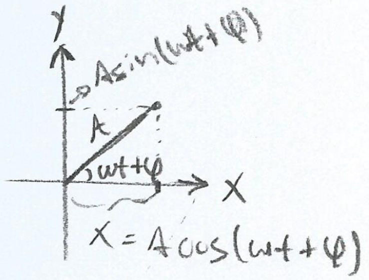
MHS e MCU

• $x(t) = A\cos(\omega t + \varphi)$

↳ Notação complexa

↳ Superposição de MHS



$$\vec{r}\begin{cases} r = cte. \\ \dot{\varphi} = \dfrac{d\varphi}{dt} = \omega = cte. \end{cases}$$

$$\Rightarrow \begin{array}{l} \ddot{x} = -\omega^2 x \Rightarrow \\ \ddot{y} = -\omega^2 y \Rightarrow \end{array} \begin{cases} \ddot{x} + \omega^2 x = 0 \\ \ddot{y} + \omega^2 y = 0 \end{cases} \Rightarrow$$

$$\Rightarrow (\ddot{x} + i\ddot{y}) + \omega^2 (x + iy) = 0 \;;\; z = x + iy \Rightarrow \boxed{\ddot{z} + \omega^2 z = 0}$$

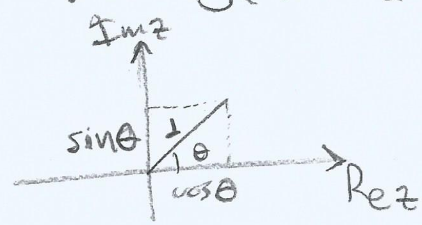
• $\ddot{z} = -\omega^2 z \overset{\text{Hipótese}}{\Rightarrow} z = z_0 e^{\rho t} \Rightarrow \dot{z} = \rho z \Rightarrow \ddot{z} = \rho^2 z;$

$(\rho^2 + \omega^2) z = 0 \Rightarrow \rho^2 + \omega^2 = 0 \Rightarrow \rho = \pm i\omega \Rightarrow z(t) = z_0 e^{\pm i\omega t};$

$f(\theta) \Rightarrow \dfrac{df}{d\theta} = i f(\theta) \Rightarrow f(\theta) = e^{i\theta} + K;\; g(\theta) = \cos(\theta) + i\sin(\theta)$

$\dfrac{dg}{d\theta} = -\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta) = i g(\theta) \;\therefore\; f(\theta) = g(\theta) + K_1$

$\Rightarrow f(0) = 1 + \cancel{K}^{0} \Rightarrow g(0) = 1;\; e^{i\theta} = \cos\theta + i\sin\theta$



$$\boxed{\begin{array}{l} z = a + ib = re^{i\theta} \\ a = r\cos\theta \\ b = r\sin\theta \\ r = \sqrt{a^2 + b^2} \\ \tan\theta = \dfrac{b}{a} \end{array}}$$

$$\boxed{e^{i\pi} = -1}$$

$\Rightarrow e^{-i\theta} = \cos\theta - i\sin\theta = (e^{i\theta})^*;$

$z = re^{i\theta} \Rightarrow |z|^2 = z \cdot z^* = r^2 \cdot e^{i\theta} \cdot e^{-i\theta} = r^2$

$z^* = a - ib$ $\quad\boxed{|z| = r}$

• $z(t) = z_0 e^{\pm i\omega t};\; z_0 = Ae^{i\varphi} \Rightarrow z(t) = A e^{\pm i(\omega t \pm \varphi)};$

$x(t) = \mathrm{Re}\, z \Rightarrow x(t) = A\cos(\omega t + \varphi)$

Funções Trigonométricas Hiperbólicas

$$\begin{cases} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{cases} \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\boxed{\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2} \qquad \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}}$$

↳ Superposição de MHS

⇒ EDO, 2ª ordem, linear, coefs ctes, homogênea

∘ Se x_1 e x_2 soluções ⇒ $ax_1 + bx_2$ é solução

$$* \begin{cases} x_1 = A_1\cos(\omega t + \varphi_1) \\ x_2 = A_2\cos(\omega t + \varphi_2) \end{cases} \quad x_1 + x_2 = \mathrm{Re}\,z_1 + \mathrm{Re}\,z_2 = \mathrm{Re}(z_1 + z_2)$$

$$\begin{cases} z_1 = A_1 e^{i(\omega t + \varphi_1)} \\ z_2 = A_2 e^{i(\omega t + \varphi_2)} \end{cases} \Rightarrow \boxed{z = e^{i(\omega t + \varphi_1)}\left[A_1 + A_2 e^{i(\varphi_2 - \varphi_1)}\right]}$$

$$\Rightarrow z = A e^{i\beta} e^{i(\omega t + \varphi_1)} = A e^{i(\omega t + \varphi_1 + \beta)} \Rightarrow x(t) = A\cos(\omega t + \varphi_1 + \beta)$$

$$Ae^{i\beta} = A_1 + A_2 e^{i(\varphi_2 - \varphi_1)} \overset{\cdot z^*}{\Rightarrow} A^2 = A_1^2 + A_2^2 + A_1A_2\left(e^{i(\varphi_2-\varphi_1)} + e^{-i(\varphi_2-\varphi_1)}\right)$$

$$= \boxed{A^2 = A_1^2 + A_2^2 + 2A_1 \cdot A_2 \cdot \cos(\varphi_2 - \varphi_1);\ (A_1 - A_2)^2 \le A^2 \le (A_1 + A_2)^2}$$

∘ $\beta = ?$ ⇒ $A\sin\beta = A_2\sin(\varphi_2 - \varphi_1)$ ⇒ $\boxed{\sin\beta = \frac{A_2}{A}\sin(\varphi_2 - \varphi_1)}$

→ Casos Particulares

↳ $\varphi = 0 \Rightarrow \frac{x^2}{A^2} + \frac{y^2}{B^2} - 2\frac{x}{A}\cdot\frac{y}{B} = 0 = \left(\frac{x}{A} - \frac{y}{B}\right)^2 \Rightarrow \frac{x}{A} = \frac{y}{B}$

↳ $\varphi = \pi \Rightarrow \frac{x^2}{A^2} + \frac{y^2}{B^2} + 2\frac{x}{A}\cdot\frac{y}{B} = 0 = \left(\frac{x}{A} + \frac{y}{B}\right)^2 \Rightarrow \frac{x}{A} = -\frac{y}{B}$

↳ $\varphi = \frac{\pi}{2} \Rightarrow \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ ↳ $\varphi = \frac{3\pi}{2} \Rightarrow \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \Rightarrow$ MCU, $A = B$

