left(-1\right) + Q\left(1\right) \right) \\
&= 2Q(1) \left(1 - 2Q\left(\frac{1}{2}\right) \right)
\end{aligned}
$$

(c) $Pr(Y > X)$

$$
Pr(Y > X) = \int_{-\infty}^{\infty} \int_{-\infty}^{y} f_{X,Y}(x,y) dx\, dy
$$

This integral is easier to do with a change of variables. Let us use the substitution

$$
\begin{aligned}
u &= x - y \\
v &= x + y \\
J\begin{pmatrix} u & v \\ x & y \end{pmatrix} &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\
\left| det \begin{bmatrix} u & v \\ x & y \end{bmatrix} \right| &= 2 \\
f_{U,V}(u,v) &= \frac{f_{X,Y}(x,y)}{2} \\
Pr(Y > X) &= Pr(U < 0) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{1}{8\pi} \exp\left(-\frac{(x-1)^2 + (y+1)^2}{8} \right) \frac{1}{2}\, du\, dv \\
&= \frac{1}{16\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{0} \exp\left(-\frac{(u-v-2)^2 + (u+v+2)^2}{32} \right) du\, dv \\
&= \frac{1}{16\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{0} \exp\left(-\frac{(u^2 + v^2 + 4 - 4u}{16} \right) du\, dv
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{16\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{0}\exp\left(-\frac{(u-2)^2+v^2}{16}\right)\,du\,dv \\
&= \frac{1}{16\pi}\int_{-\infty}^{\infty}\exp\left(-\frac{v^2}{16}\right)\,dv\int_{-\infty}^{0}\exp\left(-\frac{(u-2)^2}{16}\right)\,du \\
&= \frac{1}{16\pi}\sqrt{16\pi}\sqrt{16\pi}\,\Phi\left(\frac{0-2}{\sqrt{8}}\right) \\
&= Q(\frac{1}{\sqrt{2}})
\end{aligned}
$$

Problem 5.6

(a)

$$
\begin{aligned}
\sum_m\sum_n P_{M,N}(m,n) &= \sum_{m=0}^{L-1}\sum_{n=0}^{L-1-m} c \\
&= c\sum_{m=0}^{L-1} L-m \\
&= c\sum_{k=1}^{m} k \\
&= c\frac{L(L+1)}{2} \\
\Rightarrow\ c &= \frac{2}{L(L+1)}.
\end{aligned}
$$

(b)

$$
P_M(m) = \sum_{n=0}^{L-1-m} c = \frac{2(L-m)}{L(L+1)}, \qquad 0 \le m < L.
$$

Similarly,

$$
P_N(n) = \sum_{m=0}^{L-1-n} c = \frac{2(L-n)}{L(L+1)}, \qquad 0 \le n < L.
$$

(c) If L is an even integer, then

$$
Pr\,(M+N < L/2) = \sum_{m=0}^{L/2-1}\ \sum_{n=0}^{L/2-1-m} c = \frac{L+2}{4(L+1)}.
$$

If L is an odd integer, then

$$Pr\,(M+N<L/2)=\sum_{m=0}^{(L-1)/2}\sum_{n=0}^{(L-1)/2-m}c=\frac{L+3}{4L}.$$

Hence, we have

$$Pr\,(M+N<L/2)=\begin{cases}\frac{L+2}{4(L+1)} & \text{L even,}\\ \\ \frac{L+3}{4L} & \text{L odd.}\end{cases}$$

Problem 5.7

$$\begin{aligned}
f_{X,Y}(x,y) &= \frac{1}{2\pi\sqrt{3}}\exp\left(-\frac{x^2+2xy+4y^2}{6}\right)\\
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\\
&= \int_{-\infty}^{\infty}\frac{1}{2\pi\sqrt{3}}\exp\left(-\frac{x^2+2xy+4y^2}{6}\right)\,dy\\
&= \frac{1}{2\pi\sqrt{3}}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2+2xy+4y^2}{6}\right)\,dy\\
&= \frac{1}{2\pi\sqrt{3}}\int_{-\infty}^{\infty}\exp\left(-\frac{(2y+x/2)^2+3x^2/4}{6}\right)\,dy\\
&= \frac{1}{2\pi\sqrt{3}}\int_{-\infty}^{\infty}\exp\left(-2\frac{(y+x/2)^2}{3}\right)\exp\left(-\frac{x^2}{8}\right)\,dy\\
&= \frac{1}{2\pi\sqrt{3}}\sqrt{\frac{3\pi}{2}}\,\exp\left(-\frac{x^2}{8}\right)\\
&= \frac{1}{\sqrt{8\pi}}\,\exp\left(-\frac{x^2}{8}\right)\\
f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\\
&= \int_{-\infty}^{\infty}\frac{1}{2\pi\sqrt{3}}\,\exp\left(-\frac{x^2+2xy+4y^2}{6}\right)\,dx\\
&= \frac{1}{2\pi\sqrt{3}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x+y)^2+3y^2}{6}\right)\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi\sqrt{3}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x+y)^2}{6}\right) exp\left(-\frac{y^2}{2}\right) dx \\
&= \frac{1}{2\pi\sqrt{3}} \sqrt{6\pi} \exp\left(-\frac{y^2}{2}\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)
\end{aligned}
$$

(b) From the results of part (a), the marginal PDFs of X and Y are both Gaussian. Hence, the required means and variances can be found simply by inspecting the PDFs.

$$
\begin{aligned}
E[X] &= 0, \\
Var(X) &= 4, \\
E[Y] &= 0, \\
Var(Y) &= 1.
\end{aligned}
$$

(c) Conditional pdf $f_{X|Y}(x|y)$

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
&= \frac{\frac{1}{2\pi\sqrt{3}} \exp\left(-\frac{x^2+2xy+4y^2}{6}\right)}{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)} \\
&= \frac{1}{\sqrt{6\pi}} \exp\left(-\frac{x^2+2xy+y^2}{6}\right) \\
&= \frac{1}{\sqrt{6\pi}} \exp\left(-\frac{(x+y)^2}{6}\right)
\end{aligned}
$$

(d)$E[XY]$

$$
\begin{aligned}
E[XY] &= E_Y[Y\, E_X[X|Y]] \\
&= E_Y[Y\,(-Y)] \text{ Mean of } f_{X|Y}(x|y) \text{ as found in (c)} \\
&= E_Y[-Y^2] \\
&= -Var(Y) = -1
\end{aligned}
$$

Problem 5.8

(a) Since x and y are uniformly distributed,

$$f_{X,Y}(x,y) = c, \qquad \text{over } x^2 + 4y^2 \leq 1.$$

The constant c is the inverse of the area of the ellipse. Recall, that the area of an ellipse is given by $\pi r_1 r_2$, where r_1 and r_2 are the radii of the major and minor axes of the ellipse. In this case we get,

$$c = \frac{1}{\pi r_1 r_2} = \frac{2}{\pi}.$$

Hence,

$$\begin{aligned} f_X(x) &= \int f_{X,Y}(x,y)dy = \int_{-\sqrt{1-x^2}/2}^{\sqrt{1-x^2}/2} \frac{2}{\pi} dy \\ &= \frac{2}{\pi}\sqrt{1-x^2} \qquad -1 \leq x \leq 1. \end{aligned}$$

Similarly,

$$\begin{aligned} f_Y(y) &= \int f_{X,Y}(x,y)dx = \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} \frac{2}{\pi} dx \\ &= \frac{4}{\pi}\sqrt{1-4y^2} \qquad -1/2 \leq y \leq 1/2. \end{aligned}$$

(b) Since both marginal PDFs are symmetric about the origin,

$$E[X] = E[Y] = 0.$$

Hence, the variances are equal to the second moments.

$$Var(X) = E(X^2) = \frac{2}{\pi}\int_{-1}^{1} x^2\sqrt{1-x^2}dx = 1/4.$$

Similarly,

$$Var(Y) = E(Y^2) = \frac{4}{\pi}\int_{-1/2}^{1/2} y^2\sqrt{1-4y^2}dy = 1/16.$$

(c)

$$f_{X|Y}(x|y) = f_{X,Y}(x,y)/f_Y(y) = \frac{2/\pi}{\frac{4}{\pi}\sqrt{1-4y^2}} = \frac{1}{2\sqrt{1-4y^2}} \qquad |x| \le \sqrt{1-4y^2}.$$

$$f_{Y|X}(y|x) = f_{X,Y}(x,y)/f_X(x) = \frac{2/\pi}{\frac{2}{\pi}\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} \qquad |y| \le \frac{\sqrt{1-x^2}}{2}.$$

(d)

$$E[XY] = E_Y[YE_X[X|Y]].$$

From the results of part (c), the conditional PDFs are symmetric about the origin and hence, $E[X|Y] = 0$. It follows that

$$\begin{aligned} E[XY] &= 0, \\ Cov(X,Y) &= 0, \\ \rho_{X,Y} = & \quad 0. \end{aligned}$$

Problem 5.9

$$\rho_{X,Y} = \begin{cases} 1 & \text{if a} > 0 \\ -1 & \text{if a} < 0 \end{cases} \quad \begin{aligned} Y &= aX + b \end{aligned}$$

$$\begin{aligned}
\rho_{X,Y} &= \frac{Cov(X,Y)}{\sigma_x \sigma_y} \\
\mu_Y &= E[Y] = aE[X] + b = a\mu_X + b \\
\sigma_y^2 &= E[Y^2] - \mu_Y^2 = E[(aX+b)^2] = E[a^2X^2 + 2abX + b^2] - (a\mu_X + b)^2 \\
&= a^2E[X^2] + 2abE[X] + b^2 - (a^2\mu_X^2 + 2ab\mu_X + b^2) \\
&= a^2(\sigma_x^2 + \mu_X^2) + 2ab\mu_X + b^2 - (a^2\mu_X^2 + 2ab\mu_X + b^2) \\
&= a^2\sigma_x^2 \\
\Rightarrow \sigma_y &= |a|\sigma_x \\
Cov(X,Y) &= E[(X-\mu_X)(Y-\mu_Y)] \\
&= E[(X-\mu_X)(aX + b - a\mu_X - b)]
\end{aligned}$$

$$
\begin{aligned}
&= E[a(X-\mu_X)^2] = aE[(X-\mu_X)^2] = a\sigma_x^2 \\
\rho_{X,Y} &= \frac{a\sigma_x^2}{|a|\sigma_x\sigma_x} \\
\rho_{X,Y} &= \frac{a}{|a|} \\
&= \begin{cases} 1 & \text{if a} > 0 \\ -1 & \text{if a} < 0 \end{cases}
\end{aligned}
$$

Now we shall prove the converse of the statement that is if $\rho_{X,Y} = 1$ then we will be able to express Y as a linear function of X. That means we will be able to find constants a, b such that $Y = aX + b$ and $(a \neq 0.)$

$$
\begin{aligned}
|\rho_{X,Y}| &= 1 \\
\Rightarrow Cov(X,Y) &= \pm\sigma_X\sigma_Y = E[XY] - \mu_X\mu_Y
\end{aligned}
$$

If as we mentioned earlier $Y = aX + b$ then

$$
\begin{aligned}
E[(Y-aX-b)^2] = 0 \\
\Rightarrow E[Y^2 + a^2X^2 + b^2 - 2aXY - 2abX - 2bY] = 0
\end{aligned}
$$

$$
\begin{aligned}
E[Y^2] + a^2E[X^2] + b^2 - 2aE[XY] - 2abE[X] - 2bE[Y] = 0 \\
\sigma_Y^2 + \mu_Y^2 + a^2(\sigma_X^2 + \mu_X^2) + b^2 - 2a(\pm\sigma_X\sigma_Y + \mu_X\mu_Y) - 2ab\mu_X - 2b\mu_Y = 0 \\
(\sigma_Y \pm a\sigma_X)^2 + (\mu_Y - a\mu_X - b)^2 = 0
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow a = \pm\frac{\sigma_Y}{\sigma_X} \\
b = \mu_Y \pm \mu_X\frac{\sigma_Y}{\sigma_X}
\end{aligned}
$$

Thus we can find two constants a, b such that Y is a linear function of X if $\rho_{X,Y} = \pm 1$ and this proves what we set out to prove.

Problem 5.10

From Theorem 5.4. and its proof, we are aware that

$$
(E[XY])^2 \leq E[X^2]E[Y^2] \ ,
$$

which gives

$$E[XY] \le \sqrt{E[X^2]E[Y^2]}\ .$$

Note that $E[X^2]E[Y^2] \ge 0$. Hence,

$$E[X^2] + E[Y^2] + 2E[XY] \le E[X^2] + E[Y^2] + 2\sqrt{E[X^2]E[Y^2]}\ ,$$

which means

$$E[(X+Y)^2] \le (\sqrt{E[X^2]} + \sqrt{E[Y^2]})^2\ .$$

Therefore,

$$\sqrt{E[(X+Y)^2]} \le \sqrt{E[X^2]} + \sqrt{E[Y^2]}\ .$$

Problem 5.11

$$\begin{aligned}
\mu_X &= 2 \\
\sigma_X &= 1 \\
\Rightarrow E[X^2] &= \sigma_X^2 + \mu_X^2 = 1 + 4 = 5 \\
\mu_Y &= -1 \\
\sigma_Y &= 4 \\
\Rightarrow E[Y^2] &= \sigma_Y^2 + \mu_Y^2 = 16 + 1 = 17 \\
\rho_{X,Y} &= \frac{1}{4} \\
\Rightarrow Cov(X,Y) &= \rho_{X,Y}\sigma_X\sigma_Y = 1 \\
\Rightarrow E[XY] &= Cov(X,Y) + \mu_X\mu_Y = 1 - 2 = -1 \\
U &= X + 2Y \\
V &= 2X - Y
\end{aligned}$$

(a) $E[U]$ and $E[V]$

$$\begin{aligned}
E[U] &= E[X+2Y] = E[X] + 2E[Y] = 2 - 2 = 0 \\
E[V] &= E[2X-Y] = 2E[X] - E[Y] = 4 - (-1) = 5
\end{aligned}$$

(b)$E[U^2]$, $E[V^2]$, $Var(U)$ and $Var(V)$

$$\begin{aligned}
E[U^2] &= E[(X+2Y)^2] = E[X^2] + 4E[XY] + 4E[Y^2] = 5 + 4(-1) + 4 \cdot 17 = 69 \\
E[V^2] &= E[(2X-Y)^2] = 4E[X^2] + E[Y^2] - 4E[XY] = 4 \cdot 5 - 4(-1) + 17 = 41 \\
Var(U) &= E[U^2] - \mu_u^2 = 69 - 0 = 69 \\
Var(V) &= E[V^2] - \mu_v^2 = 41 - 25 = 16
\end{aligned}$$

(c) $E[UV]$, $Cov(U,V)$ and $\rho_{U,V}$

$$\begin{aligned}
E[UV] &= E[(X+2Y)(2X-Y)] = E[2X^2 - 2Y^2 + 3XY] \\
&= 2E[X^2] - 2E[Y^2] + 3E[XY] = 2(5) - 2(17) + 3(-1) = -27 \\
Cov(U,V) &= E[UV] - \mu_U \mu_V = -27 - (0)(3) = -27 \\
\rho_{U,V} &= \frac{Cov(U,V)}{\sigma_U \sigma_V} = -\frac{27}{\sqrt{67 \cdot 16}}
\end{aligned}$$

Problem 5.12

Since $f_X(x)$ is symmetric about the origin, we know $\mu_X = 0$, and

$$E[XY] = E[aX^3] = a \int_{-\infty}^{\infty} x^3 f_X(x) = 0 \ .$$

Hence,

$$\rho_{X,Y} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y} = 0 \ .$$

Problem 5.13

$$\begin{aligned}
f_{X,Y}(x,y) &= \begin{cases} \frac{1}{\pi ab} & \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \\ 0 & \text{otherwise} \end{cases} \\
f_X(x) &= \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} \frac{1}{\pi ab}\, dy \\
&= \frac{2}{\pi a} \sqrt{1 - \frac{x^2}{a^2}}
\end{aligned}$$

Similary we can write

$$f_Y(y) = \frac{2}{\pi b} \sqrt{1 - \frac{y^2}{b^2}}$$

We can see that the product of the marginal pdfs is not equal to the constant $\frac{1}{\pi ab}$ which means that the two variables X, Y are not independent. Let us calculate the correlation of X, Y.

$$\begin{aligned} E[XY] &= \int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} xy\frac{1}{\pi ab}\,dy\,dx \\ &= \frac{1}{\pi a}\int_{-a}^{a} x\sqrt{1-\frac{x^2}{a^2}}\,dx \end{aligned}$$

The above integral is odd and hence it evaluates to zero. Thus we have found a case where two variables which are uncorrelated and yet are dependent.

Problem 5.14

The marginal PDF of X can be found as follows:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}} \exp\left(-\left[\frac{(\frac{x-\mu_X}{\sigma_X})^2 - 2\rho_{X,Y}(\frac{x-\mu_X}{\sigma_X})(\frac{y-\mu_Y}{\sigma_Y}) + (\frac{y-\mu_Y}{\sigma_Y})^2}{2(1-\rho_{X,Y}^2)}\right]\right) dy \end{aligned}$$

To simplify notation, let $u = \frac{X-\mu_X}{\sigma_X}$ and $v = \frac{Y-\mu_Y}{\sigma_Y}$. The above equation then simplifies to

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sqrt{1-\rho_{X,Y}^2}} \exp\left(-\left[\frac{u^2 - 2\rho_{X,Y}uv + v^2}{2(1-\rho_{X,Y}^2)}\right]\right) dv \\ &= \exp\left(-\frac{u^2}{2(1-\rho_{X,Y}^2)}\right)\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sqrt{1-\rho_{X,Y}^2}} \exp\left(-\left[\frac{v^2 - 2\rho_{X,Y}uv}{2(1-\rho_{X,Y}^2)}\right]\right) dv \\ &= \frac{1}{\sqrt{2\pi\sigma_X^2}}\exp\left(-\frac{u^2}{2}\right)\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-\rho_{X,Y}^2)}} \exp\left(-\left[\frac{v^2 - 2\rho_{X,Y}uv + \rho_{X,Y}^2u^2}{2(1-\rho_{X,Y}^2)}\right]\right) dv \\ &= \frac{1}{\sqrt{2\pi\sigma_X^2}}\exp\left(-\frac{u^2}{2}\right) = \frac{1}{\sqrt{2\pi\sigma_X^2}}\exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right) \end{aligned}$$

The last step is accomplished by noting that the integrand is a properly normalized Gaussian PDF and hence integrates to 1. Following identical steps, the marginal PDF of Y is found to be

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right).$$

Problem 5.15

$$f_{X,Y}(x,y) \quad = \quad \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{X,Y}^2}}\, exp\left(-\frac{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho_{X,Y}\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2}{2(1-\rho_{X,Y}^2)}\right)$$

From the results of the previous exercise, the marginal PDF of Y is

$$f_Y(y) \quad = \quad \frac{1}{\sqrt{2\pi\sigma_y^2}}\, exp\left(-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right)$$

The conditional pdf $f_{X|Y}(x|y)$ is then given by

$$\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
f_{X|Y}(x|y) &= \frac{\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{X,Y}^2}}\ \exp\left(-\frac{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho_{X,Y}\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2}{2(1-\rho_{X,Y}^2)}\right)}{\frac{1}{\sqrt{2\pi\sigma_y^2}}\ \exp\left(-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right)} \\
&= \frac{1}{\sqrt{2\pi\sigma_x^2(1-\rho_{X,Y}^2)}}\ \exp\left(-\frac{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho_{X,Y}\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \rho_{X,Y}^2\left(\frac{y-\mu_y}{\sigma_y}\right)^2}{2(1-\rho_{X,Y}^2)}\right) \\
&= \frac{1}{\sqrt{2\pi\sigma_x^2(1-\rho_{X,Y}^2)}}\, exp\left(-\frac{\left(x-\mu_x-\rho_{X,Y}\frac{\sigma_x}{\sigma_y}(y-\mu_y)\right)^2}{2\sigma_X^2(1-\rho_{X,Y}^2)}\right)
\end{aligned}$$

The above equation is in the form of a Gaussian random variable whose mean and variance are given by

$$\mu_{X|Y} = \mu_x + \rho_{X,Y}\frac{\sigma_x}{\sigma_y}(y-\mu_y),$$

$$\sigma_{X|Y}^2 = \sigma_X^2(1-\rho_{X,Y}^2).$$

Problem 5.16

(a)

$$\begin{aligned}
\frac{d}{da}E[(Y-aX)^2] &= E[\frac{d}{da}(Y-aX)^2] = E[2(Y-aX)(-X)] = 0 \\
\Rightarrow E[XY] &= aE[X^2] \\
\Rightarrow a &= \frac{E[XY]}{E[X^2]} = \frac{\rho\sigma_X\sigma_Y + \mu_X\mu_Y}{\sigma_X^2+\mu_X^2}
\end{aligned}$$

(b)

$$\begin{aligned}
E[(Y-aX)^2] &= E[Y^2] - 2aE[XY] + a^2E[X^2] \\
&= E[Y^2] - 2\left(\frac{E[XY]}{E[X^2]}\right)E[XY] + \left(\frac{E[XY]}{E[X^2]}\right)^2 E[X^2] \\
&= E[Y^2] - \frac{E[XY]^2}{E[X^2]} \\
&= \sigma_Y^2 + \mu_Y^2 - \frac{(\rho\sigma_X\sigma_Y+\mu_X\mu_Y)^2}{\sigma_X^2+\mu_X^2} \\
&= \frac{(\sigma_Y^2+\mu_Y^2)(\sigma_X^2+\mu_X^2) - (\rho\sigma_X\sigma_Y+\mu_X\mu_Y)^2}{\sigma_X^2+\mu_X^2}.
\end{aligned}$$

Problem 5.17

(a) Characteristic function of the jointly distributed Gaussian

$$\begin{aligned}
f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{X,Y}^2}}\, exp\left(-\frac{\left(\frac{x}{\sigma_x}\right)^2 - 2\rho_{X,Y}\left(\frac{x}{\sigma_x}\right)\left(\frac{y}{\sigma_y}\right) + \left(\frac{y}{\sigma_y}\right)^2}{2(1-\rho_{X,Y}^2)}\right) \\
\Phi_{X,Y}(\omega_1,\omega_2) &= E[e^{j\omega_1 x + j\omega_2 y}] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\exp(j(\omega_1 x + \omega_2 y))\,dx\,dy \\
&= \int_{-\infty}^{\infty} f_Y(y)e^{j\omega_2 y}\left[\int_{-\infty}^{\infty} f_{X|Y}(x|y)e^{j\omega_1 x}\,dx\right]dy \\
&= E_Y[e^{j\omega_2 y}\Phi_{X|Y}(\omega_1)]
\end{aligned}$$

From Problem 5.15 we know that conditioned on Y, X is Gaussian with a mean of $\mu_{X|Y} = \rho y\sigma_X/\sigma_Y$ and a variance of $\sigma_{X|Y}^2 = \sigma_X^2(1-\rho^2)$. Hence the conditional characteristic function of X given Y is

$$\Phi_{X|Y}(\omega_1) = \exp\left(-\frac{1}{2}\omega_1^2\sigma_X^2(1-\rho^2) + j\omega_1\rho y\frac{\sigma_X}{\sigma_Y}\right)$$

The joint characteristic function is then given by

$$\begin{aligned}
\Phi_{X,Y}(\omega_1,\omega_2) &= \int_{-\infty}^{\infty} f_Y(y)e^{j\omega_2 y}\Phi_{X|Y}(\omega_1)dy \\
&= \exp\left(-\frac{1}{2}\omega_1^2\sigma_X^2(1-\rho^2)\right)\int_{-\infty}^{\infty} f_Y(y)\exp\left(j\left(\omega_2+\omega_1\rho\frac{\sigma_X}{\sigma_Y}\right)y\right)dy \\
&= \exp\left(-\frac{1}{2}\omega_1^2\sigma_X^2(1-\rho^2)\right)\Phi_Y\left(\omega_2+\omega_1\rho\frac{\sigma_X}{\sigma_Y}\right) \\
&= \exp\left(-\frac{1}{2}\omega_1^2\sigma_X^2(1-\rho^2)\right)\exp\left(-\frac{1}{2}\left(\omega_2+\omega_1\rho\frac{\sigma_X}{\sigma_Y}\right)^2\sigma_Y^2\right) \\
&= \exp\left(-\frac{1}{2}(\omega_1^2\sigma_X^2 + 2\rho\sigma_x\sigma_Y\omega^1\omega_2 + \omega_2^2\sigma_Y^2)\right).
\end{aligned}$$

(b) First, find the first few terms in the 2-D Taylor series expansion of the joint characteristic function as follows:

$$\begin{aligned}
\Phi_{X,Y}(\omega_1,\omega_2) &= \exp\left(-\frac{1}{2}\omega_1^2\sigma_X^2\right)\exp\left(-\rho\omega_1\omega_2\sigma_X\sigma_2\right)\exp\left(-\frac{1}{2}\omega_2^2\sigma_Y^2\right) \\
&= \left(1-\frac{1}{2}\omega_1^2\sigma_X^2+\frac{1}{8}\omega_1^4\sigma_X^4+\dots\right) \\
&\times \left(1-\rho\omega_1\omega_2\sigma_X\sigma_Y+\frac{1}{2}\rho\omega_1^2\omega_2^2\sigma_X^2\sigma_Y^2+\dots\right) \\
&\times \left(1-\frac{1}{2}\omega_2^2\sigma_Y^2+\frac{1}{8}\omega_2^4\sigma_Y^4+\dots\right) \\
&= 1-\frac{1}{2}\omega_1^2\sigma_X^2-\frac{1}{2}\omega_2^2\sigma_Y^2-\rho\omega_1\omega_2\sigma_X\sigma_2 \\
&+ \frac{1}{8}\omega_1^4\sigma_X^4+\frac{1}{8}\omega_2^4\sigma_Y^4+\left(\frac{1}{4}+\frac{1}{2}\rho^2\right)\sigma_X^2\sigma_Y^2\omega_1^2\omega_2^2 \\
&+ \text{higher order terms} \qquad (2)
\end{aligned}$$

