Hatcher's Algebraic Topology - Solutions

Institute for Pure and Applied Mathematics (IMPA)

Written by: Tiam Koukpari

Trying to collect the fragmented sets of solutions into one file. Here is the sequence of requisites needed for this topic:

$$
\begin{tikzcd}
 & \text{Differential Forms (de Rham Cohomology)} \arrow[d, leftrightarrow] & & \\
 & \text{Analysis in } \mathbb{R}^n \subseteq \text{Topology} \subseteq \text{Manifolds} \arrow[drr] & & \\
\text{Set Theory} \arrow[ur] \arrow[dr] & & & \text{Algebraic Topology} \\
 & \text{Groups} \subseteq \text{Rings} \subseteq \text{Fields} \arrow[r] \arrow[d, leftrightarrow] & \text{Linear Algebra} \arrow[ur] & \\
 & \text{Homological Algebra \& Categories} & &
\end{tikzcd}
$$

References, if used, are included at the end of each exercise.

If you find any mistakes or if you want to submit a solution, please email tiam.koukpari@impa.br. The remaining problems are:

Chapter 0:

3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29

Chapter 1:

1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 1.10, 1.11, 1.12, 1.13, 1.14, 1.15, 1.16, 1.17, 1.18, 1.19, 1.20
2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 2.16, 2.17, 2.18, 2.19, 2.20, 2.21, 2.22
3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.16, 3.17, 3.18, 3.19, 3.20, 3.21, 3.22, 3.23, 3.24, 3.25, 3.26, 3.27, 3.28, 3.29, 3.30, 3.31, 3.32, 3.33
A.1, A.2, A.3, A.4, A.5, A.6, A.7, A.8, A.9, A.10, A.11, A.12, A.13, A.14
B.1, B.2, B.3, B.4, B.5, B.6, B.7, B.8, B.9

Chapter 2:

1.2, 1.3, 1.6, 1.7, 1.8, 1.10, 1.12, 1.13, 1.14, 1.16, 1.17, 1.18, 1.19, 1.20, 1.21, 1.22, 1.23, 1.24, 1.26, 1.27, 1.28, 1.29, 1.30, 1.31
2.2, 2.3, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 2.16, 2.17, 2.18, 2.19, 2.22 2.23, 2.24, 2.25, 2.26, 2.27, 2.28, 2.29, 2.30, 2.31, 2.33, 2.35, 2.36, 2.38, 2.39, 2.40, 2.42, 2.43
3.1, 3.2, 3.3, 3.4
B.1, B.2, B.3, B.5, B.6, B.7, B.8, B.9, B.10, B.11
C.1, C.2, C.3, C.4, C.5, C.6, C.7, C.8, C.9

Chapter 3:

1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.11, 1.12, 1.13

2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 2.16, 2.17, 2.18

3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.17, 3.18, 3.19, 3.20, 3.21, 3.22, 3.23, 3.24, 3.25, 3.26, 3.27, 3.28, 3.29, 3.30, 3.31, 3.32, 3.33, 3.34, 3.35

A.1, A.2, A.3, A.4, A.5, A.6

B.1, B.2, B.3, B.4, B.5

C.1, C.2, C.3, C.4, C.5, C.6, C.7, C.8, C.9, C.10, C.11, C.12, C.13, C.14, C.15, C.16

D.1, D.2, D.3

E.1, E.2, E.3, E.4

F.1, F.2, F.3, F.4, F.5, F.6, F.7, F.8, F.9

H.1, H.2, H.3, H.4, H.5, H.6

Chapter 4:

1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 1.10, 1.11, 1.12, 1.13, 1.14, 1.15, 1.16, 1.17, 1.18, 1.19, 1.20, 1.21, 1.22, 1.23

2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 2.16, 2.17, 2.18, 2.19, 2.20, 2.21, 2.22 2.23, 2.24, 2.25, 2.26, 2.27, 2.28, 2.29, 2.30, 2.31, 2.32, 2.33, 2.34, 2.35, 2.36, 2.37, 2.38, 2.39

3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.16, 3.17, 3.18, 3.19, 3.20, 3.21, 3.22, 3.23, 3.24

A.1, A.2, A.3, A.4, A.5

B.1, B.2

D.1, D.2, D.3, D.4, D.5, D.6, D.7, D.8, D.9, D.10

F.1, F.2, F.3

G.1, G.2, G.3, G.4

H.1, H.2, H.3, H.4

I.1 I.2, I.3

J.1

K.1, K.2, K.3, K.4, K.5, K.6

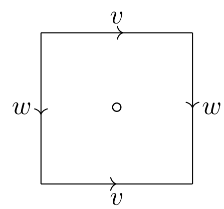
L.1, L.2, L.3, L.4, L.5

Contents

0 Some Underlying Geometric Notions

1. Construct an explicit deformation retraction of the torus with one point deleted onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