- $x = A\cos(wt)$, $y = B\sin(wt)$, $\left(\varphi = \frac{\pi}{2}\right)$
- $x = A\cos(wt)$, $y = -B\sin(wt)$, $\left(\varphi = \frac{3\pi}{2}\right)$

→ Força Restauradora em 2D

$$\vec{F} = -k_x \cdot x \cdot \hat{i} - k_y \cdot y \cdot \hat{j} \Rightarrow \begin{cases} m\ddot{x} + k_x \cdot x = 0 \Rightarrow \ddot{x} + w_x^2 \cdot x = 0 \\ m\ddot{y} + k_y \cdot y = 0 \Rightarrow \ddot{y} + w_y^2 \cdot y = 0 \end{cases}$$

$\begin{cases} x = A\cos(w_x \cdot t) \\ y = B\cos(w_y \cdot t) \end{cases}$, $\frac{w_x}{w_y} \in \mathbb{Q} \Leftrightarrow$ Periódico com trajetória fechada

↳ figuras de Lissajour

Superposição de MHS

Richard Feiman

⇒ Mesma direção e frequências diferentes

⇒ Direções ortogonais $\begin{cases} \text{mesma frequência} \\ \text{frequências diferentes} \end{cases}$

$$\begin{cases} x_1(t) = A_1 \cos(\omega_1 t + \varphi_1) \rightarrow \ddot{x}_1 + \omega_1^2 x_1 = 0 \\ x_2(t) = A_2 \cos(\omega_2 t + \varphi_2) \rightarrow \ddot{x}_2 + \omega_2^2 x_2 = 0 \end{cases} \qquad \ddot{x} + \omega^2 x = 0$$

⇒ x_1 é periódico: τ_1, x_2 é periódico: τ_2 ∴ $\boxed{m\tau_1 = n\tau_2}$

⇔ $x = x_1 + x_2$ ⇔ $\boxed{\frac{\tau_1}{\tau_2} \in \mathbb{Q}}$; Não periódico ⇔ $\boxed{\frac{\tau_1}{\tau_2} \notin \mathbb{Q}}$

⇒ não comensuráveis

→ Caso de interesse $\omega_1 \approx \omega_2$

⇒ $|\omega_1 - \omega_2| \ll \omega_1, \omega_2$

* Restrições: $\begin{cases} \theta_1(t) = \omega_1 t + \varphi_1 \\ \theta_2(t) = \omega_2 t + \varphi_2 \end{cases}$ ⇒

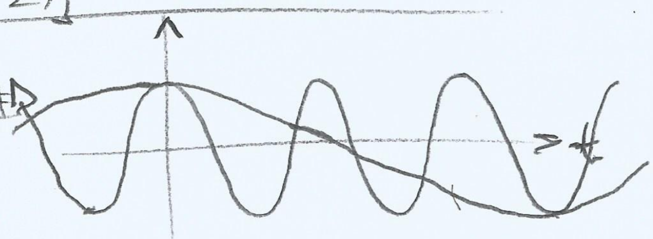
⇒ $\theta_2 - \theta_1 = (\omega_2 - \omega_1)t + \varphi_2 - \varphi_1$; $A_1 = A_2 = A$;

$$\begin{cases} \Delta\omega = \omega_2 - \omega_1 \quad \boxed{\text{hipótese } (\omega_2 > \omega_1)} \\ \bar{\omega} = \dfrac{\omega_1 + \omega_2}{2} \end{cases} \quad \begin{cases} x_1 = A\cos\left[\left(\bar{\omega} - \frac{\Delta\omega}{2}\right)t\right] \\ x_2 = A\cos\left[\left(\bar{\omega} + \frac{\Delta\omega}{2}\right)t\right] \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_1 = A\left[\cos(\bar{\omega}t)\cos\left(\frac{\Delta\omega}{2}\right) + \sin(\bar{\omega}t)\sin\left(\frac{\Delta\omega}{2}\right)\right] \\ x_2 = A\left[\cos(\bar{\omega}t)\cos\left(\frac{\Delta\omega}{2}\right) - \sin(\bar{\omega}t)\sin\left(\frac{\Delta\omega}{2}\right)\right] \end{cases} ; \quad x = x_1 + x_2 \Rightarrow$$

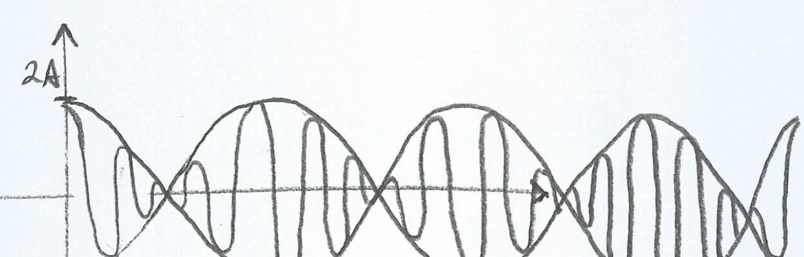
$$\Rightarrow x = 2A\cos(\bar{\omega}t)\cos\left(\frac{\Delta\omega}{2}\right)$$

$\boxed{\Delta\omega \ll \bar{\omega}}$ ⇒



$x = a(t)\cos(\bar{\omega}t)$ ∴

⇒

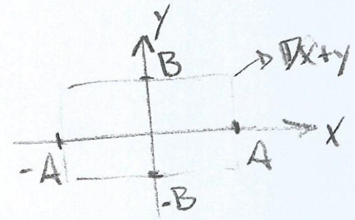


⇒ $\boxed{\text{Batimento}}$

→ Força Restauradora em 2D:

$$\vec{F} = -K\vec{r} = -K(x\hat{i} + y\hat{j}) \Rightarrow m\ddot{\vec{r}} = -K\vec{r} \quad \begin{cases} m\ddot{x} + Kx = 0 \\ m\ddot{y} + Ky = 0 \end{cases}$$

$$\begin{cases} x = A\cos(\omega t + \varphi_x) \\ y = B\cos(\omega t + \varphi_y) \end{cases}$$



sistema deve percorre uma trajetória no plano xy

⊳ Eliminar t: $\frac{x}{A} = \cos(\omega t)$; $\frac{y}{B} = \cos(\omega t)\cos(\varphi) - \sin(\omega t)\sin(\varphi) \Rightarrow$

⊳ elipse?

