SECRECY AND TREE PACKING

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Information Theoretic Security

- A complementary approach to `computational security` for *secret key* cryptosystems.
- *Unconditional Security*: A quantifiable and provable notion of security, with no assumption of "one-way" functions and no restrictions on the computational power of an adversary.
- Information theoretic perfect secrecy: Shannon, 1949.
- A modified notion: Maurer (1990, 1993), Ahlswede-Csiszár (1993)
  - an adversary does not have access to precisely the same observations as the legitimate users;
  - the legitimate plaintext messages and secret keys are, in effect, "nearly statistically independent" of the observations of the adversary.
- ? New insights: Innate connections with multiterminal data compression and points of contact with combinatorial tree packing algorithms.
- ??? New algorithms: Potential rests on advances in algorithms for multiterminal data compression and a better understanding of connections with combinatorial tree packing of multigraphs.

SECRET KEY GENERATION

Multiterminal Source Model

- The terminals in $\mathcal{M} = \{1, \ldots, m\}$ observe separate but correlated signals, e.g., different noisy versions of a common broadcast signal or measurements of a parameter of the environment.

- The terminals in a given *subset* $A \subseteq \mathcal{M}$ wish to generate a "secret key" with the cooperation of the remaining terminals, to which end *all* the terminals can communicate among themselves – possibly interactively in multiple rounds – over a public noiseless channel of unlimited capacity.

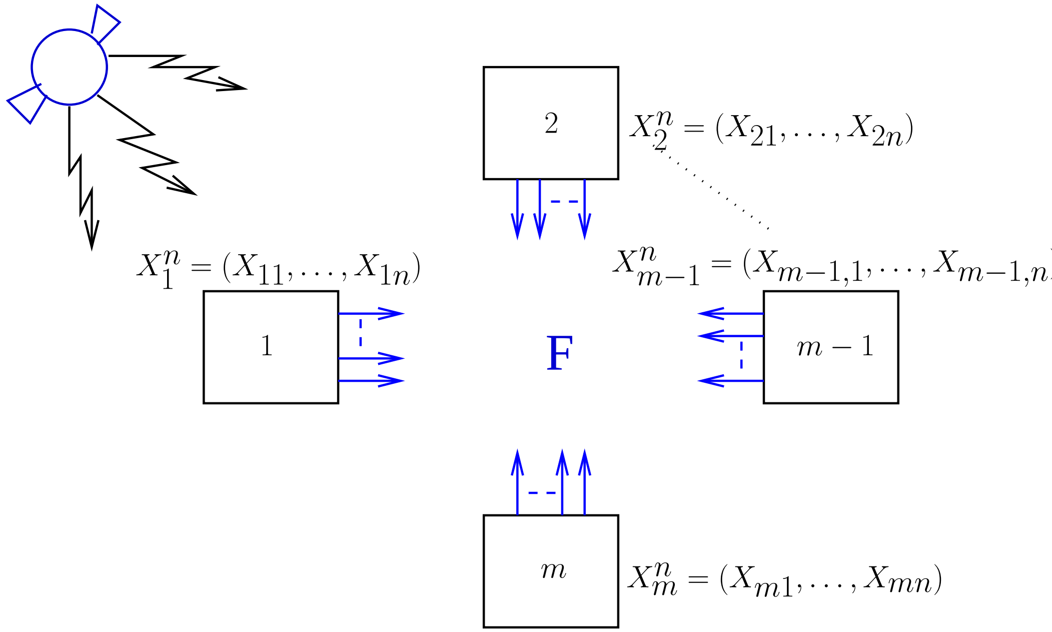
- A secret key:

  - random variables (rvs) generated at each terminal in A which agree with probability $\cong 1$; and
  - the rvs are effectively concealed from an eavesdropper with access to the public communication.

- The key generation exploits the correlated nature of the observed signals.

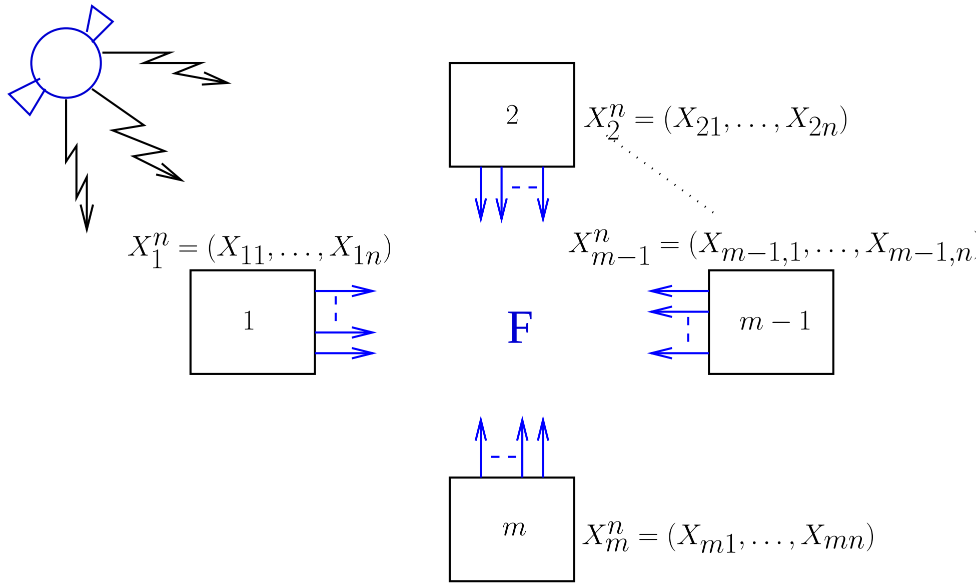
- The secret key thereby generated can be used as a *one-time pad* for secure encrypted communication among the terminals in A.

Multiterminal Source Model



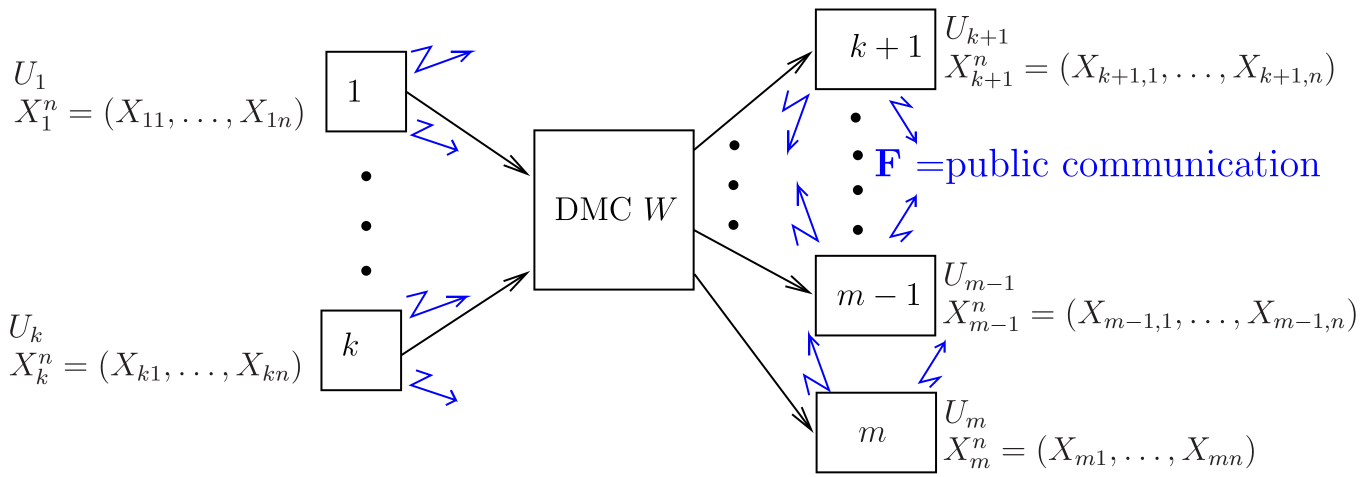
- The m legitimate terminals in $\mathcal{M} = \{1, \ldots, m\}$ cooperate in secret key generation.
- $X_1, \ldots, X_m$ are finite-valued random variables (rvs) with (known) joint distribution $P_{X_1,\ldots,X_m}$.
- Each terminal i, $i = 1, \ldots, m$, observes a signal comprising n independent and identically distributed repetitions (say, in time) of the rv X_i, namely the sequence $X_i^n = (X_{i1}, \ldots, X_{in})$.
- The signal components observed by the different terminals at successive time instants are i.i.d. according to $P_{X_1,\ldots,X_m}$.

