

# Cálculo I

## Fórmulas

### Trigonometria

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1 \\ 1 + \tan^2 x &= \frac{1}{\cos^2 x} \\ \sin(a + b) &= \sin a \cos b + \sin b \cos a \\ \cos(a + b) &= \cos a \cos b - \sin a \sin b \\ \tan(a + b) &= \frac{\tan a + \tan b}{1 - \tan a \tan b}\end{aligned}$$

$$\begin{aligned}\cdot \sin x &= \frac{2 \tan(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})}; \quad \cdot \cos x = \frac{1 - \tan^2(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})}; \quad \cdot \tan x = \frac{2 \tan(\frac{x}{2})}{1 - \tan^2(\frac{x}{2})} \\ \Rightarrow \text{Para Primitivas:} \\ \sin x &= \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}, \tan x = \frac{2t}{1-t^2}, \cot x = \frac{1-t^2}{2t}\end{aligned}$$

### Polinómios

$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$$

## Limites

### Limites Notáveis

$$\begin{aligned}\lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x &= e \\ \lim_{x \rightarrow 0} \frac{\sin(x)}{x} &= 1 \\ \lim_{x \rightarrow 0} \frac{e^x - 1}{x} &= 1 \\ \lim_{x \rightarrow 0} \frac{\ln(x+1)}{x} &= 1 \\ \lim_{x \rightarrow +\infty} \frac{\ln(x)}{x} &= 0 \\ \lim_{x \rightarrow +\infty} \frac{e^x}{x^p} &= +\infty, (p \in \mathbb{R})\end{aligned}$$

### Limites Gerais

$$\begin{aligned}\lim_{x \rightarrow a} (f(x) \pm g(x)) &= \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) \\ \lim_{x \rightarrow a} (f(x) \cdot g(x)) &= \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) \\ \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \\ \text{Indeterminações } \frac{0}{0} \text{ ou } \frac{\pm\infty}{\pm\infty}: \\ \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \\ \text{Indeterminações } 1^\infty: \\ \text{Transformar em } \lim_{x \rightarrow a} (1 + I(x))^{\frac{1}{I(x)}} \text{ infinito; } I(x) &\text{ é infinitésimo}\end{aligned}$$

## Derivadas

### Derivadas Básicas

$$\begin{aligned}f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \\ (c \cdot f)' &= c \cdot f'(x) \\ (x^n)' &= nx^{n-1} \\ (f \pm g)' &= f'(x) \pm g'(x) \\ (fg)' &= f'g + fg' \\ (\frac{f}{g})' &= \frac{f'g - fg'}{g^2} \\ (f \circ g)' &= f'(g(x)) \cdot g'(x)\end{aligned}$$

$$\begin{aligned}(f^{-1})' &= \frac{1}{f'(f^{-1})} \\ (fg)' &= (e^{g \ln(f)})' = fg(f' \frac{g}{f} + g' \ln(f))\end{aligned}$$

## Derivadas Logarítmicas e Exponenciais

$$\begin{aligned}(a^x)' &= a^x \ln a \\ (\ln x)' &= \frac{1}{x}, \quad x > 0 \\ (\log_a x)' &= \frac{1}{x \ln a}, \quad x > 0\end{aligned}$$

$$\begin{aligned}(x^x)' &= x^x (1 + \ln x) \\ (e^{f(x)})' &= f'(x) e^{f(x)} \\ ([f(x)]^n)' &= n[f(x)]^{n-1} f'(x) \\ (\ln[f(x)])' &= \frac{f'(x)}{f(x)}, \quad f(x) > 0\end{aligned}$$

### Derivadas Trigonométricas

$$\begin{aligned}(\sin x)' &= \cos x \\ (\arcsin x)' &= \frac{1}{\sqrt{1-x^2}} \\ (\sec x)' &= \sec x \tan x \\ (\operatorname{arcsec} x)' &= \frac{1}{|x| \sqrt{x^2 - 1}} \\ (\cos x)' &= -\sin x \\ (\arccos x)' &= -\frac{1}{\sqrt{1-x^2}} \\ (\csc x)' &= -\csc x \cot x \\ (\operatorname{arccsc} x)' &= -\frac{1}{|x| \sqrt{x^2 - 1}} \\ (\tan x)' &= \sec^2 x = \frac{1}{\cos^2 x} = 1 + \tan^2 x \\ (\arctan x)' &= \frac{1}{1+x^2} \\ (\cot x)' &= -\csc^2 x = -\frac{1}{\sin^2 x} = -(1 + \cot^2 x) \\ (\operatorname{arccot} x)' &= -\frac{1}{1+x^2}\end{aligned}$$

$$\begin{aligned}(\sin[f(x)])' &= f'(x) \cos[f(x)] \\ (\cos[f(x)])' &= -f'(x) \sin[f(x)] \\ (\tan[f(x)])' &= f'(x) \sec^2[f(x)] = \frac{f'(x)}{\cos^2[f(x)]}\end{aligned}$$