Solution. It is useful to visualize the torus with a 2×2 square centered at the origin:



To form a torus, fold the shape to connect v with itself, creating two copies of S^1 on w. Then fold the shape to connect w with itself, joining the two copies of S^1 on w and creating another S^1 on v. Without loss of generality, assume the deleted point is at the origin. As in the reference, construct

$$f_t(x,y) = (1-t)(x,y) + t\left(\frac{(x,y)}{\max\{|x|,|y|\}}\right).$$

Then $f_0(x,y) = (x,y)$ so that $f_0 = \mathbb{1}$, $f_1(x,y) = (x,y)/\max\{|x|,|y|\}$ so that $f_1 = S^1 \vee S^1$, and $f_1|S^1 \vee S^1 = \mathbb{1}$ since $\max\{|x|,|y|\} = 1$ on the boundary. The function is continuous since $(0,0)$ is not in its domain. ∎

Remark. This may or may not be an acceptable solution, depending on whether 'explicit' means '3-space.'

References: 1.

2. Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto S^{n-1}.

Solution. Construct

$$f_t(\mathbf{x}) = (1-t)\mathbf{x} + t\frac{\mathbf{x}}{|\mathbf{x}|}.$$

Then $f_0(\mathbf{x}) = \mathbf{x}$ so that $f_0 = \mathbb{1}$, $f_1(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|$ so that $f_1 = S^{n-1}$, and $f_t|S^{n-1} = \mathbb{1}$. The function is a straight, continuous line from $\mathbf{x}$ to a normalized $\mathbf{x}$, i.e. on the $(n-1)$-sphere. The function is continuous since $\{0\}$ is not in its domain. ∎

3. (a) Show that the composition of homotopy equivalence $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

Solution. Let $f : X \to Y$ be a homotopy equivalence and $f^{-1} : Y \to X$ its inverse. Similarly, let $g : Y \to Z$ be a homotopy equivalence and $g^{-1} : Z \to Y$ its inverse. Construct $h := g \circ f$ and $h^{-1} := f^{-1} \circ g^{-1}$. We want to show that $h \circ h^{-1} \simeq \mathbb{1}$:

$$h \circ h^{-1} = g \circ f \circ f^{-1} \circ g^{-1} \simeq g \circ \mathbb{1} \circ g^{-1} = g \circ g^{-1} \simeq \mathbb{1}.$$

(b) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

Solution. □

(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Solution. ∎

4. A **deformation retraction in the weak sense** of a space X to a subspace A is a homotopy $f_t : X \to X$ such that $f_0 = \mathbb{1}$, $f_1(X) \subset A$, and $f_t(A) \subset A$ for all t. Show that if X deformation retracts to A in this weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

Solution. ■

5. Show that if a space X deformation retracts to a point $x \in X$, then for each neighborhood U of x in X there exists a neighborhood $V \subset U$ of x such that the inclusion map $V \hookrightarrow U$ is nullhomotopic.

Solution. ■

1 The Fundamental Group

1.1 Basic Constructions

1.2 Van Kampen's Theorem

1.3 Covering Spaces

Additional Topics

1.A. Graphs and Free Groups

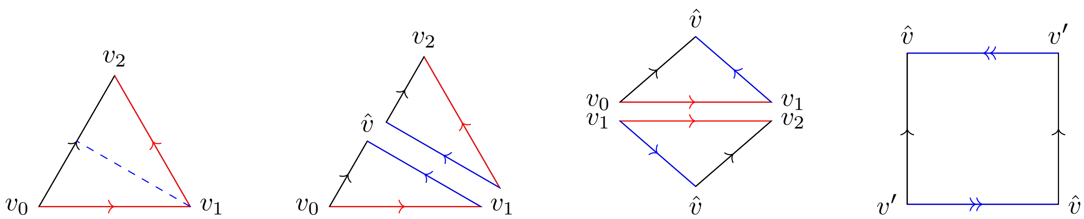
1.B. K(G,1) Spaces and Graphs of Groups

2 Homology

2.1 Simplicial and Singular Homology

1. What familiar space is the quotient Δ-complex of a 2 simplex $[v_0, v_1, v_2]$ obtained by identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$, preserving the ordering of vertices?

Solution The Möbius strip. We draw the same construction as in the reference:



The latter being the Möbius strip. ■

References: 1.