The correlation is found from the characteristic function as

$$E[XY] = -\left.\frac{\partial^2\Phi_{X,Y}(\omega_1,\omega_2)}{\partial\omega_1\partial\omega_2}\right|_{\omega_1=\omega_2=0}.$$

The required mixed partial derivative is the coefficient of $\omega_1\omega_2$ in the series expansion given in (2) above. Hence,

$$E[XY] = \rho\sigma_1\sigma_2.$$

(c) Proceeding as in part (b),

$$E[X^2Y^2] = \left.\frac{\partial^4 \Phi_{X,Y}(\omega_1,\omega_2)}{\partial\omega_1^2\partial\omega_2^2}\right|_{\omega_1=\omega_2=0}.$$

This fourth order mixed partial derivative can also be found from the series expansion given in (2) above. That is

$$\left.\frac{\partial^4 \Phi_{X,Y}(\omega_1,\omega_2)}{\partial\omega_1^2\partial\omega_2^2}\right|_{\omega_1=\omega_2=0} = 4 \times (\text{coefficient of } \omega_1^2\omega_2^2).$$

Hence,

$$E[X^2Y^2] = (1+2\rho^2)\sigma_X^2\sigma_Y^2$$

Problem 5.18

First we note that (see solution to Problem 5.17 for details)

$$\Phi_{X,Y}(\omega_1,\omega_2) = E_Y\left[e^{j\omega_2 Y}E_X[e^{j\omega_1 X}|Y]\right]. \tag{3}$$

Conditioned on Y, X is a Gaussian random variable with a mean of $\mu_X + \rho(\sigma_X/\sigma_Y)(Y-\mu_Y)$ and a variance of $\sigma_X^2(1-\rho^2)$. Hence

$$E_X[e^{j\omega_1 X}|Y] = \exp\left(j\omega_1\Big(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(Y-\mu_Y)\Big) - \frac{1}{2}\omega_1^2\sigma_X^2(1-\rho)^2)\right) \tag{4}$$

By (3), (4), we have

$$\begin{aligned}
\Phi_{X,Y}(\omega_1,\omega_2) &= E_Y\left[\exp\left(j\omega_2 Y + j\Big(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(Y-\mu_Y)\Big)\omega_1 - \frac{1}{2}\omega_1^2\sigma_X^2(1-\rho^2)\right)\right] \\
&= \exp\left(j\omega_1\left(\mu_X - j\rho\mu_Y\frac{\sigma_X}{\sigma_Y}\right) - \frac{\omega_1^2}{2}\sigma_X^2(1-\rho^2)\right) \\
&\times\ E_Y\left[\exp\left(j(\omega_2 + \rho\frac{\sigma_X}{\sigma_Y}\omega_1)Y\right)\right] \\
&= \exp\left(j\omega_1\left(\mu_X - j\rho\mu_Y\frac{\sigma_X}{\sigma_Y}\right) - \frac{\omega_1^2}{2}\sigma_X^2(1-\rho^2)\right)\Phi_Y\left((\omega_2 + \rho\frac{\sigma_X}{\sigma_Y}\omega_1)\right)
\end{aligned}$$

Since

$$\Phi_Y\left(\omega_2+\rho\frac{\sigma_X}{\sigma_Y}\omega_1\right)=\exp\left(j\mu_Y\left(\omega_2+\rho\frac{\sigma_X}{\sigma_Y}\omega_1\right)-\frac{\left(\omega_2+\rho\frac{\sigma_X}{\sigma_Y}\omega_1\right)^2}{2}\sigma_Y^2\right), \quad (5)$$

the joint characteristic function is then (after some algebraic manipulations)

$$\Phi_{X,Y}(\omega_1,\omega_2)=\exp\left(j(\omega_1\mu_X+\mu_Y\omega_2)-\frac{\omega_1^2\sigma_X^2+2\rho\sigma_X\sigma_Y\omega_1\omega_2+\omega_2^2\sigma_Y^2}{2}\right) \ .$$

Problem 5.19

$$\begin{aligned}
f_R(r)dr &= \Pr(r\le R<r+dr)\\
&= \int_0^{2\pi}\int_r^{r+dr}\frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right)dr\,d\theta\\
&= \int_r^{r+dr}\frac{r}{\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right)dr\\
&= -\exp\left(-\frac{r^2}{2\sigma^2}\right)\Bigg|_r^{r+dr}\\
&= \exp\left(-\frac{r^2}{2\sigma^2}\right)-\exp\left(-\frac{(r+dr)^2}{2\sigma^2}\right)\\
&= \exp\left(-\frac{r^2}{2\sigma^2}\right)\left[1-\exp\left(-\frac{rdr}{2\sigma^2}-\frac{dr^2}{2\sigma^2}\right)\right].
\end{aligned}$$

As $dr\to 0$ the terms involving dr^2 will be small compared to those involving dr. Hence,

$$\begin{aligned}
f_R(r)dr &\to \exp\left(-\frac{r^2}{2\sigma^2}\right)\left[1-\exp\left(-\frac{rdr}{\sigma^2}\right)\right]\\
&= \exp\left(-\frac{r^2}{2\sigma^2}\right)\left[1-\left(1-\frac{rdr}{\sigma^2}+\frac{r^2dr^2}{2\sigma^4}+\dots\right)\right]\\
&\to \frac{r}{\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right)dr\\
\Rightarrow f_R(r) &= \frac{r}{\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right)
\end{aligned}$$

Problem 5.20

$$
\begin{aligned}
f_X(x) &= be^{-bx} \\
f_Y(y) &= be^{-by} \\
Z &= X - Y \\
F_Z(z) &= Pr(Z \leq z) = Pr(X - Y \leq z) \\
&= Pr(X \leq Y + z) \\
&= \int\int_{X \leq Y+z} f_X(x) f_Y(y) dx\, dy \\
&= \int\int_{X \leq Y+z} b^2 e^{-b(x+y)} dx\, dy
\end{aligned}
$$

Here we need to consider two cases($Z > 0$ and $Z < 0$) because the limits for the area defined by Z will be different for these cases
(i) $Z < 0$

$$
\begin{aligned}
F_Z(z) &= b^2 \int_{-z}^{\infty} \int_0^{y+z} e^{-b(x+y)} dx\, dy \\
&= b^2 \int_{-z}^{\infty} \left[-\frac{e^{-bx}}{-b} \right] \Bigg|_{x=0}^{x=y+z} e^{-by}\, dy \\
&= b \int_{-z}^{\infty} (1 - e^{-(by+bz)}) e^{-by}\, dy \\
&= b \left[-\frac{e^{-by}}{b} + \frac{e^{-(2by+bz)}}{2b} \right] \Bigg|_{y=-z}^{y=\infty} \\
&= \left(e^{bz} - \frac{e^{bz}}{2} \right) \\
&= \frac{e^{bz}}{2}
\end{aligned}
$$

(ii) $Z > 0$

$$
\begin{aligned}
F_Z(z) &= b^2 \int_0^{\infty} \int_0^{y+z} e^{-b(x+y)} dx\, dy \\
&= b^2 \int_0^{\infty} \left[-\frac{e^{-bx}}{-b} \right] \Bigg|_{x=0}^{x=y+z} e^{-by}\, dy
\end{aligned}
$$

$$
\begin{aligned}
&= b\int_0^{\infty}(1 - e^{-(by+bz)})e^{-by}\,dy \\
&= b\left[-\frac{e^{-by}}{b} + \frac{e^{-(2by+bz)}}{2b}\right]\Bigg|_{y=0}^{y=\infty} \\
&= \left(1 - \frac{e^{-bz}}{2}\right)
\end{aligned}
$$

In summary the CDF of Z is given by

$$
F_Z(z) = \begin{cases} \frac{e^{bz}}{2} & Z < 0 \\ 1 - \frac{e^{-bz}}{2} & Z \geq 0 \end{cases}
$$

And the corresponding pdf is given by

$$
f_Z(z) = \begin{cases} \frac{be^{bz}}{2} & Z < 0 \\ \frac{be^{-bz}}{2} & Z \geq 0 \end{cases}
$$

Problem 5.21

This was proven in the text. In Example 5.25, a 2x2 linear transformation of two jointly Gaussain random variables X and Y produce two new jointly Gaussian random variables Z and W. Since Z and W are jointly Gaussian, Z and W are individually Gaussian as well. Hence Z is a Gaussian random variable.

Problem 5.22

$$
\begin{aligned}
X &\sim N(1,4) \\
Y &\sim N(-2,9) \\
Z &= 2X - 3Y - 5 \\
\rho_{X,Y} &= \frac{1}{3} \\
\Rightarrow Cov(X,Y) &= \frac{3 \times 2}{3} = 2
\end{aligned}
$$

We will make use of the fact that a linear transformation of the Gaussian Random variables results in a Gaussian Random variable.

$$
E[Z] = E[2X - 3Y - 5] = 2(1) - 3(-2) - 5 = 3
$$

$$\begin{aligned}
Var(Z) &= E[(Z-3)^2] = E[(2X-3Y-8)^2] \\
&= E[(2X-2-3Y-6)^2] \\
&= E[4(X-1)^2 + 9(Y+2)^2 - 12(X-1)(Y+2)] \\
&= 4Var(X) + 9Var(Y) - 12Cov(X,Y) \\
&= 4(4) + 9(9) - 12(2) = 73
\end{aligned}$$

Hence Z is a Gaussian distributed as follows

$$Z \sim N(3, 73)$$

Problem 5.23

(a) We first find the CDF of Z and then differentiate to get the PDF.

$$\begin{aligned}
\Pr(Z \leq z) &= \Pr(\max(X,Y) \leq z) \\
&= \Pr(X \leq z, Y \leq z) \\
&= F_X(z) F_Y(z) \\
&= \left(1 - \exp\left(-\frac{z^2}{2}\right)\right)^2. \\
f_Z(z) &= 2z \exp\left(-\frac{z^2}{2}\right)\left(1 - \exp\left(-\frac{z^2}{2}\right)\right).
\end{aligned}$$

(b) Proceeding in a simalar manner as in part (a),

$$\begin{aligned}
\Pr(W \geq z) &= \Pr(\min(X,Y) \geq z) \\
&= \Pr(X \geq z, Y \geq z) \\
&= (1 - F_X(z))(1 - F_Y(z)) \\
&= \left(\exp\left(-\frac{z^2}{2}\right)\right)^2 \\
&= \exp(-z^2). \\
f_Z(z) &= \frac{d}{dz}(1 - \Pr(W \geq z) \\
&= 2z \exp(-z^2).
\end{aligned}$$

Problem 5.24

$$\begin{aligned} f_X(x) &= e^{-x}U(x) \\ f_Y(y) &= e^{-y}U(y) \end{aligned}$$

For the equation $z^2 + Xz + Y = 0$ to have real roots the following condition must hold

$$X^2 - 4Y \geq 0 \Rightarrow Y \leq \frac{X^2}{4}$$

$$\begin{aligned} Pr(\text{real roots}) &= Pr(Y \leq \frac{X^2}{4}) \\ &= \int_0^\infty \int_0^{x^2/4} e^{-x}e^{-y}\, dy\, dx \\ &= \int_0^\infty (1 - e^{-x^2/4})e^{-x}\, dx \\ &= \int_0^\infty e^{-x}\, dx - \int_0^\infty e^{-x-x^2/4}\, dx \\ &= 1 - \int_0^\infty e^{-x-x^2/4}\, dx \\ &= 1 - \int_0^\infty exp(-\frac{(x+2)^2-4}{4})\, dx \\ &= 1 - e\sqrt{4\pi} \int_0^\infty \frac{1}{\sqrt{4\pi}} exp(-\frac{(x+2)^2}{4})\, dx \\ &= 1 - e\sqrt{4\pi}Q(\frac{0+2}{\sqrt{2}}) \\ &= 1 - e\sqrt{4\pi}Q(\sqrt{2}) \approx 0.77 \end{aligned}$$

(b) For the equation to have imaginary roots the condition is

$$X^2 - 4Y > 0 \Rightarrow Y < \frac{X^2}{4}$$

$$\begin{aligned} Pr(\text{imaginary roots}) &= Pr(Y > \frac{X^2}{4}) = 1 - Pr(\text{real roots}) \\ &= 1 - (1 - e\sqrt{4\pi}Q(\sqrt{2})) = 1 - e\sqrt{4\pi}Q(\sqrt{2}) \approx 0.23 \end{aligned}$$

(c) For the equation to have equal roots the condition is

$$X^2 - 4Y = 0$$

This is the equation of a parabola in the (x,y) plane. The probability of the point (X,Y) falling on a line or a curve is zero because its area is zero.

Problem 5.25

Conditioned on X=x, the transformation Z=xY is a simple linear transformation and from $f_X(x)$ we know that $x > 0$,

$$f_{Z|x}(z|x) = \frac{1}{|x|} f_Y(\frac{z}{x}) = \frac{1}{x} \frac{1}{\pi \sqrt{1 - \frac{z^2}{x^2}}} \ , \quad |z| < x$$

The PDF of Z can be found according to

$$f_Z(z) = \int f_{Z|x}(z|x) f_X(x) dx = \begin{cases} \int_z^\infty \frac{1}{\pi\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \frac{x}{\sqrt{x^2 - z^2}} dx & z > 0 \\ \int_{-z}^\infty \frac{1}{\pi\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \frac{x}{\sqrt{x^2 - z^2}} dx & z < 0 \end{cases}$$

And,

$$\begin{aligned} \int_z^\infty \frac{1}{\pi\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \frac{x}{\sqrt{x^2 - z^2}} dx &= \int_{-z}^\infty \frac{1}{\pi\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \frac{x}{\sqrt{x^2 - z^2}} dx \\ &= \int_{z^2}^\infty \frac{1}{2\pi\sigma^2} e^{-\frac{u}{2\sigma^2}} \frac{1}{\sqrt{u - z^2}} du \\ &= \frac{1}{2\pi\sigma^2} e^{-\frac{z^2}{2\sigma^2}} \int_0^\infty e^{-\frac{t}{2\sigma^2}} \frac{1}{\sqrt{t}} dt \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}} \end{aligned}$$

Hence, Z is a Gaussian random variable with a zero mean and unit variance.

Problem 5.26

$$\begin{aligned} f_X(x) &= \frac{1}{2\pi} \\ f_Y(y) &= \frac{1}{2\pi} \\ Z &= (X + Y) \, mod \, 2\pi \\ Pr(Z \le z) &= Pr(X + Y < z) + Pr(2\pi < X + Y < 2\pi + Z) \end{aligned}$$

Note that the above two events are mutually exclusive because $Z < 2\pi$, hence the first event includes the the condition $X + Y < 2\pi$

$$\begin{aligned}
Pr(Z \le z) &= \int_0^z \int_0^x \frac{1}{2\pi}\frac{1}{2\pi}\, dy\, dx + \int_0^z \int_{2\pi-x}^{2\pi} \frac{1}{2\pi}\frac{1}{2\pi}\, dy\, dx + \int_z^{2\pi} \int_{2\pi-x}^{2\pi+z-x} \frac{1}{2\pi}\frac{1}{2\pi}\, dy\, dx \\
&= (\frac{1}{2\pi})^2 \left(\int_0^z x\, dx + \int_0^z x\, dx + \int_z^{2\pi} z\, dx \right) \\
&= (\frac{1}{2\pi})^2 \left(\frac{z^2}{2} + \frac{z^2}{2} + 2\pi z - z^2 \right) = \frac{z}{2\pi} \\
\Rightarrow f_Z(z) &= \frac{1}{2\pi}
\end{aligned}$$

Problem 5.27

Conditioned on Y=y, the transformation Z=Xy is a simple linear transformation and

$$f_{Z|Y}(z|y) = \frac{1}{|y|} f_X(\frac{z}{y}) = f_X(\frac{z}{y}) \ .$$

The PDF of Z can be found according to

$$f_Z(z) = \int f_{Z|Y}(z|y) f_Y(y) dy = p f_X(z) + (1-p) f_X(-z) \ .$$

Z is a Gaussian random variable if $f_X(x)$ is symmetric about the origin, i.e. $\mu_X = 0$. Also, if $p = 0$ or $p = 1$, Z will be Gaussian.

Problem 5.28

Since the transformation is linear and X and Y are jointly Gaussian, U and V will be jointly Gaussian with

$$\begin{aligned}
E[U] &= E[X]\cos(\theta) - E[Y]\sin(\theta) = 0 \\
E[V] &= E[X]\sin(\theta) + E[Y]\cos(\theta) = 0 \\
Var(U) &= E[U^2] = E[X^2]\cos^2(\theta) + E[Y^2]\sin^2(\theta) - 2E[XY]\cos(\theta)\sin(\theta) \\
&= \cos^2(\theta) + \sin^2(\theta) = 1 \\
Var(V) &= E[V^2] = E[X^2]\sin^2(\theta) + E[Y^2]\cos^2(\theta) + 2E[XY]\cos(\theta)\sin(\theta) \\
&= \cos^2(\theta) + \sin^2(\theta) = 1 \\
Cov(U,V) &= E[UV] = E[X^2]\cos(\theta)\sin(\theta) - E[Y^2]\cos(\theta)\sin(\theta) + E[XY](\cos^2(\theta) - \sin^2(\\
&= \cos(\theta)\sin(\theta) - \cos(\theta)\sin(\theta) = 0
\end{aligned}$$

Hence, U and V are independent standard Normal random variables.

Problem 5.29

Note that any linear transformation of jointly Gaussian random variables produces jointly Gaussian random variables. So U, V are both Gaussian random variables. And if $\rho_{U,V} = 0$, U and V are independent.

$$E[U] = aE[X] + bE[Y] = 0 \ .$$

$$E[V] = cE[X] + dE[Y] = 0 \ .$$

$$E[UV] = acE[X^2] + bdE[Y^2] + (ad + bc)E[XY] = ac + bd + \rho(ad + bc) \ .$$

Thus, $\rho_{U,V} = 0$ if

$$ac + bd + \rho(ad + bc) = 0 \ .$$

So, U and V are independent, if

$$ac + bd = -\rho(ad + bc) \ .$$

Problem 5.30

$$\begin{aligned}
f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma^2} exp\left(-\frac{(x-\mu_x)^2 + (y-\mu_y)}{2\sigma^2}\right) \\
R &= \sqrt{X^2 + Y^2} \\
\Theta &= \tan^{-1}(\frac{Y}{X}) \\
X &= R\cos\theta \\
Y &= R\sin\theta \\
J\begin{pmatrix} x & y \\ r & \theta \end{pmatrix} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \\
det\left[J\begin{pmatrix} x & y \\ r & \theta \end{pmatrix}\right] &= r \\
f_{R,\Theta}(r,\theta) &= f_{X,Y}(x,y)r \\
&= \frac{r}{2\pi\sigma^2} exp\left(-\frac{(x-\mu_x)^2 + (y-\mu_y)}{2\sigma^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{r}{2\pi\sigma^2} exp\left(-\frac{(r\cos\theta-\mu_x)^2+(r\sin\theta-\mu_y)}{2\sigma^2}\right) \\
&= \frac{r}{2\pi\sigma^2} exp\left(-\frac{(r^2+\mu_x^2+\mu_y^2-2r\mu_x\cos\theta-2r\mu_y\sin\theta)}{2\sigma^2}\right) \\
&= \frac{r}{2\pi\sigma^2} exp\left(-\frac{(r^2+\mu_x^2+\mu_y^2)}{2\sigma^2}\right)\; exp\left(\frac{r\mu_x\cos\theta+r\mu_y\sin\theta}{\sigma^2}\right) \\
&= \frac{r}{2\pi\sigma^2} exp\left(-\frac{(r^2+a^2)}{2\sigma^2}\right)\; exp\left(\frac{ra\cos(\theta+\phi)}{\sigma^2}\right)
\end{aligned}
$$

where $a = \sqrt{\mu_x^2+\mu_y^2}$ and $\tan\phi = \frac{\mu_y}{\mu_x}$.

$$
\begin{aligned}
f_R(r) &= \int_0^{2\pi} f_{R,\Theta}(r,\theta)\, d\theta \\
&= \int_0^{2\pi} \frac{r}{2\pi\sigma^2} exp\left(-\frac{(r^2+a^2)}{2\sigma^2}\right)\; exp\left(\frac{ra\cos(\theta+\phi)}{\sigma^2}\right)\, d\theta \\
&= \frac{r}{2\pi\sigma^2} exp\left(-\frac{(r^2+a^2)}{2\sigma^2}\right) \int_0^{2\pi} exp\left(\frac{ra\cos(\theta+\phi)}{\sigma^2}\right)\, d\theta \\
&= \frac{r}{2\pi\sigma^2} exp\left(-\frac{(r^2+a^2)}{2\sigma^2}\right)\; I_o(\frac{ar}{\sigma^2})
\end{aligned}
$$

This is the general form of a Rician distribution and therefore $f_R(r)$ follows a rician distribution. Please note in the above integral we have made use of the following property of periodic functions.

$$
\int_0^{2\pi} e^{cos(\theta+\phi)}\, d\theta \;=\; \int_0^{2\pi} e^{cos(\theta)}\, d\theta
$$

This follows directly because of the periodicity of the cos function.

Problem 5.31

The inverse transformation is given by

$$
X = \pm\sqrt{\frac{Z+W}{2}} \tag{6}
$$

$$Y = \pm\sqrt{\frac{Z-W}{2}} \ . \tag{7}$$

$$\mathbf{J} = \begin{pmatrix} z & w \\ x & y \end{pmatrix} = det \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix} = -8xy \ .$$

Plugging these results into the general formula results in

$$\begin{aligned} f_{Z,W}(z,w) &= 4 * \frac{1}{8\sqrt{\frac{z+w}{2}}\sqrt{\frac{z-w}{2}}} \frac{1}{2\pi\sigma_X\sigma_Y} \exp\left(-\left(\frac{(\frac{z+w}{2})}{2\sigma_X^2} + \frac{(\frac{z-w}{2})}{2\sigma_Y^2}\right)\right) \\ &= \frac{1}{\sqrt{z^2-w^2}} \frac{1}{2\pi\sigma_X\sigma_Y} \exp\Big(-(\frac{z+w}{4\sigma_X^2} + \frac{z-w}{4\sigma_Y^2})\Big), \quad z > w, z > 0 \ . \end{aligned}$$

Problem 5.32

$$\begin{aligned} Z &= X^2 + Y^2 \\ W &= XY \\ X &= \frac{\sqrt{Z+2W} + \sqrt{Z-2W}}{2} \\ Y &= \frac{\sqrt{Z+2W} - \sqrt{Z-2W}}{2} \\ f_X(x) &= \frac{1}{\pi(1+x^2)} \\ f_Y(y) &= \frac{1}{\pi(1+y^2)} \\ J\begin{pmatrix} z & w \\ x & y \end{pmatrix} &= \begin{pmatrix} 2x & 2y \\ y & x \end{pmatrix} \\ \left| det\left[J\begin{pmatrix} z & w \\ x & y \end{pmatrix}\right]\right| &= 2|x^2 - y^2| \\ f_{Z,W}(z,w) &= \frac{f_X(x) f_Y(y)}{2|x^2-y^2|} \\ &= \frac{1}{2\pi^2} \frac{1}{(1+x^2)(1+y^2)|x^2-y^2|} \\ &= \frac{1}{2\pi^2} \frac{1}{(1+x^2+y^2+x^2y^2)|x^2-y^2|} \end{aligned}$$

$$= \frac{1}{2\pi^2}\frac{1}{(1+z^2+w^2)\sqrt{z^2-4w^2}}$$

Problem 5.33

(a)

$$E[Z]=\iint Ae^{j\theta}f_{A,\theta}(A,\theta)d\theta dA=\int Af_A(A)\int_0^{2\pi}e^{j\theta}\frac{1}{2\pi}\ d\theta dA=0\ .$$

(b) The variance is defined as

$$Var(Z)=\frac{1}{2}E\left[|Z-\mu_Z|^2\right]=\frac{1}{2}E[|Z|^2]=\frac{1}{2}E[A^2]\ .$$

Problem 5.34

$$\begin{aligned}
I(X;Y) &= \sum_{i=0}^{2}\sum_{j=0}^{2}p_j q_{i,j}\log\left(\frac{q_{i,j}}{p_j}\right)\\
&= \frac{1}{3}[3\cdot 0.8\cdot\log(2.4)+6\cdot 0.1\cdot\log(0.3)]\\
&= 0.8\cdot\log(2.4)+0.2\cdot\log(0.3)\\
&= 0.663 \text{ bits}
\end{aligned}$$

Problem 5.35

$$\begin{aligned}
I(X;Y) &= \sum_{i=0}^{2}\sum_{j=0}^{2}p_j q_{i,j}\log\left(\frac{q_{i,j}}{p_j}\right)\\
&= \frac{1}{3}[3\cdot 0.9\cdot\log(2.7)+3\cdot 0.1\cdot\log(0.3)]\\
&= 0.9\cdot\log(2.7)+0.1\cdot\log(0.3)\\
&= 1.116 \text{ bits}
\end{aligned}$$

Problem 5.36

$$
\begin{aligned}
I(X;Y) &= \sum_{i=0}^{2}\sum_{j=0}^{2} p_j q_{i,j} \log\left(\frac{q_{i,j}}{p_j}\right) \\
&= 9 \cdot \frac{1}{3} \cdot \frac{1}{3} \log\left(\frac{1/3}{1/3}\right) \\
&= 0 \text{ bits}
\end{aligned}
$$

The output is independent of (tells us nothing about) the input, hence the information carried by the channel is zero.