Multiterminal Source Model



- *All* the terminals are allowed to communicate over a *noiseless* channel of *unlimited capacity*, possibly interactively in several rounds.
- The communication from any terminal is observed by all the other terminals.
- The communication from a terminal is allowed to be any function of its own signal, and of all previous communication.
- Let **F** denote collectively all the communication.

Multiterminal Channel Model

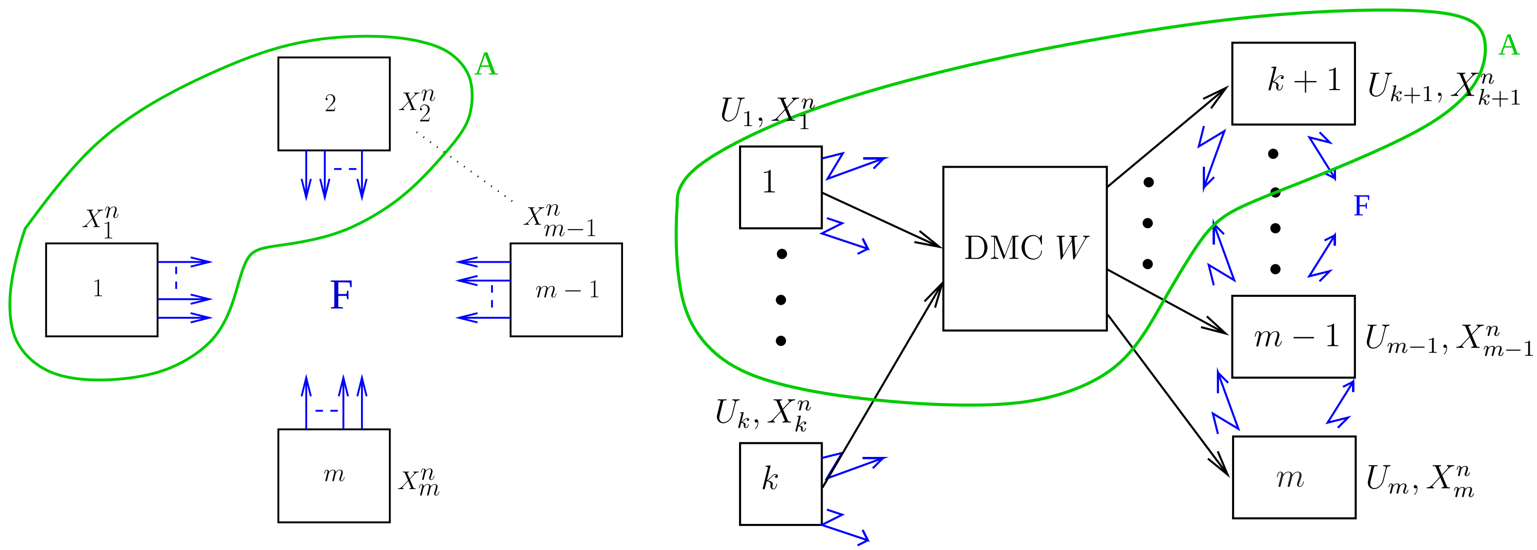


- Terminals $1, \ldots, k$ govern the inputs of a *secure* discrete memoryless channel W, with input terminal i transmitting a signal $X_i^n = (X_{i1}, \ldots, X_{in})$ of length n. Terminals $k+1, \ldots, m$ observe the corresponding output signals, with output terminal i observing X_i^n of length n.

- Following each simultaneous transmission of symbols over the channel W, communication over a *public noiseless channel of unlimited capacity* is allowed between all the terminals, perhaps interactively, and observed by all the terminals. Let $\mathbf{F}$ denote collectively all such public communication.

- Randomization at the terminals is permitted, and is modeled by the rvs $U_i, \ i = 1, \ldots, m$, which are taken to be mutually independent.

The Objective



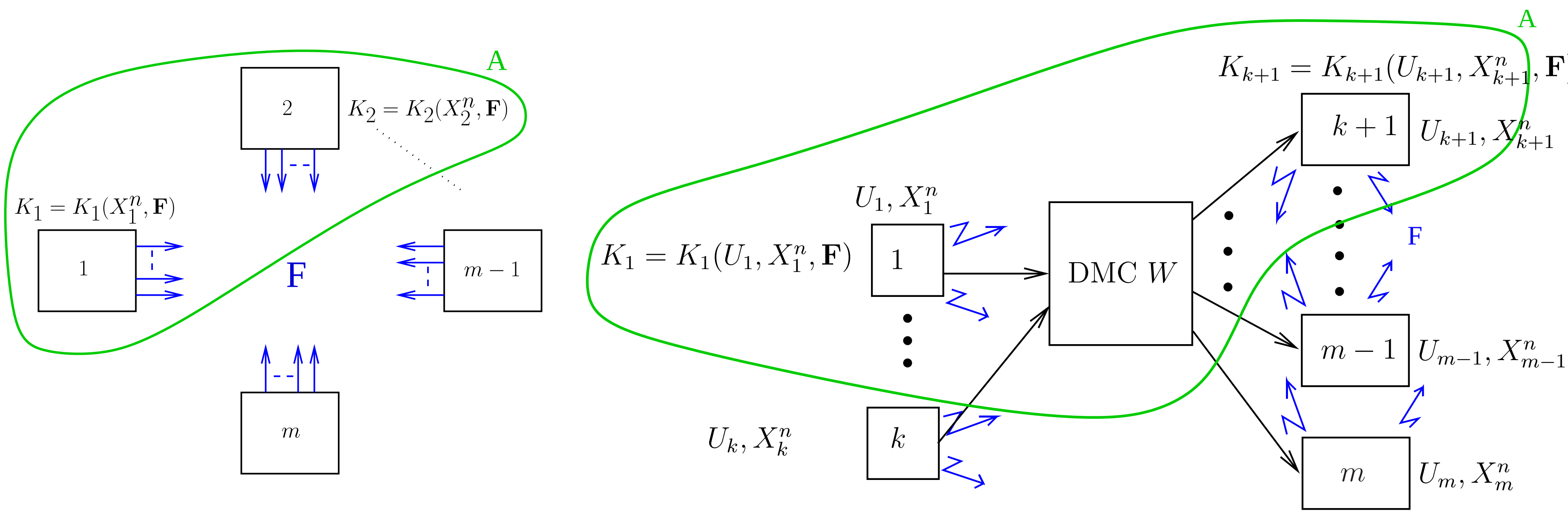
Objective: To generate a *secret key* of the largest “size” for a given set $A \subseteq \{1, \ldots, m\}$ of terminals, i.e., *common randomness* shared by the terminals in A, which is

- of near uniform distribution;
- concealed from an eavesdropper that observes the public communication **F**.

All the terminals $1, \ldots, m$ cooperate in achieving this goal.

Assume: The eavesdropper is passive and cannot wiretap.

What is a Secret Key?