## Integrais

### Integrais Básicos

$$\begin{aligned}\int c \cdot f(x) \, dx &= c \int f(x) \, dx, c \in \mathbb{R} \\ \int f(x) \pm g(x) \, dx &= \int f(x) \, dx \pm \int g(x) \, dx \\ \int_a^b f(x) \, dx &= F(x)|_a^b = F(b) - F(a), \quad F(x) = \int f(x) \, dx \\ [F(\varphi(x))]' &= F'(\varphi(x)) \varphi'(x)\end{aligned}$$

### Integrais Polinomiais

$$\begin{aligned}\int dx &= x + C, \quad C \in \mathbb{R} \\ \int k \, dx &= kx + C, \quad C \in \mathbb{R} \\ \int \frac{1}{x} \, dx &= \ln|x| + C, \quad C \in \mathbb{R} \\ \int x^n \, dx &= \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \text{ e } C \in \mathbb{R} \\ \int \frac{1}{ax+b} \, dx &= \frac{1}{a} \ln|ax+b| + C, \quad C \in \mathbb{R}\end{aligned}$$

$$\begin{aligned}\int \frac{\varphi'(x)}{\varphi(x)} \, dx &= \ln|\varphi(x)| + C, \quad C \in \mathbb{R} \\ \int \varphi'(x) [\varphi(x)]^a \, dx &= \frac{\varphi^{a+1}(x)}{a+1} + C, \quad a \neq -1 \text{ e } C \in \mathbb{R}\end{aligned}$$

### Integrais Trigonométricos

$$\begin{aligned}\int \cos x \, dx &= \sin x + C, \quad C \in \mathbb{R} \\ \int \sin x \, dx &= -\cos x + C, \quad C \in \mathbb{R} \\ \int \sec^2 x \, dx &= \int \frac{1}{\cos^2 x} = \tan x + C, \quad C \in \mathbb{R} \\ \int \csc^2 x \, dx &= \int \frac{1}{\sin^2 x} = -\cot x + C, \quad C \in \mathbb{R}\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}} \, dx &= \arcsin x + C, \quad C \in \mathbb{R} \\ \int \frac{1}{\sqrt{1-x^2}} \, dx &= -\arccos x + C, \quad C \in \mathbb{R} \\ \int \frac{1}{1+x^2} \, dx &= \arctan x + C, \quad C \in \mathbb{R} \\ \int \frac{1}{1+x^2} \, dx &= -\operatorname{arccot} x + C, \quad C \in \mathbb{R}\end{aligned}$$

$$\begin{aligned}\int \varphi'(x) \cos[\varphi(x)] \, dx &= \sin \varphi(x) + C, \quad C \in \mathbb{R} \\ \int \varphi'(x) \sin[\varphi(x)] \, dx &= -\cos \varphi(x) + C, \quad C \in \mathbb{R} \\ \int \frac{\varphi'(x)}{\cos^2 \varphi(x)} \, dx &= \tan \varphi(x) + C, \quad C \in \mathbb{R} \\ \int \frac{\varphi'(x)}{\sin^2 \varphi(x)} \, dx &= -\cot \varphi(x) + C, \quad C \in \mathbb{R} \\ \int \frac{\varphi'(x)}{\sqrt{1-\varphi(x)^2}} \, dx &= \arcsin \varphi(x) + C, \quad C \in \mathbb{R} \\ \int \frac{\varphi'(x)}{\sqrt{1-\varphi(x)^2}} \, dx &= -\arccos \varphi(x) + C, \quad C \in \mathbb{R} \\ \int \frac{\varphi'(x)}{1+\varphi(x)^2} \, dx &= \arctan \varphi(x) + C, \quad C \in \mathbb{R} \\ \int \frac{\varphi'(x)}{1+\varphi(x)^2} \, dx &= -\operatorname{arccot} \varphi(x) + C, \quad C \in \mathbb{R}\end{aligned}$$

### Integrais Exponenciais/Logarítmicos

$$\begin{aligned}\int e^x \, dx &= e^x + C, \quad C \in \mathbb{R} \\ \int a^x \, dx &= \frac{a^x}{\ln a} + C, \quad C \in \mathbb{R}\end{aligned}$$

$$\int \varphi'(x) e^{\varphi(x)} \, dx = e^{\varphi(x)} + C, \quad C \in \mathbb{R}$$

