Robótica Móvel

Localization

Part 2 - Absolute Positioning

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Summary

1. Absolute Positioning
2. Triangulation and trilateration systems
3. Global Positioning Systems

Absolute Positioning

Absolute Positioning Systems

- Principle
  - Obtain current coordinates based on information about measured points in the outside world
- Methodology
  - Triangulation and/or trilateration
- Beacons
  - Devices that allow obtaining information (coordinates,...) about those special points.
  - In general, the direction and/or distance of 3 or more beacons should be evaluated.
    - Why 3...?
    - ... for 3 coordinates we must have 3 independent pieces of information!

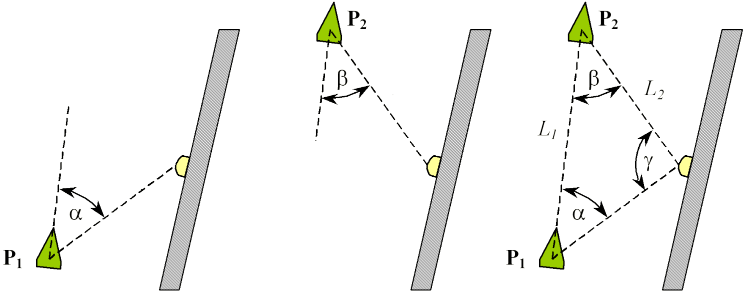
Types of beacons

- Active
  - send information about their identification by optical or radio means (eg GPS,...)
- Passive (landmarks)
  - Must be detected by the navigator system
- Landmark types (passive beacons)
  - Artificial landmarks
    - identifiers dedicated exclusively to use for localization (triangulation)
  - Natural landmarks
    - Any environment unspecific property that can be used as a reference point, if detectable and identifiable.

Triangulation and trilateration systems

Triangulation Systems - I

- The simplest triangulation system
- A beacon seen from two different points plus the robot's translation:
  - Uses two successive orientations: α and β



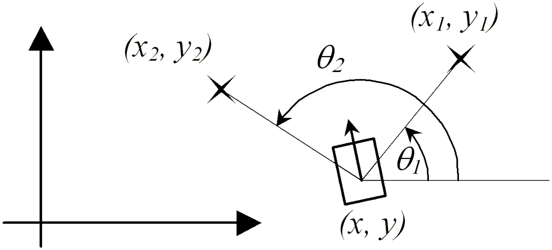
$$\frac{L_2}{\sin\alpha} = \frac{L_1}{\sin\gamma}$$

$$\frac{L_1}{\sin\gamma} = \frac{L_1}{\sin\left(\pi - (\alpha+\beta)\right)} = \frac{L_1}{\sin\left(\alpha+\beta\right)} \quad \Rightarrow \quad L_2 = L_1 \frac{\sin\alpha}{\sin\left(\alpha+\beta\right)}$$

- Limited to linear displacements

Triangulation Systems - II

- Triangulation based on the robot's orientation and the (absolute) direction of two known beacons.



$$\tan\theta_1 = \frac{y_1 - y}{x_1 - x} = \frac{\sin\theta_1}{\cos\theta_1}$$
$$\tan\theta_2 = \frac{y_2 - y}{x_2 - x} = \frac{\sin\theta_2}{\cos\theta_2}$$

$$\begin{cases} (x_1 - x)\sin\theta_1 = (y_1 - y)\cos\theta_1 \\ (x_{21} - x)\sin\theta_2 = (y_2 - y)\cos\theta_2 \end{cases} \quad \begin{bmatrix} -\sin\theta_1 & \cos\theta_1 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x_1\sin\theta_1 + y_1\cos\theta_1 \\ -x_2\sin\theta_2 + y_2\cos\theta_2 \end{bmatrix}$$

- There is always a solution as long as:

$$-\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \neq 0 \text{ or } \tan\theta_1 \neq \tan\theta_2$$

