Rabi Frequency Calculations

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1 Introduction

The goal of this document is to explain how to calculate the dipole matrix element for a given atomic transition using the Wigner-Eckart Theorem and Spectator Theorem. This, along with power and beam waist measurements, can be used to calculate the expected Rabi frequency. I will begin by defining the Rabi frequency and other relevant physical quantities needed for the calculation. I will also provide a general expression that can be used to calculate the line strength of any atomic transition.

2 Rabi Frequency, Dipole Matrix Elements and Beyond

2.1 Rabi Frequency

First, we define the Rabi frequency as:

$$\Omega = \frac{\langle i|\, \vec{D}\cdot\vec{E}\,|j\rangle}{\hbar} = \frac{\langle i|\, D\,|j\rangle\,|\vec{E}|}{\hbar} \tag{1}$$

where $\langle i|\, D\,|j\rangle$ is the dipole matrix element, $|\vec{E}|$ is the magnitude of the electric field and $\hbar$ is the reduced Planck's constant. Here, j and i refer to the initial and final states of the dipole transition, respectively. Using the following expression for the intensity of a Gaussian beam of light in vacuum,

$$I = \frac{1}{2}c\epsilon_0|\vec{E}|^2 = \frac{2P}{\pi w^2} \tag{2}$$

where c is the speed of light, ϵ_0 is the permittivity of free space, P is the total power of the beam and w is the beam waist, we can rewrite the expression for the Rabi frequency as

$$\Omega = \langle i|\, D\,|j\rangle\sqrt{\frac{2I}{c\epsilon_0\hbar^2}} = \langle i|\, D\,|j\rangle\sqrt{\frac{4P}{c\epsilon_0\pi w^2\hbar^2}}. \tag{3}$$

2.2 Dipole Matrix Element

We can write out a general hyperfine state as $|F, m_F, I, J\rangle$, where F is the total angular momentum, m_F is the projection of the total angular momentum along the quantization axis, I is the total nuclear angular momentum, and J is the total electronic angular momentum. There are other quantum numbers one could write as well, however, we only need these quantum numbers to perform the calculations that follow.

In order to calculate the dipole matrix element atomic line data is needed, and this data can be found in the NIST Atomic Spectra database (see Figure 1 for properties of the D1 and D2 lines of K). However, these properties correspond only to the electronic component of the transtion, which is why the Wigner-Eckart theorem and the spectator theorem are needed to factor out the nuclear component of the interaction.

Observed Wavelength Air (nm)	Ritz Wavelength Air (nm)	Unc. (nm)	Rel. Int. (?)	A_{ki} (s^{-1})	S_{ik} (a.u.)	Acc.	E_i (GHz)	E_k (GHz)	Lower Level Conf., Term, J	Upper Level Conf., Term, J	Type	TP Ref.	Line Ref.
766.489913	766.48991	0.00003	25f	3.779e+07	3.363e+01	AAA	0.000	- 391 016.18594	$3p^64s\ ^2S\ ^1/_2$	$3p^64p\ ^2P^o\ ^3/_2$		T9745	L8995
769.896456	769.89646	0.00003	24f	3.734e+07	1.683e+01	AAA	0.000	- 389 286.07458	$3p^64s\ ^2S\ ^1/_2$	$3p^64p\ ^2P^o\ ^1/_2$		T9745	L8995

Figure 1: Relevant K Data from NIST. A_{ki} is the Einstein A coefficient (also referred to as the natural linewidth, Γ) and S_{ki} is the line strength of the electronic portion of the transition in atomic units. This data and more can be obtained from https://physics.nist.gov/PhysRefData/ASD/lines_form.html

2.2.1 Wigner-Eckart Theorem

The Wigner-Eckart theorem for the matrix element of a spherical tensor operator of rank k and projection q is

$$\langle F_2, m_{F_2}, I_2, J_2|\, T^k_q\,|F_1, m_{F_1}, I_1, J_1\rangle = (-1)^{F_2-m_{F_2}}\begin{pmatrix} F_2 & k & F_1 \\ -m_{F_2} & q & m_{F_1}\end{pmatrix}\langle F_2, I_2, J_2|\,|T^k|\,|F_1, I_1, J_1\rangle \tag{4}$$

where the bracketed term containing 6 values is the Wigner-3j symbol, the $\langle F_2, I_2, J_2|\,|T^k|\,|F_1, I_1, J_1\rangle$ term is the reduced matrix element and the indices 1 and 2 correspond to the initial and final atomic states, respectively. For a dipole transition, the spherical tensor operator we are interested in is the dipole moment

operator D, which has a rank of $k = 1$ (for a quadrupole operator, the rank would be k=2, etc.) The projection, q, in our case, refers to the polarization of light, which has values of $q = 0, 1, -1$ for π, σ_+ and σ_- polarized light, respectively. As you can see, this factors out the projection of the total angular momentum from the dipole matrix element.