### Primitivação por partes

$$\begin{aligned}\int f'(x)g(x) \, dx &= f(x)g(x) - \int f(x)g'(x) \, dx \\ \text{Nota: } d\varphi(x) &= \varphi'(x) \, dx \\ \rightarrow \int f(x)g'(x) \, dx &= \int v \, du = uv - \int u \, dv\end{aligned}$$

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I

$$\left. \begin{aligned}\int P_k(x) \sin(bx) \, dx \\ \int P_k(x) \cos(bx) \, dx \\ \int P_k(x) e^{ax}\end{aligned}\right\} \begin{aligned}u &= P_k(x) \\ v' &= \begin{cases} \cdot \sin(bx) \\ \cdot \cos(bx) \\ \cdot e^{ax} \end{cases}\end{aligned}$$

II

$$\left. \begin{aligned}\int P_k(x) \ln(bx) \, dx \\ \int P_k(x) \arcsin x \, dx \\ \int P_k(x) \arccos x \, dx \\ \int P_k(x) \arctan x \, dx \\ \int P_k(x) \operatorname{arccot} x \, dx\end{aligned}\right\} \begin{aligned}u &= \begin{cases} \cdot \ln(bx) \\ \cdot \arcsin x \\ \cdot \arccos x \\ \cdot \arctan x \\ \cdot \operatorname{arccot} x \end{cases} \\ v' &= P_k(x)\end{aligned}$$

III - 2 vezes por partes

Hipótese 1

$$\left. \begin{aligned}\int e^{ax} \sin(bx) \, dx \\ \int e^{ax} \cos(bx) \, dx\end{aligned}\right\} \begin{aligned}u &= e^{ax}, \quad v' = \begin{cases} \cdot \sin(bx) \\ \cdot \cos(bx) \end{cases} \quad \textcircled{1} \\ & \quad \quad \quad \textcircled{2}\end{aligned}$$

Hipótese 2

$$u = \begin{cases} \cdot \sin(bx) \\ \cdot \cos(bx) \end{cases}, \quad v' = e^a x \quad \textcircled{1} \quad \textcircled{2}$$

Primitivação de Funções Racionais (por decomposição)

Função Racional:  $\frac{P(x)}{Q(x)}$ , P e Q polinômios de coeficientes reais.

·Função Racional **Própria**  $\rightarrow \text{gr}(P(x)) < \text{gr}(Q(x))$

·Função Racional **Imprópria**  $\rightarrow \text{gr}(P(x)) > \text{gr}(Q(x))$

0. Função Racional Imprópria  $\rightarrow$  Polinômio+F. R. Própria

1. Resolver  $Q(x) = 0$ , decompondo-se  $Q(x)$  em :

- Constantes ( $a$ )
- $(x - R)^l$ ,  $l \in \mathbb{N} \rightarrow l - \text{Mult. de Raizes Reais}$
- $(x^2 + px + q)^k$ ,  $k \in \mathbb{N} \rightarrow k - \text{Mult. de Raizes } \alpha \pm i\beta$

$$Q(x) = a(x - R_1)^{l_1} \cdot (x - R_2)^{l_2} \cdot \dots \cdot (x^2 + p_1x + q_1)^{k_1} \cdot (x^2 + p_2x + q_2)^{k_2} \cdot \dots$$

2.  $\frac{P(x)}{Q(x)} = \frac{P(x)}{a(x-R_1)^{l_1} \dots (x^2+p_1x+q_1)^{k_1} \dots}$
- Determinar:  
 $\frac{A_1}{x-R_1} + \frac{A_2}{(x-R_1)^2} + \frac{A_3}{(x-R_1)^3} + \dots + \frac{A_{l_1}}{(x-R_1)^{l_1}}$   
nº de parcelas =  $l_1$  (multiplicidade)
  - Determinar:  
 $\frac{E_1+D_1x}{x^2+p_1x+q} + \frac{E_2+D_2x}{(x^2+p_1x+q_1)^2} + \dots + \frac{E_{k_1}+D_{k_1}x}{(x^2+p_1x+q_1)^{k_1}}$   
nº de parcelas =  $k_1$  (multiplicidade)

$$\Rightarrow \frac{P(x)}{Q(x)} = \text{soma de todas as parcelas}$$

3. Calcular valores  $A_1, \dots, A_{l_1}$  e  $E_1, D_1, \dots, E_{k_1}, D_{k_1}$  através do método dos coeficientes indeterminados.

