Robótica Móvel

Locomotion - Inverse kinematics

Adapted from the RAM course notes

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Summary

Direct and Inverse Kinematics

Direct and Inverse Kinematics in Mobile Robots

- Direct kinematics provides the velocity (and ultimately the position by integration) of the robot considering the internal actuation (wheel velocities and steering) and mechanical parameters (wheel radius, wheel separation or distance to local coordinate frame origin, etc.)
  - $\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\theta}(t) \end{bmatrix} = \mathbf{f}(\omega_1(t), \omega_2(t), \cdots, \alpha(t))$
- Inverse kinematics concerns the determination of the robot actuated variables (wheel velocities and steering) to accomplish a given path/trajectory/posture.
  - $\begin{bmatrix} \omega_1(t) \\ \omega_2(t) \\ \vdots \\ \alpha(t) \end{bmatrix} = \mathbf{g}(x(t), y(t), \theta(t))$

Reminder of Local and Global Robot Kinematics

- A robot has its local kinematics (3 velocity expressions) that depend on individual wheel angular velocity, wheel radius and steering orientation (when applicable):
  - V_x
  - V_y
  - ω
- The velocity seen in the global reference frame is given by:
  - $$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ \omega \end{bmatrix} = \mathbf{R}^{-1}(\theta) \begin{bmatrix} V_x \\ V_y \\ \omega \end{bmatrix}$$
- And conversely, velocity in local frame relates to velocity seen from gobal frame as:
  - $$\begin{bmatrix} V_x \\ V_y \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \omega \end{bmatrix} = \mathbf{R}(\theta) \begin{bmatrix} \dot{x} \\ \dot{y} \\ \omega \end{bmatrix}$$
- It must be noticed that all these quantities $(V_x, V_y, \theta, x, y, \omega)$ depend on time t.

Kinematics Matrix of a mobile robot

Summary of direct kinematics of common robots

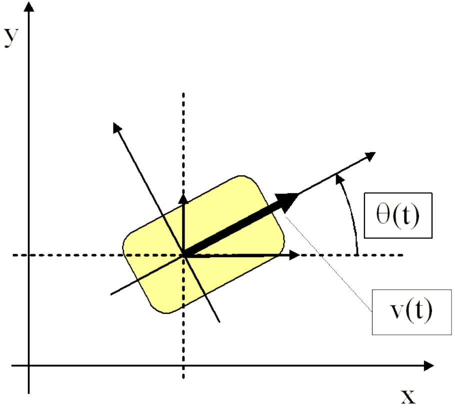
- In each type of robot, **local** velocity depends on wheel parameters:
  - $\begin{bmatrix} V_x & V_y & \omega \end{bmatrix}^\intercal = \mathbf{f}(\omega_i, r_i, \alpha_i, L_i),\ i = \{1, 2, ...\}$
    - $\omega_i \rightarrow$ angular velocity of wheel i
    - $r_i \rightarrow$ radius of wheel i
    - $\alpha_i \rightarrow$ Steering direction of wheel i (when applicable)
    - $L_i \rightarrow$ Wheel distance in local frame (meaning varies with topologies)
- Or in general matrix form:
  - $\begin{bmatrix} V_x \\ V_y \\ \omega \end{bmatrix} = \mathbf{M} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \end{bmatrix}$
- Where $\mathbf{M}$ has values that depend on individual robot tipology like:
  - Differential drive, Tricycle, Omnidirectional, etc.
  - $\mathbf{M}$ depends on fixed parameters, but may also depend on variable parameters (like steering angles).
  - We may call $\mathbf{M}$ the **kinematics matrix** of the robot (though unusual!)