$$\Rightarrow \begin{cases} \frac{y}{B} = \frac{x}{A}\cos(\varphi) \pm \sqrt{1 - \frac{x^2}{A^2}}\sin(\varphi) \\ \frac{y^2}{B^2} = \frac{x^2}{A^2}\cos^2(\varphi) + \left(1 - \frac{x^2}{A^2}\right)\sin^2(\varphi) \pm 2\frac{x}{A}\sqrt{1 - \frac{x^2}{A^2}}\sin(\varphi)\cos(\varphi) \end{cases}$$

$$\Rightarrow \frac{y^2}{B^2} - 2\frac{x}{A}\cdot\frac{y}{B}\cos(\varphi) = \frac{y^2}{B^2} - 2\frac{x^2}{A^2}\cos^2(\varphi) \pm 2\frac{x}{A}\sqrt{1 - \frac{x^2}{A^2}}\sin(\varphi)\cos(\varphi) \Rightarrow$$

$$\Rightarrow \frac{x^2}{A^2}\cos^2(\varphi) - \frac{2x^2}{A^2}\cos^2(\varphi) + \left(1 - \frac{x^2}{A^2}\right)\sin^2(\varphi) \pm 2\frac{x}{A}\sqrt{\cdots}\cdots \pm 2\frac{x}{A}\sqrt{\cdots} =$$

$$= -\frac{x^2}{A^2} + \sin^2(\varphi) \Rightarrow \boxed{\frac{y^2}{B^2} + \frac{x^2}{A^2} - 2\frac{x}{A}\cdot\frac{y}{B}\cos(\varphi) = \sin^2(\varphi)}$$

→ Energia no MHS: $\boxed{E = \frac{1}{2}m\omega^2 A^2}$

$$\frac{1}{2}mV_{max}^2 = \frac{1}{2}m\omega^2 A^2 \Rightarrow \boxed{V_{max} = \omega A}$$

Balanço de Energia

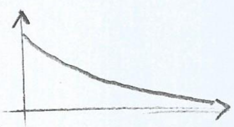
→ Energia Armazenada: $\boxed{E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2}$

$$\boxed{\frac{dE}{dt} = m \ddot{x} \cdot \dot{x} + m \omega_0^2 \dot{x} \cdot x = - F_\alpha \cdot v = - \alpha \dot{x}^2} \Rightarrow$$

$\Rightarrow \dot{x}(m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x) = 0 \hookrightarrow \frac{dE}{dt} \neq 0 \Rightarrow E = E(t)$

→ Caso Supercrítico: $x(t) = a e^{-(\frac{\gamma}{2} - \beta)t} + b e^{-(\frac{\gamma}{2} + \beta)t} \Rightarrow$

$\Rightarrow E(t) \sim e^{-(\gamma - 2\beta)t} \rightarrow$



→ Crítico: $E(t) \sim e^{-\gamma t} \nearrow$

→ Subcrítico: $x(t) = A e^{-\frac{\gamma}{2}t} \cos(\omega t + \varphi)$; $\dot{x}(t) = -\frac{\gamma}{2} x(t) - \omega A e^{-\frac{\gamma}{2}t} \sin(\omega t + \varphi)$

$$E(t) = \frac{1}{2} m A^2 e^{-\gamma t} \left[\left(\omega_0^2 + \frac{\gamma^2}{4}\right) \cos^2(\omega t + \varphi) + \omega^2 \sin^2(\omega t + \varphi) + \right.$$
$$\left. + 2 \frac{\gamma}{2} \cdot \omega \sin(\omega t + \varphi) \cos(\omega t + \varphi) \right]$$

Especial Interesse $\frac{\gamma}{2} << \omega_0$; $e^{-\gamma t}$ varia pouco comparado com $\sin(\omega t + \varphi)$, $\cos(\omega t + \varphi)$

$\boxed{\omega^2 = \omega_0^2 - \frac{\gamma^2}{4}}$

° $\bar{E}(t) = \frac{1}{\tau} \int_t^{t+\tau} E(t') dt' \approx e^{-\gamma t} \frac{1}{\tau} \int (\ ) \cos^2 + (\ ) \sin^2 + (\ ) \sin \cos$

$\bar{E}(t) \approx \frac{1}{2} m A^2 e^{-\gamma t} \left[\frac{1}{2} \left(\omega_0^2 + \frac{\gamma^2}{4}\right) + \frac{1}{2} \omega^2 \right] \Rightarrow \boxed{\bar{E}(t) = \frac{1}{2} m \omega_0^2 A^2 e^{-\gamma t}}$

$\Rightarrow E(0) e^{-\gamma t}$

$$\begin{cases} z = Ae^{i\varphi}e^{i\omega t} \\ x = \operatorname{Re} z \Rightarrow x(t) = A\cos(\omega t + \varphi) = A(\omega)\cos(\omega t + \varphi(\omega)) \end{cases}$$

$$x(t) = x_{transiente}(t) + x_p(t); \quad \text{para } \underbrace{t \gg \frac{1}{\gamma}}_{x_{trans} \to 0} \Rightarrow x(t) \to x_p(t)$$

↳ Hipótese 1: $x(t) = \underset{\downarrow \atop A(\Omega)}{A} \cos(\Omega t + \underset{\downarrow \atop \varphi(\Omega)}{\varphi})$

↳ Hipótese 2: $z(t) = x(t) + i y(t)$

$$\mathcal{F}(t) = F(t) + i F^{im}(t)$$

$$\mathcal{F}(t) = F_0 e^{i\omega t}$$

$$\ddot{z} + \gamma \dot{z} + \omega_0^2 z = \frac{\mathcal{F}(t)}{m} \quad \begin{cases} \ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F(t)}{m} \\ \ddot{y} + \gamma \dot{y} + \omega_0^2 y = \frac{F^{im}(t)}{m} \end{cases}$$

<u>Tentativa</u>: $z(t) = z_0 e^{i\omega t} \Rightarrow \dot{z} = i\Omega z, \; \ddot{z} = -\Omega^2 z \Rightarrow$

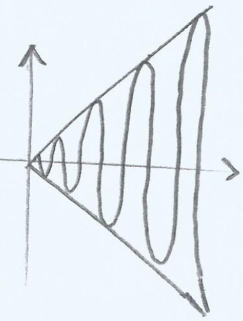
$$\Rightarrow (-\Omega^2 + i\gamma\Omega + \omega_0^2) z_0 e^{i\Omega t} = \frac{F_0}{m} e^{i\omega t} \Rightarrow z_0 = \frac{F_0}{m[(\omega_0^2 - \omega^2) + i\Omega\gamma]} = Ae^{i\varphi}$$

$$z_0 = Ae^{i\varphi} \Rightarrow z_0 \cdot z_0^* = Ae^{i\varphi} \cdot Ae^{-i\varphi} \Rightarrow A^2 = |z_0|^2$$

Solução Geral (Ajustes das condições iniciais)

$$\boxed{x(t) = a\cos(w_0 t) + b\sin(w_0 t) + \frac{F_0}{m(w_0^2 - w^2)}\cos(wt)}$$