Solutions to Chapter 6 Exercises

Problem 6.1

(a) $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(0,1,1)$, $(1,0,0)$, $(1,0,1)$, $(1,1,0)$, $(1,1,1)$
(b) Each outcome is mutually exclusive (and therefore not independent).

Problem 6.2

(a)

$$\begin{aligned}
f_{\mathbf{X}}(\mathbf{x}) &= \begin{cases} c & \|\mathbf{x}\| \le 1 \\ 0 & \|\mathbf{x}\| > 1 \end{cases} \\
\int f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} &= 1 \\
&= c \int\!\!\int\!\!\int dx_1\, dx_2\, dx_3 \\
&= c\frac{4\pi}{3} \text{ (This integral is the volume of a unit sphere)} \\
\Rightarrow c &= \frac{3}{4\pi}
\end{aligned}$$

(b) Marginal pdf $f_{X_1,X_2}(x_1, x_2)$

$$\begin{aligned}
f_{X_1,X_2}(x_1, x_2) &= \int f_{X_1,X_2,X_3}(x_1, x_2, x_3)\, dx_3 \\
&= \int_{-\sqrt{1-x_1^2-x_2^2}}^{\sqrt{1-x_1^2-x_2^2}} c\, dx_1 \\
&= \frac{3}{4\pi}\left(2\sqrt{1-x_1^2-x_2^2}\right) \\
&= \frac{3}{2\pi}\sqrt{1-x_1^2-x_2^2}
\end{aligned}$$

(c) Marginal pdf $f_{X_1}(x_1)$

$$f_{X_1}(x_1) = \int f_{X_1,X_2}(x_1, x_2)\, dx_2$$

$$
\begin{aligned}
&= \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \frac{3}{2\pi}\sqrt{1-x_1^2-x_2^2}\, dx2 \\
&= \frac{3}{4}(1-x_1^2)
\end{aligned}
$$

(d) Conditional pdfs

$$
\begin{aligned}
f_{X_1|X_2,X_3}(x_1|x_2,x_3) &= \frac{f_{X_1,X_2,X_3}(x_1,x_2,x_3)}{f_{X_2,X_3}(x_2,x_3)} \\
&= \frac{\frac{3}{4\pi}}{\frac{3}{2\pi}\sqrt{1-x_2^2-x_3^2}} \\
&= \frac{1}{2\sqrt{1-x_2^2-x_3^2}} \\
f_{X_1,X_2|X_3}(x_1,x_2|x_3) &= \frac{f_{X_1,X_2,X_3}(x_1,x_2,x_3)}{f_{X_3}(x_3)} \\
&= \frac{\frac{3}{4\pi}}{\frac{3}{4}(1-x_3^2)} \\
&= \frac{1}{\pi\sqrt{1-x_3^2}}
\end{aligned}
$$

Problem 6.3

(a) Define

$$
I_n(z) = \int\int\cdots\int_{\sum_{i=1}^n x_i \le z} dx_1 dx_2 \ldots dx_n.
$$

Then $c = I_N^{-1}(1)$. Note that

$$
\begin{aligned}
I_n(z) &= \int_0^z \left[\int\int\cdots\int_{\sum_{i=1}^{n-1} x_i \le z-x_n} dx_1 dx_2 \ldots dx_{n-1}\right] dx_n \\
&= \int_0^z I_{n-1}(z-x_n) dx_n \\
&= \int_0^z I_{n-1}(u) du.
\end{aligned}
$$

Using this iteration, we see that

$$
I_1(z) = \int_0^z du = z,
$$

$$
\begin{aligned}
I_2(z) &= \int_0^z u du = \frac{1}{2} z^2, \\
I_3(z) &= \int_0^z \frac{1}{2} u^2 du = \frac{1}{3!} z^3, \\
&\vdots \\
I_n(z) &= \frac{1}{n!} z^n.
\end{aligned}
$$

Then, $c = I_N^{-1}(1) = N!$.
(b)

$$
\begin{aligned}
f_{X_1,X_2,\ldots,X_M}(x_1,x_2,\ldots,x_M) &= \int\int \ldots \int_{x_{M+1}+\ldots+x_N \le 1-x_1-x_2-\ldots-x_M} c\, dx_N dx_{N-1} \ldots dx_{M+1} \\
&= c\, I_{N-M}(1 - x_1 - x_2 - \ldots - x_M) \\
&= \frac{N!}{(N-M)!}(1 - x_1 - \ldots - x_M)^{N-M}, \qquad \sum_{m=1}^{M} x_M \le 1, x_i \ge 0
\end{aligned}
$$

(c) From part(b), we have

$$
f_{X_i}(x_i) = N(1 - x_i)^{N-1} .
$$

Then,

$$
f_{X_1}(x_1) f_{X_1}(x_1) \ldots f_{X_N}(x_N) = N^N (1-x_1)^{N-1} \ldots (1-x_N)^{N-1} \ne f_{X_1,X_2,\ldots,X_N}(x_1,x_2,\ldots,x_N) = N!
$$

Thus, X_i are identically distributed, but they are not independent.

Problem 6.4

$$
\begin{aligned}
\mathbf{X} &= [X_1, X_2, \ldots, X_N]^T \\
\mathbf{1} &= [1, 1, \ldots, 1]^T \ n \times 1 \text{ column vector} \\
Z &= \sum_{i=1}^{N} b_i X_i = \mathbf{b}^T \mathbf{X} \\
\sum_{i=1}^{N} b_i &= 1 = \mathbf{b}^T \mathbf{1}
\end{aligned}
$$

We know that

$$
Var(Z) = Var(\mathbf{b}^T \mathbf{X}) = \mathbf{b}^T \mathbf{C_{XX}} \mathbf{b}
$$

To minimize σ_Z^2 we use lagrange multiplier

$$
\begin{aligned}
\nabla(\mathbf{b}^T\mathbf{C_{XX}}\mathbf{b}) + \lambda\nabla(\mathbf{b}^T\mathbf{1} - 1) &= 0 \\
\nabla(\mathbf{b}^T)\mathbf{C_{XX}}\mathbf{b} + \mathbf{b}^T\mathbf{C_{XX}}\nabla(\mathbf{b}) + \lambda\nabla(\mathbf{b}^T)\mathbf{1} &= 0 \\
2\nabla(\mathbf{b}^T)\mathbf{C_{XX}}\mathbf{b} + \lambda\nabla(\mathbf{b}^T)\mathbf{1} &= 0 \\
\nabla(\mathbf{b}^T) &= \mathbf{I}_{n\times n} \\
\Rightarrow \lambda\mathbf{1} &= -2\mathbf{C_{XX}}\mathbf{b} \\
\lambda\mathbf{C_{XX}^{-1}}\mathbf{1} &= -2\mathbf{b} \qquad (1) \\
\lambda\mathbf{1}^T\mathbf{C_{XX}^{-1}}\mathbf{1} &= -2\mathbf{1}^T\mathbf{b} = -2 \\
\lambda &= -\frac{2}{\mathbf{1}^T\mathbf{C_{XX}^{-1}}\mathbf{1}} \\
\text{Using this value of } \lambda \text{ in (1) we get} & \\
-2\mathbf{b} &= -\frac{2}{\mathbf{1}^T\mathbf{C_{XX}^{-1}}\mathbf{1}}\mathbf{C_{XX}^{-1}}\mathbf{1} \\
\mathbf{b} &= \frac{\mathbf{C_{XX}^{-1}}\mathbf{1}}{\mathbf{1}^T\mathbf{C_{XX}^{-1}}\mathbf{1}} \qquad (2)
\end{aligned}
$$

This solution must then be checked against the costraint that each b_i must be non-negative. If any of the b_i specified by (2) are negative, then those b_i should be set equal to zero and the problem reworked with the lower dimensionality.

Problem 6.5

A matrix $\mathbf{R}$ is non-negative definite if $\mathbf{z}^T\mathbf{R}\mathbf{z} \geq 0$ for any vector $\mathbf{z}$. If $\mathbf{R}$ is a correlation matrix, then

$$
\begin{aligned}
\mathbf{z}^T\mathbf{R}\mathbf{z} &= \mathbf{z}^T E[\mathbf{X}\mathbf{X}^T]\mathbf{z} \\
&= E[\mathbf{z}^T\mathbf{X}\mathbf{X}^T\mathbf{z}] \\
&= E[(\mathbf{z}^T\mathbf{X})^2] \geq 0.
\end{aligned}
$$

Problem 6.6

(a) The 36 possible realizations of the random vector (X, Y) are:

$$(1,1)\,(1,2)\,(1,3)\,(1,4)\,(1,5)\,(1,6)$$

$$
\begin{array}{cccccc}
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}
$$

(b) Each realization has a probability of 1/36.
(c) $E[(X, Y)] = (3.5, 3.5)$.
(d)

$$
\mathbf{C} = \begin{bmatrix} \frac{35}{12} & 0 \\ 0 & \frac{35}{12} \end{bmatrix}
$$

Problem 6.7

$$
E[(X, Y, Z)] = (3.5, 3.5, 3.5), \mathbf{C} = \begin{bmatrix} \frac{35}{12} & 0 & 0 \\ 0 & \frac{35}{12} & 0 \\ 0 & 0 & \frac{35}{12} \end{bmatrix}
$$

Problem 6.8

$$
\begin{aligned}
\mathbf{C} &= \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T \\
\mathbf{\Lambda} &= \begin{bmatrix} \frac{7+\sqrt{13}}{2} & 0 \\ 0 & \frac{7-\sqrt{13}}{2} \end{bmatrix} = \begin{bmatrix} 5.3028 & 0 \\ 0 & 1.6972 \end{bmatrix} \\
\mathbf{Q} &= \begin{bmatrix} 0.9571 & -0.2898 \\ 0.2898 & 0.9571 \end{bmatrix}
\end{aligned}
$$

The appropriate transformation is given by

$$
\mathbf{T} = \mathbf{Q}\sqrt{\mathbf{\Lambda}} = \begin{bmatrix} 2.2040 & -0.3775 \\ 0.6673 & 1.2469 \end{bmatrix}
$$

Problem 6.9

The covariance matrix of Y will be of the form,

$$C_{YY} = AC_{XX}A^T,$$

Note that C_{XX} can be decomposed into

$$C_{XX} = Q\Lambda Q^T,$$

where Λ is a diagonal matrix of the eigenvalues of C_{XX} and Q is an orthogonal matrix whose columns are the corresponding eigenvectors of C. So if we let C_{YY} be Λ, then Y will be uncorrelated, where A should be Q^T.

$$\mathbf{C_{XX}} - \lambda\mathbf{I} = \begin{bmatrix} 3-\lambda & 1 & -1 \\ 1 & 5-\lambda & -1 \\ -1 & -1 & 3-\lambda \end{bmatrix}.$$

$$\Rightarrow \det(C_{XX} - \lambda I) = -(\lambda-2)(\lambda-3)(\lambda-6) = 0$$

So,

$$\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6\ .$$

And, the eigenvector matrix is calculated to be

$$\mathbf{Q} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ 0 & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} \end{bmatrix}.$$

Hence, the appropriate transformation matrix is

$$\mathbf{A} = \mathbf{Q^T} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & -\frac{\sqrt{6}}{6} \end{bmatrix}.$$

The covariance matrix of Y is

$$\mathbf{C_{YY}} = \mathbf{\Lambda} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

Problem 6.10

(a) $E[X_1|X_2 = x_2, X_3 = x_3]$

$$\begin{aligned}
\mathbf{X}^T &= [X_1 X_2 X_3] \\
\mathbf{C_{XX}} &= \sigma^2 \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix} \qquad (3) \\
det[\mathbf{C_{XX}}] &= \sigma^6 (1-\rho)^2 (1+2\rho) \\
\mathbf{C_{XX}^{-1}} &= \frac{\sigma^{-2}}{(1-\rho)(1+2\rho)} \begin{bmatrix} 1+\rho & -\rho & -\rho \\ -\rho & 1+\rho & -\rho \\ -\rho & -\rho & 1+\rho \end{bmatrix} \qquad (4)
\end{aligned}$$

We know that $f_{X_1|X_2X_3}(x_1|x_2, x_3)$ is also a Gaussian random variable and the mean of a Gaussian Random Variable with pdf $d\,exp(-ax^2 - bx - c)$ is given by $-\frac{b}{2a}$. So all we need to do is get the coefficient of x_1^2 and x_1 in the pdf of $f_{X_1|X_2X_3}(x_1|x_2, x_3)$

$$\begin{aligned}
f_{X_1|X_2X_3}(x_1|x_2, x_3) &= \frac{f_{X_1,X_2,X_3}(x_1, x_2, x_3)}{f_{X_2,X_3}(x_2, x_3)} \qquad (5) \\
&= \frac{1}{\sqrt{(2\pi)^3 det[\mathbf{C_{XX}}]}} \frac{exp\left(-\frac{1}{2}\mathbf{X^T C_{XX}^{-1} X}\right)}{f_{X_2,X_3}(x_2, x_3)} \qquad (6)
\end{aligned}$$

We know that $f_{X_2,X_3}(x_2, x_3)$ is a function comprising only of x_2, x_3 and so it will not affect the coefficient of x_1^2 or x_1 in the above equation. Let us calculate the coefficient of x_1^2 (say a) and x_1 (say b) in the above equation. Using 4 and the above eqn we get

$$\begin{aligned}
a &= \frac{\sigma^{-2}}{2(1-\rho)(1+2\rho)}(1+\rho) \\
b &= \frac{\sigma^{-2}}{2(1-\rho)(1+2\rho)} 2\rho(x_2 + x_3)
\end{aligned}$$

Mean of the distribution given by eqn 6

$$\frac{b}{2a} = \frac{\rho}{1+\rho}(x_2 + x_3) = E[X_1|X_2 = x_2, X_3 = x_3]$$

(b) $E[X_1X_2|X_3]$

$$
\begin{aligned}
E[X_1X_2|X_3] &= E\left[(X_2E[X_1|X_2,X_3])|X_3\right] \\
E[X_1X_2|X_3] &= E[X_2\frac{\rho}{1+\rho}(X_2+X_3)|X_3] \\
&= \frac{\rho}{1+\rho}E[(X_2^2+X_2X_3)|X_3]
\end{aligned}
$$

We know from problem 5.15 that for a pair of jointly distributed Gaussian Random Variables X,Y the conditonal pdf $f_{X|Y}(x|y)$ is given by $N \sim (\mu_X + \rho_{XY}(\frac{\sigma_X}{\sigma_Y})(Y-\mu_Y), \sigma_X^2(1-\rho_{XY}^2))$

$$
\begin{aligned}
f_{X_2|X_3}(x_2|x_3) &= N \sim (\rho X_3, \sigma^2(1-\rho^2)) \\
\Rightarrow E[X_1X_2|X_3] &= \frac{\rho}{1+\rho}E[X_2^2|X_3] + E[X_2X_3|X_3] \\
&= \frac{\rho}{1+\rho}\Big(\sigma^2(1-\rho^2) + (\rho x_3)^2 + x_3\rho x_3\Big) \\
&= \frac{\rho}{1+\rho}\Big(\sigma^2(1-\rho^2) + \rho x_3^2(1+\rho)\Big) \\
&= \rho\sigma^2(1-\rho) + (\rho x_3)^2
\end{aligned}
$$

(c) $E[X_1X_2X_3]$

$$
\begin{aligned}
E[X_1X_2X_3] &= E\left[X_3E[(X_1X_2|X_3)]\right] \\
&= E[X_3(\rho\sigma^2(1-\rho) + (\rho X_3)^2)] = 0
\end{aligned}
$$

We notice the expectation is a linear combination of the odd moments of a Gaussian Random Variable and each of them is zero and consequently the above expectation evaluates to zero

Problem 6.11

$$\Phi_X(\Omega) = E[e^{j\Omega^TX}] = E[e^{jZ}] = \Phi_Z(1)\ .$$

where $Z = \Omega^TX$, And,

$$\Phi_Z(\omega) = \exp(-\frac{\omega^2\sigma_Z^2}{2}) = \exp(-\frac{\omega^2\Omega^TC_{XX}\Omega}{2})\ .$$

Hence,

$$\Phi_X(\Omega) = \Phi_Z(1) = \exp(-\frac{\Omega^T C_{XX}\Omega}{2}) \ .$$

Problem 6.12

$$\begin{aligned}
\mathbf{X} &= [X_1, X_2, \ldots, X_N]^T \\
\mathbf{\Omega} &= [\omega_1, \omega_2, \ldots, \omega_N]^T
\end{aligned}$$

From problem 6.6 we know

$$\begin{aligned}
\Phi_X(\Omega) &= \exp\left(-\frac{\Omega^T\,\mathbf{Cxx}\Omega}{2}\right) \\
\Rightarrow E[e^{j(\omega_1 x_1+\omega_2 x_2+\omega_3 x_3+\omega_4 x_4)}] &= \left(1 - \frac{\Omega^T\,\mathbf{Cxx}\Omega}{2} + \frac{1}{2!}\left(\frac{\Omega^T\,\mathbf{Cxx}\Omega}{2}\right)^2 \cdots\right)
\end{aligned}$$

Expanding both the exponentials and equating the coeffts of $\omega_1\omega_2\omega_3\omega_4$ we get

$$\begin{aligned}
E\left[\cdots + X_1X_2X_3X_4\omega_1\omega_2\omega_3\omega_4 + \cdots\right] &= \left(\cdots + \frac{1}{2!}\frac{1}{2^2}\left(C_{12}C_{34} + C_{13}C_{24} + C_{14}C_{23}\right)8\omega_1\omega_2\omega_3\omega_4 + \cdots\right) \\
\text{Where } C_{ij} = Cov(X_i, X_j) & \\
\Rightarrow E[X_1X_2X_3X_4] &= \frac{1}{2!}\frac{1}{2^2}\left(C_{12}C_{34} + C_{13}C_{24} + C_{14}C_{23}\right)8 \\
\Rightarrow E[X_1X_2X_3X_4] &= \left(C_{12}C_{34} + C_{13}C_{24} + C_{14}C_{23}\right)
\end{aligned}$$

Because X_i are zero mean Random Variables $C_{ij} = E[X_iX_j]$

$$E[X_1X_2X_3X_4] = E[X_1X_2]E[X_3X_4] + E[X_1X_3]E[X_2X_4] + E[X_1X_4]E[X_2X_3]$$

Problem 6.13

(a) The PDF of the median sequence is

$$f_{Y_k}(y) = \frac{(2k-1)!}{[(k-1)!]^2} f_X(y)[F_X(y)]^{k-1}[1 - F_X(y)]^{k-1}.$$

Then, we have

$$f_{Y_k}(y) = \frac{(2k-1)!}{[(k-1)!]^2}\frac{1}{\mu}\exp(-\frac{y}{\mu})\left[1 - \exp(-\frac{y}{\mu})\right]^{k-1}\left[\exp(-\frac{y}{\mu})\right]^{k-1} U(y) \quad (7)$$

(b)

$$\begin{aligned}
M_Y(\mu_1) = E[e^{\mu_1 Y}] &= \int_0^\infty e^{\mu_1 y} \frac{(2k-1)!}{[(k-1)!]^2} \frac{1}{\mu} \exp(-\frac{yk}{\mu}) \left[1 - \exp(-\frac{y}{\mu})\right]^{k-1} dy \\
&= \frac{(2k-1)!}{[(k-1)!]^2} \frac{1}{\mu} \int_0^\infty \sum_{m=0}^{k-1} \binom{k-1}{m} (-1)^m e^{y(\mu_1 - \frac{k}{\mu} - \frac{m}{\mu})} dy \\
&= \frac{(2k-1)!}{[(k-1)!]^2} \frac{1}{\mu} \sum_{m=0}^{k-1} \binom{k-1}{m} (-1)^m \frac{1}{\frac{k}{\mu} + \frac{m}{\mu} - \mu_1} \qquad (8)
\end{aligned}$$

$$E[Y] = \frac{d}{d\mu_1} M_Y(\mu_1)|_{\mu_1=0} = \frac{(2k-1)!}{[(k-1)!]^2} \sum_{m=0}^{k-1} \binom{k-1}{m} (-1)^m \frac{\mu}{(k+m)^2}$$

for $k \neq 1$, $E[Y] \neq \mu$. Hence, the median is a biased estimate of the mean of the underlying exponential distribution.

(c) From part(b), we have

$$E[Y^2] = \frac{d^2}{d\mu_1^2} M_Y(\mu_1)|_{\mu_1=0} = \frac{(2k-1)!}{[(k-1)!]^2} \sum_{m=0}^{k-1} \binom{k-1}{m} (-1)^m \frac{2\mu^2}{(k+m)^3}.$$

Then,

$$\begin{aligned}
Var(Y) &= E[Y^2] - E[Y]^2 \\
&= \frac{(2k-1)!}{[(k-1)!]^2} \sum_{m=0}^{k-1} \binom{k-1}{m} (-1)^m \frac{2\mu^2}{(k+m)^3} \\
&- \left(\frac{(2k-1)!}{[(k-1)!]^2} \sum_{m=0}^{k-1} \binom{k-1}{m} (-1)^m \frac{\mu}{(k+m)^2} \right)^2 .
\end{aligned}$$

Problem 6.14

$$\begin{aligned}
\frac{d}{dy} F_{Y_m}(y) &= \frac{d}{dy} \sum_{k=m}^{N} \binom{N}{k} (F_X(y))^k (1 - F_X(y))^{N-k} \\
&= \sum_{k=m}^{N} \binom{N}{k} k f_X(y) (F_X(y))^{k-1} (1 - F_X(y))^{N-k} \\
&- \sum_{k=m}^{N} \binom{N}{k} (N-k) f_X(y) (F_X(y))^k (1 - F_X(y))^{N-k-1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=m}^{N} \frac{N!}{(k-1)!(N-k)!} f_X(y)(F_X(y))^{k-1}(1-F_X(y))^{N-k} \\
&- \sum_{k=m}^{N} \frac{N!}{k!(N-k-1)!} f_X(y)(F_X(y))^{k}(1-F_X(y))^{N-k-1}
\end{aligned}
$$

In the second sum, let $n = k+1$. Then

$$
\begin{aligned}
f_{Y_m}(y) &= \sum_{k=m}^{N} \frac{N!}{(k-1)!(N-k)!} f_X(y)(F_X(y))^{k-1}(1-F_X(y))^{N-k} \\
&- \sum_{n=m+1}^{N-1} \frac{N!}{(m-1)!(N-m)!} f_X(y)(F_X(y))^{n-1}(1-F_X(y))^{N-n}
\end{aligned}
$$

The second series cancels the first except the $k=m$ term. Hence,

$$
f_{Y_m}(y) = \frac{N!}{(m-1)!(N-m)!} f_X(y)(F_X(y))^{m-1}(1-F_X(y))^{N-m}
$$

Problem 6.15

$$
f_{y_m}(y)dy = \Pr(y < Y_m < y+dy)
$$

Define the following events:

$$
\begin{aligned}
A &= \{m-1 \text{ of the } X\text{'s are } < y\} \\
B &= \{\text{one of the } X\text{'s } \in (y, y+dy)\} \\
C &= \{N-m \text{ of } X\text{'s are } > y+dy\}
\end{aligned}
$$

Then,

$$
\begin{aligned}
\Pr(y < Y_m < y+dy) &= \Pr(A,B,C) \\
&= \Pr(A)\Pr(B|A)\Pr(C|A,B)
\end{aligned}
$$

These various probabilities work out as follows:

$$
\begin{aligned}
\Pr(A) &= \binom{N}{m-1}(F_X(y))^{m-1} \\
\Pr(B|A) &= (N-m+1)f_X(y)dy \\
\Pr(C|A,B) &= (1-F_X(y+dy))^{N-m}
\end{aligned}
$$

Note that as $dy \to 0$, $F_X(y+dy) \to F_X(y)$ for any continuous random variable. Hence,

$$\begin{aligned}
\Pr(y < Y_m < y+dy) &= \binom{N}{m-1}(F_X(y))^{m-1}(N-m+1)f_X(y)dy(1-F_X(y+dy))^{N-m} \\
\Rightarrow f_{Y_m}(y) &= \frac{N!}{(m-1)!(N-m)!}f_X(y)(F_X(y))^{m-1}(1-F_X(y+dy))^{N-m}
\end{aligned}$$

Problem 6.16

(a)

$$\Pr(X=k) = \frac{\alpha^k}{k!}e^{-\alpha} = \frac{10^{11}}{11!}e^{-10} = 0.1137.$$

(b) Let X_i be the number of cars that approach the ith booth in one minute.

$$\begin{aligned}
\Pr(X_1 \le 5, X_2 \le 5, \ldots, X_N \le 5) &= \Pr(X_1 \le 5)\Pr(X_2 \le 5)\ldots\Pr(X_N \le 5) \\
&= \Pr(X_i \le 5)^N. \\
\Pr(X_i \le 5) &= \sum_{m=0}^{5}\frac{\alpha_i^m}{m!}e^{-\alpha_i}
\end{aligned}$$

Since the traffic splits evenly between all booths, we take $\alpha_i = \alpha/N$. The following table illustrates the relevant calculations.

N	α_i	$\Pr(X_i \le 5)$	$\Pr(X_i \le 5)^N$
1	30	2.2573×10^{-8}	2.2573×10^{-8}
2	15	0.0028	7.7977×10^{-6}
3	10	0.0671	3.0192×10^{-4}
4	7.5	0.2414	0.0034
5	6	0.4457	0.0176
6	5	0.6160	0.0546
7	4.2857	0.7390	0.1204
8	3.75	0.8229	0.2102

From these numbers we conclude that the manager must keep 6 booths open.

