Hatcher's Algebraic Topology - Solutions

Institute for Pure and Applied Mathematics (IMPA)

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Trying to collect the fragmented sets of solutions into one file. Here is the sequence of requisites needed for this topic:

$$
\begin{tikzcd}
 & \text{Differential Forms (de Rham Cohomology)} \arrow[d, leftrightarrow] & & \\
 & \text{Analysis in } \mathbb{R}^n \subseteq \text{Topology} \subseteq \text{Manifolds} \arrow[drr] & & \\
\text{Set Theory} \arrow[ur] \arrow[dr] & & & \text{Algebraic Topology} \\
 & \text{Groups} \subseteq \text{Rings} \subseteq \text{Fields} \arrow[r] \arrow[d, leftrightarrow] & \text{Linear Algebra} \arrow[ur] & \\
 & \text{Homological Algebra \& Categories} & &
\end{tikzcd}
$$

References, if used, are included at the end of each exercise.

If you find any mistakes or if you want to submit a solution, please email tiam.koukpari@impa.br. The remaining problems are:

Chapter 0:

1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29

Chapter 1:

1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 1.10, 1.11, 1.12, 1.13, 1.14, 1.15, 1.16, 1.17, 1.18, 1.19, 1.20
2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 2.16, 2.17, 2.18, 2.19, 2.20, 2.21, 2.22
3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.16, 3.17, 3.18, 3.19, 3.20, 3.21, 3.22, 3.23, 3.24, 3.25, 3.26, 3.27, 3.28, 3.29, 3.30, 3.31, 3.32, 3.33
A.1, A.2, A.3, A.4, A.5, A.6, A.7, A.8, A.9, A.10, A.11, A.12, A.13, A.14
B.1, B.2, B.3, B.4, B.5, B.6, B.7, B.8, B.9

Chapter 2:

1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 1.10, 1.11, 1.12, 1.13, 1.14, 1.15, 1.16, 1.17, 1.18, 1.19, 1.20, 1.21, 1.22, 1.23, 1.24, 1.26, 1.27, 1.28, 1.29, 1.30, 1.31
2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 2.16, 2.17, 2.18, 2.19, 2.20, 2.21, 2.22
2.23, 2.24, 2.25, 2.26, 2.27, 2.28, 2.29, 2.30, 2.31, 2.32, 2.33, 2.34, 2.35, 2.36, 2.37, 2.38, 2.39, 2.40, 2.41, 2.42, 2.43
3.1, 3.2, 3.3, 3.4
B.1, B.2, B.3, B.4, B.5, B.6, B.7, B.8, B.9, B.10, B.11
C.1, C.2, C.3, C.4, C.5, C.6, C.7, C.8, C.9

Chapter 3:

1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 1.10, 1.11, 1.12, 1.13

2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 2.16, 2.17, 2.18

3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.16, 3.17, 3.18, 3.19, 3.20, 3.21, 3.22, 3.23, 3.24, 3.25, 3.26, 3.27, 3.28, 3.29, 3.30, 3.31, 3.32, 3.33, 3.34, 3.35

A.1, A.2, A.3, A.4, A.5, A.6

B.1, B.2, B.3, B.4, B.5

C.1, C.2, C.3, C.4, C.5, C.6, C.7, C.8, C.9, C.10, C.11, C.12, C.13, C.14, C.15, C.16

D.1, D.2, D.3

E.1, E.2, E.3, E.4

F.1, F.2, F.3, F.4, F.5, F.6, F.7, F.8, F.9

H.1, H.2, H.3, H.4, H.5, H.6

Chapter 4:

1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 1.10, 1.11, 1.12, 1.13, 1.14, 1.15, 1.16, 1.17, 1.18, 1.19, 1.20, 1.21, 1.22, 1.23

2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 2.16, 2.17, 2.18, 2.19, 2.20, 2.21, 2.22 2.23, 2.24, 2.25, 2.26, 2.27, 2.28, 2.29, 2.30, 2.31, 2.32, 2.33, 2.34, 2.35, 2.36, 2.37, 2.38, 2.39