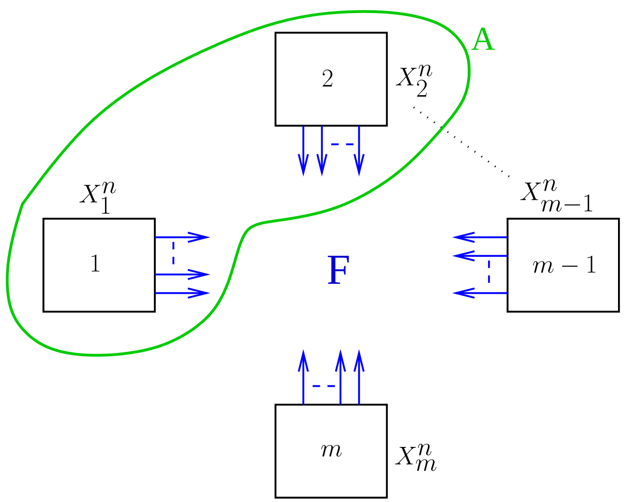


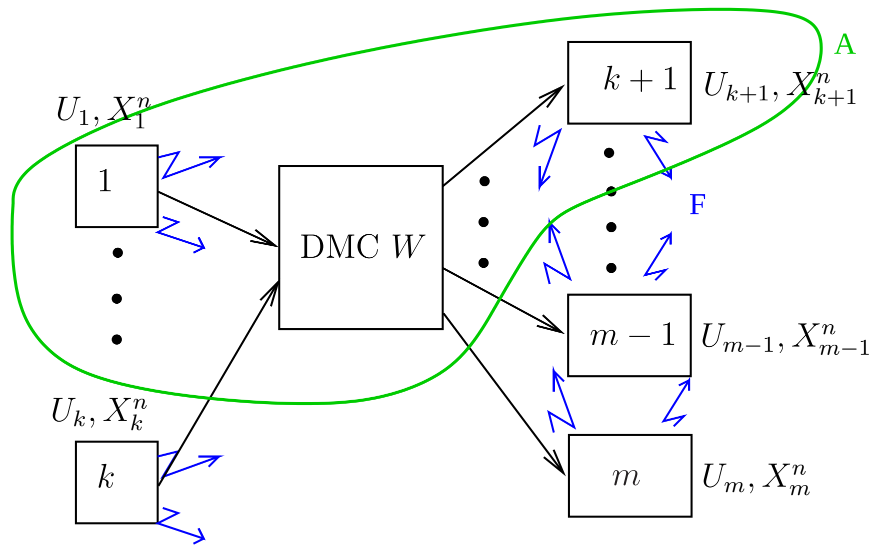
Secret Key (SK): A random variable K is a *SK* for the terminals in A, achievable with communication $\mathbf{F}$, if

- $Pr\{K = K_i,\ i \in A\} \cong 1$ (“common randomness”)
- $I(K \wedge \mathbf{F}) \cong 0$ (“secrecy”)
- $H(K) \cong \log |\text{key space}|$. (“uniformity”)

Thus, a SK is effectively concealed from an eavesdropper with access to $\mathbf{F}$, and is nearly uniformly distributed.

Secret Key Capacity



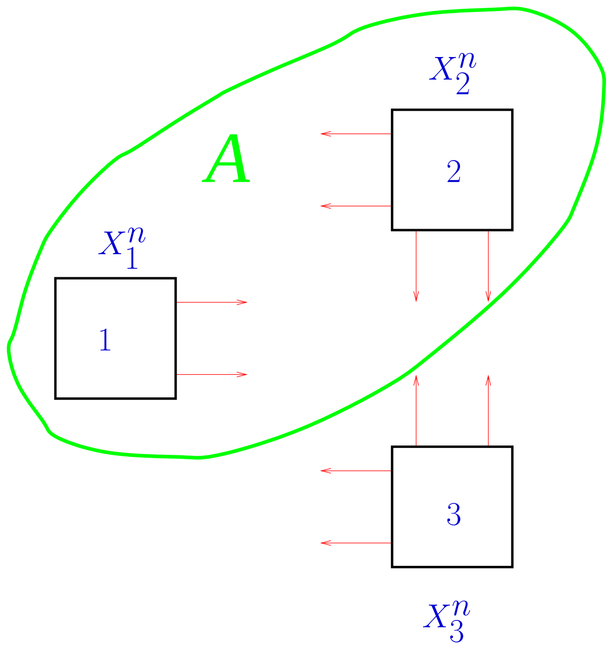


?? What is the *largest rate* $\lim_n \frac{1}{n} \log |\text{key space}|$ of such a SK for A which can be achieved with suitable communication: SK capacity $C_S(A)$?

?? How to construct such a SK?

Hereafter, we shall restrict ourselves to the multiterminal source model.

A Toy Example: $\mathcal{M} = \{1, 2, 3\}$, $A = \{1, 2\}$



- X_1^n, X_2^n are $\{0, 1\}$-valued Bernoulli $(\frac{1}{2})$ sequences, and X_1^n is independent of X_2^n.

$$X_{3t} = X_{1t} \oplus X_{2t}, \qquad t = 1, \ldots, n.$$

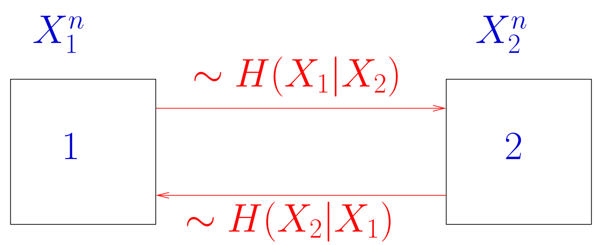
- *Scheme (using $n = 1$):* Terminal 3 communicates publicly $X_{31} = X_{11} \oplus X_{21}$.
- Terminals 1 and 2 respectively infer X_{21} and X_{11}.
- X_{11} is independent of $\mathbf{F} = X_{31} = X_{11} \oplus X_{21}$, and is uniform on $\{0, 1\}$.
- Thus, X_{11} is a *perfect* SK of rate 1.0 (optimal), so that SK capacity $C_S(A) = 1.0$.

Some Related Work

- Maurer 1990, 1991, 1993, 1994, $\cdots$
- Ahlswede - Csiszár 1993, 1994, 1998, $\cdots$
- Bennett, Brassard, Crépeau, Maurer 1995.
- Csiszár 1996.
- Maurer - Wolf 1997, 2000, 2003, $\cdots$
- Venkatesan - Anantharam 1995, 1997, 1998, 2000, $\cdots$
- Csiszár - Narayan 2000, 2004, 2005, 2007, 2008, 2009.
- Renner - Wolf 2003.
- Muramatsu 2004, 2005.
- Ye - Narayan 2004, 2005.
- Gohari-Anantharam 2007, 2008.
- Ye-Reznik, 2007.
- Nitinawarat-Ye-Barg-Narayan-Reznik, 2008, 2009.

$\vdots$

Secret Key Capacity: $\mathcal{M} = \{1, 2\} = A$



- SK capacity [Maurer '93, Ahlswede-Csiszár '93]:

$$C_S(A) = I(X_1 \wedge X_2).$$

- **An interpretation:**

$$\begin{aligned} C_S(A) &= I(X_1 \wedge X_2) \\ &= H(X_1, X_2) - [H(X_1|X_2) + H(X_2|X_1)] \\ &= \text{ Entropy rate of ``omniscience''} \ - \\ & \quad \text{Smallest aggregate rate of communication, } R_{CO}(A), \\ & \quad \text{that enables the terminals in } A \text{ to become omniscient.} \end{aligned}$$

Secret Key Capacity

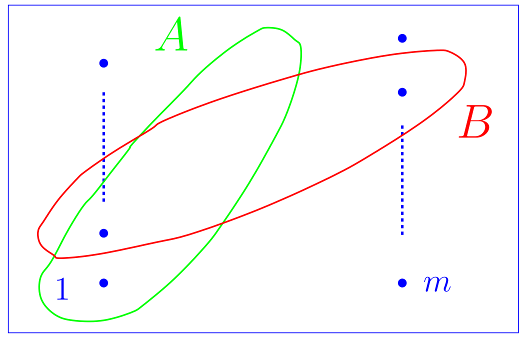
Theorem [I. Csiszár - P. N., '04, '08]:

$$\begin{aligned} C_S(A) &= H(X_1, \ldots, X_m) - \text{Smallest aggregate rate of overall} \\ &\quad \text{interterminal communication, } R_{CO}(A)\text{, that enables} \\ &\quad \textit{all} \text{ the terminals in } A \text{ to become omniscient} \\ &= H(X_1, \ldots, X_m) - \max_{\lambda \in \Lambda(A)} \sum_{B \in \mathcal{B}(A)} \lambda_B H\left(X_B | X_{B^c}\right) \end{aligned}$$

and can be achieved with noninteractive communication.