Problem 6.17

(a)

$$E[Y_n] = E\left[\frac{X_n + X_{n-1}}{2}\right] = \frac{1}{2}E[X_n] + \frac{1}{2}E[X_{n-1}] = 0.$$

$$\begin{aligned}
E[Y_nY_k] &= E\left[\left(\frac{X_n + X_{n-1}}{2}\right)\left(\frac{X_k + X_{k-1}}{2}\right)\right] \\
&= \frac{1}{4}\left(E[X_nX_k] + E[X_nX_{k-1}] + E[X_{n-1}X_k] + E[X_{n-1}X_{k-1}]\right) \\
&= \frac{1}{4}[c_{n,k} + c_{n,k-1} + c_{n-1,k} + c_{n-1,k-1}]
\end{aligned}$$

Using $c_{i,j} = \delta_{i,j}$ we get

$$E[Y_nY_k] = \begin{cases} \frac{1}{2} & n = k, \\ \frac{1}{4} & n = k \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

The correlation matrix is then of the form

$$\mathbf{R} = \frac{1}{4}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & \dots \\ 0 & 1 & 2 & 1 & 0 & \dots \\ 0 & 0 & 1 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

(b) In this case, $c_{i,j} = \sigma_X^2\delta_{i,j}$ so that

$$\mathbf{R} = \frac{\sigma_X^2}{4}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & \dots \\ 0 & 1 & 2 & 1 & 0 & \dots \\ 0 & 0 & 1 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Problem 6.18

(a)

$$\begin{aligned}
E[Y_n] &= E\left[\frac{X_n - X_{n-1}}{2}\right] = \frac{1}{2}E[X_n] - \frac{1}{2}E[X_{n-1}] = 0. \\
E[Y_nY_k] &= E\left[\left(\frac{X_n - X_{n-1}}{2}\right)\left(\frac{X_k - X_{k-1}}{2}\right)\right] \\
&= \frac{1}{4}\left(E[X_nX_k] - E[X_nX_{k-1}] - E[X_{n-1}X_k] + E[X_{n-1}X_{k-1}]\right) \\
&= \frac{1}{4}[c_{n,k} - c_{n,k-1} - c_{n-1,k} + c_{n-1,k-1}]
\end{aligned}$$

Using $c_{i,j} = \delta_{i,j}$ we get

$$E[Y_n Y_k] = \begin{cases} \frac{1}{2} & n = k, \\ -\frac{1}{4} & n = k \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

The correlation matrix is then of the form

$$\mathbf{R} = \frac{1}{4} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \dots \\ -1 & 2 & -1 & 0 & 0 & \dots \\ 0 & -1 & 2 & -1 & 0 & \dots \\ 0 & 0 & -1 & 2 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

(b) In this case, $c_{i,j} = \sigma_X^2 \delta_{i,j}$ so that

$$\mathbf{R} = \frac{\sigma_X^2}{4} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \dots \\ -1 & 2 & -1 & 0 & 0 & \dots \\ 0 & -1 & 2 & -1 & 0 & \dots \\ 0 & 0 & -1 & 2 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Problem 6.19

(a)

$$\begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} c \\ c^2 \end{bmatrix} \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix} \Rightarrow \hat{X}_n = cX_{n-1}.$$

(b)

$$\begin{aligned} E[(X_n - \hat{X}_n)^2] &= E[(X_n - cX_{n-1})^2] \\ &= E[X_n^2] + c^2 E[X_{n-1}^2] - 2cE[X_n X_{n-1}] \\ &= 1 + c^2 - 2c^2 \\ &= 1 - c^2. \end{aligned}$$

Solutions to Chapter 7 Exercises

Problem 7.1

(a)

$$E[\hat{\mu}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}\sum_{i=1}^{n} 5 = 5.$$

(b)

$$\begin{aligned}
Var(\hat{\mu}) &= E[(\hat{\mu}-5)^2] \\
&= E\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i - 5\right)^2\right] \\
&= E\left[\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - 5)\right)^2\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E[(X_i-5)(X_j-5)] \\
&= \frac{1}{n^2}\sum_{i=1}^{n} E[(X_i-5)^2] \\
&= \frac{\sigma_X^2}{n} = \frac{1}{10}.
\end{aligned}$$

(c) Since the sample variance is unbiased, $E[\hat{s}^2] = \sigma_X^2 = 1$.

Problem 7.2

In this case, $\mu_X = 1$ and $\sigma_X^2 = 1$.

$$\begin{aligned}
E[\hat{\mu}] &= \mu_X = 1 \text{ (since the sample mean is unbiased)} \\
Var(\hat{\mu}) &= \frac{\sigma_X^2}{n} = \frac{1}{2}.
\end{aligned}$$

Problem 7.3

Since the N_i are Gaussian, $\hat{\mu}$ is also Gaussian with

$$\begin{aligned} E[\hat{\mu}] &= \mu_N = 0 \\ Var(\hat{\mu}) &= \frac{\sigma_N^2}{n} = \frac{0.01}{100} = 10^{-4}. \\ \Rightarrow \hat{\mu} &\sim N(0, 10^{-4}). \end{aligned}$$

Problem 7.4

Consider a discrete random variable, $X \in \{-1, 0, 1\}$, whose PMF is

$$P_X(k) = \begin{cases} \epsilon & k = \pm 1, \\ 1 - 2\epsilon & k = 0. \end{cases}$$

For this random variable, $\mu_X = 0$ and $\sigma_X^2 = 2\epsilon$. Both the sample mean and the median will be unbiased in this case. The variance of these two estimators are as follows:

Sample Mean:

$$Var(\hat{\mu}) = \frac{\sigma_X^2}{n} = \frac{2\epsilon}{n}.$$

Median: Suppose $n = 2k - 1$ samples are taken. Then Y_k is the median.

$$\begin{aligned} \Pr(Y_k = 1) &= \Pr(k \text{ or more } X\text{'s} = 1) \\ &= \sum_{m=k}^{n} \binom{n}{m} \epsilon^m (1-\epsilon)^{n-m} \\ \Pr(Y_k = -1) &= \Pr(Y_k = 1) \\ \Pr(Y_k = 0) &= 1 - 2\Pr(Y_k = 1) \\ Var(Y_k) &= 2\Pr(Y_k = 1) = 2\sum_{m=k}^{n} \binom{n}{m} \epsilon^m (1-\epsilon)^{n-m} \end{aligned}$$

Note that for small ϵ, $Var(Y_k) \sim \epsilon^{n/2}$. Hence while the variance of the sample mean decays in an inverse linear fashion with n, the variance of the median decays exponentially in n. Hence in this case, the median would give a better (lower variance) estimate of the mean.

Problem 7.5

Since X_m (m from 1 to n) are IID sequence, assume the expected value and the variance are μ and σ respectively. Moreover, since

$$\hat{\mu} = \frac{1}{n}\sum_{m=1}^{n} X_m \ ,$$

$$\begin{aligned} \hat{\sigma^2} &= \frac{1}{n}\sum_{m=1}^{n}(X_m - \hat{\mu})^2 \\ &= \frac{1}{n}\sum_{m=1}^{n} X_m^2 - \hat{\mu}^2 \end{aligned}$$

From that, we have

$$\begin{aligned} E(\hat{\sigma^2}) &= \frac{1}{n}\sum_{m=1}^{n} E(X_m^2) - E(\hat{\mu}^2) \\ &= \frac{1}{n}\sum_{m=1}^{n}(\mu^2 + \sigma^2) - E(\hat{\mu}^2) \\ &= \mu^2 + \sigma^2 - \frac{1}{n^2}E[(\sum_{m=1}^{n} X_m)^2] \\ &= \mu^2 + \sigma^2 - \frac{1}{n^2}(n\sigma^2 + n^2\mu^2) \\ &= \frac{n-1}{n}\sigma^2 \end{aligned}$$

Hence, this estimate is biased.

Problem 7.6

$$\begin{aligned} \hat{\mu} &= \frac{1}{n}\sum_{m=1}^{n} X_m \\ \hat{s}^2 &= \frac{1}{n-1}\sum_{m=1}^{n}(X_m - \hat{\mu})^2 \\ Var(\hat{s}^2) &= E\left[(\hat{s}^2)^2\right] - \left(E\left[\hat{s}^2\right]\right)^2 \\ &= E\left[(\hat{s}^2)^2\right] - \sigma^4 \end{aligned}$$

We can write the equation for $\hat{s}^2$ a little differently using matrix tranformations. First make the following definitions:

$$\begin{aligned}
Y_m &= X_m - \hat{\mu} \\
Y_m &= [-\frac{1}{n} - \frac{1}{n} \cdots \frac{n-1}{n} \cdots - \frac{1}{n}][X_1 \cdots X_m \cdots X_n]^T \\
\mathbf{Y} &= [Y_1, Y_2, \cdots Y_n]^T = \mathbf{AX} \\
\mathbf{A} &= \begin{bmatrix} \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & \frac{n-1}{n} & \cdots & -\frac{1}{n} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & \frac{n-1}{n} \end{bmatrix}_{n\times n}
\end{aligned}$$

Note that the covariance matrix of $\mathbf{Y}$ is

$$\mathbf{C_Y} = E[\mathbf{YY}^T] = E[\mathbf{AXX}^T\mathbf{A}^T] = \mathbf{AC_XA}^T = \sigma_X^2\mathbf{AA}^T = \sigma_X^2\mathbf{A}.$$

From this we see that $Var(Y_k) = \sigma_X^2\frac{n-1}{n}$ and $Cov(Y_k, Y_m) = -\sigma_X^2/n$. Furthermore, since the Y_k are Gaussian, we have the following higher order moments (which will be needed soon):

$$\begin{aligned}
E[Y_k^4] &= 3\sigma_Y^4 = 3\sigma_X^4\frac{(n-1)^2}{n^2}, \\
E[Y_kY_m] &= E[Y_k^2]E[Y_m^2] + 2E[Y_kY_m]^2 \\
&= \sigma_Y^4 + 2Cov(Y_k, Y_m)^2 \\
&= \sigma_X^4\frac{(n-1)^2}{n^2} + 2\sigma_X^4\frac{1}{n^2} \\
&= \sigma_X^4\frac{n^2-2n+3}{n^2}.
\end{aligned}$$

The variance of the sample variance is then found according to

$$\begin{aligned}
E[(\hat{s}^2)^2] &= \frac{1}{(n-1)^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E[Y_i^2Y_j^2] \\
&= \frac{1}{(n-1)^2}\left\{nE[Y_i^4] + (n^2-n)E[Y_i^2Y_j^2]\right\} \\
&= \frac{1}{(n-1)^2}\left\{3n\sigma_X^4\frac{(n-1)^2}{n^2} + (n^2-n)\sigma_X^4\frac{n^2-2n+3}{n^2}\right\} \\
&= \sigma_X^4\frac{n+1}{n-1}.
\end{aligned}$$

$$\begin{aligned} Var(\hat{s}^2) &= E[(\hat{s}^2)^2] - E[\hat{s}^2]^2 \\ &= \sigma_X^4 \frac{n+1}{n-1} - \sigma_X^4 \\ &= \frac{2}{n-1}\sigma_X^4. \end{aligned}$$

Problem 7.7

First, define $Y_m = X_m - \hat{\mu}$ for $m = 1, 2, \ldots, n$. Note that both the $\{Y_m\}$ and $\hat{\mu}$ are Gaussian random variables. Furthermore,

$$\begin{aligned} Cov(\hat{\mu}, Y_m) &= E[(\hat{\mu} - \mu)(X_m - \hat{\mu})] \\ &= E[\hat{\mu} X_m] - E[\hat{\mu}^2] - \mu E[X_m] + \mu E[\hat{\mu}] \\ &= \left(\frac{\sigma_X^2}{n} + \mu^2\right) - \left(\frac{\sigma_X^2}{n} + \mu^2\right) - \mu^2 + \mu^2 \\ &= 0. \end{aligned}$$

Hence, $\hat{\mu}$ is independent of all the Y_m. Since $\hat{s}^2$ is a function of the Y_m, namely $\frac{1}{n-1}\sum Y_m^2$, then $\hat{s}^2$ must also be independent of $\hat{\mu}$.

Problem 7.8

Given $x_1, x_2, \ldots, x_N$ are observed, we want to minimize

$$\epsilon^2 = \frac{1}{N}\sum_{n=1}^{N}(x_n - a - bn)^2.$$

Taking derivatives with respect to a and b and setting equal to zero produces

$$\begin{aligned} \frac{\partial \epsilon^2}{\partial a} &= \frac{1}{N}\sum_{n=1}^{N}(-2)(x_n - a - bn) = 0 \\ \Rightarrow \frac{1}{N}\sum_{n=1}^{N} x_n &= a\left(\frac{1}{N}\sum_{n=1}^{N} 1\right) + b\left(\frac{1}{N}\sum_{n=1}^{N} n\right) \\ \frac{\partial \epsilon^2}{\partial b} &= \frac{1}{N}\sum_{n=1}^{N}(-2n)(x_n - a - bn) = 0 \\ \Rightarrow \frac{1}{N}\sum_{n=1}^{N} n x_n &= a\left(\frac{1}{N}\sum_{n=1}^{N} n\right) + b\left(\frac{1}{N}\sum_{n=1}^{N} n^2\right) \end{aligned}$$

To simplify the notation, define the following:

$$\begin{aligned}
\overline{n} &= \frac{1}{N}\sum_{n=1}^{N} n, \\
\overline{n^2} &= \frac{1}{N}\sum_{n=1}^{N} n^2, \\
\overline{x_n} &= \frac{1}{N}\sum_{n=1}^{N} x_n, \\
\overline{nx_n} &= \frac{1}{N}\sum_{n=1}^{N} nx_n.
\end{aligned}$$

Then, the optimum values of a and b will satisfy the following matrix equation:

$$\begin{bmatrix} 1 & \overline{n} \\ \overline{n} & \overline{n^2} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \overline{x_n} \\ \overline{nx_n} \end{bmatrix}$$

. The solution is

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\begin{bmatrix} \overline{n^2} \cdot \overline{x_n} - \overline{n} \cdot \overline{nx_n} \\ \overline{nx_n} - \overline{n} \cdot \overline{x_n} \end{bmatrix}}{\overline{n^2} - (\overline{n})^2}.$$

Problem 7.9

(a) A sequence converges in the mean square sense(MSS) if

$$\lim_{n\to\infty} E\left[|S_n - S|^2\right] = 0, \tag{1}$$

and it converges in probability if

$$\lim_{n\to\infty} Pr\left(|S_n - S| > \epsilon\right) = 0. \tag{2}$$

Applying Markov's ineqaulity to the LHS of (2) we get

$$\begin{aligned}
\lim_{n\to\infty} Pr\left(|S_n - S| > \epsilon\right) &\leq \frac{lim_{n\to\infty} E\left[|S_n - S|^2\right]}{\epsilon^2} \\
&\leq 0 \text{ (since the sequence converges in MSS)} \\
\Rightarrow \lim_{n\to\infty} Pr\left(|S_n - S| > \epsilon\right) &= 0 \text{ (since probabilities cannot be negative)}
\end{aligned}$$

(b) Consider the sequence that takes two values $0, n$ with probabilities $1-1/n$ and $1/n$ respectively. The sequence tends to 0 as n tends to infinity. Consider the convergence in probability

$$\begin{aligned}
Pr(S_n = 0) &= 1 - \frac{1}{n} \\
Pr(S_n = n) &= \frac{1}{n} \\
\lim_{n\to\infty} Pr\left(|S_n - S| > \epsilon\right) &= \lim_{n\to\infty} Pr\left(|S_n - 0| > \epsilon\right) \\
&= \lim_{n\to\infty} Pr\left(S_n \neq 0\right) \\
&= \lim_{n\to\infty} \frac{1}{n} = 0
\end{aligned}$$

So the sequence converges in probability.If we consider the convergence in MSS

$$\begin{aligned}
\lim_{n\to\infty} E\left[|S_n - S|^2\right] &= \lim_{n\to\infty} E\left[|S_n - 0|^2\right] \\
&= \lim_{n\to\infty} \left(0(1 - \frac{1}{n}) + n^2\frac{1}{n}\right) \\
&= \lim_{n\to\infty} n \neq 0
\end{aligned}$$

The sequence does not converge in MSS.

Problem 7.10

Let $Z_i = \log X_i$, then $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{Z_i - u_Z}{\sigma_Z}$ converges to a standard normal random variable as n approaches infinity. And,

$$\log Y_n = \sqrt{n}\sigma_Z \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{Z_i - u_Z}{\sigma_Z} + nu_Z\ .$$

Hence, Y_n converges to a lognormal distribution with a mean of $nE[\log(X_i)]$ and a variance of $nVar(\log(X_i))$.

$$f_{Y_n}(y) = \frac{1}{\sqrt{2\pi n s^2 y^2}}\exp\left(-\frac{(\log(y) - n\eta)^2}{2ns^2}\right)u(y),$$

where $\eta = E[\log(X_i)]$ and $s^2 = Var(\log(X_i))$.

Problem 7.11

(a) Because X_i is a Bernoulli RV

$$\begin{aligned}
\sigma_X^2 &= p_A(1-p_A) \\
\hat{p_A} &= \frac{1}{n}\sum_{i=1}^{n} X_i \\
E[\hat{p_A}] &= p_A \\
Var(\hat{p_A}) &= \frac{\sigma_X^2}{n} = \frac{p_A(1-p_A)}{n}
\end{aligned}$$

By virtue of the central limit theorem, we can write

$$\hat{p_A} \sim (p_A, \frac{p_A(1-p_A)}{n})$$

$$\begin{aligned}
Pr(|\hat{p_A} - p_A| < \varepsilon) &= 1 - 2Q\left(\frac{p_A + \varepsilon - p_A}{\sqrt{p_A(1-p_A)/n}}\right) \\
&= 1 - 2Q\left(\sqrt{\frac{n\varepsilon^2}{p_A(1-p_A)}}\right)
\end{aligned}$$

(b)

$$Pr(|\hat{p_A} - p_A| < 0.1p_A) = 0.95$$

Using the result from (a) we get

$$\begin{aligned}
1 - 2Q\left(\sqrt{\frac{n(0.1p_A)^2}{p_A(1-p_A)}}\right) &= 0.95 \\
\Rightarrow Q\left(\sqrt{\frac{0.01p_A n}{(1-p_A)}}\right) &\leq 0.025 \\
\Rightarrow \sqrt{\frac{0.01p_A n}{(1-p_A)}} &\geq 1.9597 \approx 1.96
\end{aligned}$$

Note in the last step, the inequality is reversed since $Q(x)$ is a decreasing function of x.

$$\Rightarrow n \geq 19.6^2 \frac{1-p_A}{p_A}$$

(c)

$$\begin{aligned} Y_n &= \sum_{i=1}^{n} X_i = n\hat{p_A} \\ E[Y_n] &= np_A \end{aligned}$$

Since the value of n was chosen to satisfy the constraints of (b), we can write

$$E[Y_n] = 19.6^2 \frac{1-p_A}{p_A} p_A = 19.6^2(1-p_A).$$

Strictly speaking we will have

$$E[Y_n] \geq 19.6^2(1-p_A).$$

If we assume that $p_A << 1$ we can approximate it as

$$E[Y_n] \geq 19.6^2 \approx 384.$$

Problem 7.12

(a) The probability mass function for the random variable N is

$$P_N(k) = \binom{k-1}{m-1} p_A^m (1-p_A)^{k-m}.$$

Hence,

$$\begin{aligned} E[\hat{p_A}] &= \sum_{k=m}^{\infty} \frac{m-1}{k-1} P_N(k) = \sum_{k=m}^{\infty} \frac{m-1}{k-1} \binom{k-1}{m-1} p_A^m (1-p_A)^{k-m} \\ &= \sum_{k=m}^{\infty} \binom{k-2}{m-2} p_A^m (1-p_A)^{k-m}. \end{aligned}$$

Using the substitutions, $i = k-2$ and $j = m-2$, the above expression is rewritten as

$$E[\hat{p_A}] = p_A^m \sum_{i=j}^{\infty} \binom{i}{j} (1-p_A)^{i-j}.$$

This series is evaluated using identity (E.12) in Appendix E of the text. The results is

$$E[\hat{p_A}] = \frac{p_A^m}{(1-(1-p_A))^{m+1}} = p_A.$$

Hence, the estimate of p_A is unbiased.
(b) If $\hat{p_A} = \frac{m}{N}$,

$$E[\hat{p_A}] = \sum_{k=m}^{\infty} \frac{m}{k} P_N(k) = \sum_{k=m}^{\infty} \frac{m}{k} \binom{k-1}{m-1} p_A^m (1-p_A)^{k-m} \neq p_A.$$

The estimate would be biased and hence not as good of a choice.

Problem 7.13

For this uniform random variable, $\mu_X = 0$ and $\sigma_X^2 = 1/36$. With a sufficiently large number of samples, the sample mean will be Gaussian. The mean and variance of the sample mean are $E[\hat{\mu}] = 0$ and $\sigma_{\hat{\mu}}^2 = \frac{1}{36n}$. Hence the PDF of the sample mean will be:

$$f_{\hat{\mu}}(u) = \sqrt{\frac{18n}{\pi}} \exp(-18nu^2).$$

Problem 7.14

$$\mu_X = 5\text{volts}, \sigma_X = 0.25\text{volts}.$$

For $n = 100$ samples, the sample mean will have

$$E[\hat{\mu}] = 5\text{volts}, \sigma_{\hat{\mu}} = \frac{1}{40}\text{volts}.$$

The 99% confidence interval will be $(\mu_X - \epsilon, \mu_X + \epsilon)$ where

$$\epsilon = c_{0.99}\sigma_{\hat{\mu}} = 2.58 \cdot \frac{1}{40} = 0.0645\text{volts}.$$

Hence, the 99% confidence interval is (4.9355,5.0645) volts. None of the estimates in (a)-(c) fall in this range.

Problem 7.15

<u>90% confidence</u>: $\epsilon = 1.64\sigma_X \ /\sqrt{N_1}$.
<u>99.9% confidence</u>: $\epsilon = 3.29\sigma_X \ /\sqrt{N_2}$.

For these confidence intervals to be the same length we must have

$$1.64\sigma_X \;/\sqrt{N_1} = 3.29\sigma_X \;/\sqrt{N_2}$$

$$\Rightarrow \frac{N_2}{N_1} = \left(\frac{3.29}{1.64}\right)^2 = 4.02.$$

N_2 should be about 4 times larger than N_1.

Problem 7.16

If M is an exponential random variable, then $E[M] = \mu_M$ and $Var(M) = \sigma_M^2 = \mu_M^2$. It is desired that the confidence interval have a width of $\epsilon = 0.2\mu_M$. Hence, the number of samples is determined from $\epsilon = c_{0.9}\sigma_{\hat{\mu}}$. This results in

$$\begin{aligned} 0.2\mu_M &= 1.64\sigma_M/\sqrt{n} = 1.64\mu_M/\sqrt{n} \\ \Rightarrow n &= 67.24. \end{aligned}$$

At least 68 failures need to be observed.

Problem 7.17

$$E[S_N] = E\left[\sum_{i=1}^{\infty} Y_i Z_i\right] = \sum_{i=1}^{\infty} E[Y_i Z_i]$$

Note the value of Y_i is determine by whether or not the test terminates <u>before</u> time$i$. In particular, the values of $\{Z_1, Z_2, \ldots, Z_{i-1}\}$ determine Y_i. In other words, Y_i is dependent on $\{Z_1, Z_2, \ldots, Z_{i-1}\}$ but not on $\{Z_i, Z_{i+1}, \ldots\}$. Therefore, Y_i and Z_i are independent.

$$\begin{aligned} \Rightarrow E[Y_i Z_i] &= E[Y_i]E[Z_i]. \\ E[S_N] &= \sum_{i=1}^{\infty} E[Y_i]E[Z_i] \end{aligned}$$

Note that Z_i is independent of i since the Z_i are IID. Hence,

$$E[S_N] = E[Z_i]\sum_{i=1}^{\infty} E[Y_i]$$

$$
\begin{aligned}
&= E[Z_i]E\left[\sum_{i=1}^{\infty} Y_i\right] \\
&= E[Z_i]E\left[\sum_{i=1}^{N} 1\right] \\
&= E[Z_i]E[N].
\end{aligned}
$$

Solutions to Chapter 8 Exercises

Problem 8.1

(a)

$$\mu_X(t) = \frac{1}{3}\cdot 1 + \frac{1}{3}(-3) + \frac{1}{3}\cdot\sin(2\pi t) = -\frac{2}{3} + \frac{1}{3}\sin(2\pi t).$$

(b)

$$\begin{aligned} R_{X,X}(t_1,t_2) &= E[X(t_1)X(t_2)] \\ &= \frac{1}{3}\cdot 1 + \frac{1}{3}(-3)(-3) + \frac{1}{3}\cdot\sin(2\pi t_1)\sin(2\pi t_2) \\ &= -\frac{8}{3} + \frac{1}{3}\sin(2\pi t_1)\sin(2\pi t_2). \end{aligned}$$

(c) The process is not WSS since the mean is not constant nor is the autocorrelation a function of $t_2 - t_2$.

Problem 8.2

This problem is worked out assuming that each member function occurs with equal probability.
(a)

$$\begin{aligned} \mu_X(t) &= \frac{1}{5}[-2\cos(t) - 2\sin(t) + 2(\cos(t)+\sin(t)) \\ &+ (\cos(t)-\sin(t)) - (\cos(t)-\sin(t))] \\ &= 0. \end{aligned}$$

(b)

$$\begin{aligned} R_{X,X}(t_1,t_2) &= E[X(t_1)X(t_2)] \\ &= \frac{1}{5}[4\cos(t_1)\cos(t_2) + 4\sin(t_1)\sin(t_2) + 4(\cos(t_1)+\sin(t_1))(\cos(t_2)+\sin(t_2)) \\ &+ 2(\cos(t_1)-\sin(t_1))(\cos(t_2)-\sin(t_2))] \\ &= \frac{1}{5}[10\cos(t_1)\cos(t_2) + 10\sin(t_1)\sin(t_2) + 2\cos(t_1)\sin(t_2) + 2\sin(t_1)\cos(t_2) \\ &= 2\cos(t_2-t_1) + \frac{2}{5}\sin(t_1+t_2). \end{aligned}$$

(c) The process is not WSS since the autocorrelation is not a function of $t_2 - t_2$ only. Hence, the process is also not strictly stationary.