2. Show that the Δ-complex obtained from Δ^3 by performing the order-preserving edge identifications $[v_0, v_1] \sim [v_1, v_3]$ and $[v_0, v_2] \sim [v_2, v_3]$ deformation retracts onto a Klein bottle. Also, find the other pairs of identifications of edges that produce Δ-complexes deformation retracting onto a torus, a 2-sphere, and $\mathbb{R}P^2$.

Solution. ■

3. Construct a Δ-complex structure on $\mathbb{R}P^n$ as a quotient of a Δ-complex structure on S^n having vertices the two vectors of length along each coordinate axis in $\mathbb{R}^{n+1}$.

Solution. ■

4. Compute the simplicial homology groups of the triangular parachute obtained from Δ^2 by identifying its three vertices to a single point.

Solution. The face is generated by U, edges by a, b and c, and vertex by v. The boundary operators (according to the ordering given in the reference) are given by

$$\partial U_2 = b - c + a, \quad \partial_1 a = \partial_1 b = \partial_1 c = \partial_0 v = 0.$$

The first two homology groups are then given by

$$H_0^{\Delta} = \frac{\ker(\partial_0)}{\operatorname{im}(\partial_1)} = \frac{\langle v \rangle}{0} = \mathbb{Z}, \quad H_1^{\Delta} = \frac{\ker(\partial_1)}{\operatorname{im}(\partial_2)} = \frac{\langle a, b, c \rangle}{\langle b - c + a \rangle} = \frac{\langle a, b, b - c + a \rangle}{\langle b - c + a \rangle} = \langle a, b \rangle = \mathbb{Z}^2.$$

For $k \geq 2$, $H_k^{\Delta} = 0$. ■

References: 1.

5. Compute the simplicial homology groups of the Klein bottle using the Δ-complex structure described at the beginning of this section.

Solution. The faces are generated by U and L, edges by a, b and c, and vertex by v. The boundary operators are given by

$$\partial_2 U = a + b - c, \quad \partial_2 L = a - b + c, \quad \partial_1 a = \partial_1 b = \partial_1 c = \partial_0 v = 0.$$

The first two homology groups are then given by

$$H_0^\Delta = \frac{\ker(\partial_0)}{\operatorname{im}(\partial_1)} = \frac{\langle v\rangle}{0} = \mathbb{Z}, \quad H_1^\Delta = \frac{\ker(\partial_1)}{\operatorname{im}(\partial_2)} = \frac{\langle a,b,c\rangle}{\langle a+b-c, a-b+c\rangle} = \frac{\langle a,b,c\rangle}{\langle a+b-(b-a), c\rangle} = \frac{\langle a,b\rangle}{\langle 2a\rangle} = \mathbb{Z}_2 \oplus \mathbb{Z}.$$

For $k \geq 2$, $H_k^\Delta = 0$. ∎

6. Compute the simplicial homology groups of the Δ-complex obtained from $n+1$ 2-simplices $\Delta_0^2, \dots, \Delta_n^2$ by identifying all three edges of Δ_0^2 to a single edge, and for $i > 0$ identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$ of Δ_i^2 to a single edge and the edge $[v_0, v_2]$ to the edge $[v_0, v_1]$ of Δ_{i-1}^2.

Solution. ∎

7. Find a way of identifying pairs of faces of Δ^3 to produce a Δ-complex structure on S^3 having a single 3-simplex, and compute the simplicial homology groups of this Δ-complex.

Solution. Identify any two faces together, then identify the remaining two faces together. For example, identify 123 with 023, and 012 with 013. The construction is a single 3-simplex 0123, two 2-simplices 123 = 023 and 012 = 013, three 1-simplices 02 = 03 = 12 = 13, 01 and 23, and two 0-simplices 0 = 1 and 2 = 3.

We can easily compute $\partial_3(0123) = 0$. Next we compute the lower-dimensional boundaries:

$$\partial_2(123) = \partial_2(023) = (23) - (03) + (02) = (23), \quad \partial_2(012) = \partial_2(013) = (13) - (03) + (01) = (01)$$
$$\partial_1(02) = \partial_1(03) = \partial_1(12) = \partial_1(13) = (1) - (3), \quad \partial_1(01) = 0, \quad \partial_1(23) = 0$$
$$\partial_0(0) = \partial_0(1) = 0, \quad \partial_0(2) = \partial_0(3) = 0.$$

