

# Resolução: Distribuições Contínuas

## Exercício 01

**i** Comentário — Fórmula-chave (Gamma)

Sempre que aparecer uma integral do tipo

$$\int_0^{\infty} x^k e^{-ax} dx, \quad a > 0,$$

podemos usar a identidade

$$\int_0^{\infty} x^k e^{-ax} dx = \frac{\Gamma(k+1)}{a^{k+1}}, \quad a > 0$$

(a)  $\int_0^{\infty} x^2 e^{-3x} dx$

**Passo 1 — Identificar parâmetros**

Aqui  $k = 2$  e  $a = 3$

**Passo 2 — Aplicar a fórmula**

$$\int_0^{\infty} x^2 e^{-3x} dx = \frac{\Gamma(3)}{3^3}$$

**Passo 3 — Calcular  $\Gamma(3)$**

$$\Gamma(3) = 2! = 2$$

**Conclusão**

$$\boxed{\int_0^{\infty} x^2 e^{-3x} dx = \frac{2}{27}}$$

$$(b) \int_0^{\infty} x^4 e^{-x/5} dx$$

**Passo 1 — Identificar parâmetros**

Note que  $e^{-x/5} = e^{-(1/5)x}$ . Logo  $k = 4$  e  $a = \frac{1}{5}$

**Passo 2 — Aplicar a fórmula**

$$\int_0^{\infty} x^4 e^{-x/5} dx = \frac{\Gamma(5)}{(1/5)^5}$$

**Passo 3 — Calcular  $\Gamma(5)$**

$$\Gamma(5) = 4! = 24$$

**Conclusão**

$$\int_0^{\infty} x^4 e^{-x/5} dx = 24 \cdot 5^5 = 24 \cdot 3125 = 75000$$

**⚠ Atenção — erro comum**

Não confunda o parâmetro  $a$ .

Em  $e^{-x/5}$ , vale  $a = 1/5$  (e não 5)

$$(c) \int_0^{\infty} \sqrt{x} e^{-2x/3} dx$$

**Passo 1 — Reescrever  $\sqrt{x}$  como potência**

$$\sqrt{x} = x^{1/2}$$

Logo  $k = \frac{1}{2}$  e  $a = \frac{2}{3}$ .

**Passo 2 — Aplicar a fórmula**

$$\int_0^{\infty} x^{1/2} e^{-(2/3)x} dx = \frac{\Gamma(3/2)}{(2/3)^{3/2}}$$

**Passo 3 — Calcular  $\Gamma(3/2)$**

Usando  $\Gamma(x+1) = x\Gamma(x)$  e  $\Gamma(1/2) = \sqrt{\pi}$ :

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

**Passo 4 — Simplificar**  $(2/3)^{-3/2}$

$$\frac{1}{(2/3)^{3/2}} = \left(\frac{3}{2}\right)^{3/2}$$

**Conclusão**

$$\int_0^\infty \sqrt{x} e^{-2x/3} dx = \frac{\sqrt{\pi}}{2} \left(\frac{3}{2}\right)^{3/2} = \frac{3\sqrt{3\pi}}{4\sqrt{2}}$$

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## Exercício 02

**i** Comentário — parametrização (taxa)

Neste gabarito,  $\text{Gamma}(\alpha, \lambda)$  usa  $\lambda$  **como taxa** (rate):

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0.$$

(a) Densidade  $f_X(x)$

**Passo 1 — Substituir**  $\alpha = 3$  e  $\lambda = 1/2$

$$f_X(x) = \frac{(1/2)^3}{\Gamma(3)} x^{3-1} e^{-(1/2)x} = \frac{(1/2)^3}{\Gamma(3)} x^2 e^{-x/2}$$

**Passo 2 — Calcular**  $\Gamma(3)$

Como  $\Gamma(3) = 2! = 2$ , temos:

$$f_X(x) = \frac{1/8}{2} x^2 e^{-x/2} = \boxed{\frac{1}{16} x^2 e^{-x/2}}, \quad x > 0$$