Remark: $R_{CO}(A)$ is obtained as the solution to a multiterminal data compression problem of omniscience generation that *does not involve any secrecy constraints*.

Interpretation: All the terminals cooperate – through public communication – in enabling the terminals in A to attain omniscience. Then the terminals in A extract a SK from their omniscience by purging the rate of this communication.



Proposition [I. Csiszár - P. N., '04]: The smallest aggregate rate of interterminal communication, $R_{CO}(A)$, that enables all the terminals in A to become omniscient, is

$$R_{CO}(A) = \min_{(R_1,\ldots,R_m)\in\mathcal{R}_{SW}(A)} \sum_{i=1}^{m} R_i,$$

where

$$\mathcal{R}_{SW}(A) = \left\{ (R_1, \cdots, R_m) : \sum_{i\in B} R_i \geq H(X_B|X_{B^c}),\ \forall B \subset \mathcal{M},\ B \neq \emptyset,\ A \nsubseteq B \right\},$$

and can be achieved with noninteractive communication. Furthermore

$$R_{CO}(A) \ = \ \max_{\lambda\in\Lambda(A)} \sum_{B\in\mathcal{B}(A)} \lambda_B H\left(X_B|X_{B^c}\right).$$

How Can a Secret Key be Constructed?

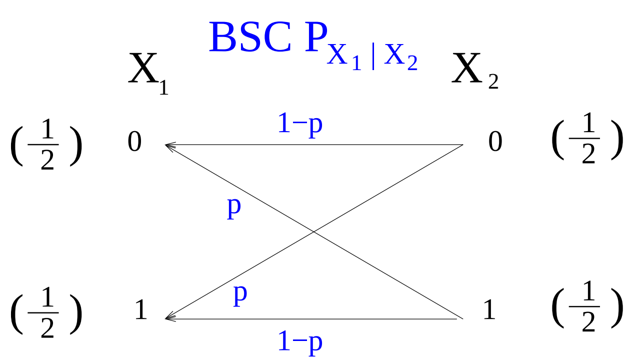
- **Step 1: Common randomness generation:** The terminals communicate over the public channel using compressed data in order to generate omniscience or some form of “common randomness.” This public communication is observed by the eavesdropper.

- **Step 2: Secret key construction:** The terminals then process this “common randomness” to extract a SK of which the eavesdropper has provably little or no knowledge.

Example: Two Terminals with Symmetrically Correlated Signals

- Terminals 1 and 2 observe, respectively, n i.i.d. repetitions of the correlated rvs X_1 and X_2, where X_1, X_2 are $\{0,1\}$-valued rvs with

$$P_{X_1 X_2}(x_1, x_2) = \frac{1}{2}(1-p)\delta_{x_1 x_2} + \frac{1}{2}p\ (1 - \delta_{x_1 x_2}), \qquad p < \frac{1}{2}.$$



- $C_S(\{1,2\}) = I(X_1 \wedge X_2) = 1 - h_b(p)$.
- Can assume: $X_1^n = X_2^n \oplus V^n$, where $V^n = (V_1, \cdots, V_n)$ is independent of X_2^n, and is a Bernoulli (p) sequence of rvs.

Step 1: Common Randomness Generation by SW Data Compression

A.D. Wyner, 1974: Scheme for reconstructing x_1^n at terminal 2

- Standard array for $(n, n-m)$ linear channel code with parity check matrix $\mathbf{P}$ for a channel with noise V^n:

$$
\begin{array}{ccccccc}
c_1^n & c_2^n & \cdots & c_j^n & \cdots & c_{2^{n-m}}^n \\
e_2^n & e_2^n + c_2^n & & e_2^n + c_j^n & & e_2^n + c_{2^{n-m}}^n \\
\vdots & & & & & \vdots \\
e_i^n & e_i^n + c_2^n & & e_i^n + c_j^n = x_1^n & & e_i^n + c_{2^{n-m}}^n \\
\vdots & & & & & \\
e_{2^m}^n & e_{2^m}^n + c_2^n & \cdots & e_{2^m}^n + c_j^n & \cdots & e_{2^m}^n + c_{2^{n-m}}^n
\end{array}
$$

- Terminal 1 communicates $\mathbf{F}$ = the syndrome $\mathbf{P}x_1^n$ to terminal 2.

- Terminal 2 computes the ML estimate $\widehat{x}_1^n = \widehat{x}_1^n(x_2^n, \mathbf{F})$ as:

$$\widehat{x}_1^n = x_2^n \oplus f_{\mathbf{P}}(\mathbf{P}x_1^n \oplus \mathbf{P}x_2^n),$$

  where $f_{\mathbf{P}}(\mathbf{P}x_1^n \oplus \mathbf{P}x_2^n)$ = most likely noise sequence v^n with syndrome

$$\mathbf{P}v^n = \mathbf{P}x_1^n \oplus \mathbf{P}x_2^n.$$

- Thus, terminal 2 reconstructs x_1^n with

$$\Pr\{\widehat{X}_1^n = X_1^n\} = \cdots = \Pr\{f_{\mathbf{P}}(\mathbf{P}V^n) = V^n\} \cong 1.$$

Step 2: Secret Key Construction

C. Ye - P.N., '05

- SK for terminals 1 and 2

  Terminal 1 sets K_1 = numerical index of x_1^n in coset containing x_1^n;

  Terminal 2 sets K_2 = numerical index of $\widehat{x}_1^n$ in coset containing x_1^n.

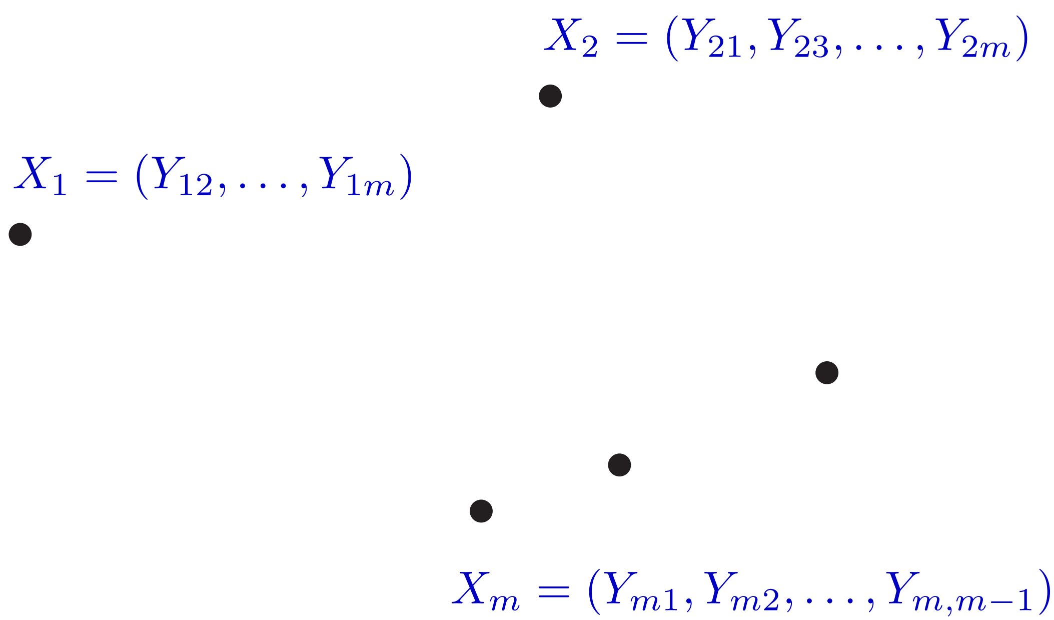
- For a systematic channel code: K_1 (resp. K_2) = first $(n-m)$ bits of x_1^n (resp. $\widehat{x}_1^n$).