The simplicial homology groups are then

$$H_0^\Delta = \frac{\ker(\partial_0)}{\operatorname{im}(\partial_1)} = \frac{\langle (0),(2)\rangle}{\langle (1)-(3), 0, 0\rangle} = \mathbb{Z}, \quad H_1^\Delta = \frac{\ker(\partial_1)}{\operatorname{im}(\partial_2)} = 0 = \frac{\ker(\partial_2)}{\operatorname{im}(\partial_3)} = H_2^\Delta$$
$$H_3^\Delta = \frac{\ker(\partial_3)}{\operatorname{im}(\partial_4)} = \frac{\langle (0123)\rangle}{0} = \mathbb{Z}. \quad ∎$$

9. Compute the homology groups of the Δ-complex X obtained from Δ^n by identifying all faces of the same dimension. Thus X has a single k simplex for each $k \leq n$.

Solution. The boundary operators are given by $\partial_k = \sum_{i=0}^k (-1)^i [v_1, \dots, \hat{v}_i, \dots, v_k] = 0$ if k is even and $[v_1, \dots, v_{k-1}]$ otherwise. If $k > n$, then $H_k^\Delta = 0$. If $k < n$, then

$$H_k^\Delta = \frac{\ker(\partial_k)}{\operatorname{im}(\partial_{k+1})} = \begin{cases} \mathbb{Z}/\mathbb{Z} = 0 & \text{if } k \text{ is even} \\ 0/0 = 0 & \text{if } k \text{ is odd.} \end{cases}$$

Finally, since the image of ∂_{n+1} is 0, if $k = n$, then

$$H_n^\Delta = \ker(\partial_n) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \quad ∎$$

11. Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.

Solution. By the long exact sequence of a pair (X, A), we have

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots \to H_0(X,A) \to 0.$$

Since A is a retract of X, $H_i(X, A) = 0$ for all i and the above reduces to

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} 0 \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots \to H_0(X) \to 0.$$

Since $0 \to A \xrightarrow{\alpha} B$ is exact iff $\ker\alpha = 0$, and the above sequence is exact, we conclude that $\ker i_* = 0$ and i_* is injective. □

Alternate. If A is a retract of X, then there exists $r : X \to A$ such that $r \circ i = \mathbb{1}_A$. Since the induced identity is the identity, we have $(r \circ i)_* = r_* \circ i_* = \mathbb{1}_* = \mathbb{1}$, which implies i_* is injective. ■

15. For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ iff the map $A \to B$ is surjective and $D \to E$ is injective. Hence for a pair of spaces (X, A), the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n.

Solution. We first label the above sequence:

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E.$$

If $C = 0$, then $\ker\beta = B$ and since the sequence is exact, $\mathrm{im}\alpha = \ker\beta = B$. So $\alpha : A \to B$ is surjective. We also have $0 = \mathrm{im}\gamma = \ker\delta$ so that $\delta : D \to E$ is injective. Conversely, if $\alpha : A \to B$ is surjective, then $B = \mathrm{im}\alpha = \ker\beta$ so that $\beta = 0$ and the above sequence reduces to

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} 0 \to C \xrightarrow{\gamma} D \xrightarrow{\delta} E.$$

Similarly, if $\delta : D \to E$ is injective, then $\mathrm{im}\gamma = \ker\delta = 0$ and the above sequence reduces to

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} 0 \to C \to 0 \xrightarrow{\gamma} D \xrightarrow{\delta} E.$$

Hence $C = 0$. The implication being for the long exact sequence

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots \to H_0(X, A) \to 0$$

with $H_n(X, A) = 0$. At each dimension, the above sequence reduces to

$$\cdots \to 0 \to H_n(A) \xrightarrow{i_*} H_n(X) \to 0 \to \cdots$$

implying that the inclusion $i : A \hookrightarrow X$ induces isomorphisms $H_n(A) \cong H_n(X)$. ■

25. Find an explicit, noninductive formula for the barycentric subdivision operator $S : C_n(X) \to C_n(X)$.

Solution. In general we have the inductive operator taking $\sigma \in C_n(X) \to C_n(X)$ by

$$B_p(\sigma) = b(\sigma)\,(B_{p-1}(\partial\sigma))$$

where b is the barycenter of σ. For $n = 1$, we have

$$\begin{aligned} B[v_0, v_1] &= b([v_0, v_1])(B\partial[v_0, v_1]) = b([v_0, v_1])(B([v_1] - [v_0])) \\ &= b([v_0, v_1])([v_1] - [v_0]) = \left[\frac{v_0 + v_1}{2}, v_1\right] - \left[\frac{v_0 + v_1}{2}, v_0\right]. \end{aligned}$$