(b) Função de distribuição  $F_X(x)$

Para  $\alpha$  inteiro, vale a forma fechada:

$$F_X(x) = 1 - e^{-\lambda x} \sum_{k=0}^{\alpha-1} \frac{(\lambda x)^k}{k!}, \quad x \geq 0$$

**Substituindo**  $\alpha = 3$  e  $\lambda = 1/2$ :

$$F_X(x) = 1 - e^{-x/2} \left(1 + \frac{x}{2} + \frac{1}{2} \left(\frac{x}{2}\right)^2\right) = 1 - e^{-x/2} \left(1 + \frac{x}{2} + \frac{x^2}{8}\right), \quad x \geq 0$$

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(c) Verificação por derivação:  $F'_X(x) = f_X(x)$

**Passo 1 — Escrever como produto**

Defina

$$g(x) = e^{-x/2}, \quad h(x) = 1 + \frac{x}{2} + \frac{x^2}{8},$$

de modo que  $F_X(x) = 1 - g(x)h(x)$

**Passo 2 — Derivar com regra do produto**

$$F'_X(x) = -(g'(x)h(x) + g(x)h'(x))$$

Como

$$g'(x) = -\frac{1}{2}e^{-x/2}, \quad h'(x) = \frac{1}{2} + \frac{x}{4},$$

segue:

$$F'_X(x) = -\left[-\frac{1}{2}e^{-x/2}\left(1 + \frac{x}{2} + \frac{x^2}{8}\right) + e^{-x/2}\left(\frac{1}{2} + \frac{x}{4}\right)\right]$$

**Passo 3 — Colocar  $e^{-x/2}$  em evidência e simplificar**

$$F'_X(x) = e^{-x/2} \left[ \frac{1}{2} \left( 1 + \frac{x}{2} + \frac{x^2}{8} \right) - \left( \frac{1}{2} + \frac{x}{4} \right) \right]$$

Note que

$$\frac{1}{2} \left( 1 + \frac{x}{2} + \frac{x^2}{8} \right) = \frac{1}{2} + \frac{x}{4} + \frac{x^2}{16},$$

logo o colchete vira  $\frac{x^2}{16}$

**Conclusão**

$$F'_X(x) = \frac{1}{16}x^2e^{-x/2} = f_X(x)$$

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**Exercício 03**

Dados:  $E(X) = 48$  e  $DP(X) = 24 \Rightarrow \text{Var}(X) = 24^2 = 576$

Para  $X \sim \text{Gamma}(\alpha, \lambda)$  (taxa):

$$E(X) = \frac{\alpha}{\lambda}, \quad \text{Var}(X) = \frac{\alpha}{\lambda^2}$$

(a) Identificar  $(\alpha, \lambda)$

**Passo 1 — Encontrar  $\lambda$** 

Da relação  $\text{Var}(X) = \alpha/\lambda^2$  e  $E(X) = \alpha/\lambda$ , obtemos:

$$\lambda = \frac{E(X)}{\text{Var}(X)} = \frac{48}{576} = \boxed{\frac{1}{12}}$$

**Passo 2 — Encontrar  $\alpha$** 

$$\alpha = E(X)\lambda = 48 \cdot \frac{1}{12} = \boxed{4}$$

Logo,

$$\boxed{X \sim \text{Gamma}(4, 1/12)}$$

**i Comentário — sobrevivência para  $\alpha$  inteiro**

Se  $\alpha$  é inteiro, então

$$S(t) = P(X > t) = e^{-\lambda t} \sum_{k=0}^{\alpha-1} \frac{(\lambda t)^k}{k!}$$

Aqui  $\alpha = 4$ , então a soma vai até  $k = 3$ .