Examples of $\mathbf{M}$ for several robots

- Differential drive
  - $$\begin{bmatrix} V_x(t) \\ V_y(t) \\ \omega(t) \end{bmatrix} = \begin{bmatrix} r/2 & r/2 \\ 0 & 0 \\ -r/L & r/L \end{bmatrix} \begin{bmatrix} \omega_L(t) \\ \omega_R(t) \end{bmatrix}$$
- Tricycle
  - $$\begin{bmatrix} V_x(t) \\ V_y(t) \\ \omega(t) \end{bmatrix} = \begin{bmatrix} r\cos\alpha(t) \\ 0 \\ r/L\sin\alpha(t) \end{bmatrix} \omega_S(t)$$
- Omnidirectional
  - $$\begin{bmatrix} V_x(t) \\ V_y(t) \\ \omega(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{r}{\sqrt{3}} & -\frac{r}{\sqrt{3}} \\ -\frac{2r}{3} & \frac{r}{3} & \frac{r}{3} \\ \frac{r}{3L} & \frac{r}{3L} & \frac{r}{3L} \end{bmatrix} \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \end{bmatrix}$$

Generic expression of direct kinematics for Diferencial Drive and similars

- Obtain global coordinates of robot after the controlled variables (velocities):
  - $\dot{x}(t) = V(t)\cos\theta(t)$
  - $\dot{y}(t) = V(t)\sin\theta(t)$
  - $\dot{\theta}(t) = \omega(t)$
- By integration we obtain:
  - $x(t) = \int_0^t V(\tau)\cos\theta(\tau)\mathrm{d}\tau + x_0$
  - $y(t) = \int_0^t V(\tau)\sin\theta(\tau)\mathrm{d}\tau + y_0$
  - $\theta(t) = \int_0^t \omega(\tau)\mathrm{d}\tau + \theta_0$
    - Where (x_0, y_0, θ_0) is the initial posture.



Implementing Inverse Kinematics

How to perform inverse kinematics?

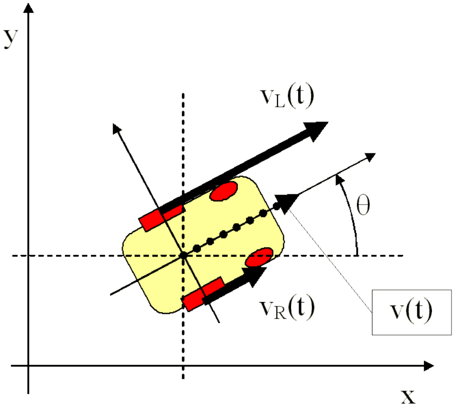
- Given a trajectory $(x(t), y(t), \theta(t))$ the challenge is then to solve or manipulate the previous equations and obtain:
  - $V(t)$
  - $\omega(t)$
- and then obtain the individual wheel angular velocities (and steering angles when applicable)
  - $\omega_1(t)$
  - $\omega_2(t)$
  - etc.

Challenge in performing inverse kinematics

- Problem in determining which wheel velocities (like $V_L(t)$ and $V_R(t)$ in differential drive robots) are required to obtain desired paths for $x(t)$, $y(t)$ and $\theta(t)$
- Complex issue for several reasons:
  - Cases of coupled inputs
    - Three inputs (x, y, θ) for only two independent controlled variables, (like V_Ls and V_R in differential drive or V_S and α in tricycle), although in ominiriectional robot this is not a problem!
  - Kinematic functions are not integrable in the general case.
  - There is an infinite number of solutions in many cases.
- Only a few simple cases can be solved, namely if the speeds are constant like these examples in differential drive:
  - Different velocities (circular movement, appropriate x and y)
  - Equal velocities (straight line movement)
  - Symmetrical velocities (rotating and turning itself in the case of the differential robot)

Particular case of differential drive robot

- Following the generic expression presented earlier we have:
  - $x(t) = \frac{1}{2}\int_0^t [V_R(\tau) + V_L(\tau)] \cos\theta(\tau) \mathrm{d}\tau$
  - $y(t) = \frac{1}{2}\int_0^t [V_R(\tau) + V_L(\tau)] \sin\theta(\tau) \mathrm{d}\tau$
  - $\theta(t) = \frac{1}{L}\int_0^t [V_R(\tau) - V_L(\tau)] \mathrm{d}\tau$