For $n = 2$, we have

$$\begin{aligned} B[v_0, v_1, v_2] &= b([v_0, v_1, v_2])(B\partial[v_0, v_1, v_2]) = b([v_0, v_1, v_2])(B([v_1, v_2] - [v_0, v_2] + [v_0, v_1])) \\ &= b([v_0, v_1, v_2])\left(\left[\frac{v_1 + v_2}{2}, v_2\right] - \left[\frac{v_1 + v_2}{2}, v_1\right] - \left[\frac{v_0 + v_2}{2}, v_2\right] + \left[\frac{v_0 + v_2}{2}, v_0\right] + \left[\frac{v_0 + v_1}{2}, v_1\right] - \left[\frac{v_0 + v_1}{2}, v_0\right]\right) \\ &= \left[\frac{v_0 + v_1 + v_2}{3}, \frac{v_1 + v_2}{2}, v_2\right] - \cdots + \left[\frac{v_0 + v_1 + v_2}{3}, \frac{v_0 + v_1}{2}, v_1\right] - \left[\frac{v_0 + v_1 + v_2}{3}, \frac{v_0 + v_1}{2}, v_0\right]. \end{aligned}$$

And now we can see a clear pattern where at each iteration, we add the barycenter of the n-th simplex to the image of the operator acting on the $(n-1)$-th simplex. We construct the non-inductive barycenter operator as

$$B(\sigma_n) := \sum_{\pi \in S_{n+1}} \operatorname{sign}(\pi) \left[\frac{\sum_{i=0}^{n} v_i}{n+1}, \frac{\sum_{i=0}^{n-1} v_i^\pi}{n}, \dots, \frac{\sum_0^1 v_i^\pi}{1}, v_0^\pi \right]$$

where S_n is the permutation group of n vertices, $\operatorname{sign}(\pi)$ is the orientation of each permutation π, and where it applies, v^π means the vertices that belong to the $(n-1)$-simplex of the π-th permutation. Note that in each element, we are summing over the i-th vertex of a permutation, and not the i-th index of σ_n. For example, in the last element, v_0^π means the 0-th element of the π-th permutation, which could mean v_0, v_1, v_2, and so on. It does not strictly mean v_0. This is exemplified in our example for $n = 2$. ∎

2.2 Computations and Applications

1. Prove the Brouwer fixed point theorem for maps $f : D^n \to D^n$ by applying degree theory to the map $S^n \to S^n$ that sends both the northern and southern hemispheres of S^n to the southern hemisphere via f. [This was Brouwer's original proof.]

Solution. Recall Brouwer's fixed point theorem states that for any continuous function f mapping a compact convex set to itself, there is a point x_0 such that $f(x_0) = x_0$. Let $g : S^n \to S^n$ be the map described in the exercise. Since g is not surjective, by degree property (b) (Page 134) $\deg g = 0$. If g did not have any fixed points, then by degree property (g) $\deg g = (-1)^{n+1}$. So g must have a fixed point, and it must be on the southern hemisphere of S^n. Any continuous map $f : D^n \to D^n$ can be described as the restriction to the southern hemisphere of some $g : S^n \to S^n$ That is, the fixed point of g is the fixed point of f. ∎

References: 1, 2.

3. Let $f : S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in $\mathbb{R}^n$ such that $F(x) \neq 0$ for all x, then there exists a point on ∂D^n where F points radially outward and another point on ∂D^n where F points radially inward.

Solution. If f did not have any fixed points, then by degree property (g) (Page 134) $\deg f = (-1)^{n+1}$. So f must have a fixed point x such that $f(x) = x$. Let $-\mathbb{1}$ be the antipodal map. Then by degree property (d) $f \circ (-\mathbb{1}) = -f$ has degree 0. So $-f$ must have a fixed point y such that $-f(y) = y$.

Incomplete...

References: 1.

4. Construct a surjective map $S^n \to S^n$ of degree zero, for each $n \geq 1$.

Solution. Construct f as the surjective map $h \circ g$ where $g : S^n \to D^n$ is the vertical projection and $h : D^n \to S^n$ is the quotient map, collapsing the border to a single point. Recall that the degree of f is d such that $f_*(\alpha) = d\alpha$ where $f_* : H_n(S^n) \to H_n(S^n)$. In our construction,

$$f_* = h_* \circ g_* : H_n(S^n) \to H_n(D^n) = 0 \to H_n(S^n).$$

Since f_* passes through 0, it has degree zero. ∎

References: 1.