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(b) Calcular  $P(X > 60)$

**Passo 1 — Calcular  $\lambda t$** 

$$\lambda t = \frac{1}{12} \cdot 60 = 5$$

**Passo 2 — Aplicar a fórmula de  $S(t)$** 

$$P(X > 60) = S(60) = e^{-5} \left( 1 + 5 + \frac{5^2}{2} + \frac{5^3}{6} \right) = e^{-5} \left( 1 + 5 + \frac{25}{2} + \frac{125}{6} \right)$$

**Passo 3 — Somar os termos**

$$1 + 5 + \frac{25}{2} + \frac{125}{6} = \frac{6 + 30 + 75 + 125}{6} = \frac{236}{6} = \frac{118}{3}$$

**Conclusão**

$$\boxed{P(X > 60) = \frac{118}{3} e^{-5}}$$


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(c) Calcular  $P(X > 60 \mid X > 40)$

**i** Comentário — por que vira razão de sobrevivências?

Como  $\{X > 60\} \subset \{X > 40\}$ , então:

$$P(X > 60 | X > 40) = \frac{P(X > 60)}{P(X > 40)} = \frac{S(60)}{S(40)}$$

**Passo 1 — Calcular  $S(40)$**

Primeiro:

$$\lambda t = \frac{1}{12} \cdot 40 = \frac{10}{3}$$

Então

$$S(40) = e^{-10/3} \left( 1 + \frac{10}{3} + \frac{(10/3)^2}{2} + \frac{(10/3)^3}{6} \right)$$

Agora:

$$\frac{(10/3)^2}{2} = \frac{100/9}{2} = \frac{50}{9}, \quad \frac{(10/3)^3}{6} = \frac{1000/27}{6} = \frac{500}{81}$$

Logo:

$$S(40) = e^{-10/3} \left( 1 + \frac{10}{3} + \frac{50}{9} + \frac{500}{81} \right)$$

Em denominador 81:

$$1 = \frac{81}{81}, \quad \frac{10}{3} = \frac{270}{81}, \quad \frac{50}{9} = \frac{450}{81}, \quad \frac{500}{81} = \frac{500}{81}$$

Somando:

$$\frac{81 + 270 + 450 + 500}{81} = \frac{1301}{81}$$

Portanto:

$$S(40) = \boxed{\frac{1301}{81} e^{-10/3}}$$

**Passo 2 — Formar a razão  $S(60)/S(40)$**

$$\frac{S(60)}{S(40)} = \frac{\frac{118}{3} e^{-5}}{\frac{1301}{81} e^{-10/3}} = \frac{118}{3} \cdot \frac{81}{1301} \cdot e^{-5+10/3}$$

Como  $\frac{81}{3} = 27$  e  $-5 + \frac{10}{3} = -\frac{5}{3}$ :

$$\boxed{P(X > 60 | X > 40) = \frac{3186}{1301} e^{-5/3}}$$

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(d) Três sistemas iid: probabilidade de pelo menos 2 passarem 60 dias

Seja

$$p = P(X > 60) = \frac{118}{3}e^{-5}$$

Se  $N$  é o número (entre 3) que passam 60 dias, então  $N \sim \text{Bin}(3, p)$ .

**Passo 1 — Escrever**  $P(N \geq 2)$

$$P(N \geq 2) = P(N = 2) + P(N = 3) = \binom{3}{2}p^2(1-p) + p^3$$

**Passo 2 — Simplificar**

$$\binom{3}{2}p^2(1-p) + p^3 = 3p^2 - 3p^3 + p^3 = \boxed{3p^2 - 2p^3}$$

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#### Exercício 04

Compare com

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$$

- (a)  $\frac{8}{3}x^3e^{-2x}$  O expoente de  $x$  é 3, então  $\alpha - 1 = 3 \Rightarrow \alpha = 4$   
O termo  $e^{-2x}$  dá  $\lambda = 2$

$$\boxed{\alpha = 4, \lambda = 2}$$

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- (b)  $\frac{1}{4}xe^{-x/2}$  Aqui  $x^1$ , então  $\alpha - 1 = 1 \Rightarrow \alpha = 2$   
E  $e^{-x/2}$  dá  $\lambda = 1/2$

$$\boxed{\alpha = 2, \lambda = \frac{1}{2}}$$

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- (c)  $\sqrt{\frac{3}{\pi x}}e^{-3x}$  Reescreva:

$$\sqrt{\frac{3}{\pi x}} = \frac{\sqrt{3}}{\sqrt{\pi}} x^{-1/2}$$

Logo  $x^{\alpha-1} = x^{-1/2} \Rightarrow \alpha - 1 = -1/2 \Rightarrow \alpha = 1/2$ , e  $e^{-3x}$  dá  $\lambda = 3$ :

$$\boxed{\alpha = \frac{1}{2}, \lambda = 3}$$

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#### Exercício 05

**i** Comentário — relação Beta-Gamma

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

(a)  $B(2, 3)$

$$B(2, 3) = \frac{\Gamma(2)\Gamma(3)}{\Gamma(5)} = \frac{1!2!}{4!} = \boxed{\frac{1}{12}}$$

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(b)  $B(3, 1)$

$$B(3, 1) = \frac{\Gamma(3)\Gamma(1)}{\Gamma(4)} = \frac{2!1}{3!} = \boxed{\frac{1}{3}}$$

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(c)  $B\left(\frac{1}{2}, 4\right)$  Usando  $\Gamma(4) = 3! = 6$  e  $\Gamma(9/2) = \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(1/2) = \frac{105}{16}\sqrt{\pi}$ :

$$B\left(\frac{1}{2}, 4\right) = \frac{\Gamma(1/2)\Gamma(4)}{\Gamma(9/2)} = \frac{\sqrt{\pi} \cdot 6}{(105/16)\sqrt{\pi}} = \boxed{\frac{32}{35}}$$

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(d)  $B\left(\frac{3}{2}, \frac{1}{2}\right)$  Como  $\Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$  e  $\Gamma(2) = 1! = 1$ :

$$B\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{\Gamma(3/2)\Gamma(1/2)}{\Gamma(2)} = \frac{(\frac{1}{2}\sqrt{\pi})\sqrt{\pi}}{1} = \boxed{\frac{\pi}{2}}$$

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### Exercício 06

A densidade da Beta é

$$f(x) = \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1}, \quad 0 < x < 1$$

(a) Densidade

Como  $B(2, 3) = 1/12$ ,

$$\boxed{f(x) = 12x(1-x)^2, \quad 0 < x < 1}$$

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(b) Função de distribuição  $F(x)$

**Passo 1 — Expandir o integrando**

$$12x(1-x)^2 = 12x(1-2x+x^2) = 12x - 24x^2 + 12x^3$$

**Passo 2 — Integrar de 0 até  $x$**

$$F(x) = \int_0^x (12t - 24t^2 + 12t^3) dt = (6t^2 - 8t^3 + 3t^4) \Big|_0^x$$

**Conclusão**

$$F(x) = 6x^2 - 8x^3 + 3x^4, \quad 0 \leq x \leq 1$$

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(c) Esperança

Para  $X \sim \text{Beta}(a, b)$ :

$$E(X) = \frac{a}{a+b}$$

Logo,

$$E(X) = \frac{2}{5} = 0,4$$

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(d) Variância

Para  $X \sim \text{Beta}(a, b)$ :

$$\text{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)}$$

Logo,

$$\text{Var}(X) = \frac{2 \cdot 3}{5^2 \cdot 6} = \frac{6}{150} = \frac{1}{25} = 0,04$$

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(e) Probabilidades

(f)  $P(X \geq E(X))$

Como  $E(X) = 0,4$ :

$$P(X \geq E(X)) = 1 - F(0,4)$$

Calcule:

$$F(0,4) = 6(0,16) - 8(0,064) + 3(0,0256) = 0,5248 = \frac{328}{625}$$

Logo:

$$P(X \geq E(X)) = 1 - \frac{328}{625} = \frac{297}{625}$$

$$(ii) P(X > E(X) \mid X < E(X) + DP(X))$$

Aqui  $DP(X) = \sqrt{0,04} = 0,2$  e  $E + DP = 0,6$ . Então:

$$P(X > 0,4 \mid X < 0,6) = \frac{P(0,4 < X < 0,6)}{P(X < 0,6)} = \frac{F(0,6) - F(0,4)}{F(0,6)}$$

Calcule:

$$F(0,6) = 6(0,36) - 8(0,216) + 3(0,1296) = 0,8208 = \frac{513}{625}$$

Portanto:

$$\frac{F(0,6) - F(0,4)}{F(0,6)} = \frac{\frac{513}{625} - \frac{328}{625}}{\frac{513}{625}} = \frac{185}{513}$$

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### Exercício 07

Denote por  $I_x(a, b)$  a Beta incompleta regularizada (a CDF da Beta).