GDOP

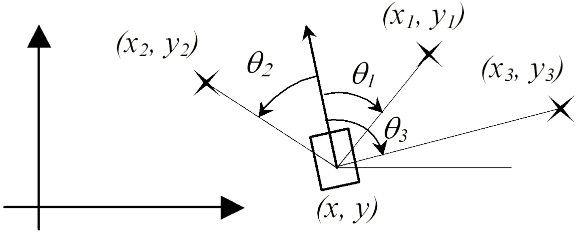
- Geometric Dilution of Precision (GDOP)
- Definition: GDOP = Position Error / Measurement Uncertainty
- Position error = GDOP $\times$ Measurement uncertainty
- In one dimension: $GDOP = \frac{\Delta x}{\Delta m}$
- Three coordinate system based on 3 measurements:

$$\begin{cases} x = F\left(m_1, m_2, m_3\right) \\ y = G\left(m_1, m_2, m_3\right) \\ \theta = H\left(m_1, m_2, m_3\right) \end{cases} \qquad \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}\theta \end{bmatrix} = \mathbf{J} \begin{bmatrix} \mathrm{d}m_1 \\ \mathrm{d}m_2 \\ \mathrm{d}m_3 \end{bmatrix} \qquad \mathbf{J} = \begin{bmatrix} \frac{\partial F}{\partial m_1} & \frac{\partial F}{\partial m_2} & \frac{\partial F}{\partial m_3} \\ \frac{\partial G}{\partial m_1} & \frac{\partial G}{\partial m_2} & \frac{\partial G}{\partial m_3} \\ \frac{\partial H}{\partial m_1} & \frac{\partial H}{\partial m_2} & \frac{\partial H}{\partial m_3} \end{bmatrix}$$

- Actually we have: $\lim\limits_{\mathrm{d}r \to 0} GDOP = |\mathbf{J}|$ (being $\mathrm{d}r = [\mathrm{d}x \ \mathrm{d}y \ \mathrm{d}\theta]^{\intercal}$)
- The GDOP translates a certain reliability of the localization estimate.

Triangulation Systems - III

- Triangulation from the relative direction of three known beacons.

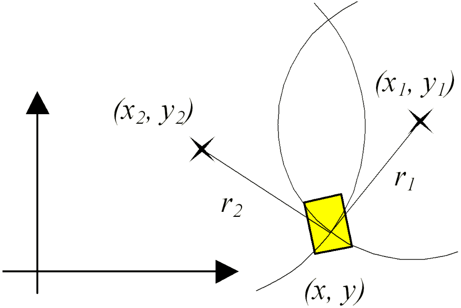


$$\tan(\theta - \theta_i) = \frac{y_i - y}{x_i - x} = \frac{\sin(\theta - \theta_i)}{\cos(\theta - \theta_i)} \text{ for } i = 1, 2, 3$$

- Results in a three-variable system (x, y, θ) with three nonlinear (trigonometric) equations.
- Analytical resolution is complex. Use numerical resolution.
  - One solution in: M. Betke and L. Gurvits, "Mobile robot localization using landmarks," in IEEE Transactions on Robotics and Automation, vol. 13, no. 2, pp. 251-263, April 1997, doi: 10.1109/70.563647.

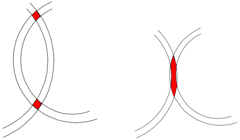
Triangulation Systems – IV

- Triangulation from distances to two known beacons (trilateration)



$$r_1 = \sqrt{(x - x_1)^2 + (y - y_1)^2}$$
$$r_2 = \sqrt{(x - x_2)^2 + (y - y_2)^2}$$

- Ambiguity in the choice of solution is resolved with an initial estimate of the localization, if any...



- Dimension of the area of uncertainty depending on the situation.
- What if r_1 and r_2 are not distinguishable...?