20. For finite CW complexes X and Y, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

Solution. The Euler characteristic $\chi(X)$ is defined by $\sum_n (-1)^n c_n(X)$ where $c_n(X)$ is the number of n-cells in X. The n-cells in $X \times Y$ are the products of i-cells in X and j-cells in Y such that $i + j = n$. So

$$\chi(X \times Y) = \sum_n (-1)^n c_n(X \times Y) = \sum_n \sum_{i+j=n} (-1)^{i+j} c_i(X) \cdot c_j(Y) = \sum_i (-1)^i c_i(X) \cdot \sum_j (-1)^j c_j(Y) = \chi(X) \cdot \chi(Y). \quad ∎$$

21. If a finite CW complex X is the union of subcomplexes A and B, show that $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.

Solution. This follows from $c_n(X) = c_n(A) + c_n(B) - c_n(A \cap B)$ where $A \cap B$ is a subcomplex consisting of the cells of X both in A and B. ∎

32. For SX the suspension of X, show by a Mayer-Vietoris sequence that there are isomorphisms $\tilde{H}_n(SX) \approx \tilde{H}_{n-1}(X)$ for all n.

Solution. Recall that $SX = X \times I$. Taking $A = X \times [0, 3/4]$ and $B = X \times [1/4, 1]$, we have $SX = A \cup B$ and $X \simeq A \cap B$. We have the Mayer-Vietoris sequence

$$\cdots \to \tilde{H}_n(A) \oplus \tilde{H}_n(B) \to \tilde{H}_n(SX) \to \tilde{H}_{n-1}(X) \to \tilde{H}_{n-1}(A) \oplus \tilde{H}_{n-1}(B) \to \cdots$$

And since A and B are both contractible, $\tilde{H}_n(A) = \tilde{H}_n(B) = 0$. The above sequence then reduces to

$$\cdots \to 0 \to \tilde{H}_n(SX) \to \tilde{H}_{n-1}(X) \to 0 \to \cdots$$

implying $\tilde{H}_n(SX) \approx \tilde{H}_{n-1}(X)$ for all n. ∎

34. [Deleted — see the errata for comments.]

37. Give an elementary derivation for the Mayer-Vietoris sequence in simplicial homology for a Δ-complex X decomposed as the union of subcomplexes A and B.

Solution. We want to show that

$$0 \to C_k^\Delta(A \cap B) \xrightarrow{\alpha} C_k^\Delta(A) \oplus C_k^\Delta(B) \xrightarrow{\beta} C_k^\Delta(X) \to 0$$

is exact with $\alpha(x) = (x, -x)$ and $\beta(x, y) = x + y$. Let $\{x_j\}$ be the set of elements in the Δ-complex of X. Then $\{x_j\} = \{a_j\} \cup \{b_j\}$ where a_j, b_j are elements of respective subcomplexes A and B. That is, we have

$$C_k^\Delta(X) = \mathbb{Z}\{x_j\}, \quad C_k^\Delta(A) = \mathbb{Z}\{a_j\}, \quad C_k^\Delta(B) = \mathbb{Z}\{b_j\}.$$

Finally, let $\{c_j\} = \{a_j\} \cap \{b_j\}$ so that $C_k^\Delta(A \cap B) = \mathbb{Z}\{c_j\}$. We want to show three things: $\ker\alpha = 0$, $\mathrm{im}\alpha = \ker\beta$, and $\mathrm{im}\beta = C_k^\Delta(X)$.

In the first instance, if $x \in C_k^\Delta(A \cap B)$ and $\alpha(x) = (0, 0)$, then $x = 0$ so that $\ker\alpha = 0$, i.e. α is injective.

Next, we have $\beta(\alpha(x)) = \beta(x, -x) = x + (-x) = 0$ so that $\mathrm{im}\alpha \subseteq \ker\beta$. Conversely, take $(x, y) \in \ker\beta \subseteq C_k^\Delta(A) \oplus C_k^\Delta(B)$. Then $0 = \beta(x, y) = x + y$ so that $x = -y$ and $(x, y) = (x, -x) \in \mathrm{im}\alpha$. So $\mathrm{im}\alpha \supseteq \ker\beta$ and $\mathrm{im}\alpha = \ker\beta$. Note here that $x = -y$ implies both x and y are in $C_k^\Delta(A)$ and $C_k^\Delta(B)$, i.e. they are both in $C_k^\Delta(A \cap B)$.

