Hatcher's Algebraic Topology - Solutions

Institute for Pure and Applied Mathematics (IMPA)

Written by: Tiam Koukpari

Trying to collect the fragmented sets of solutions into one file. Here is the sequence of requisites needed for this topic:

$$\begin{tikzcd}
 & \text{Differential Forms (de Rham Cohomology)} \arrow[d, leftrightarrow] & & \\
 & \text{Analysis in } \mathbb{R}^n \subseteq \text{Topology} \subseteq \text{Manifolds} \arrow[drr] & & \\
\text{Set Theory} \arrow[ur] \arrow[dr] & & & \text{Algebraic Topology} \\
 & \text{Groups} \subseteq \text{Rings} \subseteq \text{Fields} \arrow[r] \arrow[d, leftrightarrow] & \text{Linear Algebra} \arrow[ur] & \\
 & \text{Homological Algebra \& Categories} & &
\end{tikzcd}$$

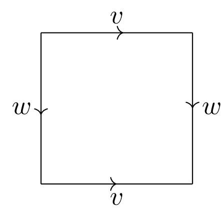
If you find any mistakes, please email tiam.koukpari@impa.br.

Contents

0 Some Underlying Geometric Notions

1. Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

Solution. It is useful to visualize the torus with



To form a torus from the above, fold the shape to connect v with itself, creating two copies of S^1 on w. Then fold the shape to connect w with itself, joining the two copies of S^1 on w and creating another S^1 on v...
■

2. Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto S^{n-1}.

Solution. Construct

$$f_t(\mathbf{x}) = (1-t)\mathbf{x} + t\frac{\mathbf{x}}{|\mathbf{x}|}.$$

Then $f_0(\mathbf{x}) = \mathbf{x}$ so that $f_0 = \mathbb{1}$, $f_1(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|$ so that $f_1 = S^{n-1}$, and $f_t|S^{n-1} = \mathbb{1}$. The function is a straight, continuous line from $\mathbf{x}$ to a normalized $\mathbf{x}$, i.e. on the $(n-1)$-sphere. The function is continuous since $\{0\}$ is not in its domain. ■

3. (a) Show that the composition of homotopy equivalence $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

Solution. □

(b) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

Solution. □

(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Solution. ■

4.

1 The Fundamental Group

1.1 Basic Constructions

1.2 Van Kampen's Theorem

1.3 Covering Spaces

Additional Topics

1.A. Graphs and Free Groups

1.B. K(G,1) Spaces and Graphs of Groups

2 Homology

2.1 Simplicial and Singular Homology

2.2 Computations and Applications

2.3 The Formal Viewpoint

Additional Topics

2.A. Homology and Fundamental Group

2.B. Classical Applications

2.C. Simplicial Approximation

3 Cohomology

3.1 Cohomology Groups

3.2 Cup Product

3.3 Poincaré Duality

Additional Topics

3.A. Universal Coefficients for Homology

3.B. The General Künneth Formula

3.C. H–Spaces and Hopf Algebras

3.D. The Cohomology of SO(n)

3.E. Bockstein Homomorphisms

3.F. Limits and Ext

3.G. Transfer Homomorphisms

3.H. Local Coefficients

4 Homotopy Theory

4.1 Homotopy Groups

4.2 Elementary Methods of Calculation

4.3 Connections with Cohomology

Additional Topics

4.A. Basepoints and Homotopy

4.B. The Hopf Invariant

4.C. Minimal Cell Structures

4.D. Cohomology of Fiber Bundles

4.E. The Brown Representability Theorem

4.F. Spectra and Homology Theories

4.G. Gluing Constructions

4.H. Eckmann-Hilton Duality

4.I. Stable Splittings of Spaces

4.J. The Loopspace of a Suspension

4.K. The Dold-Thom Theorem

4.L. Steenrod Squares and Powers