Extension of trilateration to 3 or more beacons

- In case of 3 or more beacons with coordinates (x_i, y_i), with $i = 1, 2, 3, \ldots, n$, the equations become:

$$\begin{cases} d_1^2 = (x - x_1)^2 + (y - y_1)^2 = x^2 - 2xx_1 + x_1^2 + y^2 - 2yy_1 + y_1^2 \\ d_2^2 = (x - x_2)^2 + (y - y_2)^2 = x^2 - 2xx_2 + x_2^2 + y^2 - 2yy_2 + y_2^2 \\ \vdots \\ d_n^2 = (x - x_n)^2 + (y - y_n)^2 = x^2 - 2xx_n + x_n^2 + y^2 - 2yy_n + y_n^2 \end{cases}$$

- Subtracting the first equation to equations 2 to n yields:

$$\begin{cases} d_2^2 - d_1^2 = 2x\left(x_1 - x_2\right) + x_2^2 - x_1^2 + 2y\left(y_1 - y_2\right) + y_2^2 - y_1^2 \\ d_3^2 - d_1^2 = 2x\left(x_1 - x_3\right) + x_3^2 - x_1^2 + 2y\left(y_1 - y_3\right) + y_3^2 - y_1^2 \\ \vdots \\ d_n^2 - d_1^2 = 2x\left(x_1 - x_n\right) + x_n^2 - x_1^2 + 2y\left(y_1 - y_n\right) + y_n^2 - y_1^2 \end{cases}$$

- continues...

Extension of trilateration to 3 or more beacons

- ... continued...

$$\begin{cases} 2\left(x_1 - x_2\right)x + 2\left(y_1 - y_2\right)y = d_2^2 - d_1^2 + x_1^2 - x_2^2 + y_1^2 - y_2^2 \\ 2\left(x_1 - x_3\right)x + 2\left(y_1 - y_3\right)y = d_3^2 - d_1^2 + x_1^2 - x_3^2 + y_1^2 - y_3^2 \\ \vdots \\ 2\left(x_1 - x_n\right)x + 2\left(y_1 - y_n\right)y = d_n^2 - d_1^2 + x_1^2 - x_n^2 + y_1^2 - y_n^2 \end{cases}$$

- Or in matricial form:

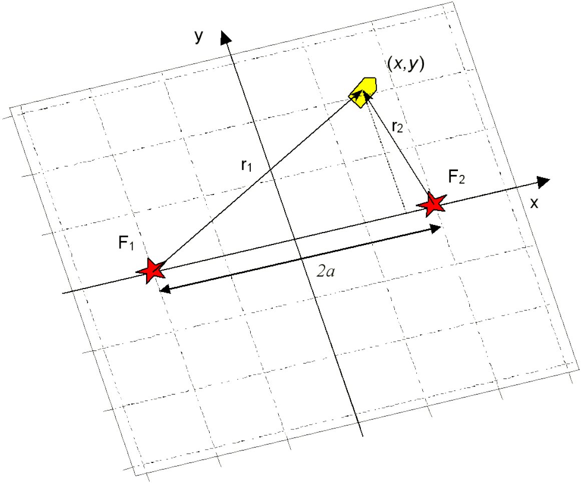
$$\begin{bmatrix} 2\left(x_1 - x_2\right) & 2\left(y_1 - y_2\right) \\ 2\left(x_1 - x_3\right) & 2\left(y_1 - y_3\right) \\ \vdots & \vdots \\ 2\left(x_1 - x_n\right) & 2\left(y_1 - y_n\right) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d_2^2 - d_1^2 + x_1^2 - x_2^2 + y_1^2 - y_2^2 \\ d_3^2 - d_1^2 + x_1^2 - x_3^2 + y_1^2 - y_3^2 \\ \vdots \\ d_n^2 - d_1^2 + x_1^2 - x_n^2 + y_1^2 - y_n^2 \end{bmatrix}$$

