1.) $B = \left(\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right), (0,0,1) \right)$

a) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right) \cdot (0,0,1) = \frac{\sqrt{2}}{2} \cdot 0 + \frac{\sqrt{2}}{2} \cdot 0 + 0 \cdot 1 = 0$

$\left\| \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right) \right\| = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = \sqrt{1} = 1$

$\| (0,0,1) \| = \sqrt{1^2} = 1$

Os vetores são ortogonais e têm norma 1, logo a base é ortonormada

b) $proj_F (1,2,3) = \left((1,2,3) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right) \right) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right) + \left((1,2,3) \cdot (0,0,1) \right) \cdot (0,0,1)$

$= \left((1,2,3) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right)^2 \right) + \left((1,2,3) \cdot (0,0,1)^2 \right)$

$= (1,2,3) \cdot \left(\frac{1}{2}, \frac{1}{2}, 0 \right) + (1,2,3) \cdot (0,0,1)$

$= \left(\frac{1}{2}, 1, 0 \right) + (0,0,3) = \left(\frac{1}{2}, 1, 3 \right)$

$proj_W (1,2,3) = v + y \Leftrightarrow y = v - proj_W (1,2,3)$

$\Leftrightarrow y = (1,2,3) - \left(\frac{1}{2}, 1, 3 \right) = \left(\frac{1}{2}, 1, 0 \right)$

c) $\begin{cases} x - y = 1 \\ x - y = 2 \\ y = 3 \end{cases}$ $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$ $x = \begin{bmatrix} x \\ y \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\hat{B} = proj_{\mathcal{C}(A)} B$ $\mathcal{C}(A) = \langle (1,1,0), (-1,-1,1) \rangle$

$proj_{\mathcal{C}(A)} B = \left((1,2,3) \cdot (1,1,0) \right) \cdot (1,1,0) + \left((1,2,3) \cdot (-1,-1,1) \right) \cdot (-1,-1,1)$

$= (1,2,0) + (1,2,3) = (2,4,3)$

$\hat{B} = (2,4,3)$ erro $\| \hat{B} - B \| = \| (2,4,3) - (1,2,3) \| = \| (1,2,0) \| = \sqrt{5}$

$$A\hat{x} = \hat{B}$$

$$\begin{cases} x - y = 2 \\ x - y = 4 \\ y = 3 \end{cases} \qquad \begin{cases} x = 2 + y \\ x = 4 + y \\ y = 3 \end{cases} \qquad \left.\begin{cases} x = 5 \\ x = 6 \\ y = 3 \end{cases}\right] \text{ impossível}$$

$$A^T A x = A^T B$$

$$A^T A = \underset{2\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}} \underset{3\times 2}{\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix}$$

$$A^T B = \underset{2\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 5 \\ 6 \\ 3 \end{bmatrix}} = \begin{bmatrix} 11 & -8 \end{bmatrix}$$

Modelo Leontief 2023

3)

	M	A	S
M	0,2	0,4	0,2
A	0,7	0,4	0,2
S	0	0,1	0,5

$$d = \begin{bmatrix} 10 \\ 5 \\ 10 \end{bmatrix}$$

$$x = Cx + d$$

$$(x - Cx) = d \iff (I - C)x = d$$

$$C = \begin{bmatrix} 0,2 & 0,4 & 0,2 \\ 0,7 & 0,4 & 0,2 \\ 0 & 0,1 & 0,5 \end{bmatrix} \qquad I - C = \begin{bmatrix} 0,8 & 0,4 & 0,2 \\ 0,7 & 0,6 & 0,2 \\ 0 & 0,1 & 0,5 \end{bmatrix}$$

2)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & c & 2 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 1 & a & 2-\lambda \end{bmatrix} \begin{matrix} 1-\lambda & 0 \\ 1 & 2-\lambda \\ 1 & c \end{matrix}$$

$$\det(A - \lambda I) = (1-\lambda)(2-\lambda)^2$$

$$(1-\lambda)(2-\lambda)^2 = 0 \iff 1-\lambda = 0 \lor (2-\lambda)^2 = 0$$

$$\iff \underbrace{\lambda = 1 \lor \lambda = 2}_{\text{Valores próprios de } A}$$

b)

$$U_1 : \quad (A - 1I) = \begin{bmatrix} 1-1 & 0 & 0 \\ 1 & 2-1 & 0 \\ 1 & c & 2-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & c & 1 \end{bmatrix}$$

$$(A - 1I)x = 0 \iff \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & c & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{cases} x + y = 0 \\ x + ay + z = 0 \end{cases}$$

$$\iff \begin{cases} x = -y \\ z = -x - cy \end{cases} \quad \begin{cases} x = -y \\ z = y - ay \end{cases} \quad U_1 = \{ (-y, y, y - cy) : y \in \mathbb{R} \}$$

$$\downarrow$$

$$\frac{(-1, 1, 1-c)}{\| (-1, 1, 1-c) \|}$$

$$U_2 : \quad (A - 2I) = \begin{bmatrix} 1-2 & 0 & 0 \\ 1 & 2-2 & 0 \\ 1 & a & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & c & 0 \end{bmatrix}$$

$$(A - 2I)x = 0 \iff \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & c & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{cases} -x = 0 \\ x = 0 \\ x - cy = 0 \end{cases}$$

$(=) \begin{cases} x = 0 \\ x = -ay \end{cases}$ Se $a = 0$ U_2 terá $\dim U_2 = 2$