(a) Média

Para Beta( $a, b$ ):

$$E(X) = \frac{a}{a+b}$$

Logo:

$$E(X) = \frac{3/2}{3/2+3} = \frac{1}{3}$$

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(b) Probabilidade de “fora do padrão”

Se

$$A = \{X > 0, 5\} \cup \{X < 0, 1\},$$

então:

$$P(A) = P(X < 0, 1) + P(X > 0, 5) = I_{0,1} \left( \frac{3}{2}, 3 \right) + \left[ 1 - I_{0,5} \left( \frac{3}{2}, 3 \right) \right]$$

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(c)  $P(X < \text{média} \mid A)$

A média é  $1/3$  e, dentro do evento  $A$ , o único trecho abaixo de  $1/3$  é  $X < 0, 1$ . Logo:

$$P \left( X < \frac{1}{3} \mid A \right) = \frac{P(X < 0, 1)}{P(A)}$$

Assim:

$$P \left( X < \frac{1}{3} \mid A \right) = \frac{I_{0,1} \left( \frac{3}{2}, 3 \right)}{I_{0,1} \left( \frac{3}{2}, 3 \right) + 1 - I_{0,5} \left( \frac{3}{2}, 3 \right)}$$

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(d) Até encontrar 3 fora do padrão: média de “dentro do padrão”

Seja  $p = P(A)$  a probabilidade de “fora do padrão”.

O número esperado de observações “dentro do padrão” antes de ocorrerem 3 “fora do padrão” é:

$$E(\text{dentro}) = \frac{3(1-p)}{p}, \quad p = P(A)$$

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### Exercício 08

Compare com:

$$f(x) = \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1}, \quad 0 < x < 1$$

(a)  $105x^2(1-x)^4$  Aqui  $a-1 = 2 \Rightarrow a = 3$  e  $b-1 = 4 \Rightarrow b = 5$ :

$$(a, b) = (3, 5)$$

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(b)  $20(x^3 - x^4) = 20x^3(1-x)$  Aqui  $a-1 = 3 \Rightarrow a = 4$  e  $b-1 = 1 \Rightarrow b = 2$ :

$$(a, b) = (4, 2)$$

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(c)  $f(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$  Aqui  $a - 1 = -1/2 \Rightarrow a = 1/2$  e  $b - 1 = 0 \Rightarrow b = 1$ :

$$(a, b) = \left(\frac{1}{2}, 1\right)$$

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### Exercício 09

A Weibull (escala  $\alpha$ , forma  $\beta$ ) tem:

$$F(x) = 1 - e^{-(x/\alpha)^\beta}, \quad x \geq 0,$$

$$f(x) = \frac{\beta}{\alpha} \left(\frac{x}{\alpha}\right)^{\beta-1} e^{-(x/\alpha)^\beta}, \quad x > 0$$

(a) Densidade

Substituindo  $\alpha = 2$ ,  $\beta = 2$ :

$$f(x) = \frac{2}{2} \left(\frac{x}{2}\right)^1 e^{-(x/2)^2} = \boxed{\frac{x}{2} e^{-x^2/4}}, \quad x > 0$$

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(b) Distribuição

$$\boxed{F(x) = 1 - e^{-x^2/4}}, \quad x \geq 0$$

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(c) Esperança

Para Weibull:

$$E(X) = \alpha \Gamma\left(1 + \frac{1}{\beta}\right)$$

Logo:

$$E(X) = 2\Gamma\left(\frac{3}{2}\right) = 2 \cdot \frac{\sqrt{\pi}}{2} = \boxed{\sqrt{\pi}}$$

 **Atenção** — correção importante

$\Gamma(3/2) = \sqrt{\pi}/2$ , portanto  $2\Gamma(3/2) = \sqrt{\pi}$  (não  $\pi$ ).