2.2.2 Spectator Theorem

Now that we have factored out the projection of the total angular momentum, we can use the spectator theorem to factor out the nuclear contribution to the dipole matrix element. We want to do this because when coupling two atomic states via an optical transition, we are really only changing the electronic portion of the state, since the nuclear angular momentum I remains unaffected. In a way, the nuclear component is merely a "spectator" of the optical transition. This theorem is given by the following expression

$$\langle F_2, I_2, J_2 | \, |T^k| \, | F_1, I_1, J_1 \rangle = \delta_{I_1,I_2} (-1)^{J_2+I_2+F_1+k} \sqrt{(2F_1+1)(2F_2+1)} \begin{Bmatrix} F_2 & k & F_1 \\ J_1 & I_1 & J_2 \end{Bmatrix} \langle J_2 | \, |T^k| \, | J_1 \rangle \tag{5}$$

where the curly bracketed term containing 6 values is the Wigner-6j symbol, and the $\langle J_2 | \, |T^k| \, | J_1 \rangle$ is the further reduced matrix element, containing only the electronic contribution. The square of this further reduced matrix element, $\langle J_2 | \, |T^k| \, | J_1 \rangle^2$, is often called the "line strength", S, of the transition, which has units of Cm, and can be found in the NIST database. It can also be calculated from the natural linewidth of the transition using:

$$S = \frac{3h\epsilon_0 \lambda^3 (2J+1)}{16\pi^3} \Gamma \tag{6}$$

where λ is the wavelength of the transition, J is the total electronic angular momentum of the excited state, and Γ is the natural linewidth (angular frequency units). It is still a little unclear to me whether to use the specific values for potassium-40 or just the values listed in NIST, since those values listed in NIST, I think, only take into account the electronic degrees of freedom. I think it is more correct to use the "electronic" values, since we have factored out the nuclear components already.

2.3 Our System: EIT + RSC

Now I will calculate the expected Rabi Frequencies for our EIT probes, F pump and Raman beams used for sideband cooling during fluorescence imaging. See the diagram below (Fig. 2 for the relevant atomic transitions.

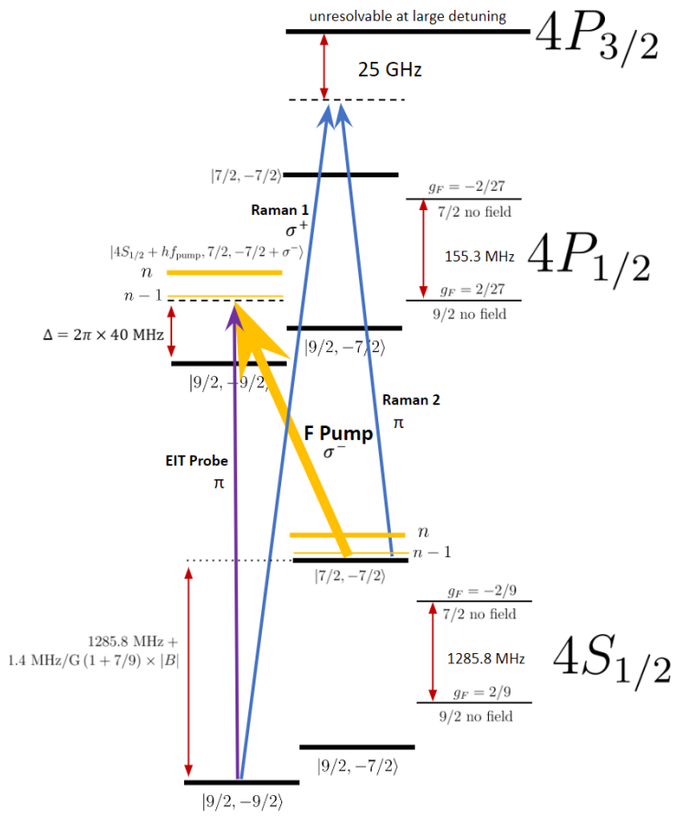

Figure 2: EIT + RSC Atomic Transitions

We can see here that the EIT probes (D1 light) optically couple the $|F = 9/2, m_F = -9/2, I = 4, J = 1/2\rangle$ and $|F' = 9/2, m_{F'} = -9/2, I' = 4, J' = 1/2\rangle$ states, which requires π polarization. Assuming a beam power of 3 μW and a beam waist of 300 μm, the expected Rabi frequency is $2\pi \times 2.7$MHz. For the F pump beam (also D1 light), which couples the $|F = 7/2, m_F = -7/2, I = 4, J = 1/2\rangle$ and $|F' = 9/2, m_{F'} = -9/2, I' = 4, J' = 1/2\rangle$

states, it is required to have σ_- polarization. Assuming that the F-pump power is 13 uW, and the beam waist is 200 μm, the expected Rabi frequency is $2\pi \times 11.3$MHz.

Not yet sure how to treat the Raman beams, since the excited states are unresolvable. Still need to think about this...