Extension of trilateration to 3 or more beacons

- Finally, the general format of the equations is $\mathbf{AX} = \mathbf{B}$, where
  - $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$
- and
  - $\mathbf{A} = \begin{bmatrix} 2(x_1 - x_2) & 2(y_1 - y_2) \\ 2(x_1 - x_3) & 2(y_1 - y_3) \\ \vdots & \vdots \\ 2(x_1 - x_n) & 2(y_1 - y_n) \end{bmatrix}$
  - $\mathbf{B} = \begin{bmatrix} d_2^2 - d_1^2 + x_1^2 - x_2^2 + y_1^2 - y_2^2 \\ d_3^2 - d_1^2 + x_1^2 - x_3^2 + y_1^2 - y_3^2 \\ \vdots \\ d_n^2 - d_1^2 + x_1^2 - x_n^2 + y_1^2 - y_n^2 \end{bmatrix}$
- The solution is overdefined for $n > 3$ but a least-square approximation can be found by using the pseudo-inverse of $\mathbf{A}$:
  - $\mathbf{X} = \mathbf{A}^{+} \cdot \mathbf{B} = \left(\mathbf{A}^{\intercal} \cdot \mathbf{A}\right)^{-1} \mathbf{A}^{\intercal} \cdot \mathbf{B}$

Triangulation Systems – V

- Consider a system with two beacons where the difference of distances is measured and not the absolute distances



- Beacons are separated by $2a$
- We denote the difference of measurements [seen from point (x, y)] by $2b$
- Thus, and also taking into account the figure, the following expressions can be written:
  - $r_1 - r_2 = 2b$
  - ${r_1}^2 = (x+a)^2 + y^2$
  - ${r_2}^2 = (x-a)^2 + y^2$

Triangulation Systems – V (cont.)

- Developing the previous expressions results in the following:

$$r_1 - r_2 = 2b \;, \quad b = (r_1 - r_2)/2 \;, \quad r_1 = 2b + r_2$$

$$r_1 = \sqrt{(x+a)^2 + y^2} \;, \quad r_2 = \sqrt{(x-a)^2 + y^2} \;, \quad r_1^2 = 4b^2 + r_2^2 + 4br_2$$

$$(x+a)^2 + y^2 = 4b^2 + (x-a)^2 + y^2 + 4b\sqrt{(x-a)^2 + y^2}$$

$$x^2 + a^2 + 2ax = 4b^2 + x^2 + a^2 - 2ax + 4b\sqrt{(x-a)^2 + y^2}$$

$$4ax - 4b^2 = 4b\sqrt{(x-a)^2 + y^2}$$

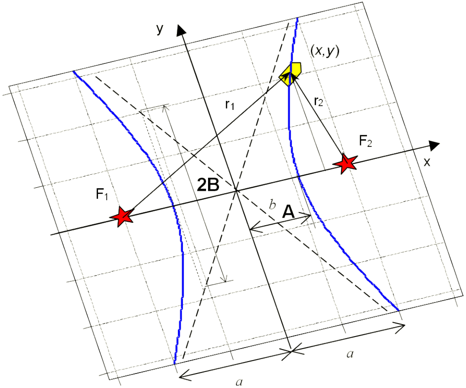
$$\left(\tfrac{a}{b}x - b\right) = \sqrt{(x-a)^2 + y^2} \;, \quad \left(\tfrac{a}{b}x - b\right)^2 = (x-a)^2 + y^2$$

$$\left(\tfrac{a}{b}\right)^2 x^2 + b^2 - 2ax = x^2 + a^2 - 2ax + y^2 \;, \quad \left(\tfrac{a^2}{b^2} - 1\right) x^2 = \left(a^2 - b^2\right) + y^2$$

$$\frac{x^2}{b^2} - \frac{y^2}{a^2 - b^2} = 1$$

Triangulation Systems – V (Hyperbolic)

- The expression obtained is the equation of a hyperbola whose generic form is given by: $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$