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(d) Variância

Para Weibull:

$$\text{Var}(X) = \alpha^2 \left[ \Gamma \left( 1 + \frac{2}{\beta} \right) - \Gamma^2 \left( 1 + \frac{1}{\beta} \right) \right]$$

Aqui:

$$\text{Var}(X) = 4 \left[ \Gamma(2) - \Gamma^2 \left( \frac{3}{2} \right) \right] = 4 \left[ 1 - \left( \frac{\sqrt{\pi}}{2} \right)^2 \right] = \boxed{4 - \pi}$$

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(e) Probabilidades

$$P(X > 1) = e^{-(1/2)^2} = \boxed{e^{-1/4}}$$

E

$$P(X > 1 \mid X < 2) = \frac{P(1 < X < 2)}{P(X < 2)} = \frac{F(2) - F(1)}{F(2)} = \boxed{\frac{e^{-1/4} - e^{-1}}{1 - e^{-1}}}$$

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### Exercício 10

(a) Esperança

$$\boxed{E(X) = \alpha \Gamma \left( 1 + \frac{1}{4} \right) = \frac{1}{2} \Gamma \left( \frac{5}{4} \right)}$$

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(b) Desvio-padrão

Primeiro:

$$\text{Var}(X) = \alpha^2 \left[ \Gamma \left( 1 + \frac{2}{4} \right) - \Gamma^2 \left( 1 + \frac{1}{4} \right) \right] = \frac{1}{4} \left[ \Gamma \left( \frac{3}{2} \right) - \Gamma^2 \left( \frac{5}{4} \right) \right]$$

Logo:

$$\boxed{DP(X) = \frac{1}{2} \sqrt{\Gamma(3/2) - \Gamma(5/4)^2}}$$

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(c)  $P(X > 10)$

$$P(X > 10) = \exp(-(10/0.5)^4) = \exp(-20^4) = \boxed{e^{-160000}}$$

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(d)  $P(X > 8 \mid X > 5)$

$$P(X > 8 \mid X > 5) = \frac{S(8)}{S(5)} = \exp\left(-\left(\frac{8}{0.5}\right)^4 + \left(\frac{5}{0.5}\right)^4\right) = \exp(-(16^4 - 10^4)) = \boxed{e^{-55536}}$$

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(e) Percentil 90 Se  $F(t) = 0.9$ , então  $e^{-(t/\alpha)^\beta} = 0.1$  e:

$$(t/\alpha)^\beta = \ln 10 \quad \Rightarrow \quad \boxed{t = \alpha(\ln 10)^{1/\beta} = \frac{1}{2}(\ln 10)^{1/4}}$$

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### Exercício 11

(a)  $\frac{3}{8}x^2e^{-x^3/8}$  Como o expoente no termo exponencial é  $x^3/8$ , temos  $\beta = 3$  e  $\alpha^3 = 8 \Rightarrow \alpha = 2$ :

$$\boxed{(\alpha, \beta) = (2, 3)}$$

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(b) Caso do tipo  $f(x) = \frac{1}{4}x^{-1/2}e^{-\sqrt{x}/2}$  Se o expoente é  $-(x/\alpha)^{1/2} = -\sqrt{x}/2$ , então  $\sqrt{x/\alpha} = \sqrt{x}/2 \Rightarrow \alpha = 4$  e  $\beta = 1/2$ :

$$\boxed{(\alpha, \beta) = \left(4, \frac{1}{2}\right)}$$

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(c)  $18xe^{-9x^2}$  Como o expoente é  $-9x^2$ , temos  $\beta = 2$  e  $(x/\alpha)^2 = 9x^2 \Rightarrow 1/\alpha^2 = 9 \Rightarrow \alpha = 1/3$ :

$$\boxed{(\alpha, \beta) = \left(\frac{1}{3}, 2\right)}$$

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### Exercício 12

Considere

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}(x-a)^{\alpha-1}e^{-\beta(x-a)}\mathbf{1}_{(a,\infty)}(x)$$