3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.16, 3.17, 3.18, 3.19, 3.20, 3.21, 3.22, 3.23, 3.24

A.1, A.2, A.3, A.4, A.5

B.1, B.2

D.1, D.2, D.3, D.4, D.5, D.6, D.7, D.8, D.9, D.10

F.1, F.2, F.3

G.1, G.2, G.3, G.4

H.1, H.2, H.3, H.4

I.1 I.2, I.3

J.1

K.1, K.2, K.3, K.4, K.5, K.6

L.1, L.2, L.3, L.4, L.5

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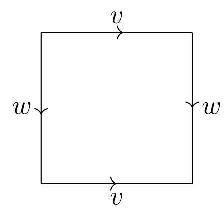
0 Some Underlying Geometric Notions

1. Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

Solution. It is useful to visualize the torus with



To form a torus from the above, fold the shape to connect v with itself, creating two copies of S^1 on w. Then fold the shape to connect w with itself, joining the two copies of S^1 on w and creating another S^1 on v...
■

2. Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto S^{n-1}.

Solution. Construct

$$f_t(\mathbf{x}) = (1-t)\mathbf{x} + t\frac{\mathbf{x}}{|\mathbf{x}|}.$$

Then $f_0(\mathbf{x}) = \mathbf{x}$ so that $f_0 = \mathbb{1}$, $f_1(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|$ so that $f_1 = S^{n-1}$, and $f_t|S^{n-1} = \mathbb{1}$. The function is a straight, continuous line from $\mathbf{x}$ to a normalized $\mathbf{x}$, i.e. on the $(n-1)$-sphere. The function is continuous since $\{0\}$ is not in its domain. ■

3. (a) Show that the composition of homotopy equivalence $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

Solution. Let $f : X \to Y$ be a homotopy equivalence and $f^{-1} : Y \to X$ its inverse. Similarly, let $g : Y \to Z$ be a homotopy equivalence and $g^{-1} : Z \to Y$ its inverse. Construct $h := g \circ f$ and $h^{-1} := f^{-1} \circ g^{-1}$. We want to show that $h \circ h^{-1} \simeq \mathbb{1}$:

$$h \circ h^{-1} = g \circ f \circ f^{-1} \circ g^{-1} \simeq g \circ \mathbb{1} \circ g^{-1} = g \circ g^{-1} \simeq \mathbb{1}.$$

(b) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

Solution. □

(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Solution. ■

4. A **deformation retraction in the weak sense** of a space X to a subspace A is a homotopy $f_t : X \to X$ such that $f_0 = \mathbb{1}$, $f_1(X) \subset A$, and $f_t(A) \subset A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

Solution. ■

5. Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of x in X there exists a neighborhood $V \subset U$ of x such that the inclusion map $V \hookrightarrow U$ is nullhomotopic.

Solution. ■

1 The Fundamental Group

1.1 Basic Constructions

1.2 Van Kampen's Theorem

1.3 Covering Spaces

Additional Topics

1.A. Graphs and Free Groups

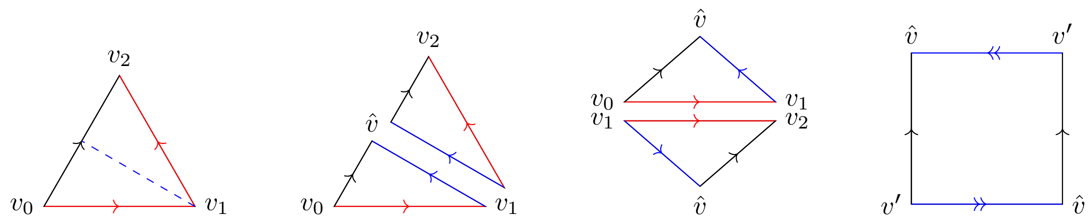
1.B. K(G,1) Spaces and Graphs of Groups

2 Homology

2.1 Simplicial and Singular Homology

1. What familiar space is the quotient δ-complex of a 2 simplex $[v_0, v_1, v_2]$ obtained by identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$, preserving the ordering of vertices?