In the last instance, let $x \in C_k^\Delta(X) = \mathbb{Z}\{x_j\} = \{a_j\} \cup \{b_j\}$. So

$$x = \sum_{j=1}^n n_j a_j + m_j b_j$$

for $n_j, m_j \in \mathbb{Z}$. Then

$$\beta : C_k^\Delta(A) \oplus C_k^\Delta(B) \to C_k^\Delta(X)$$
$$(x, y) \mapsto x + y$$

applied to the individual components of x gives

$$\beta\left(\sum_{j=1}^n n_j a_j, \sum_{j=1}^n m_j b_j\right) = \sum_{j=1}^n n_j a_j + m_j b_j = x$$

so that $x \in \text{im}\beta$ and $\text{im}\beta = C_k^{\Delta}(X)$. ∎

41. For X a finite CW complex and F a field, show that the Euler characteristic $\chi(X)$ can also be computed by the formula $\chi(X) = \sum_n (-1)^n \dim H_n(X;F)$ the alternating sum of the dimensions of the vector spaces $H_n(X;F)$.

Solution. There are two cases: when $\text{Char}(F) = 0$ and when $\text{Char}(F) = p$ where p is prime. In the first case, the torsion of F is empty and by the universal coefficient theorem for homology, we have

$$0 \to H_i(X;\mathbb{Z}) \otimes F \to H_i(X;F) \to \text{Tor}(H_{i-1}(X;\mathbb{Z}), F) \to 0.$$

Since Tor = 0 when F is torsion free, we have the isomorphism $H_i(X;\mathbb{Z}) \otimes F \cong H_i(X;F)$. Now since X is a finite (n-dimensional) CW complex, for all $m > n$, $H_m(X) = 0$ and for all $i \le n$ we have

$$H_i(X) = \mathbb{Z}^{\alpha_i} \oplus \sum_{k=1}^{m(i)} \mathbb{Z}_{\beta_k^i}.$$

Then

$$H_i(X;F) \cong H_i(X) \otimes F \cong F^{\alpha_i}$$

since $\mathbb{Z}_n \otimes F = 0$ and $\mathbb{Z}^{\alpha_i}$ is separated in the tensor product with $\mathbb{Z} \otimes F = F$. This isomorphism is given by

$$\begin{array}{c} F^{\alpha_i} \to \mathbb{Z}^{\alpha_i} \otimes F \hookrightarrow \left(\mathbb{Z}^{\alpha_i} \oplus \sum_k^{m(i)} \mathbb{Z}_{\beta_k^i} \right) \otimes F \xrightarrow{\phi \otimes \mathbb{1}} H_i(X) \otimes F \to H_i(X;F) \\ (v_1, \dots, v_{\alpha_i}) \to \sum_k e_k \otimes v_k \to \sum_k e_k \otimes v_k \to \sum_k \phi(e_k) \otimes v_k \to \sum_k v_k x_k \end{array}$$

where $e_k = (0, \dots, 1, \dots, 0)$ on the k-th element, $x_k \in [\phi(e_k)]$ is a fixed element, and $\phi : \mathbb{Z}^{\alpha_i} \oplus \sum_k^{m(i)} \mathbb{Z}_{\beta_k^i} \to H_n(X)$ is an isomorphism. So we have a vector space isomorphism and $\dim H_n(X;F) = \alpha_n = \text{rank}(H_n(X))$. Then by definition of the Euler characteristic, we have

$$\chi(X) = \sum_n (-1)^n \text{rank}(H_n(X)) = \sum_n (-1)^n \dim H_n(X;F).$$

Next we consider the case where $\text{Char}(F) = p$ where p is prime. Here we use the following lemma:

$$\text{Tor}(\mathbb{Z}_m, F) = \begin{cases} F & \text{if } p \mid m \\ 0 & \text{otherwise.} \end{cases}$$

By this lemma, we have

$$\text{Tor}(H_{i-1}(X), F) = \text{Tor}\left(\mathbb{Z}^{\alpha_{i-1}} \oplus \sum_k^{m(i-1)} \mathbb{Z}_{\beta_k^{i-1}}, F \right) = \sum_k^{m(i-1)} \text{Tor}(\mathbb{Z}_{\beta_k^{i-1}}, F) = \oplus_{p \mid \beta_k^{i-1}} F.$$

As in the first case, we have

$$0 \to H_i(X;\mathbb{Z}) \otimes F \to H_i(X;F) \to \text{Tor}(H_{i-1}(X;\mathbb{Z}), F) \to 0.$$

The maps are vector space homomorphisms. We expand the above to find

$$H_i(X;F) \cong H_i(X;\mathbb{Z}) \oplus \text{Tor}(H_{i-1}(X;\mathbb{Z}), F).$$