Simple case of inverse kinematics in differential drive

- Consider a differential drive robot with wheel velocities V_L and V_R and wheel separation L, where $\omega = \frac{V_R - V_L}{L}$ and $V = \frac{V_L + V_R}{2}$;
- If we restrict to: $V_L(t) = V_L$ (const.) and $V_R(t) = V_R$ (const.) and $V_L \neq V_R$ we have the following:
  - $\theta(t) = \int_0^t \omega \mathrm{d}\tau = \omega \int_0^t \mathrm{d}\tau = \omega t$
  - $x(t) = V \int_0^t \cos(\omega\tau)\mathrm{d}\tau = \frac{V}{\omega}\sin(\omega t)$
  - $y(t) = V \int_0^t \sin(\omega\tau)\mathrm{d}\tau = -\frac{V}{\omega}\left[\cos(\omega\tau)\right]_0^t = \frac{V}{\omega}\left(1 - \cos(\omega t)\right)$
- We assumed that $(x_0, y_0, \theta_0) = (0, 0, 0)$
- Note: In the previous expressions, the three coordinates $x(t), y(t), \theta(t)$ cannot be specified independently. Only two of them can be set independently and the third will be derived from them:
  - $x(t), y(t) \rightarrow \theta(t)$
  - $\theta(t), x(t) \rightarrow y(t)$
  - $\theta(t), y(t) \rightarrow x(t)$

Example exercise of a specific calculation

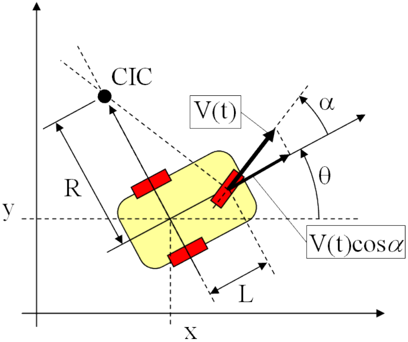
- It is intended that a differential drive robot, with $L = 50\,\text{cm}$ between wheels, starting at (0,0,0) arrives in $t = 5\,\text{s}$, $2\,\text{m}$ further ahead (x) with a final orientation (θ) of $+30°$. What should be the constant speeds of the two wheels, V_L and V_R?
- Base expressions for differential drive in these particular conditions:
  - $\theta(t) = \omega t$
  - $x(t) = \frac{V}{\omega}\sin(\omega t)$
  - $y(t) = \frac{V}{\omega}(1 - \cos(\omega t))$
  - $\omega = \frac{V_R - V_L}{L}$
  - $V = \frac{V_L + V_R}{2}$
- Calculations are:
  - $\theta(5) = 5\omega \Leftrightarrow \frac{\pi}{6} = 5\omega \Leftrightarrow \omega = \frac{\pi}{30}$
  - $x(5) = 2 \Leftrightarrow 2 = \frac{V}{\omega}\sin(5\omega) \Leftrightarrow V = 2\pi/15$
  - $\begin{cases} \frac{V_R+V_L}{2} = \frac{2\pi}{15} \\ \frac{V_R-V_L}{0.5} = \frac{\pi}{30} \end{cases}, \begin{cases} V_R + V_L = \frac{4\pi}{15} \\ V_R - V_L = \frac{\pi}{60} \end{cases}, \begin{cases} 2V_R = \frac{17\pi}{60} \\ V_L = V_R - \frac{\pi}{60} \end{cases}$
- And finally: $V_R \approx 0.445\,\text{m/s}$, $V_L \approx 0.393\,\text{m/s}$.
- Notice that, as previously mentioned, we can not impose simultaneously $y(t)$ because variables are coupled!