- K_1 or K_2 forms an optimal rate SK, since:
  - $\Pr\{K_1 = K_2\} = \Pr\{\widehat{X}_1^n = X_1^n\} \cong 1$; (common randomness)
  - $I(K_1 \wedge \mathbf{F}) = 0$; (secrecy)

    as K_1 conditioned on $\mathbf{F} = \mathbf{P}X_1^n \sim$ uniform $\{1, \cdots, 2^{n-m}\}$;
  - $K_1 \sim$ uniform $\{1, \cdots, 2^{n-m}\}$; (uniformity)
  - $\frac{1}{n}H(K_1) = \frac{n-m}{n} \cong 1 - h_b(p)$. (SK capacity)

TREE PACKING

Pairwise Independent Network (PIN) Model



A special form of a multiterminal source model in which

- $X_i = (Y_{i\,j}, j \in \{1, \ldots, m\} \backslash \{i\}),\ i = 1, \ldots, m$;
- $Y_{i\,j}$ is correlated with $Y_{j\,i}$, $\;1 \leq i \neq j \leq m$;
- the pairs $\{(Y_{i\,j}, Y_{j\,i})\}$ are mutually independent across $1 \leq i < j \leq m$.

Secret Key Capacity for the PIN Model

Proposition [Nitinawarat *et al*, '08]: For a PIN model, the SK capacity for a set of terminals $A \subseteq \mathcal{M} = \{1, \ldots, m\}$ is

$$C_S(A) = \min_{\lambda \in \Lambda(A)} \left[\sum_{1 \le i < j \le m} \left(\sum_{\substack{B \in \mathcal{B}(A): \\ i \in B,\ j \in B^c}} \lambda_B \right) I(Y_{ij} \wedge Y_{ji}) \right].$$

Proof: By an application of a general result [I. Csiszár - P.N., '04] to the PIN model.

Remarks:

(i) $C_S(A)$ depends on the underlying joint probability distribution only through a linear combination of $\{I(Y_{ij} \wedge Y_{ji})\}_{i \neq j}$.

(ii) For $i \neq j$, $I(Y_{ij} \wedge Y_{ji})$ is the *best pairwise SK rate* for the pair of terminals i, j. [Maurer (1993), Ahlswede-Csiszár (1993)].

(iii) The corresponding pairwise SKs are mutually independent.

Secret Key Generation and Tree Packing

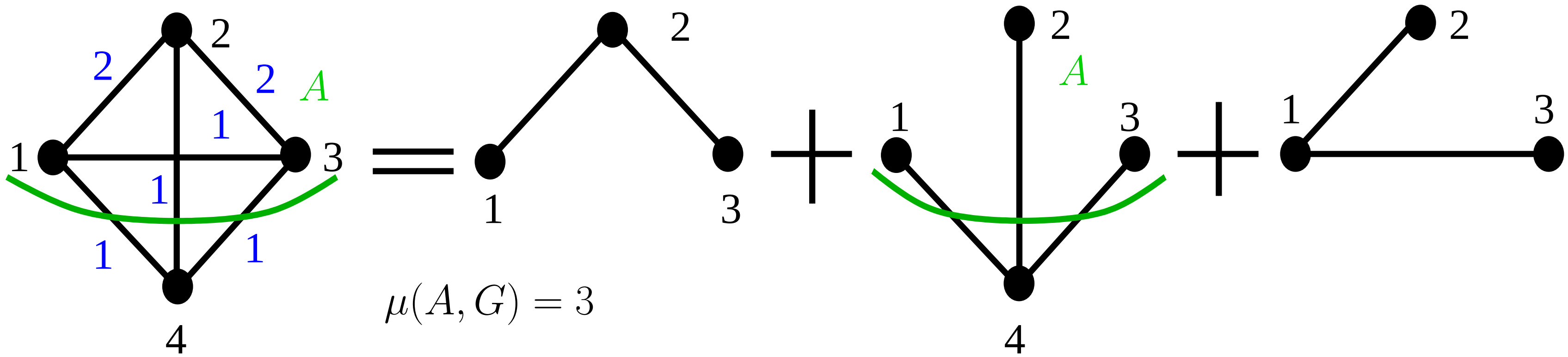
$$C_S(A) = \min_{\lambda \in \Lambda(A)} \left[\sum_{1 \le i < j \le m} \left(\sum_{\substack{B \in \mathcal{B}(A): \\ i \in B,\ j \in B^c}} \lambda_B \right) I(Y_{ij} \wedge Y_{ji}) \right]$$

?? Can a SK for the set of terminals A be formed by propagating *independent* and *locally generated pairwise* SKs in some manner ??

We shall see that

- SK generation for the PIN model can be viewed in terms of tree packing in an associated multigraph;
- reciprocally, a combinatorial problem of tree packing in a multigraph can be viewed in terms of SK generation for an appropriate PIN model.

Steiner Tree Packing



$\mu(A, G) = 3$

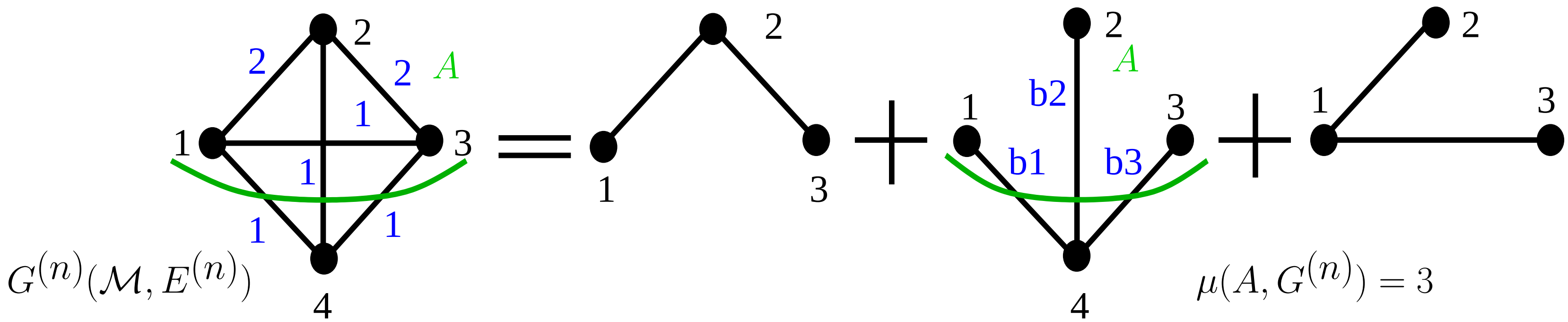
$G(\mathcal{M}, E)$ = multigraph with vertex set $\mathcal{M}$ and edge set E.

Definition

- For $A \subseteq \mathcal{M}$, a *Steiner tree* of G is a subgraph of G which is a tree and whose vertex set contains A.
- A *Steiner tree packing* of G is any collection of *edge*-disjoint Steiner trees of G. Let $\mu(A, G)$ denote the maximum size of such a packing.

How to Generate a Secret Key by Steiner Tree Packing?

- Given a PIN model, calculate $\{I(Y_{ij} \wedge Y_{ji})\}_{i \neq j}$.
- With the given PIN model, associate a multigraph $G^{(n)}(\mathcal{M}, E^{(n)})$ with vertex set $\mathcal{M} = \{1, \ldots, m\}$ and edge set $E^{(n)} = \{e_{ij}^{(n)} = nI(Y_{ij} \wedge Y_{ji})\}_{i \neq j}$.



- *Local SK generation*: For every pair of vertices $(i, j) \in G^{(n)}$, the terminals i, j generate a pairwise SK of size $nI(Y_{ij} \wedge Y_{ji})$ bits; these pairwise SKs are mutually independent.
- *SK propagation by Steiner tree packing*:
  - Claim: *Every Steiner tree corresponds to 1 bit of SK for the terminals in A.*
  - A Steiner packing of size p yields p SK bits shared by the terminals in A.

Remark: For m fixed, this algorithm can be implemented in linear time (in n).

Secret Key Capacity and Maximal Steiner Tree Packing

Theorem [Nitinawarat *et al*, '08]: For a PIN model, the SK capacity satisfies

$$\frac{1}{2}C_S(A) \le \sup_n \frac{1}{n}\ \mu(A, G^{(n)}) \le C_S(A).$$

with the second "$\le$" holding with "$=$" when $|A| = 2$ and $|A| = m$.