$\dim U_2 + \dim U_1 = 3 \to$ diagonizável

$0 = -ay \Leftrightarrow a = 0$

Para ser diagonizável $\dim U_2 + \dim U_1 = 3$, o que não acontece

---

3)

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 1 & 3 & 1 \\ -3 & 1 & 20 \end{bmatrix} \begin{matrix} 1 & 1 \\ 1 & 3 \\ -3 & 1 \end{matrix} \qquad [x\ y\ z]^T [A] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\left.\begin{aligned} \Delta_1 &= \det \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 3 - 1 = 2 > 0 \\ \Delta_2 &= \det |1| = 1 > 0 \\ \Delta_3 &= \det(A) = 60 - 3 - 7 + 27 - 1 - 20 \\ &= 60 - 37 = 23 > 0 \end{aligned}\right] \text{ é definida positiva}$$

b) Como A é simétrica é garantido que os valores próprios são reais

4) $5x^2 - 4xy + y^2 + \sqrt{2}x + \sqrt{2}y = 0$

$$[x\ y] \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + [\sqrt{2}\ \ \sqrt{2}] \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$(=)$ $X^T A X + BX = 0$

$\uparrow$

diagonalizar para remover xy

$A \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$

$\det(A - \lambda I) = 0 \iff \det\left(\begin{bmatrix} 5-\lambda & -2 \\ -2 & 5-\lambda \end{bmatrix}\right) = 0 \iff (5-\lambda)^2 - 4 = 0$

$\iff (5-\lambda)^2 = 4$

$\iff 5-\lambda = \pm 2$

$\iff \lambda = 7 \vee \lambda = 3$

$U_3 : \quad (A - 3I)x = 0 \iff \begin{bmatrix} 5-3 & -2 \\ -2 & 5-3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\iff \begin{cases} 2x - 2y = 0 \\ -2x + 2y = 0 \end{cases} \quad \{ x = y$

$U_3 = \{ (y, y) : y \in \mathbb{R} \} \longrightarrow \dfrac{(1,1)}{\|(1,1)\|} = \left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$

$U_7 : \quad (A - 7I)x = 0 \iff \begin{bmatrix} 5-7 & -2 \\ -2 & 5-7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\iff \begin{cases} -2x - 2y = 0 \\ -2x - 2y = 0 \end{cases} \quad \{ x = -y$

$U_7 = \{ (-y, y) : y \in \mathbb{R} \} \longrightarrow \dfrac{(-1,1)}{\|(-1,1)\|} = \left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$

$P = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} = P^T A P$

M.D $\quad X = PY \; , \; Y = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$(PY)^T A \; PY + B\,PY = 0 \iff Y^T P^T A P Y + BPY = 0$

$\iff Y^T D Y + BPY = 0$

$\iff \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 0$

$$\boxed{BP = \underset{2\times 1}{\begin{bmatrix} \sqrt{2} & \sqrt{2} \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}} = \begin{bmatrix} 1 & 0 \end{bmatrix}}$$

$\iff 3x'^2 + 7y'^2 + x' = 0$

$\iff 3x'^2 + x + 7y'^2 = 0$

$\iff 3\left(x'^2 + \frac{1}{3}x'\right) + 7y'^2 = 0 \iff 3\left(x'^2 + \frac{1}{3}x' + \frac{1}{36} - \frac{1}{36}\right) + 7y'^2 = 0$

$\iff 3\left(x' + \frac{1}{6}\right)^2 - \frac{1}{36} + 7y'^2 = 0$

M.D $\begin{cases} x'' = x' + \frac{1}{6} \\ y'' = 7y' - \frac{1}{36} \end{cases}$ $\qquad \iff 3x''^2 + y''^2$

$\iff \dfrac{x''^2}{\frac{1}{3}} + y''^2 = 0$

5) $L(x,y,z) = (x-z \, , -y + 2z \, , x - y + z)$

c)

b) $L(x,y,z) = (x-z, -y+2z, x-y+z)$

$L(1,-1,0) = (1-0, 1+0, 1+1) = (1,1,2)$

$L(1,0,1) = (0, 2, 2)$

$L(0,0,1) = (-1, 2, 1)$

$(1,1,2) = \alpha(1,-1,0) + \beta(1,0,1) + \gamma(0,0,1)$

$\begin{cases} \alpha + \beta = 1 \\ -\alpha = 1 \\ \beta + \gamma = 2 \end{cases} \quad \begin{cases} \beta = 0 \\ \alpha = 1 \\ \gamma = 2 \end{cases} \qquad [L(1,1,2)]_B = (1,0,2)$

$$(0,2,2) = \alpha(1,-1,0) + \beta(1,0,1) + \gamma(0,0,1)$$

$$\begin{cases} \alpha + \beta = 0 \\ -\alpha = 2 \\ \beta + \gamma = 2 \end{cases} \qquad \begin{cases} \beta = 2 \\ \alpha = -2 \\ \gamma = 0 \end{cases} \qquad [L(0,2,2)]_B = (-2, 2, 0)$$

$$(-1,2,1) = \alpha(1,-1,0) + \beta(1,0,1) + \gamma(0,0,1)$$

$$\begin{cases} \alpha + \beta = -1 \\ -\alpha = 2 \\ \beta + \gamma = 1 \end{cases} \qquad \begin{cases} \beta = 1 \\ \alpha = -2 \\ \gamma = 0 \end{cases} \qquad [L(-1,2,1)]_B = (-2, 1, 0)$$

$$[L]_{B,B} = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 2 & 1 \\ 2 & 0 & 0 \end{bmatrix}$$