* Cond. Iniciais: $x(0) = 0$; $\dot{x}(0) = 0$



$$x(0) = a\cos(w_0 \cdot 0) + b\sin(w_0 \cdot 0) + \frac{F_0}{m(w_0^2 - w^2)}\cos(w_0 \cdot 0)$$

$$\boxed{x(t) = t\sin(wt)}$$

$$\Rightarrow a = -\frac{F_0}{m(w_0^2 - w^2)}$$

$$\dot{x}(0) = -w_0 \cdot a \cdot \sin(0) + w_0 \cdot b \cdot \cos(0) - \frac{F_0 \cdot w}{m(w_0^2 - w^2)}\sin(0)$$

$$x(t) = \frac{-F_0}{m(w_0 + w)}\left[\frac{\cos(wt) - \cos(w_0 t)}{w - w_0}\right] ; \lim_{w \to w_0}\left[\frac{\cos(wt) - \cos(w_0 t)}{w - w_0}\right] =$$

$$x(t) = \frac{d}{dw}\cos(wt)\Big|_{w = w_0} ; \quad \boxed{\lim_{w \to w_0} x(t) = \frac{F_0 \cdot t}{m(w + w_0)}\sin(wt)}$$

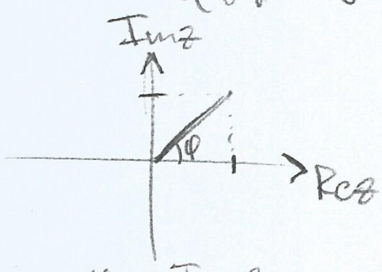
$\triangleright$ Def: $z(t) = x(t) + i(t)$, $\mathcal{F}(t) = F(t) + i F_{im}(t)$;

$$\ddot{z} + \gamma\dot{z} + w_0^2 z = \frac{\mathcal{F}(t)}{m} \Rightarrow \begin{cases} \ddot{x} + \gamma\dot{x} + w_0^2 x = \frac{F(t)}{m} \\ \ddot{y} + \gamma\dot{y} + w_0^2 y = \frac{F_{im}(t)}{m} \end{cases}$$

$$\begin{cases} \mathcal{F}(t) = F_0 e^{iwt} \\ \ddot{z} + \gamma\dot{z} + w_0^2 z = \frac{F_0}{m} e^{iwt} \end{cases}$$

Hip: $z = z_\rho e^{iwt} \Rightarrow (-w^2 + i\gamma w + w_0^2) z_\rho e^{iwt} = \frac{F_0}{m} e^{iwt}$

$$z_\rho = \frac{F_0}{m(w_0^2 + i\gamma w - w^2)} \in \mathbb{C} = Ae^{i\varphi} \Rightarrow |z_\rho|^2 = A^2 = \frac{F_0}{m^2(w_0^2 - w^2)^2 + \gamma^2 w^2}$$



$z_\rho = \frac{F_0}{m} z_2^{-1}$; $z_2 = w_0^2 - w^2 + i\gamma w$

$z_\rho = \frac{F_0}{m|z_2|} e^{-i\theta_2}$; $z_2 = |z_2| e^{i\theta_2}$

$$z = \frac{z_1}{z_2} = \frac{A_1 e^{i\varphi_1}}{A_2 e^{i\varphi_2}}$$

$\tan\varphi = \frac{Im z}{\ldots}$ $\quad \ldots e^{i\varphi} \Rightarrow \varphi = -\theta_2 = -\tan^{-1}\left(\frac{\gamma w}{\ldots}\right)$

Oscilações Forçados sem Amortecimento

$$\boxed{x = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)}$$

↳ Força Restauradora: $F_{res} = -Kx$

↳ Força Dissipativa: $F_{dis} = -\alpha \dot{x}$

↳ Força Externa: $F_{ext} = F(t)$

$$m\ddot{x} = -Kx + F(t) \Rightarrow \boxed{\ddot{x} + \omega_0^2 x = \frac{F(t)}{m}}$$

→ Abordagem Geral $\forall F(t)$

↳ Solução Geral: $x(t) = x_h(t) + x_p(t)$

Tratamento de $F(t)$ Harmônica $\Rightarrow F(t) = F_0 \cos(\omega t)$

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$$

Hipótese:

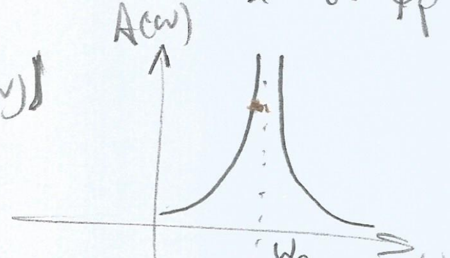
$$\begin{cases} x(t) = x_p \cos(\omega t) \\ \dot{x}(t) = -\omega \cdot x_p \cdot \sin(\omega t) \\ \ddot{x}(t) = -\omega^2 x_p \cos(\omega t) \end{cases}$$

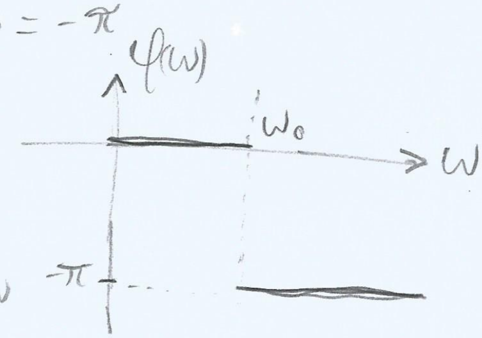
$$\boxed{\begin{array}{l} (-\omega^2 + \omega_0^2) x_p \cos(\omega t) = \frac{F_0}{m}\cos(\omega t) \\ \text{↳ Válido } \forall t \Rightarrow (-\omega^2 + \omega_0^2) x_p = \frac{F_0}{m} \end{array}}$$

$$\rightarrow x_p = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

$$x^{(p)}(t) = \frac{F_0}{m|\omega_0^2 - \omega^2|}\cos(\omega t + \varphi_p) \Rightarrow \begin{array}{l} \omega < \omega_0 : \varphi_p = 0 \\ \omega > \omega_0 : \varphi_p = -\pi \end{array}$$

$$x^{(p)}(t) = A(\omega)\cos(\omega t + \varphi(\omega))$$





$\omega \ll \omega_0$

$$-\omega^2 + \omega_0^2 \approx \omega_0^2 \Rightarrow F(t) = K x(t)$$

$\omega \gg \omega_0$

$$-\omega^2 + \omega_0^2 \approx -\omega^2 \Rightarrow m\ddot{x} \approx F(t)$$

Oscilador Harmônico

→ Sistema Conservativo

↳ O que acontece se há amortecimento?