Problem 8.3

(a)

$$\mu_X[n] = E[X[n]] = \frac{1}{6}(1+2+3+4+5+6) = 3.5.$$

(b)

$$R_{X,X}[k_1, k_2] = E[X[k_1]X[k_2]]$$

If $k_1 \neq k_2$ then $R_{X,X}[k_1, k_2] = E[X[k_1]]E[X[k_2]] = (3.5)^2$.
If $k_1 = k_2$ then $R_{X,X}[k_1, k_2] = E[X^2[k_1]] = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2) = \frac{91}{6}$.

$$R_{X,X}[k_1, k_2] = \begin{cases} \frac{49}{4} & k_1 \neq k_2, \\ \frac{91}{6} & k_1 = k_2. \end{cases}$$

Problem 8.4

Case 1: n_1 even, n_2 even

$$R_{Y,Y}[n_1, n_2] = E\left[X\left[\frac{n_1}{2}\right]X\left[\frac{n_2}{2}\right]\right] = \begin{cases} 1 & n_1 = n_2, \\ 0 & n_1 \neq n_2. \end{cases}$$

Case 2: n_1 odd, n_2 odd

$$R_{Y,Y}[n_1, n_2] = E\left[X\left[n_1+1\right]X\left[n_2+1\right]\right] = \begin{cases} 1 & n_1 = n_2, \\ 0 & n_1 \neq n_2. \end{cases}$$

Case 3: n_1 odd, n_2 even

$$R_{Y,Y}[n_1, n_2] = E\left[X\left[n_1+1\right]X\left[\frac{n_2}{2}\right]\right] = \begin{cases} 1 & n_2 = 2(n_1+1), \\ 0 & n_2 \neq 2(n_1+1). \end{cases}$$

Case 4: n_1 even, n_2 odd

$$R_{Y,Y}[n_1, n_2] = E\left[X\left[\frac{n_1}{2}\right]X\left[n_2+1\right]\right] = \begin{cases} 1 & n_1 = 2(n_2+1), \\ 0 & n_1 \neq 2(n_2+1). \end{cases}$$

In summary,

$$R_{Y,Y}[n_1, n_2] = \begin{cases} 1 & n_1 = n_2, \\ 1 & n_2 = 2(n_1+1) \ (n_1 \text{ odd}), \\ 1 & n_1 = 2(n_2+1) \ (n_2 \text{ odd}), \\ 0 & \text{otherwise}. \end{cases}$$

Problem 8.5

(a)

$R_{Y,Y}[n_1,n_2] = E[(X[n_1]+c)(X[n_2]+c)] = R_{X,X}[n_1,n_2]+c\mu_X[n_1]+c\mu_X[n_2]+c^2$

Since $X[n]$ is WSS, $\mu_X[n] = \mu_X$ and $R_{X,X}[n_1,n_2] = R_{X,X}[n_2-n_1]$.

$$\Rightarrow R_{Y,Y}[n_1,n_2] = R_{X,X}[n_2-n_1] + 2c\mu_X + c^2.$$

(b)

$$\begin{aligned} E[X[n_1]Y[n_2]] &= E[X[n_1](X[n_2]+c)] = R_{X,X}[n_2-n_1] + c\mu_X. \\ E[X[n_1]]E[Y[n_2]] &= \mu_X(\mu_X+c) = \mu_X^2 + c\mu_X. \end{aligned}$$

The processes are not orthogonal (since $R_{X,Y}[n_1,n_2] \neq 0$).
The processes are not uncorrelated (since $R_{X,Y}[n_1,n_2] \neq \mu_x\mu_Y$).
The processes are not independent (since not uncorrelated and since $Y[n] = X[n]+c$).

Problem 8.6

$$\begin{aligned} E[Y[n]] &= E[(X[n+m] - X[n-m])^2] \\ &= E[X^2[n+m]] + E[X^2[n-m]] - 2E[X[n+m]X[n-m]] \\ &= 2R_{X,X}(0) - 2R_{X,X}(2m) \end{aligned}$$

Problem 8.7

(a)

$$\mu_X(t) = \mu_A\cos(\omega t) + \mu_B\sin(\omega t) = 0.$$

(b)

$$\begin{aligned} R_{X,X}(t_1,t_2) &= E[A^2]\cos(\omega t_1)\cos(\omega t_2) + E[B^2]\sin(\omega t_1)\sin(\omega t_2) \\ &+ E[AB]\cos(\omega t_1)\sin(\omega t_2) + E[AB]\sin(\omega t_1)\cos(\omega t_2) \\ &= \frac{E[A^2]+E[B^2]}{2}\cos(\omega(t_2-t_1)) + \frac{E[A^2]-E[B^2]}{2}\cos(\omega(t_1+t_2)). \end{aligned}$$

(c) $X(t)$ will be WSS if $E[A^2] = E[B^2] \Rightarrow \sigma_A^2 = \sigma_B^2$.

Problem 8.8

Define $X(t)$ and $Y(t)$ according to:

$$\begin{aligned}
X(t) &= A(t)\cos(t), \\
Y(t) &= B(t)\sin(t), \\
\mu_A(t) &= \mu_B(t) = 0, \\
R_{A,A}(\tau) &= R_{B,B}(\tau) = R(\tau), \\
R_{A,B}(\tau) &= 0.
\end{aligned}$$

Then,

$$\begin{aligned}
E[X(t)] &= 0. \\
E[Y(t)] &= 0. \\
R_{X,X}(t_1,t_2) &= E[A(t_1)A(t_2)\cos(t_1)\cos(t_2)] \\
&= R(t_2-t_1)\cos(t_1)\cos(t_2) \\
R_{Y,Y}(t_1,t_2) &= E[B(t_1)B(t_2)\sin(t_1)\sin(t_2)] \\
&= R(t_2-t_1)\sin(t_1)\sin(t_2) \\
R_{Z,Z}(t_1,t_2) &= E[(X(t_1)+Y(t_1))(X(t_2)+Y(t_2))] \\
&= R_{X,X}(t_1,t_2)+R_{Y,Y}(t_1,t_2)+R_{X,Y}(t_1,t_2)+R_{Y,X}(t_1,t_2) \\
&= R(t_2-t_1)[\cos(t_1)\cos(t_2)+\sin(t_1)\sin(t_2)] \text{ (since} R_{X,Y}(\tau)=R_{Y,X}(\tau)=0) \\
&= R(t_2-t_1)\cos(t_2-t_1).
\end{aligned}$$

Therefore, for this example, $Z(t)=X(t)+Y(t)$ is WSS, while $X(t)$ and $Y(t)$ are not WSS.

Problem 8.9

$$\begin{aligned}
X(t) &= A(t)\cos(\omega_0 t+\Theta) \\
E[X(t)] &= E[A(t)]E[\cos(\omega_0 t+\Theta)] = 0 \\
R_{X,X}(t_1,t_2) &= E[A(t_1)A(t_2)]E[\cos(\omega_0 t_1+\Theta)\cos(\omega_0 t_2+\Theta)] \\
&= \frac{1}{2}R_{A,A}(t_2-t_1)\{E[\cos(\omega_0(t_2-t_1))]+E[\cos(\omega_0(t_1+t_2)+2\Theta)]\} \\
&= \frac{1}{2}R_{A,A}(t_2-t_1)\cos(\omega_0(t_2-t_1)).
\end{aligned}$$

Similarly,

$$\begin{aligned}
E[Y(t)] &= 0 \\
R_{Y,Y}(t_1,t_2) &= \frac{1}{2}R_{A,A}(t_2-t_1)\cos((\omega_0+\omega_1)(t_2-t_1)).
\end{aligned}$$

Hence, both $X(t)$ and $Y(t)$ are WSS. If $Z(t) = X(t) + Y(t)$,

$$\begin{aligned}
E[z(t)] &= E[X(t)] + E[Y(t)] = 0, \\
R_{Z,Z}(t_1,t_2) &= R_{X,X}(t_1,t_2) + R_{Y,Y}(t_1,t_2) + R_{X,Y}(t_1,t_2) + R_{Y,X}(t_1,t_2), \\
R_{X,Y}(t_1,t_2) &= E[A(t_1)A(t_2)]E[\cos(\omega_0 t_1 + \Theta)\cos((\omega_0+\omega_1)t_2 + \Theta)], \\
&= \frac{1}{2}R_{A,A}(t_2-t_1)\cos(\omega_0(t_2-t_1) + \omega_1 t_2), \\
R_{Y,X}(t_1,t_2) &= R_{A,A}(t_2-t_1)\cos(\omega_0(t_1-t_2) + \omega_1 t_1).
\end{aligned}$$

Since $R_{Z,Z}(t_1,t_2)$ is not a function of $t_1 - t_2$, $Z(t)$ is not WSS.

Problem 8.10

(a)

$$\begin{aligned}
\Pr(X(t)=1) &= \Pr(X(t)=1|X(0)=1)\Pr(X(0)=1) \\
&+ \Pr(X(t)=1|X(0)=-1)\Pr(X(0)=-1) \\
&= p\cdot\Pr(\text{even number of switches in } (0,t)) \\
&+ (1-p)\cdot\Pr(\text{odd number of switches in } (0,t)) \\
&= p\left(\frac{1}{2}+\frac{1}{2}\exp(-2\lambda t)\right) + (1-p)\left(\frac{1}{2}-\frac{1}{2}\exp(-2\lambda t)\right) \\
&= \frac{1}{2} - \frac{1-2p}{2}\exp(-2\lambda t). \\
\Pr(X(t)=-1) &= 1-\Pr(X(t)=1) = \frac{1}{2} + \frac{1-2p}{2}\exp(-2\lambda t).
\end{aligned}$$

(b)

$$E[X(t)] = 1\cdot\Pr(X(t)=1) + (-1)\cdot\Pr(X(t)=-1) = (1-2p)\exp(-2\lambda t).$$

(c) First assume $t_2 > t_1$. To calculate $E[X(t_1)X(t_2)]$, note that

$$X(t_1)X(t_2) = \begin{cases} 1 & \text{if even number of switches in } (t_1,t_2), \\ -1 & \text{if odd number of switches in } (t_1,t_2). \end{cases}$$

Then,

$$\begin{aligned}
R_{X,X}(t_1,t_2) &= 1\cdot \Pr(\text{even number of switches in } (t_1,t_2)) \\
&+ (-1)\cdot \Pr(\text{odd number of switches in } (t_1,t_2)) \\
&= \left(\frac{1}{2}+\frac{1}{2}\exp(-2\lambda(t_2-t_1))\right) - \left(\frac{1}{2}+\frac{1}{2}\exp(-2\lambda(t_2-t_1))\right) \\
&= \exp(-2\lambda(t_2-t_1)).
\end{aligned}$$

Similarly, if $t_1 > t_2$,

$$R_{X,X}(t_1,t_2) = \exp(-2\lambda(t_1-t_2)).$$

Putting these results together, we get that

$$R_{X,X}(t_1,t_2) = \exp(-2\lambda|t_1-t_2|)$$

(d) Although $R_{X,X}(t_1,t_2)$ is only a function of t_2-t_1, the mean is not necessarily constant. Hence, the process is WSS only if $p = 1/2$ (so that $\mu_X(t) = 0$).

Problem 8.11

(a) Since T is uniformly distributed over one period of $s(t)$, for any time instant t, $X(t) = s(t-T)$ will be equally likely to take on any of the values in one period of $s(t)$. Since $s(t)$ is 1 half of the time and -1 half of the time, we get

$$\Pr(X(t)=1) = \Pr(X(t)=-1) = \frac{1}{2}.$$

(b)

$$E[X(t)] = (1)\cdot \Pr(X(t)=1) + (-1)\cdot \Pr(X(t)=-1) = 0.$$

This can also be seen in an alternative manner:

$$E[X(t)] = E[s(t-T)] = \int s(t-u) f_T(u)du = \int_0^1 s(t-u)du.$$

Since the integral is over one period of $s(t)$, $E[X(t)]$ is just the d.c. value (time average) of $s(t)$ which is zero.
(c)

$$R_{X,X}(t_1,t_2) = E[s(t_1-T)s(t_2-T)]$$

$$
\begin{aligned}
&= \int_0^1 s(t_1-u)s(t_2-u)du \\
&= \int_0^1 s(v)s(v+t_2-t_1)dv \\
&= s(t)*s(-t)\bigg|_{t=t_2-t_1}.
\end{aligned}
$$

This is the time correlation of a square wave with itself which will result in the periodic triangle wave shown in Figure 1.

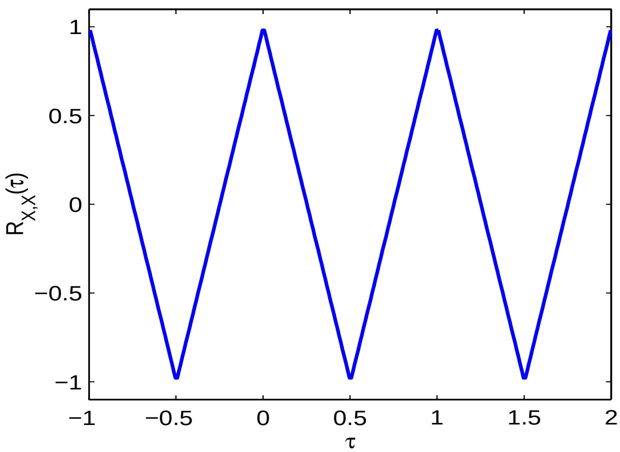

Figure 1: Autocorrelation function for process of Exercise 8.11

(d) The process is WSS.

Problem 8.12

(a) Since T is uniformly distributed over one period of $s(t)$, for any time instant t, $X(t) = s(t-T)$ will be equally likely to take on any of the values in one period of $s(t)$. Given the linear functional form of $s(t)$, $X(t)$ will be uniform over $(-1,1)$.

$$
f_X(x;t) = \begin{cases} \frac{1}{2} & -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}
$$

(b) $E[X(t)] = 0$ since the PDF above is symmetric about zero.
(c)

$$\begin{aligned} R_{X,X}(t_1,t_2) &= E[s(t_1-T)s(t_2-T)] \\ &= \int_0^1 s(t_1-u)s(t_2-u)du \\ &= \int_0^1 s(v)s(v+t_2-t_1)dv \\ &= s(t)*s(-t)\Big|_{t=t_2-t_1}. \end{aligned}$$

This is the time correlation of a triangle wave with itself which will result in the periodic signal shown in Figure 2.

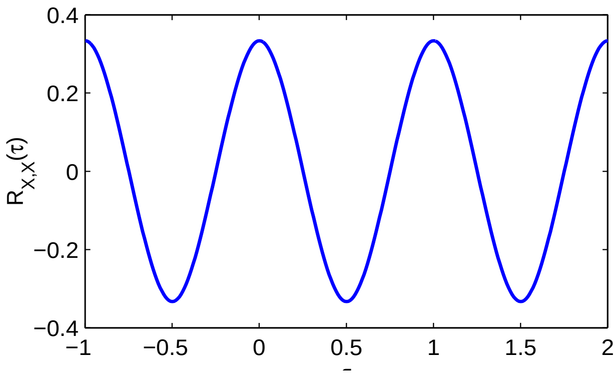

Figure 2: Autocorrelation function for process of Exercise 8.12

(d) The process is WSS.

Problem 8.13

(a)

$$E[X(t)] = \Pr(X(t)=+1) - \Pr(X(t)=-1) = \frac{1}{2} - \frac{1}{2} = 0.$$

(b)

$$X(t_1)X(t_2) = \begin{cases} 1 & \text{if no pulse transitions in } (t_1,t_2), \\ \pm 1 & \text{with equal prob. if 1 or more pulse transitions in } (t_1,t_2). \end{cases}$$

$E[X(t)] = 0 \cdot \Pr(\geq 1 \text{ transitions in } (t_1, t_2)) + 1 \cdot \Pr(\text{no transitions in } (t_1, t_2))$.

For $|t_2 - t_1| < \Delta$,

$$\Pr(\text{no transitions in } (t_1, t_2)) = 1 - \frac{|t_2 - t_1|}{\Delta}.$$

Therefore,

$$R_{X,X}(t_1, t_2) = \begin{cases} 1 - \frac{|t_2 - t_1|}{\Delta} & |t_2 - t_1| < \Delta, \\ 0 & |t_2 - t_1| > \Delta. \end{cases}$$

(c) $X(t)$ is WSS.

Problem 8.14

(a)

$$f_X(x;t) = \left. \frac{f_A(a)}{\left|\frac{dX}{dA}\right|} \right|_{A = -\frac{1}{t}\ln(x)} = \frac{f_A(-\frac{1}{t}\ln(x))}{tx}$$

(b)

$$E[X(t)] = E[e^{-At}] = \int_0^\infty e^{-at} e^{-a} da = \frac{1}{1+t}.$$

$$R_{X,X}(t_1, t_2) = E[X(t_1)X(t_2)] = E[e^{-A(t_1+t_2)}] = \frac{1}{1 + t_1 + t_2}.$$

The process is not WSS.

Problem 8.15

(a)

$$\begin{aligned} E[X[n]] &= E[pX[n-1]] + E[W_n] \\ \Rightarrow \mu_X[n] &= p\mu_X[n-1] \\ \Rightarrow \mu_X[n] &= p^n \mu_X[0] \end{aligned}$$

Since $\mu_X[0] = E[W_0] = 0$, then $\mu_X[n] = 0$ for all n.

(b) Assume $n_2 > n_1$. Then,

$$\begin{aligned}
X[n_2] &= pX[n_2-1] + W_{n_2} \\
&= p^2X[n_2-2] + W_{n_2} + pW_{n_2-1} \\
&= p^3X[n_2-3] + W_{n_2} + pW_{n_2-1} + p^2W_{n_2-2} \\
&\vdots \\
&= p^{n_2-n_1}X[n_1] + \sum_{m=0}^{n_2-n_1-1} p^m W_{n_2-2} \\
&\vdots \\
&= \sum_{m=0}^{n_2} p^m W_{n_2-m} = \sum_{i=0}^{n_2} p^{n_2-i} W_i \\
E[X[n_1]X[n_2]] &= \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} p^{n_1-i}p^{n_2-j}E[W_iW_j] \\
&= \sum_{i=0}^{n_1} p^{n_1+n_2-2i}E[W_i^2] \\
&= \sigma_W^2 p^{n_1+n_2}\sum_{i=0}^{n_1}(p^{-2})^i \\
&= \sigma_W^2 p^{n_1+n_2}\frac{1-(p^{-2})^{n_1+1}}{1-p^{-2}} \\
&= \sigma_W^2\frac{p^{n_1+n_2+2}-p^{n_2-n_1}}{p^2-1}.
\end{aligned}$$

Similarly, if $n_1 > n_2$, then

$$E[X[n_1]X[n_2]] = \sigma_W^2\frac{p^{n_1+n_2+2}-p^{n_1-n_2}}{p^2-1}.$$

Putting these two results together gives

$$R_{X,X}[n_1,n_2] = \sigma_W^2\frac{p^{n_1+n_2+2}-p^{|n_2-n_1|}}{p^2-1}.$$

Problem 8.16

$$\begin{aligned}
E[X^2(t_1)Y^2(t_2)] &= E[X^2(t_1)]E[Y^2(t_2)] + 2E[X(t_1)Y(t_2)]^2 \\
&= R_{X,X}(0)R_{Y,Y}(0) + 2R_{X,Y}^2(t_2-t_1).
\end{aligned}$$

Problem 8.17

Using the result of Exercise 6.12,

$$\begin{aligned} E[X(t_1)X^3(t_2)] &= 3E[X(t_1)X(t_2)]E[X^2(t_2)] \\ &= 3R_{X,X}(t_2-t_1)R_{X,X}(0). \end{aligned}$$

Problem 8.18

$$\begin{aligned} R_{Z,Z}[k] &= R_{X,X}[k]+R_{Y,Y}[k]+R_{X,Y}[k]+R_{Y,X}[k]. \\ R_{X,Y}[k] &= E[X[n]Y[n+k]=\mu_X[n]\mu_Y[n+k]=0. \\ \Rightarrow R_{Z,Z}[k] &= R_{X,X}[k]+R_{Y,Y}[k]. \end{aligned}$$

Figure 3 shows the various autocorrelation functions.

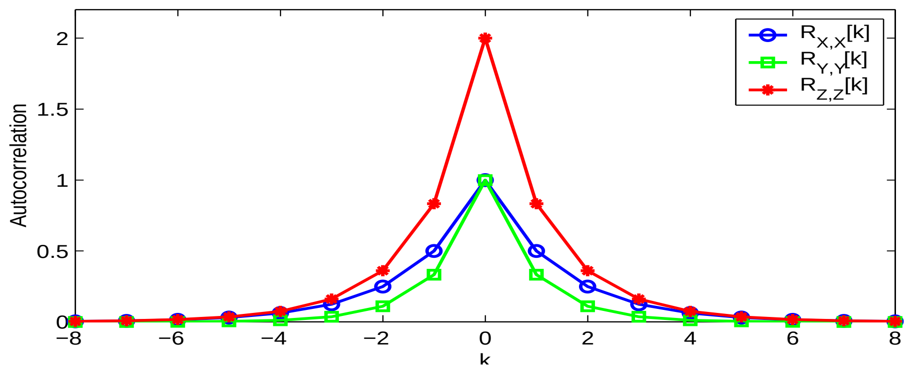

Figure 3: Autocorrelation functions for processes of Exercise 8.18

Problem 8.19

Since

$$\lim_{k\to\infty} R_{X,X}[k]=\mu_X=0,$$

the process is ergodic in the mean.

Problem 8.20

$$\begin{aligned}\langle Y(t)\rangle &= \langle C\cdot X(t)\rangle = C\langle X(t)\rangle = C\mu_X.\\ E[Y(t)] &= E[C\cdot X(t)] = E[C]\mu_X.\end{aligned}$$

Since these two quantities are not equal, the process is not ergodic in the mean.

$$\begin{aligned}\langle Y(t_1)Y(t_2)\rangle &= \langle C^2\cdot X(t_1)X(t_2)\rangle = C^2\langle X(t_1)X(t_2)\rangle = C^2R_{X,X}(t_1,t_2).\\ E[Y(t_1)Y(t_2)] &= E[C^2\cdot X(t_1)X(t_2)] = E[C^2]R_{X,X}(t_1,t_2).\end{aligned}$$

Since these two quantities are not equal, the process is not ergodic in the mean.

Problem 8.21

The first equation is a first order constant coefficient differential equation whose solution is

$$P_X(0;t) = c\cdot e^{-\lambda t}.$$

The constant c is resolved by noting that since $P_X(i;t)$ is a counting process, at time $t=0$,

$$P_X(i;0) = \Pr(X(0)=i) = \begin{cases} 1 & i=0,\\ 0 & i>0.\end{cases}$$

Since $P_X(0;0)=1$, we get $c=1 \Rightarrow P_X(0;t)=e^{-\lambda t}$. The second differential equation is also first order, contant coefficient (although not homogeneous). It can be solved using the integrating factor technique to be

$$P_X(i;t) = \lambda\int_0^t P_X(i-1;u)\exp(-\lambda(t-u))du + c\cdot e^{-\lambda t}.$$

Using the initial condition $P_X(i;0)=0$ for $i>0$ we get $c=0$. Therefore,

$$P_X(i;t) = \lambda\int_0^t P_X(i-1;u)\exp(-\lambda(t-u))du$$

For

$$\begin{array}{ll} i=1, & P_X(1;t) = \lambda \int_0^t e^{-\lambda t} du = \lambda t e^{-\lambda t}, \\ i=2, & P_X(2;t) = \lambda \int_0^t \lambda u e^{-\lambda t} du = \frac{(\lambda t)^2}{2} e^{-\lambda t}, \\ i=3, & P_X(3;t) = \lambda \int_0^t \frac{(\lambda u)^2}{2} e^{-\lambda t} du = \frac{(\lambda t)^3}{3!} e^{-\lambda t}, \\ & \vdots \end{array}$$

In general

$$P_X(i;t) = \frac{(\lambda t)^i}{i!} e^{-\lambda t}.$$

Problem 8.22

(a) Consider a time instant, t, such that $0 < t < t_o$.

$$\begin{aligned} \Pr(X(t)=1|X(t_o)=1) &= \frac{\Pr(X(t_o)=1|X(t)=1)\Pr(X(t)=1)}{\Pr(X(t_o)=1)} \\ &= \frac{\lambda t e^{-\lambda t}}{\lambda t_o e^{-\lambda t_o}} \Pr(\text{no arrivals in } (0,t)) \\ &= \frac{t}{t_o} \exp(-\lambda(t-t_o)) \exp(-\lambda(t_o-t)) \\ &= \frac{t}{t_o}. \end{aligned}$$

Let S_1 be the arrival time of the first arrival. Then $\{X(t)=1\} \Leftrightarrow \{S_1 \le t\}$. Hence, given that there is one arrival in $(0, t_o)$, that is $X(t_o)=1$,

$$\Pr(X(t)=1|X(t_o)=1) = \Pr(S_1 \le t|X(t_o)=1) = F_{S_1}(t|X(t_o)=1) = \frac{t}{t_o}$$

$$\Rightarrow f_{S_1}(t|X(t_o)=1) = \frac{1}{t_o}, \qquad 0 \le t < t_o.$$

(b) Let $0 \le t_1 \le t_2 \le t_o$. Also define

$$\begin{aligned} S_1 &= \text{arrival time of first arrival}, \\ S_2 &= \text{arrival time of second arrival}. \end{aligned}$$

The joint distribution of the two arrival times is found according to:

$$\begin{aligned} f_{S_1,S_2}(t_1,t_2|X(t_o)=2) &= f_{S_1|S_2}(t_1|S_2=t_2, X(t_o)=2) f_{S_2}(t_2|X(t_o)=2) \\ &= f_{S_1|S_2}(t_1|S_2=t_2) f_{S_2}(t_2|X(t_o)=2) \end{aligned}$$

To find $f_{S_2}(t_2)$, proceed as in part (a).

$$\begin{aligned}
F_{S_2}(t_2|X(t_o)=2) &= \Pr(X(t_2)=2|X(t_o)=2) \\
&= \Pr(X(t_o)=2|X(t_2)=2)\frac{\Pr(X(t_2)=2)}{\Pr(X(t_o)=2)} \\
&= \exp(-\lambda(t_o-t_2))\frac{\frac{(\lambda t_2)^2}{2}e^{-\lambda t_2}}{\frac{(\lambda t_0)^2}{2}e^{-\lambda t_o}} \\
&= \left(\frac{t_2}{t_o}\right)^2. \\
\Rightarrow f_{S_2}(t_2|X(t_o)=2) &= \frac{2t_2}{t_o^2}, \qquad 0 \le t_2 \le t_o.
\end{aligned}$$

Given $S_2 = t_2$ there is one arrival between 0 and t_2. From the results of part (a), we know S_1 is uniform over $(0, t_2)$ given $S_2 = t_2$. Therefore

$$f_{S_1|S_2}(t_1|t_2) = \frac{1}{t_2}, \qquad 0 \le t_1 \le t_2.$$

Putting the two previous results together we get

$$\begin{aligned}
f_{S_1,S_2}(t_1,t_2|X(t_o)=2) &= f_{S_1|S_2}(t_1|S_2=t_2)f_{S_2}(t_2|X(t_o)=2) \\
&= \frac{2t_2}{t_o^2}\cdot\frac{1}{t_2} \\
&= \frac{2}{t_o^2}, \qquad 0 \le t_1 \le t_2 \le t_o.
\end{aligned}$$

The two arrival times S_1 and S_2 are uniformly distributed over $0 \le t_1 \le t_2 \le t_o$.