A consequence of independent interest: Given a multigraph $G = G^{(1)}$,

- the SK capacity of an associated PIN model with

$$I(Y_{ij} \wedge Y_{ji}) = e_{ij}, \quad 1 \le i < j \le m$$

  provides new (information theoretic) upper and lower bounds for the maximum rate of Steiner tree packing $\sup_n \frac{1}{n}\ \mu(A, G^{(n)})$;

- the bounds are not tight, in general.

Secret Key Capacity and Maximal Spanning Tree Packing

When $A = \mathcal{M} = \{1, \ldots, m\}$, a Steiner tree becomes a spanning tree.

Theorem [Nitinawarat *et al*, '08]: For a PIN model,

$$C_S(\mathcal{M}) = \sup_n \frac{1}{n}\, \mu(\mathcal{M}, G^{(n)}).$$

Idea of proof: By a result of [Nash-Williams, '61] and [Tutte, '61],

$$\sup_n \frac{1}{n}\, \mu(\mathcal{M}, G^{(n)}) = \min_{\mathcal{P}:\ \mathcal{P} \text{ a partition of } \mathcal{M}} \frac{1}{|\mathcal{P}| - 1} \Big(\text{No. of edges of } G^{(1)} \text{that cross } \mathcal{P}\Big),$$

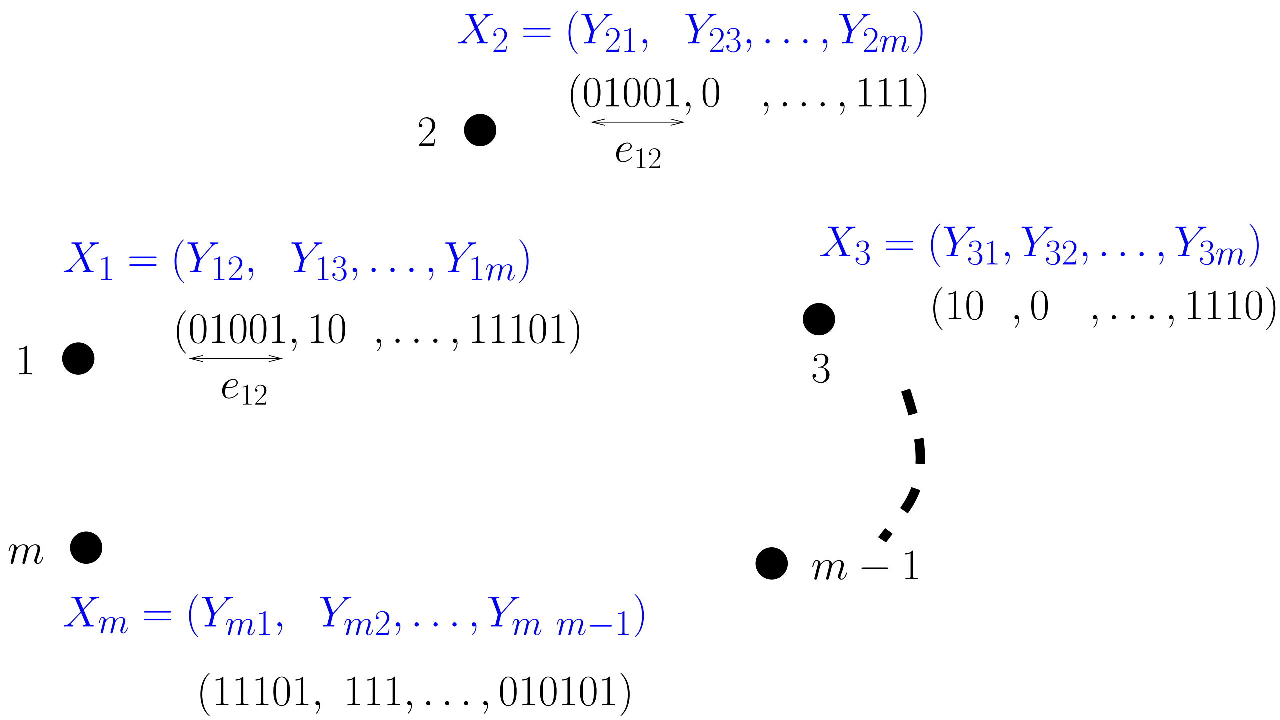
which coincides with an upper bound for $C_S(\mathcal{M})$ in [I. Csiszár-P.N., '04].

Remarks:

(i) Thus, maximal spanning tree packing attains the SK capacity $C_S(\mathcal{M})$.

(ii) There exists a polynomial-time algorithm (in both m, n) for finding a maximal collection of edge-disjoint spanning trees for $G^{(n)}$ [Gabor-Westermann] and forming an optimal rate SK.

Perfect Secret Key for the PIN Model



- $X_i = (Y_{i\,j}, j \in \{1, \ldots, m\} \backslash \{i\}), \ i = 1, \ldots, m;$
- $Y_{ij} = Y_{ji}, \ 1 \leq i < j \leq m$, and $Y_{ij} \ \sim$ unif. $\{0,1\}^{e_{ij}}$;
- $Y_{ij}, \ 1 \leq i < j \leq m$ are mutually independent.

Perfect SK: $Pr\{K = K_i, \ i \in A\} = 1, \ \ I(K \wedge \mathbf{F}) = 0, \ \ H(K) = \log |\text{key space}|.$

Perfect Secret Key Capacity for the PIN Model

Theorem [S. Nitinawarat-P.N, '09]: For a PIN model, the *perfect* SK capacity for a set of terminals $A \subseteq \mathcal{M} = \{1, \ldots, m\}$ is

$$C_S(A) = \min_{\lambda \in \Lambda(A)} \left[\sum_{1 \le i < j \le m} \left(\sum_{\substack{B \in \mathcal{B}(A): \\ i \in B,\ j \in B^c}} \lambda_B \right) e_{ij} \right] = \sum_{i,j} e_{ij} - OMN(A),$$

where $OMN(A)$ is the *minimum rate of linear noninteractive communication for the terminals in A to attain perfect omniscience.* Furthermore, this perfect SK capacity can be achieved with *linear noninteractive* communication.

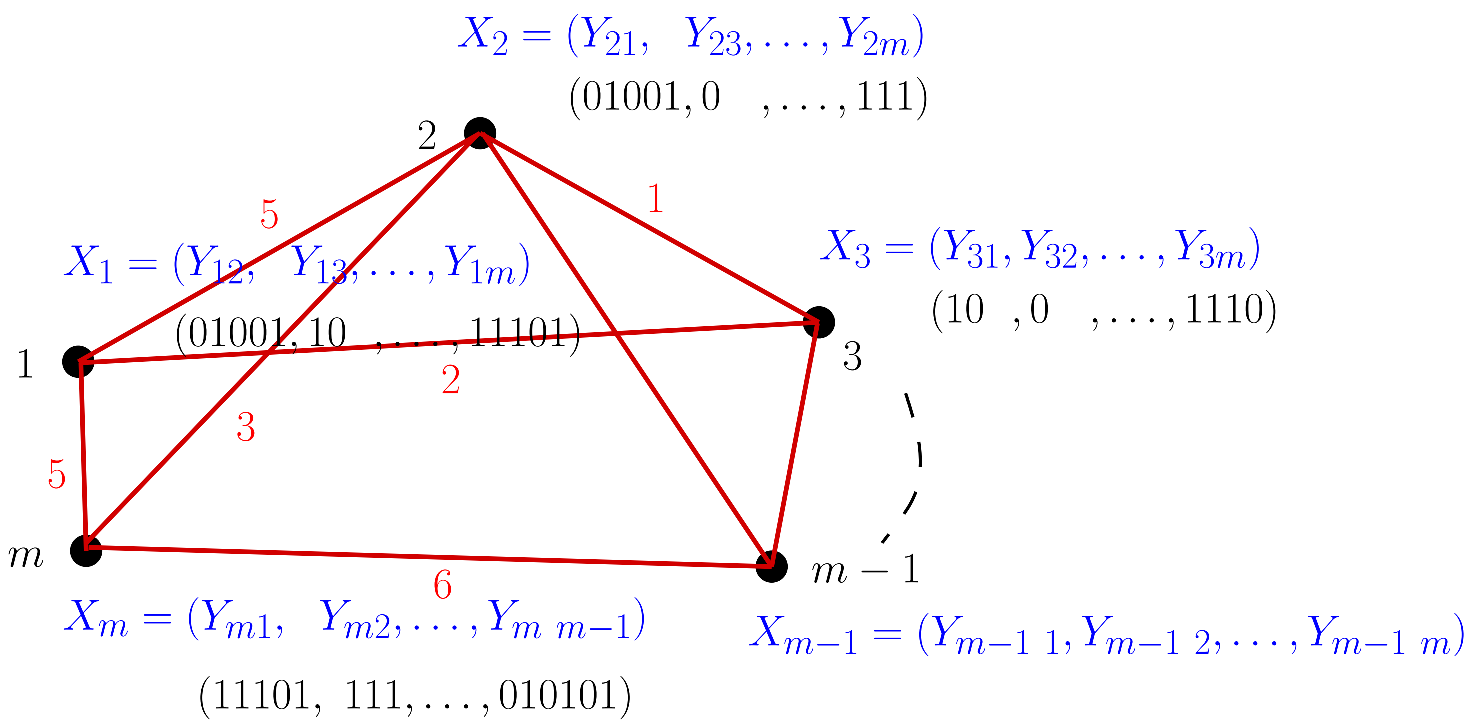
Remarks:

(i) The formula for $C_S(A)$ coincides with that of the (standard) SK capacity.

(ii) The complexity of perfect SK generation in the theorem is prohibitive (exponential in n).

(iii) ?? Is there an efficient algorithm for perfect SK generation? If so, how well does it perform?

PIN Model $\iff$ Multigraph



The PIN model can be described equivalently by a multigraph $G(\mathcal{M}, E)$ with vertex set $\mathcal{M} = \{1, \dots, m\}$ and edge set E consisting of e_{ij} edges connecting the vertices i and j, $1 \le i < j \le m$.

Theorem [S. Nitinawarat-P.N, '09]: For every $A \subseteq \mathcal{M} = \{1, \ldots, m\}$, it holds that

$$\frac{1}{2} C_S(A) \ \leq \ \lim_{n \to \infty} \frac{1}{n} \mu(A, G^{(n)}) \ \leq C_S(A),$$

where $C_S(A)$ is the perfect SK capacity of the PIN model with equivalent multigraph G. Furthermore, for the special cases $|A| = 2$ and $A = \mathcal{M}$,

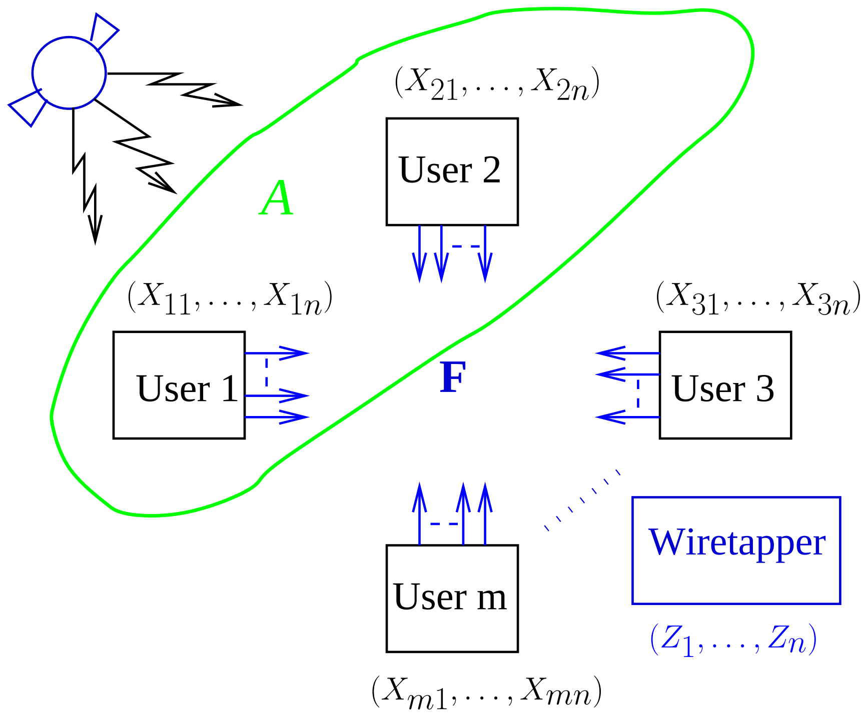
$$\lim_n \frac{1}{n} \mu(A, G^{(n)}) = C_S(A).$$

Outline of proof:

- Based on the fact that every Steiner tree in a packing serves to generate 1 bit of perfect SK for the terminals in A, and all the SK bits so generated are mutually independent.
- Equality when $|A| = 2$ follows from the max-flow min-cut theorem; equality when $A = \mathcal{M}$ follows from a classic result on the maximum size of spanning tree packing in a multigraph [Nash-Williams, '61 and Tutte, '61].

$$\boxed{\textbf{VARIANT MODELS FOR SECRET KEY GENERATION}}$$

Multiterminal Source Model with Wiretapper



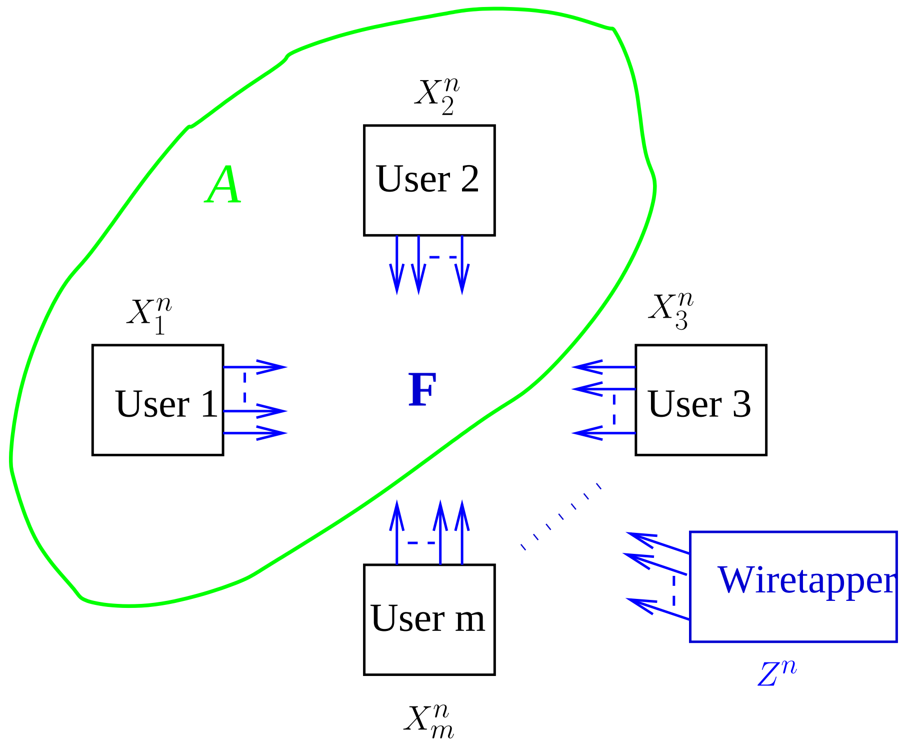
The legitimate user terminals in A wish to generate a secret key K with the cooperation of the remaining legitimate terminals, which is concealed from an eavesdropper with access to the public interterminal communication $\mathbf{F}$ and wiretapped side information $Z^n = (Z_1, \ldots, Z_n)$.

The *secrecy condition* is now strengthened to

$$I(K \wedge \mathbf{F}, Z^n) \cong 0.$$

??? Largest rate of a *wiretap secret key* for A: Unknown in general but for special cases and bounds.