Primitivas de Funções Racionais

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan(\frac{x}{a}) + C, \quad C \in \mathbb{R}$$

$$\int \frac{1}{(x\pm b)^2+a^2} dx = \frac{1}{a} \arctan(\frac{x\pm b}{a}) + C, \quad C \in \mathbb{R}$$

Se  $\exists \int f(x) dx = F(x) + C$

$$\Rightarrow \int f(ax+b) dx = \frac{1}{a} F(ax+b) + C, \quad C \in \mathbb{R}$$

$$\int \frac{1}{[\varphi(x)]^2-a^2} \varphi'(x) dx = \frac{1}{2} \ln \left| \frac{\varphi(x)-a}{\varphi(x)+a} \right| + C, \quad C \in \mathbb{R}$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)} + C, \quad C \in \mathbb{R}$$

$$\int \frac{x}{a^2+x^2} dx = \frac{1}{2} \ln|a^2+x^2| + C, \quad C \in \mathbb{R}$$

$$\int \frac{1}{ax^2+bx+c} dx = \frac{2}{\sqrt{4ac-b^2}} \arctan(\frac{2ax+b}{\sqrt{4ac-b^2}}) + C, \quad C \in \mathbb{R}$$

$$\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln(\frac{a+x}{b+x}) + C, \quad a \neq b \text{ e } C \in \mathbb{R}$$

Primitivação por Mudança de Variável

Seja  $f$  uma função contínua em  $[a, b]$  e  $x = \varphi(t)$  uma aplicação com derivada contínua e que não anula:

$$P_x(f(x)) = P_t(f(\varphi(t))) \cdot \varphi'(t)|_{t=\varphi^{-1}(x)}$$

$$\rightarrow \int f(x) dx = \int f(\varphi(t)) d\varphi(t) = \int f(\varphi(t)) \cdot \varphi'(t) dt|_{t=\varphi^{-1}(x)}$$

Substituições

Primitivas	Substituição
$\int f(e^x) dx$	$t = e^x \Rightarrow x = \ln t$
$\int f(\ln x) dx$	$t = \ln x \Rightarrow x = e^t$
$\int f(x^{\frac{p}{q}}, x^{\frac{r}{s}}, \dots) dx$	$t = x^{\frac{1}{m}} \Rightarrow x = t^m,$ com $m = m.m.c.(q, s, \dots)$
$\int f(x, (ax+b)^{\frac{p}{q}}, (ax+b)^{\frac{r}{s}}, \dots) dx$	$t = (ax+b)^{\frac{1}{m}} \Rightarrow ax+b = t^m,$ com $m = m.m.c.(q, s, \dots)$
$\int f(x, \sqrt{ax^2+bx+c}), a > 0$	$\sqrt{ax^2+bx+c} = t + x\sqrt{a}$
$\int f(x, \sqrt{ax^2+bx+c}), c > 0$	$\sqrt{ax^2+bx+c} = tx + \sqrt{c}$
$\int f(x, \sqrt{ax^2+bx+c}), b^2-4ac > 0$	$\sqrt{ax^2+bx+c} = (x-\alpha)t,$ $\alpha$ é raiz de $ax^2+bx+c$

Primitivação de Funções Trigonômétricas

1.  $\int f(\sin^k x, \cos^m x) dx$

(a)  $k - \text{par}, m - \text{ímpar}$   
substituição  $t = \sin x$

(b)  $k - \text{ímpar}, m - \text{par}$   
substituição  $t = \cos x$

(c)  $k, m - \text{ímpares}$   
substituição  $t = \sin x$  ou  $t = \cos x$   
+ fórm.:  $\sin x \cos x = \frac{1}{2} \sin(2x)$

(d)  $k, m - \text{pares}$   
"baixar" ordem de  $\sin x$  e  $\cos x$ :  
 $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$   
 $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$

2.  $\int f(\tan^k x) dx$  ou  $\int f(\cot^k x) dx$

Fórmulas:

$$- \tan^2 x = \frac{1}{\cos^2 x} - 1$$

$$- \cot^2 x = \frac{1}{\sin^2 x} - 1$$

3.  $\int f(\sin x, \cos x) dx, \int f(\tan x) dx, \int f(\cot x) dx$   
Substituição "Universal":  
 $t = \tan(\frac{x}{2}) \Rightarrow x = 2 \arctan t \Rightarrow dx = \frac{2}{1+t^2} dt$

Primitivação de Funções Irracionais

$\rightarrow$  Substituir usando Fórmulas Trigonômétricas

1.  $\int f(\sqrt{a^2-b^2x^2}) dx$

$$\sqrt{a^2-b^2x^2} = \sqrt{a^2(1-(\frac{b}{a}x)^2)} = a\sqrt{1-(\frac{b}{a}x)^2}$$