Problem 8.23

$$\begin{aligned}
\Pr(N(t)=k|N(t+\tau)=m) &= \Pr(N(t+\tau)=m|N(t)=k)\frac{\Pr(N(t)=k)}{\Pr(N(t+\tau)=m)} \\
&= \frac{\frac{(\lambda\tau)^{m-k}}{(m-k)!}e^{-\lambda\tau}\frac{(\lambda t)^k}{k!}e^{-\lambda t}}{\frac{(\lambda(t+\tau))^m}{m!}\exp(-\lambda(t+\tau))} \\
&= \binom{m}{k}\frac{t^k\tau^{m-k}}{(t+\tau)^m}.
\end{aligned}$$

Problem 8.24

(a)

$$E[Y(t) = \frac{1}{t_o}(E[X(t+t_o)] - E[X(t)]) = \frac{\mu_X - \mu_X}{t_o} = 0.$$

(b)

$$\begin{aligned}
R_{Y,Y}(t_1,t_2) &= E\left[\frac{(X(t_1+t_o) - X(t_1)(X(t_2+t_o) - X(t_2))}{t_o^2}\right] \\
&= \frac{R_{X,X}(t_2-t_1) - R_{X,X}(t_2-t_1-t_o) - R_{X,X}(t_2-t_1+t_o) + R_{X,X}(t_2-t_1)}{t_o^2} \\
&= \frac{2R_{X,X}(\tau) - R_{X,X}(\tau-t_o) - R_{X,X}(\tau+t_o)}{t_o^2}
\end{aligned}$$

The process $Y(t)$ is WSS.

Problem 8.25

Using probability generating functions:

$$\begin{aligned}
H_X(z) &= E[z^{X(t)}] = E[z^{\sum_{i=1}^n X_i(t)}] = E\left[\prod_{i=1}^n z^{X_i(t)}\right] = \prod_{i=1}^n E[z^{X_i(t)}] = \prod_{i=1}^n H_{X_i}(z). \\
\Pr(X_i(t)=k) &= \frac{(\lambda_i t)^k}{k!}e^{-\lambda_i t}. \\
H_{X_i}(z) &= \sum_{k=0}^{\infty} \frac{(\lambda_i z t)^k}{k!}e^{-\lambda_i t} = e^{\lambda_i z t}e^{-\lambda_i t} = e^{\lambda_i t(z-1)}. \\
H_X(z) &= \prod_{i=1}^n e^{\lambda_i t(z-1)} = \exp\left(\left(\sum_{i=1}^n \lambda_i\right) t(z-1)\right).
\end{aligned}$$

Define $\lambda = \sum_{i=1}^n \lambda_i$. Then $H_X(z) = \exp(\lambda t(z-1))$ which is the probability generating function of a Poisson random variable. Therefore $X(t)$ is a Poisson proces with arrival rate $\lambda = \sum_{i=1}^n \lambda_i$.

Problem 8.26

(a)

$$\Pr(N(t)=k) = \Pr(k \text{ of the } T_i \text{ are } < t, n-k \text{ are } > t)$$

$$
\begin{aligned}
&= \binom{n}{k}(\Pr(T_i < t))^k(\Pr(T_i > t))^{n-k} \\
&= \binom{n}{k}(1-e^{-\lambda t})^k(e^{-\lambda t})^{n-k}
\end{aligned}
$$

Therefore, $N(t)$ is a Binomial process.
(b)

$$\Pr(N(t) \geq 1) = 1 - \Pr(N(t) = 0) = 1 - \binom{n}{0}e^{-n\lambda t} = 1 - e^{-n\lambda t}.$$

Using $n = 10$, $\lambda = (250 \text{ days})^{-1}$, and $t = 90$ days, we get

$$\Pr(N(t) \geq 1) = 1 - \exp(-\frac{900}{250}).$$

Problem 8.27

$$\Pr(N(t) < 10) = \sum_{k=0}^{9} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

(a)

$$\lambda = 0.1, t = 10 \Rightarrow Pr(N(t) < 10) = \sum_{k=0}^{9} \frac{(1)^k}{k!} e^{-1} \approx 1.$$

(b)

$$\lambda = 10, t = 10 \Rightarrow Pr(N(t) < 10) = \sum_{k=0}^{9} \frac{(100)^k}{k!} e^{-100} \approx 0.$$

(c)

$$
\begin{aligned}
\Pr(\text{1 call in 10 minutes}) &= 1 \cdot e^{-1} = 0.3679. \\
\Pr(\text{2 calls in 10 minutes}) &= \frac{1^2}{2!} \cdot e^{-1} = 0.1839. \\
\Pr(\text{1 call, 2 calls}) &= \Pr(\text{1 call}) \Pr(\text{2 calls}) \\
&= \frac{1^3}{2!1!} e^{-2} = 0.0677.
\end{aligned}
$$

Problem 8.28

Expected number of strikes is st.
(a) In one minute, $st = \frac{1}{3} \cdot 1 = \frac{1}{3}$.
(b) In ten minute, $st = \frac{1}{3} \cdot 10 = \frac{10}{3}$.
(c) Average time between strikes is $\frac{1}{s} = 3$ minutes.

Problem 8.29

(a) Figure 4 shows a typical realization.

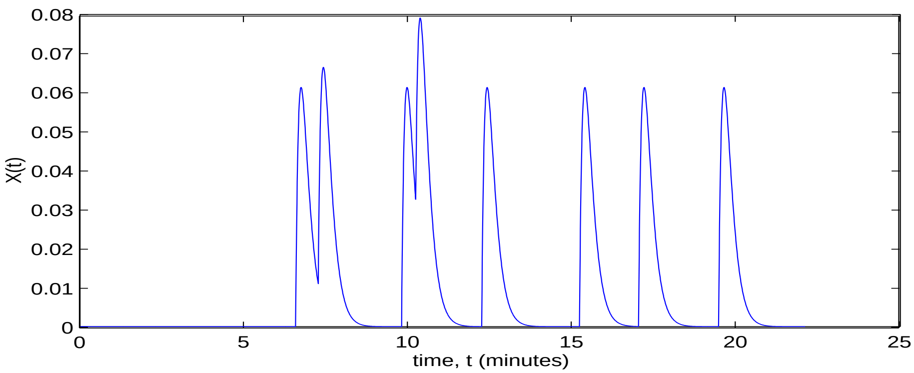

Figure 4: Sample realization for process of Exercise 8.29

(b)

$$E[X(t)] = s \int_0^t t \exp(-at) dt = \frac{s}{a^2}[1 - (1 + at)e^{-at}].$$

(c)

$$\begin{aligned}
C_{X,X}(t, t+\tau) &= s \int_0^t t \exp(-at)(t+\tau) \exp(-a(t+\tau)) dt \\
&= se^{-a\tau} \left[\int_0^t t^2 e^{-2at} dt + \tau \int_0^t te^{-2at} dt \right] \\
&= se^{-a\tau} \left\{ \frac{1}{4a^3}[1 - (1 + 2at + 2a^2t^2)e^{-2at}] + \frac{\tau}{4a^2}[1 - (1 + 2at)e^{-2at}] \right\} \\
R_{X,X}(t, t+\tau) &= C_{X,X}(t, t+\tau) + E[X(t)]E[X(t+\tau)]
\end{aligned}$$

Problem 8.30

Following the same procedure used in the text, approximate the shot noise process as

$$X(t) = \sum_{n=0}^{\infty} A_n V_n h(t - n\Delta t),$$

where the A_n are IID and independent of the $V_n \in (0,1)$ which are also IID with $\Pr(V_n = 1) = \lambda\Delta t$.

(a)

$$\begin{aligned} E[X(t)] &= \sum_{n=0}^{\infty} E[A_n]E[V_n]h(t - n\Delta t) \\ &= \lambda E[A] \sum_{n=0}^{\infty} h(t - n\Delta t)\Delta t \\ &= \lambda E[A] \int_0^{\infty} h(t-u)du \\ &= \lambda E[A] \int_0^{t} h(v)dv. \end{aligned}$$

(b)

$$\begin{aligned} R_{X,X}(t, t+\tau) &= E\left[\sum_{n=0}^{\infty} A_n V_n h(t - n\Delta t) \sum_{m=0}^{\infty} A_m V_m h(t + \tau - m\Delta t)\right] \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} E[A_n A_m]E[V_n V_m]h(t - n\Delta t)h(t + \tau - m\Delta t) \end{aligned}$$

Note that

$$E[V_n V_m] = \begin{cases} \lambda\Delta t & n = m, \\ (\lambda\Delta t)^2 & n \neq m. \end{cases}$$

Using this, the autocorrelation becomes:

$$\begin{aligned} R_{X,X}(t, t+\tau) &= E[A]^2 \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (\lambda\Delta t)^2 h(t - n\Delta t)h(t + \tau - m\Delta t) \\ &+ E[A^2] \sum_{n=0}^{\infty} (\lambda\Delta t) h(t - n\Delta t)h(t + \tau - n\Delta t) \\ &- E[A]^2 \sum_{n=0}^{\infty} (\lambda\Delta t)^2 h(t - n\Delta t)h(t + \tau - n\Delta t). \end{aligned}$$

In the limit as $\Delta t \to 0$, the last term will be insignificant because of the $(\Delta t)^2$ term. The autocorrelation then becomes:

$$\begin{aligned} R_{X,X}(t,t+\tau) &= \lambda^2 E[A]^2 \int_0^\infty h(t-u)du \int_0^\infty h(t+\tau-v)dv \\ &+ \lambda E[A^2] \int_0^\infty h(t-u)h(t+\tau-u)du \\ &= \mu_X(t)\mu_X(t+\tau) + \lambda E[A^2] \int_0^\infty h(v)h(v+\tau)dv. \end{aligned}$$

Solutions to Chapter 9 Exercises

Problem 9.1

$$\mathbf{P} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}.$$

Performing an eigen decomposition, we get $\mathbf{P} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$ where

$$\mathbf{Q} = \begin{bmatrix} 1 & p \\ 1 & -q \end{bmatrix}, \mathbf{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 1-p-q \end{bmatrix}, \mathbf{Q}^{-1} = \frac{1}{p+q}\begin{bmatrix} q & p \\ 1 & -1 \end{bmatrix}.$$

Then

$$\begin{aligned}
\mathbf{P}^n &= \mathbf{Q}\mathbf{\Lambda}^n\mathbf{Q}^{-1} \\
&= \frac{1}{p+q}\begin{bmatrix} 1 & p \\ 1 & -q \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (1-p-q)^n \end{bmatrix}\begin{bmatrix} q & p \\ 1 & -1 \end{bmatrix} \\
&= \frac{1}{p+q}\begin{bmatrix} q+p(1-p-q)^n & p-p(1-p-q)^n \\ q-q(1-p-q)^n & p+q(1-p-q)^n \end{bmatrix}.
\end{aligned}$$

Problem 9.2

(a)

$$\begin{aligned}
\Pr(X_2=1, X_1=0) &= \Pr(X_2=1|X_1=0)\Pr(X_1=0) = p\cdot\Pr(X_1=0). \\
\Pr(X_1=0) &= \Pr(X_1=0|X_0=0)\Pr(X_0=0) + \Pr(X_1=0|X_0=1)\Pr(X_0=1) \\
&= (1-p)s + q(1-s). \\
\Rightarrow \Pr(X_2=1, X_1=0) &= p(1-p)s + pq(1-s).
\end{aligned}$$

(b)

$$\begin{aligned}
\Pr(X_1=1|X_0=0, X_2=0) &= \frac{\Pr(X_2=0|X_1=1, X_0=0)\Pr(X_1=1|X_0=0)}{\Pr(X_2=0|X_0=0)} \\
&= \frac{P_{1,0}^{(1)}P_{0,1}^{(1)}}{P_{0,0}^{(2)}} \\
&= \frac{qp}{(1-p)^2+pq}.
\end{aligned}$$

(c)

$$\begin{aligned}
\Pr(X_2 = X_1) &= \Pr(X_2 = 0, X_1 = 0) + \Pr(X_2 = 1, X_1 = 1) \\
&= (1-p)\Pr(X_1 = 0) + (1-q)\Pr(X_1 = 1) \\
&= (1-p)[(1-p)s + q(1-s)] + (1-q)[ps + (1-q)(1-s)] \\
&= (1-p)^2 s + q(1-p)(1-s) + p(1-q)s + (1-q)^2(1-s). \\
\Pr(X_1 = X_0) &= \Pr(X_0 = 0, X_1 = 0) + \Pr(X_0 = 1, X_1 = 1) \\
&= (1-p)s + (1-q)(1-s).
\end{aligned}$$

These expressions are not the same, so $\Pr(X_1 = X_0) \neq \Pr(X_2 = X_1)$.

Problem 9.3

To simplify notation, define $\mathbf{Y} = [X_{k+2}, X_{k+3}, \ldots, X_{k+m}]$. Then,

$$\begin{aligned}
\Pr(X_k|X_{k+1}, X_{k+2}, \ldots, X_{k+m}) &= \Pr(X_k|X_{k+1}, \mathbf{Y}) \\
&= \frac{\Pr(X_{k+1}, \mathbf{Y}|X_k)\Pr(X_k)}{\Pr(X_{k+1}, \mathbf{Y})} \\
&= \frac{\Pr(\mathbf{Y}|X_{k+1}, X_k)\Pr(X_{k+1}|X_k)\Pr(X_k)}{\Pr(\mathbf{Y}|X_{k+1})\Pr(X_{k+1})}
\end{aligned}$$

Because the process is a Markov chain, $\Pr(\mathbf{Y}|X_{k+1}, X_k) = \Pr(\mathbf{Y}|X_{k+1})$. Therefore,

$$\Pr(X_k|X_{k+1}, \mathbf{Y}) = \frac{\Pr(X_{k+1}|X_k)\Pr(X_k)}{\Pr(X_{k+1})} = \Pr(X_k|X_{k+1}).$$

The statement is true. That is, a Markov chain viewed in reverse time still possesses the memoryless property.

Problem 9.4

For a general 2x2 transition matrix,

$$\mathbf{P} = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix},$$

the two-step transition matrix is

$$\mathbf{P}^2 = \begin{bmatrix} (1-p)^2 + pq & p(1-p) + p(1-q) \\ q(1-p) + q(1-q) & (1-q)^2 + pq \end{bmatrix}.$$

Note that $\mathbf{P}^2$ is a stochastic matrix as each element is ≤ 1 (for $0 \leq p, q \leq 1$) and each row sums to 1. Summing the diagonal elements we note that

$$\begin{aligned} P_{1,1}^{(2)} + P_{2,2}^{(2)} &= (1-p)^2 + pq + (1-q)^2 + pq \\ &= 2 - 2p - 2q + p^2 + q^2 + 2pq \\ &= 1 + 1 - 2(p+q) + (p+q)^2 \\ &= 1 + (1-p-q)^2 \\ &\geq 1. \end{aligned}$$

Since any two-step transition matrix must be of this general form, any two-step transition matrix must have the sum of its diagonal elements ≥ 1.

Next, consider a general stochastic matrix

$$\mathbf{A} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}.$$

The matrix $\mathbf{A}$ will have an eigen decompostion of the form $\mathbf{A} = \mathbf{Q\Lambda Q}^{-1}$ where the matrix of eigenvalues is of the form

$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 1-\alpha-\beta \end{bmatrix}.$$

For this to be a valid two-step transition matrix, $\sqrt{\mathbf{A}}$ must be a stochastic matrix. Note that $\sqrt{\mathbf{A}} = \mathbf{Q}\sqrt{\mathbf{\Lambda}}\mathbf{Q}^{-1}$ where

$$\sqrt{\mathbf{\Lambda}} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\alpha-\beta} \end{bmatrix}.$$

This will result in a valid stochastic matrix if $\sqrt{1-\alpha-\beta}$ is real. Hence, we must have $\alpha + \beta \leq 1$ or equivalently $(1-\alpha) + (1-\beta) \geq 1$. Therefore the matrix $\mathbf{A}$ will be a valid two-step transition matrix if the sum of its diagonal components is ≥ 1.

Problem 9.5

(a) The states of the system can be the values of the voltage between switches. Hence there are three states, namely -1, 0, and +1. With this representation, the process is a random walk with absorbing boundaries. The corresponding state transition matrix is

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix}.$$

(b)For this process $P_{i,i}^{(n)} > 0$ only if n is even. Therefore, the process is periodic with period 2.
(c) For $(i-1)t_s \le t < it_s$ and i odd, $X(t) = 0$. For $(i-1)t_s \le t < it_s$ and i even,

$$\begin{aligned} \Pr(X(t) = 1) &= p, \\ \Pr(X(t) = -1) &= q = 1 - p. \end{aligned}$$

Problem 9.6

Given there are i black balls (and $n-i$ white balls) in Urn A, there must be $n-i$ black balls (and i white balls) in Urn B. Represent this state by $X_t = i$. Given $X_t = i$, there are three possibilities for X_{t+1}:

$$X_{t+1} = \begin{cases} i-1 & \text{if black selected from A and white selected from B,} \\ i+1 & \text{if white selected from A and black selected from B,} \\ i & \text{same color selected from both urns.} \end{cases}$$

The transition probabilites are then calculated as follows:

$$\begin{aligned} \Pr(X_{t+1} = i-1 | X_t = i) &= \Pr(\text{A=black, B=white} | X_t = i) \\ &= \frac{i}{n} \cdot \frac{i}{n} = \left(\frac{i}{n}\right)^2, \\ \Pr(X_{t+1} = i+1 | X_t = i) &= \Pr(\text{A=white, B=black} | X_t = i) \\ &= \frac{n-i}{n} \cdot \frac{n-i}{n} = \left(\frac{n-i}{n}\right)^2, \\ \Pr(X_{t+1} = i | X_t = i) &= 1 - \left(\frac{i}{n}\right)^2 - \left(\frac{n-i}{n}\right)^2. \end{aligned}$$

Problem 9.7

(a)

$$\pi(1) = \pi(0)\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \end{bmatrix}$$

Therefore, $\Pr(+1 \text{ after 1 step}) = \frac{3}{8}$.

(b)

$$\pi(1) = \pi(0)\mathbf{P} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Therefore, $\Pr(+1 \text{ after 1 step}) = \frac{2}{3}$.

(c)

$$\pi(2) = \pi(0)\mathbf{P}^2,$$

where

$$\mathbf{P}^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{bmatrix}.$$

With $\pi(0) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$,

$$\pi(2) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{bmatrix} = \begin{bmatrix} \frac{11}{32} & \frac{21}{32} \end{bmatrix}.$$

Therefore, $\Pr(+1 \text{ after 1 step}) = \frac{11}{32}$.

With $\pi(0) = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}$,

$$\pi(2) = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{3}{8} & \frac{5}{8} \\ \frac{5}{16} & \frac{11}{16} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Therefore, $\Pr(+1 \text{ after 1 step}) = \frac{1}{3}$.

Problem 9.8

(a)

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} \text{go} & \text{skip} \end{array} \\ \begin{array}{c} \text{go} \\ \text{skip} \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} \end{array}$$

(b)

$$\Pr(\text{go Friday}|\text{went Wednesday}) = \frac{1}{2}.$$

(c)

$$\mathbf{P}^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{9}{16} & \frac{7}{16} \end{bmatrix}$$

$$\Pr(\text{go Friday}|\text{went Monday}) = P_{0,0}^{(2)} = \frac{5}{8}.$$

(d) $\mathbf{P} = \mathbf{Q\Lambda Q}^{-1}$ where

$$\mathbf{Q} = \frac{1}{4}\begin{bmatrix} 4 & 2 \\ 4 & -3 \end{bmatrix}, \mathbf{\Lambda} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{Q}^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 2 \\ 4 & -4 \end{bmatrix}.$$

The limiting form of the eigenvalue matrix is

$$\lim_{k\to\infty} \mathbf{\Lambda}^k = \lim_{k\to\infty} \begin{bmatrix} 1 & 0 \\ 0 & \left(-\frac{1}{4}\right)^k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, the limiting form of the k-step transition matrix is

$$\lim_{k\to\infty} \mathbf{P}^k = \frac{1}{20}\begin{bmatrix} 4 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 4 & -4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}.$$

Since both rows are equal, a steady state distribution does exist and it is given by

$$\pi = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} \end{bmatrix}.$$

Problem 9.9

(a)

$$\mathbf{P} = \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.4 & 0.6 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \mathbf{Q\Lambda Q}^{-1}.$$

Using MATLAB we found,

$$\mathbf{Q} = \begin{bmatrix} -0.4197 & 0.5774 & 0.2792 \\ 0.1570 & 0.5774 & -0.3981 \\ 0.8940 & 0.5774 & 0.8738 \end{bmatrix}, \quad \mathbf{\Lambda} = \begin{bmatrix} -0.4695 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.3195 \end{bmatrix}.$$

The limiting form of the k-step transition probability matrix is then found as follows:

$$\begin{aligned}\lim_{k\to\infty}\mathbf{\Lambda}^k &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \\ \lim_{k\to\infty}\mathbf{P}^k &= \mathbf{Q}\lim_{k\to\infty}\mathbf{\Lambda}^k\mathbf{Q}^{-1} = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0.4 & 0.5 & 0.1 \end{bmatrix}.\end{aligned}$$

The steady-state distribution is then

$$\pi = \begin{bmatrix} 0.4 & 0.5 & 0.1 \end{bmatrix}.$$

(b) Using MATLAB to calculate $\mathbf{P}^{100}$, we get the same matrix found in part (a):

$$\mathbf{P}^{100} = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0.4 & 0.5 & 0.1 \end{bmatrix}.$$

$$\Rightarrow P_{1,3}^{(100)} = 0.1.$$

The interpretation of this result is that for all practical purposes, the process has reached steady state after 100 steps.
(c)

$$\pi(3) = \pi(0)\mathbf{P}^3 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.4 & 0.6 & 0 \\ 1 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0.4274 & 0.4808 & 0.0919 \end{bmatrix}$$

$$\Pr(\text{in state 3 after 3rd step}) = 0.0919.$$

Problem 9.10

$$\begin{aligned}\Pr(\text{A C T}) &= \frac{1}{3}\cdot 0.6\cdot 0.2 = 0.04. \\ \Pr(\text{C A T}) &= \frac{1}{3}\cdot 0.7\cdot 0.3 = 0.07. \\ \Pr(\text{T A T}) &= \frac{1}{3}\cdot 0.8\cdot 0.3 = 0.08.\end{aligned}$$

Therefore,

$$\Pr(\text{proper english word}) = 0.04 + 0.07 + 0.08 = 0.19.$$

Problem 9.11

(a)

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{12} & 0 & \frac{7}{12} & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{12} & 0 & \frac{7}{12} & 0 & 0 & 0 \\ 0 & 0 & \frac{5}{12} & \frac{1}{6} & \frac{5}{12} & 0 & 0 \\ 0 & 0 & 0 & \frac{7}{12} & 0 & \frac{5}{12} & 0 \\ 0 & 0 & 0 & 0 & \frac{7}{12} & 0 & \frac{5}{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

(b)

$$\Pr(\text{loses in 3 tosses}) = \Pr(3 \to 2 \to 1 \to 0) = (5/12)^3 = 0.0723.$$

(c)

$$\Pr(\text{loses in 4 tosses}) = \Pr(3 \to 3 \to 2 \to 1 \to 0) = \frac{1}{6}\left(\frac{5}{12}\right)^3 = 0.0121.$$

$$\begin{aligned} \Pr(\text{loses in 5 tosses}) &= \Pr(3 \to 3 \to 3 \to 2 \to 1 \to 0) \\ &+ \Pr(3 \to 4 \to 3 \to 2 \to 1 \to 0) \\ &+ \Pr(3 \to 2 \to 3 \to 2 \to 1 \to 0) \\ &+ \Pr(3 \to 2 \to 1 \to 2 \to 1 \to 0) \\ &= \left(\frac{1}{6}\right)^2\left(\frac{5}{12}\right)^3 + 3\frac{7}{12}\left(\frac{5}{12}\right)^4 = 0.0548. \end{aligned}$$

$$\Pr(\text{loses in 5 or fewer tosses}) = 0.0723 + 0.0121 + 0.0548 = 0.1392.$$

We can verify this solution using MATLAB by noting that the probability of interest is found by finding the entry in the 4th row and 1st column of $\mathbf{P}^5$.