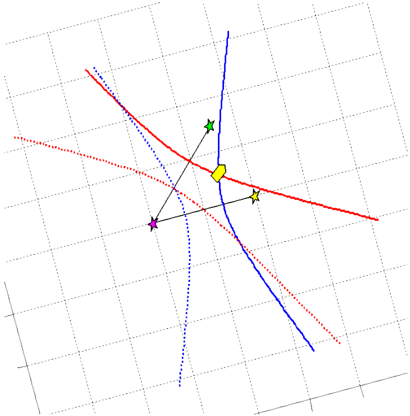
- Where $2A$ is the distance between the vertices and $2B$ is the height of the rectangle defined by the asymptotes.
- By the figure, the definition of hyperbola ensures that:

$$A = b\ , \quad 2B = 2\sqrt{a^2 - b^2}\ \frac{x^2}{b^2} - \frac{y^2}{a^2 - b^2} = 1$$

- The hyperbola indicates the set of possible locations
- Obviously a pair of beacons is insufficient, even if it is known which of the signals reaches the system first (that may indicate in which branch of the hyperbola the vehicle is)!

Triangulation Systems – V (conc.)

- In summary, more than two beacons must be used in order to have more than one difference in distances

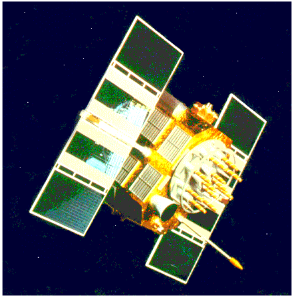


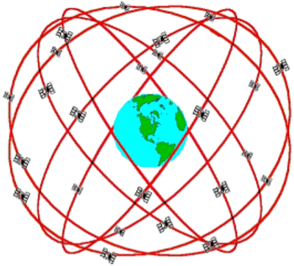
- With 2 hyperbolas there are 4 possible positions:
  - Possible ambiguity can be resolved by knowing...
  - ...which signal arrives first (distinguish the beacons from each other), and thus eliminate branches of the hyperbolas (dashed in the figure)
  - ...or other complementary information (in the case of boats, the coastline is usually a delimiter)
- With 3 hyperbolas there would be no ambiguity.
- System widely used in cellular communications systems

Global Positioning Systems

Global Positioning System - GPS

- System of 24 orbiting satellites called "Space Vehicles" – SV
- The principle is that of triangulation (trilateration) by the distances to the satellites
- They emit digital signals by radio waves in the 2 GHz band and send codes that include their position (ephemeris)
- Its distribution allows 5 to 8 satellites to be seen from any point on earth.
- New (replacement) satellites are placed occasionally





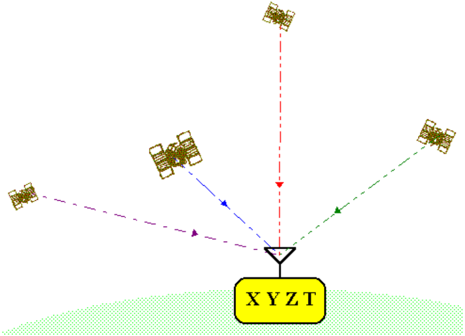
GPS Nominal Constellation
24 Satellites in 6 Orbital Planes
4 Satellites in each Plane
20,200 km Altitudes, 55 Degree Inclination

GPS – monitoring earth stations

- There is a central control station at Falcom AFB
- There are several monitoring stations spread across the planet

Principle of GPS system

- Three position variables (in 3D space): (x, y, z)
  - 3 elements of information are needed with precise synchronization between satellites and receiver.
- The 3 equations if we know the propagation time from each satellite:
  - For $i = 1, 2, 3$ the distances to the satellites are $r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}$
- But it is impractical to use atomic clocks in ordinary receivers to ensure high precise measurements of propagation times (i.e. r_i)...
- Alternative: use 4 variables (thus, 4 satellites)
  - x, y, z, T : (T- GPS time, i.e., the time frame of the satellites).
  - Details follow next...