Subst.:  $\frac{b}{a}x = \sin t \Rightarrow dx = \frac{a}{b} \cos t dt$

$$\int f(\sqrt{a^2-b^2x^2}) dx = \int f(a\sqrt{1-\sin^2 t}) \cdot \frac{a}{b} \cdot \cos t dt$$
  
$$\Rightarrow \int f(a \cdot \cos t) \cdot \frac{a}{b} \cos t dt + C, \quad C \in \mathbb{R}$$

2.  $\int f(\sqrt{a^2+b^2x^2}) dx$

$$\sqrt{a^2+b^2x^2} = \sqrt{a^2(1+(\frac{b}{a}x)^2)} = a\sqrt{1+(\frac{b}{a}x)^2}$$

Subst.:  $\frac{b}{a}x = \tan t \Rightarrow dx = \frac{a}{b} \frac{1}{\cos^2 t} dt$

$$\int f(\sqrt{a^2+b^2x^2}) dx = \int f(a\sqrt{1+\tan^2 t}) \cdot \frac{a}{b} \cdot \frac{1}{\cos^2 t} dt$$
  
$$\Rightarrow \int f(a \cdot \frac{1}{\cos t}) \cdot \frac{a}{b} \cdot \frac{1}{\cos^2 t} dt + C, \quad C \in \mathbb{R}$$

3.  $\int f(\sqrt{a^2x^2-b^2}) dx$

$$\sqrt{a^2x^2-b^2} = \sqrt{b^2((\frac{a}{b}x)^2-1)} = b\sqrt{(\frac{a}{b}x)^2-1}$$

Subs.:  $\frac{a}{b}x = \frac{1}{\cos t} \Rightarrow dx = \frac{b}{a} \cdot \frac{\sin t}{\cos^2 t} dt$

$$\int f(\sqrt{a^2x^2-b^2}) dx = \int f(b\sqrt{(\frac{1}{\cos t})^2-1}) \cdot \frac{b}{a} \cdot \frac{\sin t}{\cos^2 t} dt$$
  
$$\Rightarrow \int f(b \tan t) \cdot \frac{b}{a} \cdot \frac{\sin t}{\cos^2 t} dt + C, \quad C \in \mathbb{R}$$

Integrais de Riemann

Integral de Riemann é o limite da soma de Riemman.

Soma de Riemann:

$$\text{Sf}(P, C) = \sum_{i=1}^n f(c_i) \cdot \Delta x_i, \quad \Delta x_i = x_i - x_{i-1}$$

$$\Rightarrow \text{Integral de Riemann} = \lim_{x \rightarrow \infty} \sum_{i=1}^n f(c_i) \cdot \Delta x_i = I$$

$$I = \int_a^b f(x) dx = \lim_{\Delta P \rightarrow 0} \text{Sf}(P, C)$$

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

Geometria de integral de Riemann

$f : [a, b] \rightarrow \mathbb{R} - \text{integrável}, a < b$

- $f(x) > 0, \forall x \in [a, b]$   
 $I = \int_a^b f(x) dx$
- $c \in ]a, b[: f(c) = 0$   
 $I = \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$
- $f, g : [a, b] - \mathbb{R} - \text{integráveis}, \text{ e } f > g, \forall x \in [a, b]$   
 $I = \int_a^b |f(x) - g(x)| dx$

Propriedades

$f, g : [a, b] \rightarrow \mathbb{R} - \text{integráveis e } \alpha, \beta \in \mathbb{R}$

- $\int_a^b [\alpha f(x) + \beta g(x)] dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx$
- Se  $a < c < b \Rightarrow \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$
- Se  $f(x) \geq 0, \forall x \in [a, b] \Rightarrow \int_a^b f(x) dx \geq 0$
- Se  $f(x) \geq g(x), \forall x \in [a, b] \Rightarrow \int_a^b f(x) dx \geq \int_a^b g(x) dx$
- Se  $m \leq f(x) \leq M, \forall x \in [a, b]$   
 $\Rightarrow m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$
- $|\int_a^b f(x) dx| \leq \int_a^b |f(x)| dx$

Critérios de Integrabilidade

Se  $f : [a, b] \rightarrow \mathbb{R}$  é integrável (no sentido de Riemann)

$\Rightarrow f$  é limitada em  $[a, b]$

\*Importante\*  $\Rightarrow$  Se  $f$  não é limitada em  $[a, b] \Rightarrow$

$f$  não é integrável em  $[a, b]$  (no sentido de Riemann)