↳ Força de Amortecimento

- Viscosa: $\vec{F} = -\alpha \vec{v}$

→ Oscilador Harmônico com Amortecimento

$$F = \frac{m}{m}\frac{d^2x}{dt^2} = -\underbrace{\frac{\alpha}{m}}_{\gamma}\frac{dx}{dt} - \overbrace{\frac{K}{m}}^{\omega_0^2}x \Rightarrow \boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0}$$

$\Downarrow$

Hipótese: $x = x_0 e^{\rho t}$; $\dot{x} = \rho x$; $\ddot{x} = \rho^2 x \Rightarrow (\rho^2 + \gamma\rho + \omega_0^2)x = 0 \Rightarrow$

$\Rightarrow \rho^2 + \gamma\rho + \omega_0^2 = 0 \Rightarrow \gamma^2 - 4\omega_0^2 \geq 0$?

- Se as raízes forem complexas, $z = z_0 e^{\rho t}$, $z \in \mathbb{C}$
- Se $z = x + iy \Rightarrow Re_z = x(t)$ ∴ $\ddot{z} = \ddot{x} + i\ddot{y}$, se $\ddot{z} + \gamma\dot{z} + \omega_0^2 z = 0$

porque? → Linear e homogênea; $\dot{z} = \dot{x} + i\dot{y}$ $\Rightarrow \begin{cases} \ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \\ \ddot{y} + \gamma\dot{y} + \omega_0^2 y = 0 \end{cases}$

↳ $\Delta > 0$: $\beta = \frac{\gamma^2}{4} - \omega_0^2 > 0$ (supercrítico)

↳ $\Delta = 0$: $\frac{\gamma^2}{4} - \omega_0^2 = 0$ (crítico)

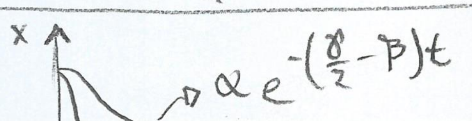
↳ $\Delta < 0$: $\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} > 0$ (subcrítico)

→ Amortecimento Supercrítico: $\beta^2 > 0$, $\rho^{\pm} = -\frac{\gamma}{2} \pm \beta \Rightarrow$

$$\begin{cases} x_+(t) = e^{-(\frac{\gamma}{2} - \beta)t} \\ x_-(t) = e^{-(\frac{\gamma}{2} + \beta)t} \end{cases} \quad x(t) = a e^{-(\frac{\gamma}{2} - \beta)t} + b e^{-(\frac{\gamma}{2} + \beta)t}, \; a, b \in \mathbb{R}$$

$\boxed{x(0) = x_0 = a + b}$; $\dot{x}(0) = v_0 = \left(-\frac{\gamma}{2} - \beta\right)a + \left(-\frac{\gamma}{2} - \beta\right)b$

$\Rightarrow \boxed{v_0 = \dot{x}(0) = -\frac{\gamma}{2}(a+b) + \beta(a-b)}$, $\frac{\gamma}{2} - \beta < \frac{\gamma}{2} + \beta$



→ Amortecimento Crítico

$\beta = 0 \Rightarrow x_1 = e^{-\frac{\gamma}{2}t}$ ↳ Escrever com $\lim\limits_{\beta \to 0}$ do caso supercrítico

↳ Escolher coeficientes $a(\beta)$, $b(\beta)$

↳ Escolher $a = \frac{1}{2}\beta$; $b = -\frac{1}{2\beta} \Rightarrow x(t) = \lim\limits_{\beta \to 0} e^{-\frac{\gamma}{2}t}\left(\frac{e^{\beta t} - e^{-\beta t}}{2\beta}\right) \overset{L'H}{\Rightarrow}$

$$\Rightarrow x(t) = e^{-\frac{\gamma}{2}t} \lim_{\beta \to 0} \frac{te^{\beta t} + te^{-\beta t}}{2} \Rightarrow \boxed{x(t) = te^{-\frac{\gamma}{2}t}}$$

↳ $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$; $te^{-\frac{\gamma}{2}t}$ é solução se $\frac{\gamma^2}{4} - \omega_0^2 = 0 \Rightarrow \omega_0 = \frac{\gamma}{2}$

$$\Rightarrow \begin{cases} \dot{x} = e^{-\frac{\gamma}{2}t} - \frac{\gamma}{2}te^{-\frac{\gamma}{2}t} \\ \ddot{x} = -\frac{\gamma}{2}e^{-\frac{\gamma}{2}t} - \frac{\gamma}{2}e^{-\frac{\gamma}{2}t} + \frac{\gamma^2}{4}te^{-\frac{\gamma}{2}t} \end{cases}$$

$$\Rightarrow \frac{\gamma^2}{4}te^{-\frac{\gamma}{2}t} - \cancel{\gamma e^{-\frac{\gamma}{2}t}} + \cancel{\gamma e^{-\frac{\gamma}{2}t}} - \frac{\gamma^2}{2}te^{-\frac{\gamma}{2}t} + \omega_0^2 te^{-\frac{\gamma}{2}t} \Rightarrow$$

$$\Rightarrow \left(-\frac{\gamma^2}{4} + \omega_0^2\right)te^{-\frac{\gamma}{2}t} = 0$$

→ Amortecimento Subcrítico: $\frac{\gamma^2}{4} - \omega_0^2 < 0 \Rightarrow \omega^2 = \omega_0^2 - \frac{\gamma^2}{4} > 0$

↳ $\frac{\gamma^2}{4} - \omega_0^2 < 0 \Rightarrow \omega^2 = \omega_0^2 - \frac{\gamma^2}{4} > 0 \Rightarrow$ fornece solução para $z(t)$

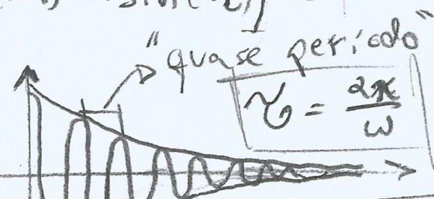
$$\Rightarrow \rho^{\pm} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2} = -\frac{\gamma}{2} \pm \sqrt{-\left(\omega_0^2 - \frac{\gamma^2}{4}\right)} \Rightarrow \rho^{\pm} = -\frac{\gamma}{2} \pm i\omega$$

$$z(t) = e^{-\frac{\gamma}{2}t}\left[z_{0+}e^{i\omega t} + z_{0-}e^{-i\omega t}\right], \quad z_{0+}, z_{0-} \in \mathbb{C}$$