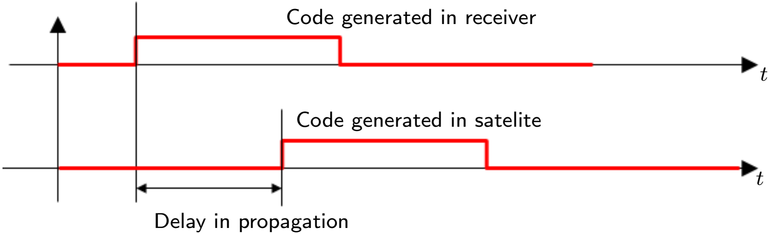
GPS - about the clocks

- Satellite clocks
  - May accumulate an error of 1 ns every 3 hours!
  - A ground station corrects this uncertainty
    - Based on a terrestrial network of several atomic clocks (more than 10)
- Receiver clocks are inaccurate
  - The drift of a common receiver clock is on the order of $1\,\mu s/s$ and therefore must be corrected (synchronized)
  - The knowledge of clocks of multiple satellites (at least 4) allows the synchronization of the local clock.
    - This adjustment is made once per second.

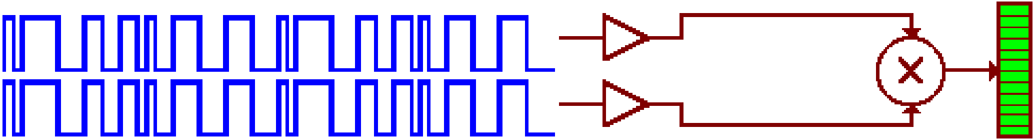
GPS - about the Codes

- Satellites periodically generate pulse sequences (codes) in which the receivers also know the structure.
  - Codes are complex and also include data from each satellite
  - Two main types of sent codes (pseudo-random)
    - C/A – equivalent to $300\,\mathrm{km}$ in length (ambiguity given that the satellite is at $20\,200\,\mathrm{km}$)
    - P – unambiguous because its repeatability corresponds to the length traveled by light in a week: $181 \times 10^9\,\mathrm{km}$

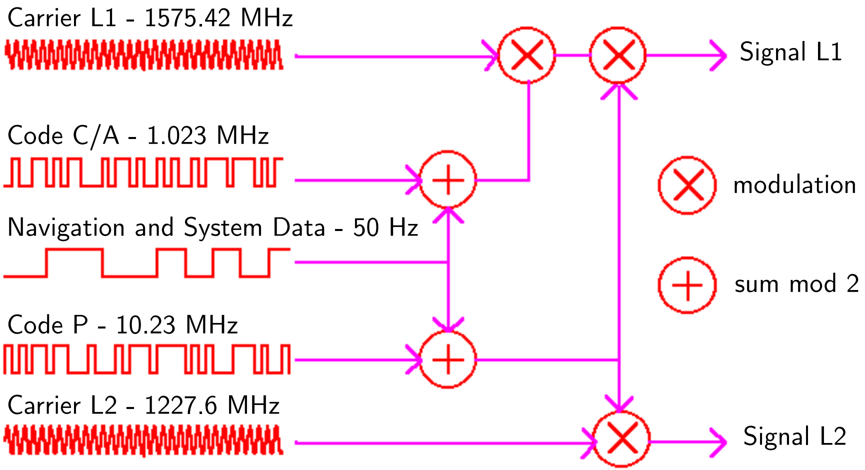
- When the systems (satellite and receiver) are synchronized (equal clocks) the code arrival delay will give the propagation time from the satellite to the receiver



- The GPS receiver reproduces the C/A and/or P codes locally for a given SV and tries to detect the sequence in the data arriving at the antenna by correlation.



GPS – Sent signals/codes



NB: Latest developments in GPS are going to provide new signals: L2C, L5 and L1C.

Information from GPS satellites

- Satellites are uniquely identified by a serial number called space vehicle number (SVN) which does not change during its lifetime.
- In addition, all operating satellites are numbered with a space vehicle identifier (SV ID) and pseudorandom noise number (PRN number) which uniquely identifies the ranging codes that a satellite uses.
- In addition to the PRN ranging codes, a receiver needs to know the time and position of each active satellite.
- GPS encodes this information into the navigation message and modulates it onto both the C/A and P(Y) ranging codes at 50 bit/s. The navigation message format is called LNAV data (for legacy navigation).
- The navigation message conveys information of three types:
  - The GPS date and time and the satellite's status.
  - The ephemeris: precise orbital information for the transmitting satellite.
  - The almanac: status and low-resolution orbital information for every satellite.