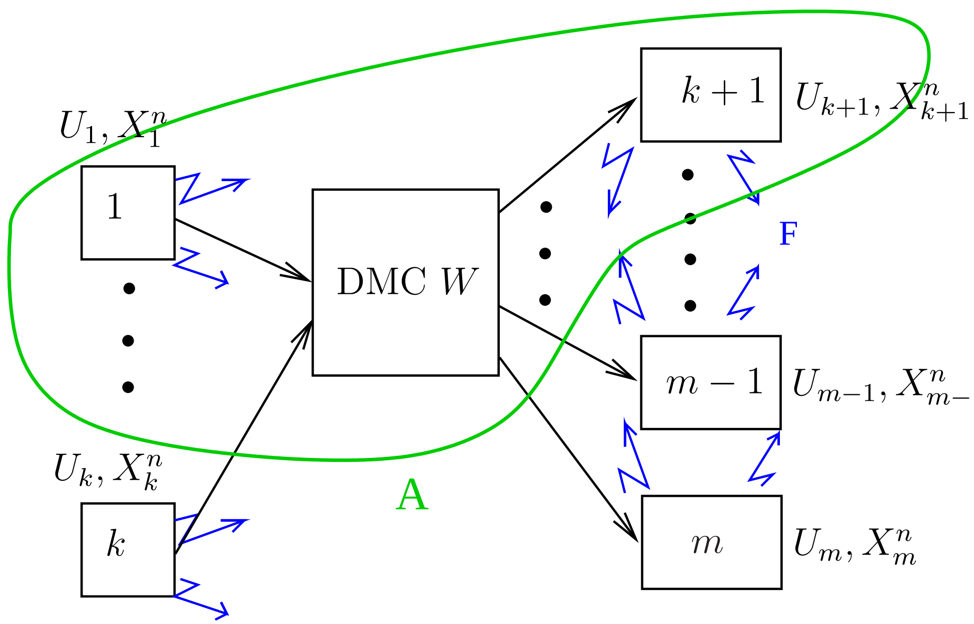
Multiterminal Source Model with Cooperative Wiretapper: Private Key



The wiretapped terminal *cooperates* in the secrecy generation by "revealing" its observations to all the legitimate terminals; the resulting key must be concealed from the eavesdropper which knows $(\mathbf{F}, Z^n)$.

??? Largest rate of a *private key* for A: Known.

Multiterminal Channel Model



??? Largest rate of a *secret key* for A: Unknown in general but for special cases and bounds.

IN CLOSING

A Few Questions

- Information theoretic secrecy generation in a network is intertwined with multiterminal data compression and channel coding for certain network models.
  - What are the *explicit connections* for general network models?
  - What are the corresponding best rates of secret keys?
  - New algorithms for secret key construction?
- Multiuser secrecy generation for the PIN model has connections to the combinatorial problem of tree packing in multigraphs.
  - Tree packing algorithms for global secret key generation?
  - Information theoretic tools for tackling combinatorial tree packing problems?

Idea of Proof of SK Capacity Theorem

Achievability

If L represents "common randomness" for all the terminals in A, achievable with communication F for some (signal) observation length n, then $\frac{1}{n}H(L|\mathbf{F})$ is an achievable SK rate for the terminals in A.

- The terminals communicate publicly using compressed data in order to generate common randomness for the terminals in A equalling $L \cong$ omniscience $= (X_1^n, \ldots, X_m^n)$, with $\mathbf{F} = \mathbf{F}_{CO} = \mathbf{F}_{CO}(X_1^n, \ldots, X_m^n)$.
- The terminals in A then process this L to extract a SK of rate
$$\frac{1}{n}H(L|\mathbf{F}) \cong \frac{1}{n}H(X_1^n, \ldots, X_m^n|\mathbf{F}_{CO}) = H(X_1, \ldots, X_m) - \frac{1}{n}H(\mathbf{F}_{CO})$$
and of which the eavesdropper has provably little or no knowledge.

Converse

Tricky, since interactive communication is not excluded a priori.

Decomposition interpretation:

$$\text{Omniscience} = (X_1^n, \ldots, X_m^n) \cong (\text{Optimum secret key for A}, \ \mathbf{F}_{CO}).$$

Private Key Capacity

Theorem [I. Csiszár - P. N., '04, '08]:

$$
\begin{aligned}
C_P(A|Z) &= H(X_1, \ldots, X_m, Z) - H(Z) - \text{Smallest aggregate rate of} \\
&\quad \text{public communication which enables the terminals in } A \text{ to} \\
&\quad \text{become omniscient when all terminals additionally know } Z^n \\
&= H(X_1, \ldots, X_m | Z) - \max_{\lambda \in \Lambda(A|Z)} \sum_{B \in \mathcal{B}(A|Z)} \lambda_B H\left(X_B | X_{B^c}, Z\right)
\end{aligned}
$$

and can be achieved with noninteractive communication.

Remarks:

- Clearly, WSK capacity $C_W(A|Z) \leq$ PK capacity $C_P(A|Z)$ with equality in special cases.
- Better upper bounds on WSK capacity are available due to Renner-Wolf ('03) and Gohari-Anantharam ('07, '08).

Private Key Generation

Achievability:

- The terminals in A generate common randomness L such that

$$(L, Z^n) \cong (\text{omniscience}, Z^n) = (X_1^n, \ldots, X_m^n, Z^n),$$

  using public interterminal communication $\mathbf{F}_{CO} = \mathbf{F}_{CO}(X_1^n, \ldots, X_m^n, Z^n)$ that is independent of Z^n.

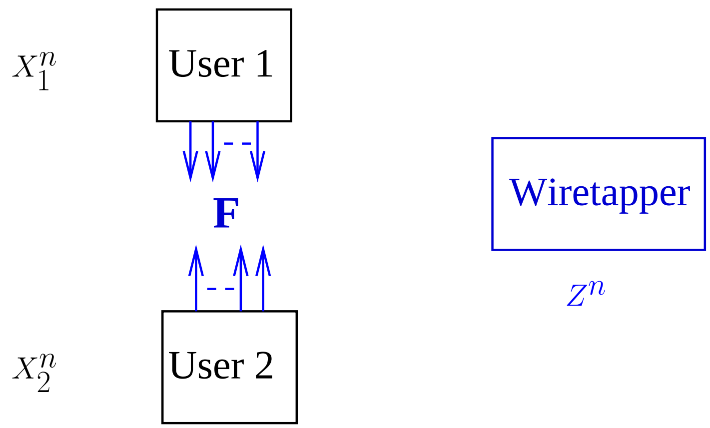
- The terminals in A then extract secrecy of rate

$$\frac{1}{n} H(L, Z^n | \mathbf{F}_{CO}, Z^n) \;\cong\; \cdots \;\cong\; H(X_1, \ldots, X_m | Z) - \frac{1}{n} H(\mathbf{F}_{CO}).$$

Decomposition interpretation:

$$(X_1^n, \ldots, X_m^n, Z^n) \;\cong\; (\text{Optimum private key for A},\ \mathbf{F}_{CO},\ Z^n)\,.$$

Open Problem: The General Wiretapper Model with $\mathcal{M} = \{1, 2\} = A$



Gohari-Anantharam, '07, '08

- Terminals $1, 2$ generate common randomness L using public interterminal communication $\mathbf{F} = \mathbf{F}(X_1^n, X_2^n)$, such that

$$(L, Z^n) \cong (\text{omniscience}, Z^n) = (X_1^n, X_2^n, Z^n).$$

  Note that that $\mathbf{F}$ is *not* a function of Z^n.

- The "nonsingle-letter" characterization of WSK capacity is

$$C_W(A|Z) = \lim_n \max_{L,\mathbf{F}} \frac{1}{n} H(L|\mathbf{F}, Z^n) = \cdots = H(X_1, X_2|Z) - \lim_n \min_{\mathbf{F}} \frac{1}{n} H(\mathbf{F}|Z^n).$$

Question: If L is the *Slepian-Wolf codeword* for the *joint source* (X_1^n, X_2^n) with "decoder side information" Z^n, what is $\lim_n \min_{\mathbf{F}} \frac{1}{n} H(\mathbf{F}|Z^n)$ where $\mathbf{F}$ is the interterminal communication needed to form L by distributed processing?