$e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t)$, $z_{0+} = a$, $z_{0-} = -ib$

$$z(t) = e^{-\frac{\gamma}{2}t}\left[a(\cos(\omega t) + i\sin(\omega t)\right] + ib(\cos(\omega t) - i\sin(\omega t)) =$$

↳ $\boxed{x(t) = \mathrm{Re}\, z(t) = \underbrace{Ae^{-\frac{\gamma}{2}t}}_{a(t)}\cos(\omega t + \varphi)}$



$z_p = \frac{F_0}{m} z_2^{-1}$; $z_2 = \omega_0^2 - \omega^2 + i\omega\gamma$

$= \frac{F_0}{m|z_2|} e^{-i\theta_2}$; $z_2 = |z_2| e^{i\theta_2} \Rightarrow Ae^{i\varphi} \Rightarrow \theta_2 = \varphi$, p/ $\varphi = \varphi_1 - \varphi_2$

$\varphi = -\tan^{-1}\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right) \Rightarrow z = \frac{z_1}{z_2} = Ae^{i\varphi} = \frac{A_1 e^{i\varphi_1}}{A_2 e^{i\varphi_2}} \Rightarrow A = \frac{A_1}{A_2}$

$z = Ae^{i\varphi} e^{i\omega t} \Rightarrow x = \mathrm{Re}\, z \Rightarrow x(t) = A\cos(\omega t + \varphi) = A(\omega)\cos(\omega t + \varphi(\omega))$

$x(t) = x_{transiente} + x_p(t) \Rightarrow$ para $t \gg \frac{1}{\gamma} \Rightarrow x(t) \to x_p(t)$

Oscilador Amortecido

$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0} \Rightarrow \omega_0 > \frac{\gamma}{2}$: Subcrítico $\left(\omega^2 = \omega_0^2 + \frac{\gamma^2}{4}\right)$

$\boxed{x(t) = Ae^{-\frac{\gamma}{2}t}\cos(\omega t + \varphi)}$

$\omega_0 < \frac{\gamma}{2}$; Supercrítico $\Rightarrow x(t) = ae^{-\left(\frac{\gamma}{2} - \beta\right)t} + be^{-\left(\frac{\gamma}{2} + \beta\right)t}$, $\beta^2 = \frac{\gamma^2}{4} - \omega_0^2$

$\omega_0 = \frac{\gamma}{2}$: Crítico $\Rightarrow x(t) = e^{-\frac{\gamma}{2}t}(A + Bt)$

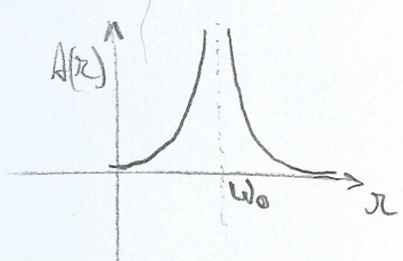
Determinação de (A, φ, ou a e b) através de $\begin{cases} x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases}$

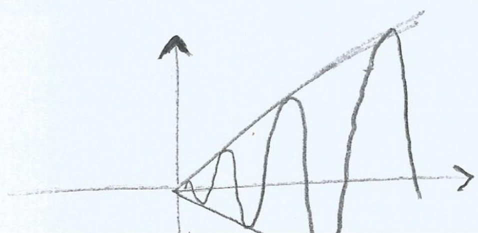
*OBS: p/ t suficientemente longo $x(t) \to 0$, $t \gg \frac{1}{\gamma}$

Oscilador Forçado (harmonicamente) s/ amortecido

$\boxed{\ddot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\Omega t)}$ $\boxed{x(t) = A\cos(\omega_0 t + \varphi) + \frac{F_0}{m(\omega_0^2 - \Omega^2)}\cos(\Omega t)}$

* Se $x(0) = 0$, $\dot{x}(0) = 0 \Rightarrow x(t) \approx t\sin(\Omega t)$





→ Problema Completo

↳ 3 Forças: $\begin{cases} \text{Restauradora: } F = -kx = -m\omega_0^2 x \\ \text{Dissipativa: } F = -\alpha \dot{x} = -m\gamma \dot{x} \\ \text{Ext. Harm: } F = F_0 \cos(\Omega t) \end{cases}$ ⇒

$$\Rightarrow \boxed{\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\Omega t)}$$

⇒ Achar sol. particular Estacionária

• <u>Hipótese 1</u>: $x(t) = \underset{A(\Omega)}{A}\cos(\Omega t + \underset{\varphi(\Omega)}{\varphi})$

• <u>Hipótese 2</u>: Def. $\begin{cases} z(t) = x(t) + i\,y(t) \\ F(t) = F(t) + i\,F^{im}(t) \end{cases}$ $\Rightarrow \ddot{z} + \gamma \dot{z} + \omega_0^2 z = \frac{F(t)}{m} \Rightarrow$

$$\Rightarrow \begin{cases} \ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F(t)}{m} \\ \ddot{y} + \gamma \dot{y} + \omega_0^2 y = \frac{F^{im}(t)}{m} \end{cases}$$

$* F(t) = F_0 e^{i\omega t}$

Sol. tentativa: $z(t) = z_0 e^{i\Omega t}$

$\dot{z} = i\Omega z$, $\ddot{z} = -\Omega^2 z \Rightarrow$

$$\Rightarrow (-\Omega^2 + i\gamma\Omega + \omega_0^2) z_0 e^{i\Omega t} = \frac{F_0}{m} e^{i\Omega t} \Rightarrow z_0 = \frac{F_0}{m[(\omega_0^2 - \Omega^2) + i\gamma\Omega]} = A e^{i\Omega}$$

$$z_0 = A e^{i\varphi} \Rightarrow z_0 \cdot z_0^* = A e^{i\varphi} \cdot A e^{-i\varphi} \Rightarrow A^2 = |z_0|^2 = \frac{F_0^2}{m^2[(\omega_0^2 - \Omega^2)^2 + \gamma^2\Omega^2]} \Rightarrow$$

$$\boxed{A(\omega) = \frac{F_0}{m[(\omega_0^2 - \Omega^2)^2 + \Omega^2\gamma^2]^{1/2}}} \; ; \; tg\,\varphi = \frac{Im\, z_0}{Re\, z_0} \; ; \; z_0 = \frac{F_0}{m z_1} = \frac{F_0}{m|z_1|} e^{-i\theta_1} \; ;$$

$$z_1 = |z_1| e^{i\theta_1} \Rightarrow \varphi = -\theta_1 \Rightarrow \varphi =$$

$$\boxed{\varphi = -tg^{-1}\left(\frac{\gamma\Omega}{\omega_0^2 - \Omega^2}\right)}$$