Solution The Möbius strip. We draw the same construction as in the reference:



The latter being the Möbius strip. ■

References: 1.

2. Show that the δ-complex obtained from δ^3 by performing the order-preserving edge identifications $[v_0, v_1] \sim [v_1, v_3]$ and $[v_0, v_2] \sim [v_2, v_3]$ deformation retracts onto a Klein bottle. Also, find the other pairs of identifications of edges that produce δ-complexes deformation retracting onto a torus, a 2-sphere, and $\mathbb{R}P^2$.

Solution. ■

11. Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.

Solution. ■

15. For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ iff the map $A \to B$ is surjective and $D \to E$ is injective. Hence for a pair of spaces (X, A), the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n.

Solution. ■

25. Find an explicit, noninductive formula for the barycentric subdivision operator $S : C_n(X) \to C_n(X)$.

Solution. In general we have the inductive operator taking $\sigma \in C_n(X) \to C_n(X)$ by

$$B_p(\sigma) = b(\sigma)\,(B_{p-1}(\partial\sigma))$$

where b is the barycenter of σ. For $n = 1$, we have

$$\begin{aligned} B[v_0, v_1] &= b([v_0, v_1])(B\partial[v_0, v_1]) = b([v_0, v_1])(B([v_1] - [v_0])) \\ &= b([v_0, v_1])([v_1] - [v_0]) = \left[\frac{v_0 + v_1}{2}, v_1\right] - \left[\frac{v_0 + v_1}{2}, v_0\right]. \end{aligned}$$

For $n = 2$, we have

$$B[v_0, v_1, v_2] = b([v_0, v_1, v_2])(B\partial[v_0, v_1, v_2]) = b([v_0, v_1, v_2])(B([v_1, v_2] - [v_0, v_2] + [v_0, v_1]))$$

$$= b([v_0, v_1, v_2])\left(\left[\frac{v_1 + v_2}{2}, v_2\right] - \left[\frac{v_1 + v_2}{2}, v_1\right] - \left[\frac{v_0 + v_2}{2}, v_2\right] + \left[\frac{v_0 + v_2}{2}, v_0\right] + \left[\frac{v_0 + v_1}{2}, v_1\right] - \left[\frac{v_0 + v_1}{2}, v_0\right]\right)$$

$$= \left[\frac{v_0 + v_1 + v_2}{3}, \frac{v_1 + v_2}{2}, v_2\right] - \cdots + \left[\frac{v_0 + v_1 + v_2}{3}, \frac{v_0 + v_1}{2}, v_1\right] - \left[\frac{v_0 + v_1 + v_2}{3}, \frac{v_0 + v_1}{2}, v_0\right].$$

And now we can see a clear pattern where at each iteration, we add the barycenter of the n-th simplex to the image of the operator acting on the $(n-1)$-th simplex. We construct the non-inductive barycenter operator as

$$B(\sigma_n) := \sum_{\pi \in S_{n+1}} \operatorname{sign}(\pi) \left[\frac{\sum_{i=0}^{n} v_i}{n+1}, \frac{\sum_{i=0}^{n-1} v_i^{\pi}}{n}, \dots, \frac{\sum_{0}^{1} v_i^{\pi}}{1}, v_0^{\pi} \right]$$

where S_n is the permutation group of n vertices, $\operatorname{sign}(\pi)$ is the orientation of each permutation π, and where it applies, v^{π} means the vertices that belong to the $(n-1)$-simplex of the π-th permutation. Note that in each element, we are summing over the i-th vertex of a permutation, and not the i-th index of σ_n. For example, in the last element, v_0^{π} means the 0-th element of the π-th permutation, which could mean v_0, v_1, v_2, and so on. It does not strictly mean v_0. This is exemplified in our example for $n = 2$. ■

2.2 Computations and Applications

2.3 The Formal Viewpoint

Additional Topics

2.A. Homology and Fundamental Group

2.B. Classical Applications

2.C. Simplicial Approximation

3 Cohomology

3.1 Cohomology Groups

3.2 Cup Product

3.3 Poincaré Duality

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