Information Theory: Principles and Applications

Homework 1 - Due: March 26, 2010

1. A family has two children. One of them is a boy. What is the probability of the other being a girl?

2. Three events E_1, E_2 and E_3, defined on the same sample space Ω, have probabilities $P(E_1) = P(E_2) = P(E_3) = 1/4$. Let E_0 be the event that one or more of the events E_1, E_2, E_3 occurs.

   (a) Find $P(E_0)$ when:

      i. The events E_1, E_2 and E_3 are disjoint.

      ii. The events E_1, E_2 and E_3 are independent.

      iii. The events E_1, E_2 and E_3 are, in fact, three names for the same event.

   (b) Find the maximum value $P(E_0)$ can assume when:

      i. Nothing is known about the independence or disjointness of E_1, E_2 and E_3.

      ii. The events E_1, E_2 and E_3 are pairwise independent, that is, $P(E_i \cap E_j) = P(E_i)P(E_j), 1 \leq i \neq j \leq 3$, but nothing is known about $P(E_1 \cap E_2 \cap E_3)$.

3. Let $X_1, X_2, \dots X_n$ be a sequence of n binary independent and identically distributed random variables. Assume that $P(X_m = 0) = P(X_m = 1) = 1/2$. Let Z be a parity check on $X_1, \dots, X_n$, that is, $Z = X_1 \oplus X_2 \oplus \dots \oplus X_n$, where $0 \oplus 0 = 1 \oplus 1 = 0$ and $0 \oplus 1 = 1 \oplus 0 = 1$.

   (a) Is Z independent of X_1? (assume $n > 1$);

   (b) Are $Z, X_1, \dots, X_{n-1}$ independent?

   (c) Are $Z, X_1, \dots, X_n$ independent?

   (d) Is Z independent of X_1 if $P(X_i = 1) \neq 1/2$? (you may take $n = 2$ here).

   Justify your answers.

4. Two players throw (one player after the other) a pair of dice. The first one to get a 7 wins the game. What is the probability of the player that made the first throw win the game? Repeat the problem with 3 players.

5. Let X and Y be discrete random variables defined on some probability space with a joint probability density $p_{XY}(x, y)$.

   (a) Prove that $E[X+Y] = E[X] + E[Y]$. Do not assume independence.

   (b) Prove that if X and Y are independent, then $E[XY] = E[X]E[Y]$.

   (c) Assume that X and Y are not independent. Find an example where $E[XY] \neq E[X]E[Y]$ and another example where $E[XY] = E[X]E[Y]$.

   (d) Assume that X and Y are independent and let σ_X^2 and σ_Y^2 be the variances of X and Y respectively. Show that the variance of $X+Y$ is given by $\sigma_X^2 + \sigma_Y^2$.

6. A fair coin is flipped until the first head occurs. Let X denote the number of flips required. Find the entropy $H(X)$ in bits.

7. Consider two random variables $X \in \{0, 1\}$ and $Y \in \{0, 1\}$ with joint probability distribution $p_{XY}(x, y)$ given by:

$$p_{XY}(0,0) = 1/2 \quad p_{XY}(0,1) = 1/6 \quad p_{XY}(1,0) = 0 \quad p_{XY}(1,1) = 1/3.$$

   Calculate $H(X, Y)$, $H(X)$, $H(Y)$, $H(X|Y = 0)$, $H(X|Y)$, $H(Y|X)$, and $I(X; Y)$.

8. A source X produces letters from a three-symbol alphabet with the probability assignment $P(X = 0) = 1/4$, $P(X = 1) = 1/4$, and $P(X = 2) = 1/2$. Each source letter is transmitted simultaneously through two channels with outputs Y and Z (both assuming value 0 or 1) and the following transition probabilities:

$$P(Y = 0|X = 0) = P(Y = 1|X = 1) = 1, \quad P(Y = 0|X = 2) = P(Y = 1|X = 2) = 1/2$$

$$P(Z = 0|X = 0) = P(Z = 0|X = 1) = P(Z = 1|X = 2) = 1$$

   Calculate $H(X)$, $H(Y)$, $H(Z)$, $H(Y, Z)$, $I(X; Y)$, $I(X; Z)$, $I(X; Y|Z)$, $I(X; Y, Z)$, and interpret the mutual information expressions.

   Note: This could be considered as a single channel with output given by the random vector $[Y, Z]$.

9. Show that $\ln x \leq x - 1$, with equality occuring in $x = 1$. Using this result, show that $D(p_X(x)||q_X(x)) \geq 0$.

10. Consider the following challenge: Four identical balls are presented to you. They can all be of the same weight or one of the balls is either slightly lighter or heavier than the others. One balance with two pans is available in order to, after a given number of weighings identify:

    - If all four balls have the same weight;
    - If one ball is heavier than the others (and identify this ball);
    - If one ball is lighter than the others (and identity this ball).

    Your goal is to devise an strategy that minimize the number of weighings. In the first stage, k balls are put in the left pan (pan 1), k balls in the right pan (pan 2) and $(4 - 2k)$ balls are put aside. The result of this experiment can be modeled as a random variable Y which can assume the following values:

    - $Y = 0$, if the pans do not move.
    - $Y = 1$, if the balance tilts torwards pan 1.
    - $Y = 2$, if the balance tilts torwards pan 2.

    Which value of k should be chosen at the first weighing in order obtain more information? Give an information-theoretic argument.

11. (Extra) An ensemble X has the non-negative integers as its sample space. Find the probability assignment $P(X = i) = p_i$, $i = 0, 1, \ldots$ such that the entropy $H(X)$ is maximized subject to the constraint on the mean value of X

$$E[X] = \sum_{i=0}^{\infty} i p_i = A$$

Useful formulas

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, |r| < 1 \qquad \sum_{n=0}^{\infty} n r^n = \frac{r}{(1-r)^2}, |r| < 1$$