More info on the GPS system

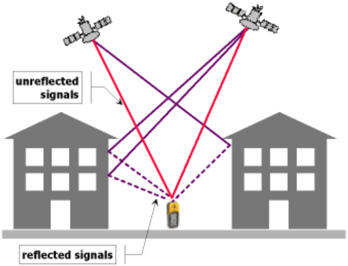
- A GPS receiver estimates the three position variables (e.g. Latitude, Longitude, and Altitude) and the time offset between the receiver's internal clock-source and the atomic clocks in the GPS satellites.
- Each GPS satellite has an atomic clock and **all** atomic clocks in the GPS satellites are synchronized periodically by the control segment of the GPS, which monitors clock errors and updates them to maintain the accuracy of the GPS system.
- Each GPS satellite transmits its own unique PRN (Pseudo Random Number) which identifies the satellite itself at the exact start of each **millisecond**.
- A GPS receiver on the surface of Earth needs at least four PRNs to fix its position.
- More details of GPS theory and operation can be found in
    - Guochang, 2003; Kaplan and Hegarty, 2005.

Calculation of GPS positioning

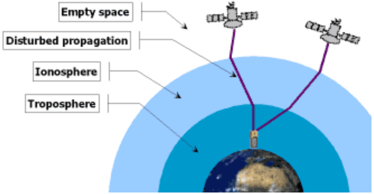
- 4 equations for four variables: $(x, y, z, \Delta t)$:
  - $(r_i - c\Delta t)^2 = (x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2$
    - with $i = 1, 2, 3, 4$
  - being
    - r_i the (pseudo-)range of satellite i extracted from the information sent by the satellite.
    - (x_i, y_i, z_i) the 3D coordinates of satellite i;
    - Δt the time difference between the satellites and receiver clocks!
    - c the speed of light ($299\,792\,458$ m/s)
- The known information is: r_i, (x_i, y_i, z_i) and c;
- The unknown information is: $(x, y, z, \Delta t)$
- The 4 equations can be solved in three main approaches:
  - Analytic closed form
  - Numerical methods
  - Using Kalman filters

Some common GPS error sources

- Multipath effect
  - Multipath effect is caused by reflection of satellite signals on objects.
  - For GPS signals this effect mainly appears near large buildings or other elevations.
  - The reflected signal takes more time to reach the receiver than the direct signal.



- Atmospheric effects
  - Influenced propagation of radio waves through the earth's atmosphere
  - This source of inaccuracy is due to a reduction of speed of propagation in the troposphere and ionosphere.



GPS - Errors

- Intentional Errors: in the codes, in the clocks, in the ephemeris;
  - Intentional errors (Selective Availability - S/A) eliminated in May 2000
- Natural: non-uniform gravitational field, sun and moon tides (well known), solar radiation pressure (difficult to model and is the largest unknown source of error), propagation in atmosphere.
- Transmission on two frequencies L1 and L2
  - Because the effects of the ionosphere on propagation (variables over an 11-year period with solar activity) affect frequencies differently.
  - Measuring the propagation difference of the two frequencies and using the mathematical model of the ionospheric delay with the frequency corrects this effect.
  - Access to L2 codes is **reserved** for the US Department of Defense.
  - Most receivers only use L1.
  - L1 occurs at the 1575.42 MHz frequency
  - L2 occurs at the 1227.60 MHz frequency

Estimated values of GPS error sources

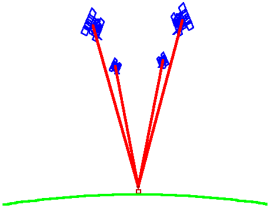
- Ionospheric effects:	± 5 meters
- Shifts in the satellite orbits:	± 2.5 meters
- Satellites' clocks errors:	± 2 meters
- Multipath effect:	± 1 meters
- Tropospheric effects:	± 0.5 meters
- Calculation and rounding errors:	± 1 meters