**Passo 1 — Integrar em  $\mathbb{R}$  (só conta  $x > a$ )**

$$\int_{-\infty}^{\infty} f(x) dx = \int_a^{\infty} f(x) dx$$

**Passo 2 — Substituição  $u = x - a$**

$$\int_a^{\infty} f(x) dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{\infty} u^{\alpha-1} e^{-\beta u} du$$

**Passo 3 — Substituição  $v = \beta u$**  Se  $v = \beta u$ , então  $du = dv/\beta$  e:

$$\frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{\infty} \left(\frac{v}{\beta}\right)^{\alpha-1} e^{-v} \frac{dv}{\beta} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} v^{\alpha-1} e^{-v} dv = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1$$

**Conclusão**

$$\boxed{\int_{-\infty}^{\infty} f(x) dx = 1}$$

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### Exercício 13

Para  $a = b = 1$ :

$$f(x) = \frac{1}{B(1,1)} x^0 (1-x)^0 \mathbf{1}_{(0,1)}(x)$$

Como

$$B(1,1) = \frac{\Gamma(1)\Gamma(1)}{\Gamma(2)} = \frac{1 \cdot 1}{1} = 1,$$

segue:

$$\boxed{f(x) = 1, \quad 0 < x < 1,}$$

que é a densidade de  $U(0,1)$ .

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### Exercício 14

Se  $X \sim \text{Gamma}(\alpha, \lambda)$  (taxa), então:

$$E(X) = \int_0^{\infty} x \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\infty} x^\alpha e^{-\lambda x} dx$$

Pela identidade:

$$\int_0^{\infty} x^{\alpha} e^{-\lambda x} dx = \frac{\Gamma(\alpha + 1)}{\lambda^{\alpha+1}}$$

Logo:

$$E(X) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha + 1)}{\lambda^{\alpha+1}} = \frac{\alpha}{\lambda}$$

Agora:

$$E(X^2) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha+1} e^{-\lambda x} dx = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha + 2)}{\lambda^{\alpha+2}} = \frac{\alpha(\alpha + 1)}{\lambda^2}$$

Então:

$$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{\alpha(\alpha + 1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \boxed{\frac{\alpha}{\lambda^2}}$$

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### Exercício 15

Se  $X \sim \text{Beta}(a, b)$ :

$$E(X) = \int_0^1 x \frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1} dx = \frac{1}{B(a, b)} \int_0^1 x^a (1-x)^{b-1} dx = \frac{B(a+1, b)}{B(a, b)}$$

Usando  $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ :

$$E(X) = \frac{\Gamma(a+1)\Gamma(b)\Gamma(a+b)}{\Gamma(a)\Gamma(b)\Gamma(a+b+1)} = \boxed{\frac{a}{a+b}}$$

Além disso:

$$E(X^2) = \frac{B(a+2, b)}{B(a, b)} = \frac{a(a+1)}{(a+b)(a+b+1)}$$

Logo:

$$\text{Var}(X) = E(X^2) - E(X)^2 = \boxed{\frac{ab}{(a+b)^2(a+b+1)}}$$

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### Exercício 16

Se  $X \sim \text{Weibull}(\alpha, \beta)$ , defina

$$Y = \left(\frac{X}{\alpha}\right)^{\beta}$$

Então  $Y \sim \text{Exp}(1)$  e  $X = \alpha Y^{1/\beta}$

Para  $r > -1$ :

$$E(X^r) = \alpha^r E(Y^{r/\beta}) = \alpha^r \Gamma\left(1 + \frac{r}{\beta}\right)$$

Em particular:

$$\boxed{E(X) = \alpha \Gamma\left(1 + \frac{1}{\beta}\right)}$$

e

$$E(X^2) = \alpha^2 \Gamma\left(1 + \frac{2}{\beta}\right) \quad \Rightarrow \quad \boxed{\text{Var}(X) = \alpha^2 \left[ \Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma^2\left(1 + \frac{1}{\beta}\right) \right]}$$