Variant of the previous example

- What if the geometric specification would be in $y(t)$ and not in $x(t)$? For example, $y(5) = 1\,\text{m}$?
- Using the equations presented before ($V = \frac{V_R+V_L}{2}$, $\omega = \frac{V_R-V_L}{0.5}$), the calculations would be as follows:
  - $\theta(5) = 5\omega \Leftrightarrow \frac{\pi}{6} = 5\omega \Leftrightarrow \omega = \frac{\pi}{30}$ (the same as before)
  - $y(5) = \frac{V}{\omega}\left(1 - \cos(5\omega)\right) = 1 \Leftrightarrow \omega = V\left(1 - \frac{\sqrt{3}}{2}\right) \approx 0.13397V$
  - $\begin{cases} V_R + V_L \approx 1.5632 \\ V_R - V_L \approx 0.0524 \end{cases}$
- And finally: $V_R \approx 0.8078\,\text{m/s}$, $V_L \approx 0.7554\,\text{m/s}$.
  - These velocities are slightly different from the earlier case, which is expectable because the final position is different, although the orientation is the same.

Inverse kinematics of a tricycle

- The direct kinematics of the tricycle is conceptually simpler than the differential drive:
  - $$\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\\ \dot{\theta}(t)\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} r\cos\alpha(t)\\ 0\\ r/L\sin\alpha(t)\end{bmatrix}\omega_S(t)$$
- Or, expanding [being $V(t) = \omega_S(t)r$]:
  - $$\begin{cases}\dot{x}(t) = V(t)\cos\alpha(t)\cos\theta(t)\\ \dot{y}(t) = V(t)\cos\alpha(t)\sin\theta(t)\\ \dot{\theta}(t) = \frac{V(t)}{L}\sin\alpha(t)\end{cases}$$
- Giving the following inverse kinematics
  - $$\begin{cases}\theta(t) = \frac{1}{L}\int\limits_0^t V(\tau)\sin\alpha(\tau)\mathrm{d}\tau\\ x(t) = \int\limits_0^t V(\tau)\cos\alpha(\tau)\cos\theta(\tau)\mathrm{d}\tau\\ y(t) = \int\limits_0^t V(\tau)\cos\alpha(\tau)\sin\theta(\tau)\mathrm{d}\tau\end{cases}$$



Inverse kinematics of tricycle

- It is clear that these equations are unsolvable analitically for the general case
  - $$\begin{cases} \theta(t) = \frac{1}{L} \int\limits_0^t V(\tau) \sin\alpha(\tau) \mathrm{d}\tau \\ x(t) = \int\limits_0^t V(\tau) \cos\alpha(\tau) \cos\theta(\tau) \mathrm{d}\tau \\ y(t) = \int\limits_0^t V(\tau) \cos\alpha(\tau) \sin\theta(\tau) \mathrm{d}\tau \end{cases}$$
- But if velocity and steering angle are made constant, some solutions are possible (but of course, it is still impossible to impose all three variables x, y and θ at the same time to calculate V and α!).
  - $$\begin{cases} \theta(t) = \frac{V}{L} \sin\alpha \int\limits_0^t \mathrm{d}\tau = t\frac{V}{L} \sin\alpha \\ x(t) = V \cos\alpha \int\limits_0^t \cos\theta(\tau) \mathrm{d}\tau \\ y(t) = V \cos\alpha \int\limits_0^t \sin\theta(\tau) \mathrm{d}\tau \end{cases}$$

The kinematics controller

- What has been presented is called the kinematics control of a robot:
- This usage of inverse kinematics is only applicable in simplified situations;
- Path divided in segments (linear and circular, mostly);
- Each segment is executed as the examples given.

- But this aproach for trajectory planning has some disadvantages (Siegwart et al. 2011):
    - It is not at all an easy task to precompute a feasible trajectory if all limitations and constraints of the robot's velocities and accelerations have to be considered.
    - The robot will not automatically adapt or correct the trajectory if dynamic changes of the environment occur.
    - The resulting trajectories are usually not smooth, because the transitions from one trajectory segment to another are, for most of the commonly used segments (e.g., lines and part of circles), not smooth. This means there is a discontinuity in the robot's acceleration.