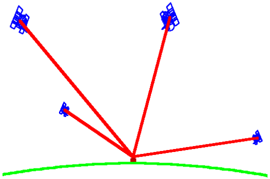
- In summary:
  - Altogether this could sum up to an error of $\pm$ 15 meters.
  - With the SA still activated, the error was in the range of ± 100 meter.
  - Corrections by systems like WAAS and EGNOS, which mainly reduce ionospheric effects, but also improve orbits and clock errors, the overall error is reduced to approximately ± 3–5 meters.

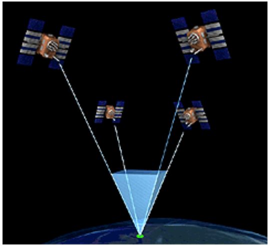
[Cf. `https://www.kowoma-gps.de/errors/`]

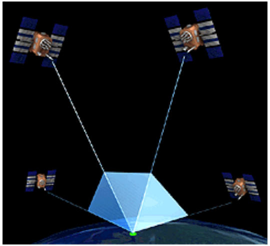
The GDOP in GPS

- The GDOP is better (smaller) the more "open" the tetrahedron defined by the four satellites and the observation point.

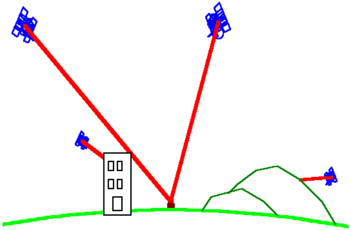






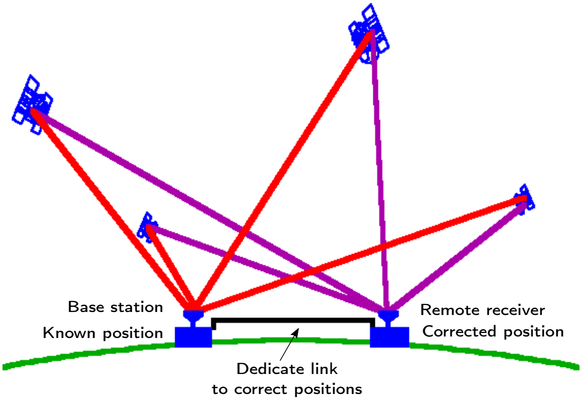


- Eample of trade-off for a favorable GDOP, but with poorer visibility at reception



Differential GPS (DGPS)

- Technique to obtain greater precision and correct errors.
- A base station and a nearby receiver (few km) are linked together (dedicate link).
- It is assumed that they receive the same errors and that the differential approach allows to eliminate, leaving only the real difference between the two points.
- In this way, a high-precision positioning relative to the base is obtained.



More about GPS and related systems

- GNSS - Global Navigation Satelite System...
- ... is the generic name of systems such as:
  - GPS (USA)
  - GLONASS (Russian Navigation Satellite System)
  - Galileo (European Navigation System)
  - Beidou (Chinese satellite navigation system)
- WAAS – Wide-Area Augmentation System
  - Uses additional satellites to send correction data such as ionospheric delays, individual satellite drifts, etc.
- EGNOS (European geostationary navigation overlay system)
  - Europe's regional satellite-based augmentation system (SBAS).
  - Used to improve the performance of GNSSs such as GPS, etc.
- Galileo Positioning System
  - European project for satellite navigation system.
  - Not yet fully operational (planned for 30 satellites)
  - Independent from the US and open for all military and civilian uses.

Comparison of dead-reckoning e absolute positioning

Attribute	Dead-reckoning	Absolute positioning
Process	Integration of kinematic variables	Solving nonlinear systems of equations
Requires Initial Conditions?	Yes	No
Errors	Time Dependent	Position Dependent
Update frequency	Determined by desired accuracy	Determined only by update requirements
Error propagation	Depends on previous estimates	Does not depend on previous estimates
Requires special marks in the environment?	No	Yes