Problem 9.12

(a)

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & \ldots & n \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ \frac{1}{n} & 0 & \frac{n-1}{n} & 0 & \ldots & 0 \\ \frac{2}{n} & 0 & 0 & \frac{n-2}{n} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \end{array}$$

(b)

$$\begin{aligned} \Pr(M = m | X_0 = 0) &= \Pr(X_1 = 1 | X_0 = 0) \Pr(X_2 = 2 | X_1 = 1) \ldots \\ &\ldots \Pr(X_{m-1} = m-1 | X_{m-2} = m-2) \Pr(X_m = 0 | X_{m-1} = m-1) \\ &= 1 \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-m+2}{n} \cdot \frac{m-1}{n} \\ &= \frac{(m-1)n!}{n^m (n-m+1)!} \end{aligned}$$

Problem 9.13

$$p_{i,j} = \begin{cases} p^i & j = 2i \\ 1 - p^i & j = 0 \end{cases}$$

$$\begin{aligned} \Pr(1 \text{ person infected}) &= 1 - p \\ \Pr(2 \text{ people infected}) &= p(1 - p^2) \\ \Pr(4 \text{ people infected}) &= p \cdot p^2 (1 - p^4) = p^3 (1 - p^4) \\ \Pr(8 \text{ people infected}) &= p \cdot p^2 \cdot p^4 (1 - p^8) = p^7 (1 - p^8) \\ &\vdots \\ \Pr(2^k \text{ people infected}) &= p^{2^k - 1} (1 - p^{2^k}) \end{aligned}$$

Problem 9.14

(a) Let $Y_n = X_n \bmod 3$ be the states of a Markov chain. Then

$$\{X_n \text{ is a multiple of } 3\} \Leftrightarrow \{Y_n = 0\}.$$

The transition matrix for this Markov chain is

$$\mathbf{P} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Since $\mathbf{P}^2 = \mathbf{P}$, $\mathbf{P}^n = \mathbf{P} \Rightarrow \pi = \frac{1}{3}[1, 1, 1]$. Therefore

$$\lim_{n\to\infty} \Pr(Y_n = 0) = \frac{1}{3}.$$

(b) Let $Y_n = X_n \bmod 5$ be the states of a Markov chain. Then

$$\{X_n \text{ is a multiple of } 5\} \Leftrightarrow \{Y_n = 0\}.$$

The transition matrix for this Markov chain is

$$\mathbf{P} = \frac{1}{6}\begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The steady state distribution is $\pi = \frac{1}{5}[1, 1, 1, 1, 1]$ since with this distribution $\pi = \pi\mathbf{P}$. Therefore

$$\lim_{n\to\infty} \Pr(Y_n = 0) = \frac{1}{5}.$$

Problem 9.15

The eigendecomposition of the given transition matrix is

$$\begin{aligned}
\mathbf{P} &= \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.05 & 0.7 & 0.25 \\ 0.05 & 0.5 & 0.45 \end{bmatrix} \\
&= \begin{bmatrix} -0.9931 & 0.5774 & 0.6509 \\ 0.0828 & 0.5774 & -0.3906 \\ 0.0828 & 0.5774 & 0.6509 \end{bmatrix} \begin{bmatrix} 0.35 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix} \begin{bmatrix} -0.9295 & 0 & 0.9295 \\ 0.1332 & 1.0825 & 0.5163 \\ 0 & -0.9601 & 0.9601 \end{bmatrix}.
\end{aligned}$$

From this it is easily determined that the limiting form on the n-step transition matrix is

$$\lim_{n\to\infty} \mathbf{P}^n = \begin{bmatrix} -0.9931 & 0.5774 & 0.6509 \\ 0.0828 & 0.5774 & -0.3906 \\ 0.0828 & 0.5774 & 0.6509 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.9295 & 0 & 0.9295 \\ 0.1332 & 1.0825 & 0.5163 \\ 0 & -0.9601 & 0.9601 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0769 & 0.6250 & 0.2981 \\ 0.0769 & 0.6250 & 0.2981 \\ 0.0769 & 0.6250 & 0.2981 \end{bmatrix}.$$

Since all rows are identical, there is a steady state distribution and it is given by

$$\pi = \begin{bmatrix} 0.0769 & 0.6250 & 0.2981 \end{bmatrix}.$$

Problem 9.16

(a) If the process is currently in state (X,Y) it must transition to a state of the form (?,X). This eliminates half of the possible transitions and hence the transition matrix must have at least half of its entries equal to zero.
(b) $\pi = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ is the solution to $\pi\mathbf{P} = \pi$.
(c)

$$\begin{aligned} \Pr(\text{process is in state A}) &= \Pr((A,A) \cup (A,B)) \\ &= \Pr(A,A) + \Pr(A,B) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \end{aligned}$$

Problem 9.17

(a) The first three states communicate with period 1.
(b) All states communicate with period = 3.
(c) All states communicate with period = 2.

Problem 9.18

Suppose $d(j) = n$. Further suppose $i \leftrightarrow j$ and a transition for i to j is possible in k steps and from j to i in m steps. Then transitions from i to i are possible in steps of $k+m+n$, $k+m+2n$, $k+m+3n$, ... Thus

$$\begin{aligned} d(i) &= \text{GCD}\{\text{numbers which include } k+m+n, k+m+2n, \ldots\} \\ \Rightarrow d(i) &\leq n \\ \text{or } d(i) &\leq d(j). \end{aligned}$$

Using a similar argument, we conclude that $d(j) \leq d(i)$ and therefore we must have $d(i) = d(j)$.

Problem 9.19

Starting with (9.19) in the text,

$$p_{i,i}^{(n)} = \sum_{m=0}^{n} p_{i,i}^{(n-m)} f_{i,i}^{(m)}, \qquad n = 1, 2, 3, \ldots.$$

Multiplying each equation by z^n and summing over n results in

$$\sum_{n=1}^{\infty} p_{i,i}^{(n)} z^n = \sum_{n=1}^{\infty} \sum_{m=0}^{n} p_{i,i}^{(n-m)} f_{i,i}^{(m)} z^n.$$

We can add the $n = 0$ term to the series on the right hand side since $f_{i,i}^{(0)} = 0$ and hence the $n = 0$ term contributes nothing. On the left hand side, the $n = 0$ term is $p_{i,i}^{(0)} = 1$ and hence we must adjust the left hand side by 1 if we include the $n = 0$ term. Doing so we get

$$\sum_{n=0}^{\infty} p_{i,i}^{(n)} z^n - 1 = \sum_{n=0}^{\infty} \sum_{m=0}^{n} p_{i,i}^{(n-m)} f_{i,i}^{(m)} z^n.$$

Changing the order of sums on the right hand side produces

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{i,i}^{(n)} z^n - 1 &= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} p_{i,i}^{(n-m)} f_{i,i}^{(m)} z^n \\
&= \sum_{m=0}^{\infty} f_{i,i}^{(m)} z^m \sum_{n=m}^{\infty} p_{i,i}^{(n-m)} z^{n-m} \\
P_{i,i}(z) - 1 &= \sum_{m=0}^{\infty} f_{i,i}^{(m)} z^m P_{i,i}(z) \\
&= F_{i,i}(z) P_{i,i}(z).
\end{aligned}$$

Problem 9.20

Start with the relationship

$$p_{i,j}^{(n)} = \sum_{m=0}^{n} f_{i,j}^{(m)} p_{j,j}^{(n-m)}, \qquad n = 1, 2, 3, \ldots.$$

Taking $f_{i,j}^{(0)} = 0$ and $p_{i,j}^{(0)} = 0$ (for $i \neq j$), the above relationship holds for $n = 0$ as well. Multiplying by z^n and summing results in

$$\sum_{n=0}^{\infty} p_{i,j}^{(n)} z^n = \sum_{n=0}^{\infty} \sum_{m=0}^{n} f_{i,j}^{(m)} p_{j,j}^{(n-m)} z^n.$$

Exchanging order of sums on the right hand side and also writing $z^n = z^m z^{n-m}$ results in

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{i,j}^{(n)} z^n &= \sum_{m=0}^{\infty} f_{i,j}^{(m)} z^m \sum_{n=m}^{\infty} p_{j,j}^{(n-m)} z^{n-m} \\
P_{i,j}(z) &= \sum_{m=0}^{\infty} f_{i,j}^{(m)} z^m P_{j,j}(z) \\
P_{i,j}(z) &= F_{i,j}(z) P_{j,j}(z).
\end{aligned}$$

Problem 9.21

Suppose $i \leftrightarrow j$ such that i can be reached from j in m steps and j can be reached from i in k steps. That is $p_{j,i}^{(m)} > 0$ and $p_{i,j}^{(k)} > 0$. Then,

$$\begin{aligned}
p_{j,j}^{(n)} &\geq p_{j,i}^{(m)} p_{i,i}^{(n-m-k)} p_{i,j}^{(k)} \\
\Rightarrow \sum_{n=1}^{\infty} p_{j,j}^{(n)} &\geq p_{j,i}^{(m)} p_{i,j}^{(k)} \sum_{n=1}^{\infty} p_{i,i}^{(n-m-k)}
\end{aligned}$$

If i is recurrent, then

$$\begin{aligned}
\sum_{n}^{\infty} p_{i,i}^{(n-m-k)} &= \sum_{n}^{\infty} p_{i,i}^{(n)} = \infty \\
\Rightarrow \sum_{n=1}^{\infty} p_{j,j}^{(n)} &= \infty.
\end{aligned}$$

Therefore state j is recurrent as well. Alternatively, suppose state j is transient $\Rightarrow \sum_{n=1}^{\infty} p_{j,j}^{(n)} < \infty$, then

$$\sum_{n=1}^{\infty} p_{i,i}^{(n-m-k)} \leq \frac{\sum_{n=1}^{\infty} p_{j,j}^{(n)}}{p_{j,i}^{(m)} p_{i,j}^{(k)}} < \infty.$$

Therfore state i is also transient.

Problem 9.22

(a)

$$p_{i,j} = \begin{cases} p & j = i+1 \\ 1-p & j = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{P} = \begin{bmatrix} 1-p & p & 0 & 0 & \cdots \\ 1-p & 0 & p & 0 & \cdots \\ 1-p & 0 & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

(b)

$$f_{0,0}^{(n)} = \Pr(n-1 \text{ successes followed by 1 failure}) = p^{n-1}(1-p).$$

(c)

$$f_{0,0} = \sum_{n=1}^{\infty} f_{0,0}^{(n)} = \sum_{n=1}^{\infty} p^{n-1}(1-p) = \frac{1-p}{1-p} = 1.$$

Therefore state 0 is recurrent.

Problem 9.23

We start by noting that the n-step transition probabilities are given by

$$p_{i,i}^{(n)} = \begin{cases} (1-p)p^i & n = i+1, i+2, \ldots, \\ 1 & n = 0, \\ 0 & n = 1, 2, \ldots, i. \end{cases}$$

From this we can determine the generating function

$$P_{i,i}(z) = \sum_{n=0}^{\infty} p_{i,i}^{(n)} z^n = 1 + \sum_{n=i+1}^{\infty} p^i (1-p) z^n = 1 + \frac{(1-p)p^i z^{i+1}}{1-z} = \frac{1-z+(1-p)p^i z^{i+1}}{1-z}.$$

Using the relation of (9.22) we get

$$F_{i,i}(z) = 1 - \frac{1}{P_{i,i}(z)} = 1 - \frac{1-z}{1-z+(1-p)p^i z^{i+1}} = \frac{(1-p)p^i z^{i+1}}{1-z+(1-p)p^i z^{i+1}}.$$

The mean time to first return is then

$$\mu_i = \left. \frac{df_{i,i}(z)}{dz} \right|_{z=1} = \left. \frac{d}{dz} \frac{(1-p)p^i z^{i+1}}{1-z+(1-p)p^i z^{i+1}} \right|_{z=1} = \frac{1}{p^i(1-p)}.$$

From this the steady state distribution is obtained:

$$\pi_i = \frac{1}{\mu_i} = p^i(1-p).$$

Problem 9.24

Starting with (9.33)

$$p_{i,j}(t+s) = \sum_k p_{i,k}(t)p_{k,j}(s).$$

Letting $s = t$ and $t = \Delta t$ produces

$$p_{i,j}(t+\Delta t) = \sum_k p_{i,k}(\Delta t)p_{k,j}(t).$$

Applying (9.34) results in

$$\begin{aligned} p_{i,j}(t+\Delta t) &= \lambda_i \Delta t p_{i+1,j}(t) + (1-(\lambda_i+\mu_i)\Delta t)p_{i,j}(t) \\ &+ \mu_i \Delta t p_{i-1,j}(t) + o(\Delta t) \\ \Rightarrow \frac{p_{i,j}(t+\Delta t) - p_{i,j}(t)}{\Delta t} &= \lambda_i p_{i+1,j}(t) - (\lambda_i+\mu_i)p_{i,j}(t) + \mu_i p_{i-1,j}(t) + \frac{o(\Delta t)}{\Delta t}. \end{aligned}$$

Passing to the limit as $\Delta t \to 0$ gives the desired result

$$\frac{dp_{i,j}(t)}{dt} = \lambda_i p_{i+1,j}(t) - (\lambda_i+\mu_i)p_{i,j}(t) + \mu_i p_{i-1,j}(t).$$

Problem 9.25

(a) Given $\Phi(\omega,t) = \int_{-\infty}^{\infty} f(x,t)e^{j\omega x}dx$, then using integration by parts

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial x} f(x,t)e^{j\omega x}dx = e^{j\omega x}f(x,t)\Big|_{-\infty}^{\infty} - j\omega \int_{-\infty}^{\infty} f(x,t)e^{j\omega x}dx = -j\omega\Phi(\omega,t).$$

Similarly

$$\int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2} f(x,t)e^{j\omega x}dx = (-j\omega)^2\Phi(\omega,t).$$

Now, multiply both sides of (9.63) by $e^{j\omega x}$ and integrate with respect to x over $(-\infty,\infty)$ to produce

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial t} f(x,t)e^{j\omega x}dx == -2c\int_{-\infty}^{\infty} \frac{\partial}{\partial x} f(x,t)e^{j\omega x}dx + D\int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2} f(x,t)e^{j\omega x}dx.$$

Using the above identities, this can be written in terms of $\Phi(\omega,t)$ as

$$\frac{\partial}{\partial t}\Phi(\omega,t) = (2cj\omega - D\omega^2)\Phi(\omega,t).$$

Given that $X(0) = 0$, then

$$\Phi(\omega, 0) = E[e^{j\omega X(0)}] = 1.$$

(b) Since this is a first order constant coefficient differential equation, the solution is of the form

$$\Phi(\omega, t) = k \exp((2cj\omega - D\omega^2)t).$$

Applying the initial condition, we determine that $k = 1$ so that

$$\Phi(\omega, t) = \exp((2cj\omega - D\omega^2)t).$$

(c) The characteristic function found in part (b) is that of a Gaussian random variable with mean $\mu = 2ct$ and variance $\sigma^2 = 2Dt$ (see Example 4.20 for the characteristic function of a Gaussian random variable). Hence,

$$f(x, t) = \frac{1}{\sqrt{4\pi Dt}} \exp(-\frac{(x - 2ct)^2}{4Dt}).$$

Problem 9.26

(a) Let $N =$ number of transmissions.

$$\Pr(N = n) = q^{n-1}(1 - q) n = 1, 2, 3, \ldots$$

$$E[N] = \sum_{n=1}^{\infty} nq^{n-1}(1 - q) = \frac{1}{1 - q}.$$

(b)

$$T = (T_t + T_a)N - T_a$$

$$E[T] = (T_t + T_a)E[N] - T_a = \frac{T_t + T_a}{1 - q} - T_a.$$

(c)

$$T = (T_t + T_a)(N_1 + N_2) - 2T_a$$

$$E[T] = (T_t + T_a)(E[N_1] + E[N_2]) - 2T_a = \frac{2(T_t + T_a)}{1 - q} - 2T_a.$$

Solutions to Chapter 10 Exercises

Problem 10.1

$$
\begin{aligned}
R_{X,X}(t,t+\tau) &= E[X(t)X(t+\tau)] \\
&= b^2 E[\cos(2\pi\Psi t+\Theta)\cos(2\pi\Psi(t+\tau)+\Theta)] \\
&= \frac{b^2}{2}E[\cos(2\pi\Psi\tau)] + \frac{b^2}{2}E[\cos(2\pi\Psi(2t+\tau)+2\Theta)] \\
&= \frac{b^2}{4}E[\exp(j2\pi\Psi\tau)] + \frac{b^2}{4}E[\exp(-j2\pi\Psi\tau)] \\
R_{X,X}(\tau) &= \frac{b^2}{4}\Phi_\Psi(2\pi\tau) + \frac{b^2}{4}\Phi_\Psi(-2\pi\tau) \\
S_{X,X}(f) &= \frac{b^2}{4}FT[\Phi_\Psi(2\pi\tau)] + \frac{b^2}{4}FT[\Phi_\Psi(-2\pi\tau)]
\end{aligned}
$$

where FT refers to the Fourier Transform operator. Note that

$$\Phi_\Psi(2\pi\tau) = \int_{-\infty}^{\infty} f_\Psi(\phi)e^{j2\pi\phi\tau}d\phi = FT^{-1}[f_\Psi(\phi)].$$

Therefore,

$$FT[\Phi_\Psi(2\pi\tau)] = FT[FT^{-1}[f_\Psi(\phi)]] = f_\Psi(f).$$

Likewise, $FT[\Phi_\Psi(-2\pi\tau)] = f_\Psi(-f)$. Thus,

$$S_{X,X}(f) = \frac{b^2}{4}f_\Psi(f) + \frac{b^2}{4}f_\Psi(-f).$$

Hence for any valid PSD, $S(f)$, we can construct a process $X(t) = b\cos(2\pi\Psi t+\Theta)$ which will have a PSD equal to $S(f)$ by choosing Ψ to have a PDF whose even part is proportional to $S(f)$. The constant b is adjusted to make the total power match that specified by $S(f)$.

Problem 10.2

$$X(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$
\begin{aligned}
E[X(t)] &= E[A]\cos(\omega t) + E[B]\sin(\omega t) = 0 \\
R_{X,X}(t, t+\tau) &= E[(A\cos(\omega t) + B\sin(\omega t))(A\cos(\omega(t+\tau)) + B\sin(\omega(t+\tau)))] \\
&= E[A^2]\cos(\omega t)\cos(\omega(t+\tau)) + E[B^2]\sin(\omega t)\sin(\omega(t+\tau))
\end{aligned}
$$

Since A and B are identically distributed, $E[A^2] = E[B^2] = \sigma^2$, so that

$$
R_{X,X}(t, t+\tau) = \sigma^2 \cos(\omega\tau).
$$

Therefore, $X(t)$ is WSS.

$$
\begin{aligned}
E[X^3(t)] &= E[(A\cos(\omega t) + B\sin(\omega t))^3] \\
&= E[A^3]\cos^3(\omega t) + E[B^3]\sin^3(\omega t) \\
&= E[A^3]\{\cos^3(\omega t) + \sin^3(\omega t)\}.
\end{aligned}
$$

Since $\cos^3(\omega t) + \sin^3(\omega t)$ is not constant, the process will not be strictly stationary for any $randomvariable A$ such that $E[A^3] \neq 0$.

Problem 10.3

(a)

$$
\begin{aligned}
R_{X,X}(t, t+\tau) &= E[(\sum_{n=1}^{N} a_n \cos(\omega_n t + \theta_n))(\sum_{m=1}^{N} a_m \cos(\omega_m(t+\tau) + \theta_m))] \\
&= \sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_m E[\cos(\omega_n t + \theta_n)\cos(\omega_m(t+\tau) + \theta_m)]
\end{aligned}
$$

The expected value in the above expression is zero for all $m \neq n$. Therefore

$$
\begin{aligned}
R_{X,X}(t, t+\tau) &= \sum_{n=1}^{N} a_n^2 E[\cos(\omega_n t + \theta_n)\cos(\omega_n(t+\tau) + \theta_n)] \\
R_{X,X}(\tau) &= \frac{1}{2}\sum_{n=1}^{N} a_n^2 \cos(\omega_n \tau).
\end{aligned}
$$

(b)

$$
S_{X,X}(f) = FT[R_{X,X}(\tau)] = \frac{1}{4}\sum_{n=1}^{N} a_n^2\{\delta(f - f_n) + \delta(f + f_n)\}.
$$

Problem 10.4

(a)

$$\begin{aligned}R_{X,X}(t,t+\tau) &= E[(\sum_{n=1}^{\infty} A_n \cos(n\omega t) + B_n \sin(n\omega t))\\
&\quad (\sum_{m=1}^{\infty} A_m \cos(m\omega(t+\tau)) + B_m \sin(m\omega(t+\tau)))]\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\{E[A_nA_m]\cos(n\omega t)\cos(m\omega(t+\tau))\\
&+ E[A_nB_m]\cos(n\omega t)\sin(m\omega(t+\tau))\\
&+ E[B_nA_m]\sin(n\omega t)\cos(m\omega(t+\tau)\\
&+ E[B_nB_m]\sin(n\omega t)\sin(m\omega(t+\tau)\}\\
&= \sum_{n=1}^{\infty}\{E[A_n^2]\cos(n\omega t)\cos(n\omega(t+\tau)) + E[B_n^2]\sin(n\omega t)\sin(n\omega(t+\tau))\}\end{aligned}$$

(b) If $E[A_n^2] = E[B_n^2] = \sigma_n^2$, then the above simplifies to

$$R_{X,X}(\tau) = \sum_{n=1}^{\infty} \sigma_n^2 \cos(n\omega t),$$

and the process is WSS.

Problem 10.5

$$S_{X,X}(f) = FT[R_{X,X}(\tau)] = FT[1] = \delta(f).$$

That is, all power in the process is at d.c.

Problem 10.6

(a)

$$\begin{aligned}R_{Y,Y}(t,t+\tau) &= E[Y(t)Y(t+\tau)] = E[X^2(t)X^2(t+\tau)]\\
&= E[X^2(t)]E[X^2(t+\tau)] + 2E[X(t)X(t+\tau)]^2\\
&= R_{X,X}^2(0) + 2R_{X,X}^2(\tau)\\
S_{Y,Y}(f) &= R_{X,X}^2(0)\delta(f) + S_{X,X}(f) * S_{X,X}(f)\\
&= \left[\int_{-\infty}^{\infty} S_{X,X}(f)df\right]^2(0)\delta(f) + S_{X,X}(f) * S_{X,X}(f)\end{aligned}$$

(b)

$$\begin{aligned}
R_{X,X}(0) &= \int_{-\infty}^{\infty} S_{X,X}(f)df = 2B. \\
S_{X,X}(f) * S_{X,X}(f) &= 2B\text{tri}\left(\frac{f}{2B}\right). \\
S_{Y,Y}(f) &= 4B^2\delta(f) + 4B\text{tri}\left(\frac{f}{2B}\right).
\end{aligned}$$

(c)

$$\begin{aligned}
E[Y(t)] &= E[X^2(t)] = \sigma_X^2 = \text{constant} \\
R_{Y,Y}(t, t+\tau) &= R_{X,X}^2(0) + 2R_{X,X}^2(\tau)
\end{aligned}$$

Since $E[Y(t)]$ is constant and $R_{Y,Y}(t, t+\tau)$ is a function of τ only, the process is WSS.

Problem 10.7

$$\begin{aligned}
R_{X,X}(t, t+\tau) &= E[b^2 \cos(\omega t + \Theta)\cos(\omega(t+\tau) + \Theta)] \\
&= \frac{b^2}{2}\cos(\omega\tau) + \frac{b^2}{2}E[\cos(\omega(2t+\tau) + 2\Theta)] \\
R_{X,X}(\tau) &= \langle R_{X,X}(t, t+\tau)\rangle = \frac{b^2}{2}\cos(\omega\tau) \\
S_{X,X}(f') &= \frac{b^2}{4}\delta(f' - f) + \frac{b^2}{4}\delta(f' + f)
\end{aligned}$$

This PSD is independent of the distribution of Θ. This is expected because the process has all its power at frequency, f, regardless of the phase Θ.

Problem 10.8

Let

$$s(t) = \sum_{k=-\infty}^{\infty} s_k \exp(j2\pi f_o t).$$

Then

$$
\begin{aligned}
s(t-T) &= \sum_{k=-\infty}^{\infty} s_k \exp(j2\pi f_o(t-T)) \\
R_{X,X}(\tau) &= E\left[\sum_k \sum_m s_k s_m^* \exp(j2\pi k f_o(t-T) \exp(-j2\pi m f_o(t+\tau-T)\right] \\
&= \sum_k \sum_m s_k s_m^* \exp(j2\pi(k-m)f_o t) \exp(-j2\pi m f_o \tau) \\
&\quad E[\exp(j2\pi(k-m)f_o T)] \\
E[\exp(j2\pi(k-m)f_o T)] &= \frac{1}{t_o}\int_0^{t_o} \exp(j2\pi(k-m)f_o u)du \\
&= \begin{cases} 0 & k \neq m \\ 1 & k = m \end{cases} \\
R_{X,X}(\tau) &= \sum_{k=-\infty}^{\infty} |s_k|^2 \exp(-j2\pi k f_o \tau) \\
S_{X,X}(f) &= \sum_{k=-\infty}^{\infty} |s_k|^2 \delta(f - k f_o)
\end{aligned}
$$

Hence the process $X(t) = s(T-T)$ has a line spectrum and the height of each line is given by the magnitude squared of the Fourier Series coefficients.