<u>Ressonância</u>: $\Omega \to \omega_0$, $|\omega_0 - \Omega| \ll \omega_0$

<u>Amortecim. Fraco</u>: $\gamma \ll \omega_0$

$$\begin{cases} \omega_0^2 - \Omega^2 = (\omega_0 + \Omega)(\omega_0 - \Omega) \cong 2\omega_0(\omega_0 - \Omega) \\ \gamma\Omega \cong \gamma\omega_0 \end{cases} \Rightarrow A^2(\omega) \cong \frac{F_0^2}{m^2[4\omega_0^2(\omega_0 - \Omega)^2 + \gamma^2\omega_0^2]}$$

Solução Estacionária

$$\boxed{x(t) = A(\omega)\cos(\Omega t + \varphi(\omega))}$$

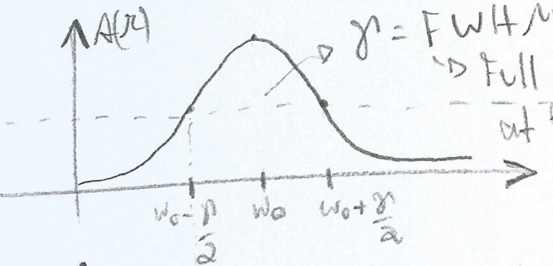
$$\Rightarrow A^2(\omega) = \frac{F_0^2}{4m^2\omega_0^2\left[(\omega_0 - \Omega)^2 + \frac{\gamma^2}{4}\right]}$$

$$\varphi(\omega) \approx -tg^{-1}\left[\frac{\gamma\omega_0}{2\omega_0(\omega_0 - \Omega)}\right] = -tg^{-1}\left(\frac{\gamma}{2(\omega_0 - \Omega)}\right)$$

↳ Amplitude Máxima: $A^2_{max}(\Omega) \Rightarrow \Omega = \omega_0 : \Rightarrow \boxed{A^2_{max}(\Omega) = \dfrac{F_0^2}{m^2\omega_0^2 \cdot \gamma^2}}$

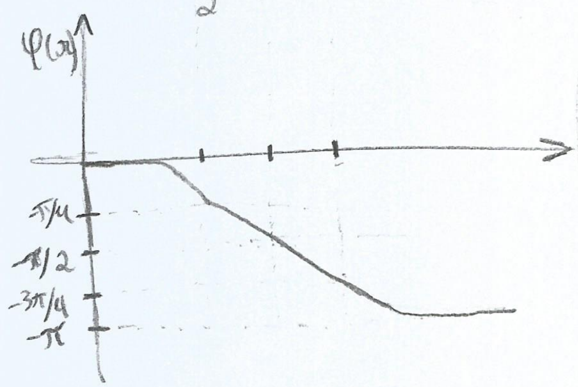
p/ qual Ω, $A^2(\Omega) = \dfrac{A^2_{max}(\Omega)}{2} \Rightarrow \Omega = \omega_0 \pm \Delta\omega \Rightarrow$

$$\frac{A^2_{max}}{2} = \frac{F_0^2}{4m^2\omega_0^2\left[(\pm\Delta\omega)^2 + \frac{\gamma^2}{4}\right]} \Rightarrow \Delta\omega^2 = \frac{\gamma^2}{4} \Rightarrow \boxed{A^2\left(\Omega = \omega_0 \pm \frac{\gamma}{2}\right) = \frac{A^2_{max}}{2}}$$



→ γ = FWHM
→ Full Width at Half Maximum

→ Espectro



Oscilador Forçado s/ Amortecimento

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\Omega t)$$

$$x = \frac{F_0}{m(\omega_0^2 - \Omega^2)}\cos(\Omega t) = \frac{F_0}{m|\omega_0^2 - \Omega^2|}\cos(\Omega t + \varphi)$$

$$\varphi = \begin{cases} 0, & \Omega < \omega_0 \\ -\pi, & \Omega > \omega_0 \end{cases}$$

Balanço de Energia

$F_{ext} = 0$; $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$; $F_{dissip.} = -\alpha\dot{x}$

* Se Sist. Conserv. $\Leftrightarrow \exists\, U(x)$ tq. $F(x) = -\dfrac{dU(x)}{dx} \Rightarrow \Delta U = -\displaystyle\int_{x_0}^{x} F(x)dx$

$\Delta U = -\Delta K \Rightarrow \Delta K + \Delta U = 0 \Rightarrow \exists\ \underset{\text{conserv.}}{E = K + U}$; $\Delta E = 0$

* Com $F_{el} = -Kx$ é conserv., podemos definir $U_{el} = \frac{1}{2}Kx^2 + cte$

→ Energia <u>Armazenada</u> no oscilador

$E = K + U_{el} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2 \Rightarrow \dfrac{dE}{dt} \neq 0 \Rightarrow \dfrac{dE}{dt} < 0$, dissipação

$$\frac{dE}{dt} = F_{dissip.} \cdot \dot{x} = -\alpha \cdot \dot{x}^2$$

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + m\omega_0^2\dot{x}x = -\alpha\dot{x}^2$$

$\Rightarrow m\ddot{x} + \alpha\dot{x} + m\omega_0^2 x = 0$

$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$

→ Amortecimento Crítico e Supercrítico

↳ Crítico: $E \propto x^2 \sim e^{-\gamma t}$ ↳ Supercrítico: $E \propto x^2 \sim e^{-(\gamma - 2\beta)t}$

→ Subcrítico $\boxed{x(t) = A e^{-\frac{\gamma}{2}t} \cos(\omega t + \varphi)}$ → $\boxed{\dot{x}(t) = -\frac{\gamma}{2} A e^{-\frac{\gamma}{2}t} \cos(\omega t + \varphi) - \omega A e^{-\frac{\gamma}{2}t} \sin(\omega t + \varphi)}$

$$E(t) = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \omega_0^2 x^2 =$$

$$E(t) = \frac{1}{2} m A^2 e^{-\gamma t} \left[\left(\omega_0^2 + \frac{\gamma^2}{4}\right) \cos^2(\omega t + \varphi) + \omega^2 \sin^2(\omega t + \varphi) + \gamma \omega \sin(\omega t + \varphi) \cos(\omega t + \varphi) \right]$$

↳ Expr. Interessante se $\gamma \ll \omega_0$. Def $\bar{E}(t) = \underbrace{\left(\frac{\omega}{2\pi}\right)}_{1/\tau} \int_t^{t + \frac{2\pi}{\omega}} E(t')\, dt'$

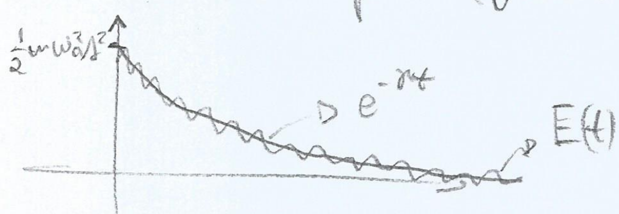
$\boxed{\omega = \omega_0^2 - \frac{\gamma^2}{4}}$