Problem 10.9

Write $Y(t) = b\cos(\omega_o t + \Omega t + \Theta)$ where $\Omega = 2\pi f_o V/c$ and $\omega_o = 2\pi f_o$. Note that Ω is uniformly distributed over $(\omega_o \nu_o/c, -\omega_o \nu_o/c)$. For simplicity, define $z_o = \omega_o \nu_o/c$.

$$
\begin{aligned}
R_{Y,Y}(t,t+\tau) &= b^2 E[\cos((\omega_o+\Omega)t+\Theta)\cos((\omega_o+\Omega)(t+\tau)+\Theta)] \\
&= \frac{b^2}{2}E[\cos((\omega_o+\Omega)\tau)] + \frac{b^2}{2}E[\cos((\omega_o+\Omega)(2t+\tau)+2\Theta)]
\end{aligned}
$$

Assuming Θ is uniform over $[0, 2\pi)$ the second expectation is zero. Hence

$$
\begin{aligned}
R_{Y,Y}(\tau) &= \frac{b^2}{2}\frac{1}{2z_o}\int_{-z_o}^{z_o} \cos(\omega_o+\omega)\tau)d\omega \\
&= \frac{b^2}{4\omega_o\tau}[\sin((z_o+\omega_o)\tau) + \sin((z_o-\omega_o)\tau)]
\end{aligned}
$$

$$\begin{aligned}
&= \frac{b^2\nu_o}{c}\frac{\sin(\omega_o\tau)}{\omega_o\tau}\cos(\omega_o\tau) \\
S_{Y,Y}(f) &= \frac{b^2}{4f_o}\left\{rect\left(\frac{\pi(f-f_o)}{z_o}\right)+rect\left(\frac{\pi(f+f_o)}{z_o}\right)\right\}
\end{aligned}$$

The power of the signal is now spread over a range of frequencies around $\pm f_o$.

Problem 10.10

(a)

$$R_{Z,Z}[k] = R_{X,X}[k] + R_{Y,Y}[k] = \left(\frac{1}{2}\right)^{|k|} + \left(\frac{1}{3}\right)^{|k|}$$

(see Exercise 8.18 for details)
(b) For a funcion of the form $R[k] = p^{|k|}$, the Fourier Transform is (t_o is the time between samples of the discrete time process)

$$\begin{aligned}
S(f) &= \sum_k R[k]e^{-j2\pi kft_o} \\
&= 1 + \sum_{k=1}^{\infty} p^k\{e^{-j2\pi kft_o} + e^{j2\pi kft_o}\} \\
&= 1 + \frac{pe^{-j2\pi kft_o}}{1-pe^{-j2\pi kft_o}} + \frac{pe^{j2\pi kft_o}}{1-pe^{j2\pi kft_o}} \\
&= \frac{1-p^2}{1+p^2-2p\cos(2\pi ft_o)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
S_{X,X}(f) &= \frac{3/4}{5/4-\cos(2\pi ft_o)} \\
S_{Y,Y}(f) &= \frac{8/9}{10/9-(2/3)\cos(2\pi ft_o)} \\
S_{Z,Z}(f) &= S_{X,X}(f) + S_{Y,Y}(f).
\end{aligned}$$

Problem 10.11

For a discrete time random process: $S(f) = \sum_{k=-\infty}^{\infty} R[k]e^{-j2\pi kft_o}$.

The inverse relationship is: $R[k] = t_o \int_{-\frac{1}{2t_o}}^{\frac{1}{2t_o}} S(f) e^{j2\pi k f t_o} df$.
The average power in the process is:

$$P_{avg} = \frac{1}{t_o} E[X^2[k]] = \frac{1}{t_o} R_{X,X}[0] = \int_{-\frac{1}{2t_o}}^{\frac{1}{2t_o}} S(f) df.$$

Problem 10.12

$$R_{X,X}(t_1, t_2) = E[\cos(\omega_c t_1 + B[n_1]\pi/2)\cos(\omega_c t_2 + B[n_2]\pi/2)],$$

where n_1 and n_2 are integers such that $n_1 T \leq t_1 < (n_1+1)T$ and $n_2 T \leq t_2 < (n_2+1)T/$. For t_1, t_2 such that $n_1 \neq n_2$,

$$R_{X,X}(t_1, t_2) = E[\cos(\omega_c t_1 + B[n_1]\pi/2)]E[\cos(\omega_c t_2 + B[n_2]\pi/2)] = 0,$$

while for t_1, t_2 such that $n_1 = n_2$,

$$\begin{aligned} R_{X,X}(t_1, t_2) &= \frac{1}{2}\cos(\omega_c(t_2 - t_1)) + \frac{1}{2}E[\cos(\omega_c(t_2 + t_1) + \pi B[n_1])] \\ &= \frac{1}{2}\cos(\omega_c(t_2 - t_1)) - \frac{1}{2}\cos(\omega_c(t_2 + t_1)) \end{aligned}$$

Since this autocorrelation depends on more than just $t_1 - t_2$, the process is not WSS.
(b) From part (a),

$$R_{X,X}(t, t+\tau) = \begin{cases} 0 & \text{if } t, t+\tau \text{ are in different intervals,} \\ \frac{1}{2}\cos(\omega_c \tau) - \frac{1}{2}\cos(\omega_c(2t+\tau)) & \text{if } t, t+\tau \text{ are in in the same intervals.} \end{cases}$$

Since the process is not WSS we must take time averages.

$$R_{X,X}(\tau) = \langle R_{X,X}(t, t+\tau)\rangle = (1-p(\tau))\langle 0\rangle + p(\tau)\langle \frac{1}{2}\cos(\omega_c\tau) - \frac{1}{2}\cos(\omega_c(2t+\tau))\rangle,$$

where $p(\tau)$ is the fraction of the values of t that lead to t and $t+\tau$ being in the same interval. This function is given by

$$p(\tau) = \begin{cases} 0 & |\tau| > T, \\ 1 - \frac{|\tau|}{T} & |\tau| < T. \end{cases}$$

Therefore,

$$
\begin{aligned}
p(\tau) &= \text{tri}(t/T) \\
R_{X,X}(\tau) &= \frac{1}{2}\text{tri}(\tau/T)\cos(\omega_c\tau) \\
S_{X,X}(f) &= \frac{1}{2}FT[\text{tri}(\tau/T)] * FT[\cos(\omega_c\tau]
\end{aligned}
$$

using Table E.1 in Appendix E in the text,

$$
\begin{aligned}
S_{X,X}(f) &= \frac{1}{4}T\text{sinc}^2(fT) * (\delta(f-f_c)+\delta(f+f_c)) \\
&= \frac{T}{4}(\text{sinc}^2((f-f_c)T)) + \text{sinc}^2((f+f_c)T))
\end{aligned}
$$

Problem 10.13

$$B_{rms}^2 = \frac{\int_{-\infty}^{\infty} f^2 S(f)df}{\int_{-\infty}^{\infty} S(f)df}$$

Recall that $R(\tau) = \int_{-\infty}^{\infty} S(f)e^{j2\pi f\tau}df$. Thus the denominator is simply $R(0)$. Also note that

$$\frac{d^2R(\tau)}{d\tau^2} = (j2\pi f)^2\int_{-\infty}^{\infty} S(f)e^{j2\pi f\tau}df.$$

Therefore, the numerator is

$$\int_{-\infty}^{\infty} f^2S(f)df = -\left(\frac{1}{2\pi}\right)^2 \left.\frac{d^2R(\tau)}{d\tau^2}\right|_{\tau=0}.$$

Therefore,

$$B_{rms}^2 = -\frac{1}{(2\pi)^2R(0)}\left.\frac{d^2R(\tau)}{d\tau^2}\right|_{\tau=0}.$$

Problem 10.14

(a) The absolute BW is ∞ since $S(f) > 0$ for all $|f| < \infty$.
(b) The 3dB BW, f_3 satisfies

$$
\begin{aligned}
\frac{1}{(1+(f_3/B)^2)^3} &= \frac{1}{2} \\
\Rightarrow f_3 &= B\sqrt{2^{1/3}-1} = 0.5098B.
\end{aligned}
$$

(c)

$$
\begin{aligned}
\int_{-\infty}^{\infty} f^2 S(f)df &= \int_{-\infty}^{\infty} \frac{f^2}{(1+(f/B)^2)^3}df = B^3 \int_{-\infty}^{\infty} \frac{z^2}{(1+z^2)^3}dz = \frac{\pi}{8}B^3 \\
\int_{-\infty}^{\infty} S(f)df &= \int_{-\infty}^{\infty} \frac{1}{(1+(f/B)^2)^3}df = B \int_{-\infty}^{\infty} \frac{1}{(1+z^2)^3}dz = \frac{3\pi}{8}B \\
B_{rms}^2 &= \frac{\frac{\pi}{8}B^3}{\frac{3\pi}{8}B} = \frac{B^2}{3} \\
B_{rms} &= \frac{B}{\sqrt{3}}.
\end{aligned}
$$

Problem 10.15

(a) The absolute BW is ∞ since $S(f) > 0$ for all $|f| < \infty$.
(b) The peak value of the PSD occurs at $f = B/\sqrt{2}$ and has a value of $S_{max} = 4/27$. Next we seek the values of f which satisfy

$$
\begin{aligned}
\frac{(f/B)^2}{(1+(f/B)^2)^3} &= \frac{1}{2}S_{max} \\
\Rightarrow (f/B)^2 &= \frac{2}{27}(1+(f/B)^2)^3.
\end{aligned}
$$

The two solutions are $f_1 = \sqrt{\frac{3\sqrt{3}-5}{2}}B$ and $f_2 = \sqrt{2}B$. The 3dB bandwidth is then

$$
f_3 = f_2 - f_1 = \left(\sqrt{2} - \sqrt{\frac{3\sqrt{3}-5}{2}}\right)B = 1.1010B.
$$

(c) Since this is a bandpass process, the definition of (10.24) in the text is used. First we must find

$$
f_o = \frac{\int_0^{\infty} fS(f)df}{\int_0^{\infty} S(f)df}
$$

$$
\begin{aligned}
\int_0^{\infty} fS(f)df &= \int_0^{\infty} \frac{f(f/B)^2}{(1+(f/B)^2)^3}df = B^2 \int_0^{\infty} \frac{z^3}{(1+z^2)^3}dz = \frac{1}{4}B^2 \\
\int_0^{\infty} S(f)df &= \int_0^{\infty} \frac{(f/B)^2}{(1+(f/B)^2)^3}df = B \int_0^{\infty} \frac{z^2}{(1+z^2)^3}dz = \frac{\pi}{16}B \\
f_o &= \frac{\frac{1}{4}B^2}{\frac{\pi}{16}B} = \frac{4}{\pi}B.
\end{aligned}
$$

Next, we find the RMS bandwidth according to

$$\begin{aligned}
B_{rms}^2 &= \frac{4\int_0^\infty (f-f_o)^2 S(f)df}{\int_0^\infty S(f)df} \\
\int_0^\infty (f-f_o)^2 S(f)df &= B^3 \int_0^\infty \frac{(z-4/\pi)^2 z^2}{(1+z^2)^3} dz = 0.2707B^3 \\
B_{rms}^2 &= \frac{4\cdot 0.2707B^3}{\pi/16\cdot B} = 5.5154B^2 \\
B_{rms} &= 2.3485B.
\end{aligned}$$

Problem 10.16

$$\begin{aligned}
R_{Y,Y}(t,t+\tau) &= E[X(t)X(t+\tau)]E[\cos(\omega_o t+\Theta)\cos(\omega_o(t+\tau)+\Theta)] \\
R_{Y,Y}(\tau) &= R_{X,X}(\tau)\cdot\frac{1}{2}\cos(\omega_o\tau) \\
S_{Y,Y}(f) &= \frac{1}{2}S_{X,X}(f)\{\frac{1}{2}\delta(f-f_o)+\frac{1}{2}\delta(f+f_o)\} \\
&= \frac{1}{4}S_{X,X}(f-f_o)+\frac{1}{4}S_{X,X}(f+f_o)
\end{aligned}$$

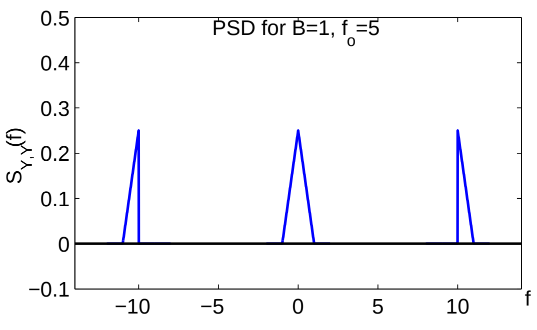

Figure 1: PSD for Problem 10.16; $B=1, f_o=5$.

Problem 10.17

$$X[n] = \frac{1}{2}X[n-1]+E[n]. \tag{1}$$

Taking expectations of both sides of (1) results in

$$\mu[n] = \frac{1}{2}\mu[n-1], \qquad n = 1, 2, 3, \ldots.$$

Hence $\mu[n] = (1/2)^n \mu[0]$. Noting that $X(0) = 0$, then $\mu[0] = 0 \Rightarrow \mu[n] = 0$. Multiply both sides of (1) by $X[k]$ and then take expected values to produce

$$E[X[k]X[n]] = \frac{1}{2}E[X[k]X[n-1]] + E[X[k]E[n]].$$

Assuming $k < n$, $X[k]$ and $E[n]$ are independent. Thus, $E[X[k]E[n]] = 0$ and therefore

$$\begin{aligned} R_{X,X}[k,n] &= \frac{1}{2}R_{X,X}[k,n-1]. \\ \Rightarrow R_{X,X}[k,n] &= \left(\frac{1}{2}\right)^{n-k} R_{X,X}[k,k], \qquad n = k, k+1, k+2, \ldots. \end{aligned}$$

Following a similar procedure, it can be shown that if $k > n$

$$R_{X,X}[k,n] = \left(\frac{1}{2}\right)^{k-n} R_{X,X}[k,k].$$

Hence in general

$$R_{X,X}[k,n] = \left(\frac{1}{2}\right)^{|n-k|} R_{X,X}[m,m], \text{ where } m = \min(n,k).$$

Note that $R_{X,X}[m,m]$ can be found as follows:

$$\begin{aligned} R_{X,X}[m,m] &= E[X^2[m]] = E[(\frac{1}{2}X[m-1] + E[m])^2] \\ &= \frac{1}{4}R_{X,X}[m-1,m-1] + E[X[m-1]E[m]] + E[E^2[m]]. \end{aligned}$$

Since $X[m-1]$ and $E[m]$ are uncorrelated, we have the following recursion

$$\begin{aligned} R_{X,X}[m,m] &= \frac{1}{4}R_{X,X}[m-1,m-1] + \sigma_E^2 \\ \Rightarrow R_{X,X}[m,m] &= \left(\frac{1}{4}\right)^m R_{X,X}[0,0] + \sigma_E^2 \sum_{i=0}^{m-1} \left(\frac{1}{4}\right)^i. \end{aligned}$$

Note that since $X(0)=0$, $R_{X,X}(0,0)=0$. Therefore

$$\begin{aligned} R_{X,X}[m,m] &= \sigma_E^2 \frac{1-(1/4)^m}{1-1/4} = \frac{4\sigma_E^2}{3}(1-(1/4)^m) \\ \Rightarrow R_{X,X}[k,n] &= \frac{4\sigma_E^2}{3}(1-(1/4)^m)\left(\frac{1}{2}\right)^{|n-k|}. \end{aligned}$$

Since $m=$min(n,k) is not a function of $n-k$, the process is not WSS.

Problem 10.18

(a) Given

$$Y[n] = a_1 Y[n-1] + a_2 Y[n-2] + X[n], \tag{2}$$

then multiplying (2) by $Y[n-k]$ and taking expectations produces

$$\begin{aligned} E[Y[n]Y[n-k]] &= a_1 E[Y[n-1]Y[n-k]] \\ &+ a_2 E[Y[n-2]Y[n-k]] + E[X[n]Y[n-k]]. \end{aligned} \tag{3}$$

Note that (for $k>0$) $X[n]$ and $Y[n-k]$ are independent since $Y[n-k]$ depends on $X[n-k], x[n-k-1], \ldots$, but not on $X[n]$. Hence,

$$E[X[n]Y[n-k]] = E[X[n]]E[Y[n-k]] = 0.$$

Then (3) becomes

$$R_{Y,Y}[k] = a_1 R_{Y,Y}[k-1] + a_2 R_{Y,Y}[k-2]. \tag{4}$$

(b) Squaring both sides of (2) and taking expectations gives

$$\begin{aligned} E[Y^2[n]] &= E[(a_1 Y[n-1] + a_2 Y[n-2] + X[n])^2] \\ R_{Y,Y}[0] &= (a_1^2 + a_2^2) R_{Y,Y}[0] + 2a_1 a_2 R_{Y,Y}[1] + R_{X,X}[0] \\ \Rightarrow \sigma_X^2 &= (1 - a_1^2 - a_2^2) R_{Y,Y}[0] - 2a_1 a_2 R_{Y,Y}[1]. \end{aligned}$$

From (4) with $k=1$, we get

$$R_{Y,Y}[1] = a_1 R_{Y,Y}[0] + a_2 R_{Y,Y}[-1].$$

Since $R_{Y,Y}[1] = R_{Y,Y}[-1]$ we have

$$(1-a_2) R_{Y,Y}[1] - a_1 R_{Y,Y}[0] = 0.$$

Thus we have the following 2x2 linear equations:

$$\begin{aligned}
\begin{bmatrix} 1-a_1^2-a_2^2 & 2a_1a_2 \\ -a_1 & 1-a_2 \end{bmatrix} \begin{bmatrix} R_{Y,Y}[0] \\ R_{Y,Y}[1] \end{bmatrix} &= \begin{bmatrix} \sigma_X^2 \\ 0 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} R_{Y,Y}[0] \\ R_{Y,Y}[1] \end{bmatrix} &= \frac{\sigma_X^2}{\Delta} \begin{bmatrix} 1-a_2 \\ a_1 \end{bmatrix}
\end{aligned}$$

where $\Delta = (1-a_2)(1-a_1^2-a_2^2)$. This then provides the initial conditions for the difference equation in (4).
(c) The general solution of (4) will be of the form

$$R_{Y,Y}[k] = k_1 b_1^{|k|} + k_2 b_2^{|k|}, \text{ where } b_1, b_2 = \frac{a_1 \pm \sqrt{a_1^2+4a_2}}{2} \tag{5}$$

The constants k_1 and k_2 are found using the initial conditions:

$$\begin{aligned}
R_{Z,Z}[0] &= k_1 + k_2 = \frac{\sigma_X^2}{\Delta}(1-a_2) \\
R_{Z,Z}[1] &= k_1 b_1 + k_2 b_2 = \frac{\sigma_X^2}{\Delta} a_1 \\
\Rightarrow \begin{bmatrix} 1 & 1 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} &= \frac{\sigma_X^2}{\Delta} \begin{bmatrix} 1-a_2 \\ a_1 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} &= \frac{1}{\Delta(b_2-b_1)} \begin{bmatrix} b_2(1-a_2)-a_1 \\ -b_1(1-a_2)+a_1 \end{bmatrix}.
\end{aligned}$$

(d) Taking the DFT of (5) gives

$$S_{Y,Y}(f) = \frac{k_1(1-b_1^2)}{1+b_1^2-2b_1\cos(2\pi f)} + \frac{k_2(1-b_2^2)}{1+b_2^2-2b_2\cos(2\pi f)}.$$

Problem 10.19

(a)

$$\begin{aligned}
E[\epsilon^2] &= E[(Y[n+1]-a_1Y[n]-a_2Y[n-1])^2] \\
&= R_{Y,Y}[0](1+a_1^2+a_2^2) - 2a_1(1-a_2)R_{Y,Y}[1] - 2a_2R_{Y,Y}[2]
\end{aligned}$$

(b)

$$\frac{\partial E[\epsilon^2]}{\partial a_1} = 2a_1R_{Y,Y}[0] - 2(1-a_2)R_{Y,Y}[1] = 0$$

$$
\begin{aligned}
&\Rightarrow \quad R_{Y,Y}[0]a_1 + R_{Y,Y}[1]a_2 = R_{Y,Y}[1] \\
\frac{\partial E[\epsilon^2]}{\partial a_2} &= 2a_2 R_{Y,Y}[0] - 2R_{Y,Y}[2] + 2a_1 R_{Y,Y}[1] = 0 \\
&\Rightarrow \quad R_{Y,Y}[1]a_1 + R_{Y,Y}[0]a_2 = R_{Y,Y}[2] \\
&\Rightarrow \quad \begin{bmatrix} R_{Y,Y}[0] R_{Y,Y}[1] & \\ R_{Y,Y}[1] & R_{Y,Y}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} R_{Y,Y}[1] \\ R_{Y,Y}[2] \end{bmatrix} \\
\Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= \frac{1}{R^2_{Y,Y}[0] - R^2_{Y,Y}[1]} \begin{bmatrix} R_{Y,Y}[0]R_{Y,Y}[1] - R_{Y,Y}[1]R_{Y,Y}[2] \\ R_{Y,Y}[0]R_{Y,Y}[2] - R^2_{Y,Y}[1] \end{bmatrix}
\end{aligned}
$$

Problem 10.20

(a)

$$
\begin{aligned}
E[\epsilon^2] &= E[(Y[n+1] - \sum_{k=1}^{p} a_k Y[n-k+1])^2] \\
&= E[Y^2[n+1]] - 2\sum_{k=1}^{p} a_k E[Y[n+1]Y[n+1-k]] \\
&+ \sum_{k=1}^{p}\sum_{m=1}^{p} a_k a_m E[Y[n+1-k]Y[n+1-m]] \\
&= R_{Y,Y}[0] - 2\sum_{k=1}^{p} a_k R_{Y,Y}[k] + \sum_{k=1}^{p}\sum_{m=1}^{p} a_k a_m R_{Y,Y}[m-k]
\end{aligned}
$$

To simplify the notation, introduce the following vectors and matrices:

$$
\begin{aligned}
\mathbf{r} &= [\, R_{Y,Y}[1] \quad R_{Y,Y}[2] \quad \ldots \quad R_{Y,Y}[p]\,]^T, \\
\mathbf{a} &= [\, a_1 \quad a_2 \quad \ldots \quad a_p\,]^T, \\
\mathbf{R} &= p \text{ x } p \text{ matrix whose } (k,m)\text{th element is } R_{Y,Y}[m-k].
\end{aligned}
$$

Then the mean squared error is

$$E[\epsilon^2] = R_{Y,Y}[0] - 2\mathbf{r}^T\mathbf{a} + \mathbf{a}^T\mathbf{R}\mathbf{a}.$$

(b)

$$
\begin{aligned}
\nabla_{\mathbf{a}} &= -2\mathbf{r} + 2\mathbf{R}\mathbf{a} = 0 \\
\Rightarrow \mathbf{a} &= \mathbf{R}^{-1}\mathbf{r}
\end{aligned}
$$

Problem 10.21

See solutions to Problem 10.18

Problem 10.22

$$\begin{aligned} S_{N,N}(f) &= \frac{kt_k}{2}\frac{z}{e^z-1}, \qquad \text{where } z = \frac{h|f|}{kt_k} \\ \frac{z}{e^z-1} &= \frac{z}{(1+z+z^2/2+z^3/3!+\ldots)-1} \\ &= \frac{1}{1+z/2+z^2/6+\ldots}. \end{aligned}$$

As $|f| \to 0, z \to 0$. Clearly, as $z \to 0$, $\frac{z}{e^z-1} \to 1$ so that

$$\lim_{|f|\to 0} s_{N,N}(f) = \frac{kt_k}{2} = \frac{N_o}{2}.$$

Problem 10.23

Noting that $V = V_1 + V_2$,

$$\begin{aligned} V_{rms}^2 &= E[V^2] = E[(V_1+V_2)^2] = E[V_1^2] + E[V_2^2] + 2E[V_1V_2] = V_{1,rms}^2 + V_{2,rms}^2 \\ \Rightarrow V_{rms} &= \sqrt{V_{1,rms}^2 + V_{2,rms}^2} \\ 4kt_e(r_1+r_2)\Delta f &= \sqrt{(4kt_1r_1\Delta f)^2 + (4kt_2r_2\Delta f)^2} \\ &= 4k\Delta f\sqrt{(t_1r_1)^2 + (t_2r_2)^2} \\ t_e &= \frac{\sqrt{(t_1r_1)^2 + (t_2r_2)^2}}{r_1+r_2} \end{aligned}$$

Note if $t_1 = t_2 = t_o$, then the effective temperature is

$$t_e = t_o\frac{\sqrt{r_1^2 + r_2^2}}{r_1 + r_2}$$

which in general is not equal to the physical temperature (unless $r_1 = r_2$).

Problem 10.24

In the case of parallel resistors,

$$V = \left(\frac{V_1}{r_1} + \frac{V_2}{r_2}\right)\frac{r_1 r_2}{r_1 + r_2}.$$

Let r be the paraalel combination of the resistances (i.e., $r = \frac{r_1 r_2}{r_1+r_2}$). Then

$$\begin{aligned}
V_{rms}^2 &= E[V^2] = \left(\frac{E[V_1^2]}{r_1^2} + \frac{E[V_2^2]}{r_2^2}\right) r^2 = \left(\frac{V_{1,rms}^2}{r_1^2} + \frac{V_{2,rms}^2}{r_2^2}\right) r^2 \\
(4kt_e r\Delta f)^2 &= ((4kt_1\Delta f)^2 + (4kt_1\Delta f)^2) r^2 \\
\Rightarrow t_e^2 &= t_1^2 + t_2^2 \\
t_e &= \sqrt{t_1^2 + t_2^2}.
\end{aligned}$$

In this case, the effective temperature is the same as the physical temperature if $t_1 = t_2$.