Aproximação: $e^{-\gamma t} \approx$ const. p/ $\Delta t \ll \frac{1}{\gamma}$

$$\Rightarrow \bar{E}(t) \approx \frac{1}{2} m A^2 e^{-\gamma t} \cdot \frac{1}{\tau} \int_t^{t+\tau} (\sim)\, dt' = \frac{1}{2} m A^2 e^{-\gamma t} \left[\frac{1}{2}\left(\omega_0^2 + \frac{\gamma^2}{4}\right) + \frac{1}{2}\omega^2 \right]$$

→ Oscilador c/ amortecimento muito fraco ($\gamma \ll \omega_0$)

$$\boxed{E(t) = \frac{1}{2} m \omega_0^2 A^2 e^{-\gamma t}}$$



$$E(t) = E(0) e^{-\gamma t} \Rightarrow \frac{dE}{dt} = -\gamma E \; ; \; \frac{1}{E}\frac{dE}{dt} = -\gamma \; ; \; \Delta t \ll \frac{1}{\gamma} \approx \Delta E = \frac{dE}{dt} \Delta t =$$

$$= -\gamma E \Delta t \Rightarrow \Delta t = \tau = \frac{2\pi}{\omega} \ll \frac{1}{\gamma} \Rightarrow \Delta E = -\gamma E \cdot \tau$$

→ Fator de Qualidade (Mérito); $Q = 2\pi \left(\frac{\text{En. armazenada}}{\text{En. dissip. p/ período}} \right) =$

$$= 2\pi \frac{E}{\gamma E \tau} = \boxed{\frac{\omega_0}{\gamma} = Q} \gg 1 \text{ (Amortecimento muito fraco)}$$

→ Osc. forçado Amortecido c/ $\gamma \ll \omega_0$; $\boxed{A_{max} = \frac{F_0}{m \omega_0 \gamma}}$

$$A^2(\Omega) = \frac{F_0^2}{m^2 \left[(\omega_0^2 - \Omega^2)^2 + \gamma^2 \Omega^2 \right]}$$

$$A(0) = \frac{F_0}{m \omega_0^2} \; ; \; \boxed{\frac{A_{max}}{A(0)} = \frac{F_0}{m \omega_0 \gamma} \cdot \frac{m \omega_0^2}{F_0} = \frac{\omega_0}{\gamma} = Q}$$

Balanço de Energia - Oscilador Forçado e Amortecido

$$\begin{cases} \ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\Omega t) \Rightarrow \ddot{z} + \gamma \dot{z} + \omega_0^2 z = \frac{F_0}{m} e^{i\Omega t}; \\ z_p = z_0 e^{i\omega t} = A e^{i(\omega t + \varphi)} \end{cases} \Rightarrow$$

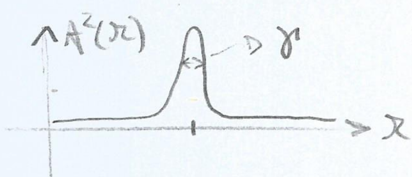
$$\Rightarrow (-\Omega^2 + i\gamma\Omega + \omega_0^2) z_0 e^{i\Omega t} = \frac{F_0}{m} e^{i\Omega t} \Rightarrow \boxed{z_0 = \frac{F_0}{m[(\omega_0^2 - \Omega^2) + i\Omega\gamma]}}$$

$$A = |z_0| = \sqrt{z_0 \cdot z_0^*} \Rightarrow \boxed{A^2 = \frac{F_0^2}{m^2[(\omega_0^2 - \Omega^2)^2 + \gamma^2\Omega^2]}}$$

$$z_0 = \frac{F_0}{m z_1} \Rightarrow z_1 = |z_1| e^{i\theta_1} \Rightarrow z_0 = \frac{F_0}{m|z_1|} e^{i\theta_1} \Rightarrow \varphi = \theta_1$$

$$z_1 = (\omega_0^2 - \Omega^2) + i\Omega\gamma \Rightarrow \boxed{\theta_1 = \tan^{-1}\left(\frac{\gamma\Omega}{\omega_0^2 - \Omega^2}\right) = -\varphi}$$

* $\gamma \ll \omega_0$ (Amortecimento fraco)



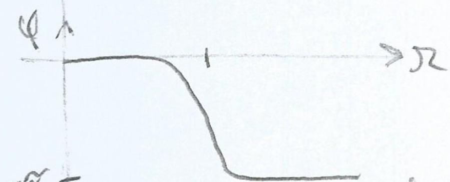
$$\boxed{\frac{A_{max}}{A(0)} = \frac{\omega_0}{\gamma} = Q}$$

$\Rightarrow \dot{x} = -\Omega A \sin(\Omega t + \varphi)$

Parece Conservativo (?)

$\rightarrow$ Sol. Estacionária: $x_{est}(t) = A(\Omega)\cos(\Omega t + \varphi(\Omega))$

$\hookrightarrow$ Em média, energia recebida = energia perdida



• Período $\tau = \frac{2\pi}{\Omega}$

• Energia Armazenada

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m\omega_0^2 x^2$$

$$\frac{dE}{dt} = \dot{x}(\overbrace{m\ddot{x} + m\omega_0^2 x}^{-m\gamma\dot{x} + F(t)}) = -m\gamma\dot{x}^2 + P(t) \rightarrow F(t)\cdot\dot{x} \rightarrow \text{Potência}$$

$\rightarrow$ Regime Estacionário

$$\begin{cases} F = F_0 \cos(\Omega t) \\ x(t) = A\cos(\Omega t + \varphi) \end{cases} \qquad \boxed{P(t) = F(t)\cdot\dot{x}(t) = -\Omega A F_0 \sin(\Omega t + \varphi)\cos(\Omega t)}$$

$$\ddot{x} = -\Omega^2 x$$

$$\frac{dE}{dt} = m\dot{x}(-\Omega^2 + \omega_0^2)x = m(\omega_0^2 - \Omega^2)A\cdot(-\Omega A)\sin(\Omega t + \varphi)\cos(\Omega t + \varphi)$$

$$\overline{\frac{dE}{dt}} = -m(\omega_0^2 - \Omega^2)\Omega A^2 \overline{\sin(\Omega t + \varphi)\cos(\Omega t + \varphi)} \Rightarrow \overline{\frac{dE}{dt}} = 0 \Leftrightarrow \overline{P} = m\gamma\overline{\dot{x}^2}$$