Since

$$\mathbb{Z}_m \otimes F \cong F/mF = \begin{cases} F & \text{if } p \mid m \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$H_i(X;F) \cong F^{\alpha_i} \oplus F^{\gamma_i} \oplus F^{\gamma_{i-1}}$$

where γ_i is the number of times $p \mid m$ in the i-th homology group. Since $H_k(X) = 0$ for $k > n$, again using $\operatorname{rank}(H_n(X)) = \alpha_n$, we have

$$\sum_n (-1)^n \dim H_n(X;F) = \alpha_0 + \gamma_0 + \sum_{k=1}^{n} (-1)^k(\alpha_k + \gamma_k + \gamma_{k-1}) + (-1)^{n+1}(\gamma_n) = \sum_{k=0}^{n} (-1)^n \alpha_k$$

where the second last equality follows from the telescopic sum of γ_i. Since α_k is the rank of $H_n(X)$, the equality holds for the $\operatorname{Char}(F) = p$ where p is prime. ■

2.3 The Formal Viewpoint

Additional Topics

2.A. Homology and Fundamental Group

No exercises in this subsection.

2.B. Classical Applications

4. In the unit sphere $S^{p+q-1} \subset \mathbb{R}^{p+1}$ let S^{p-1} and S^{q-1} be the subspheres consisting of points whose last q and first p coordinates are zero, respectively.

(a) Show that $S^{p+q-1} - S^{p-1}$ deformation retracts onto S^{q-1}, and is in fact homeomorphic to $S^{q-1} \times \mathbb{R}^p$.

Solution. We can take the homeomorphism

$$\phi : S^{q-1} \times \mathbb{R}^p \to S^{p+q-1} - S^{p-1}$$
$$(s_{p+1}, \ldots, s_{p+q}, v_1, \ldots, v_p) \mapsto \frac{(v_1, \ldots, v_p, s_{p+1}, \ldots, s_{p+q})}{\sqrt{s_{p+1}^2 + \cdots + s_{p+q}^2 + v_1^2 + \cdots + v_p^2}}$$

where in the domain, $\mathbf{v} \in \mathbb{R}^p$ is a vector. Note here that the image is correct because we have a $p+q$ vector such that the last q coordinates are *not* zero. If they were zero, then it would imply the first p coordinates *and* the last q coordinates in S^{q-1} are zero, which implies we just have the zero vector. But the zero vector is not in any S^i. Clearly we have a continuous map (with continuous inverse), since the denominator in the image is never zero. Since $S^{q-1} \times \mathbb{R}^p$ deformation retracts to $S^{q-1} \times \{0\} = S^{q-1}$, by the above homeomorphism we have a deformation retraction from $S^{p+q-1} - S^{p-1}$ to S^{q-1}. □

(b) Show that S^{p-1} and S^{q-1} are not the boundaries of any pair of disjointly embedded disks D^p and D^q in D^{p+q}. [The preceding exercise may be useful.]

Solution. Let $D^p \cap D^q = \varnothing$ for D^p and D^q in D^{p+q}. The assumption of the question states that $S^{p-1} = D^p \cap S^{p+q-1}$ and/or $S^{q-1} = D^q \cap S^{p+q-1}$. We consider the case of S^{p-1}. We have

$$\begin{array}{ccc} S^{p-1} \subseteq & & D^p \\ \cap\!\! \shortmid & & \cap\!\! \shortmid \\ S^{p+q-1} \backslash S^{q-1} \subseteq & & D^{p+q} \backslash D^q \end{array} \quad \Longrightarrow \quad \begin{tikzcd} H_*(S^{p-1}) \arrow[r] \arrow[d, "\cong"] & H_*(D^p) \arrow[d] \\ H_*(S^{p+q-1} \backslash S^{q-1}) \arrow[r, "\cong"] & H_*(D^{p+q} \backslash D^q). \end{tikzcd}$$

The top down inclusion of spaces comes from the assumption $S^{p-1} = D^p \cap S^{p+q-1}$. The top-down isomorphism in homology is from part (a), while the left-right isomorphism is from the previous exercise (2.B.3). But $H_{p-1}(S^{p-1}) = \mathbb{Z}$ implies $H_{p-1}(D^{p+q} \backslash D^q) = \mathbb{Z}$ which contradicts $H_{p-1}(D^p) = 0$ (contractible). The same is true for the case with q. So S^{p-1} and S^{q-1} are not the boundaries of any pair of disjointly embedded disks D^p and D^q. ■

2.C. Simplicial Approximation

3 Cohomology

3.1 Cohomology Groups

9. Show that if $f : S^n \to S^n$ has degree d then $f^* : H^n(S^n;G) \to H^n(S^n;G)$ is multiplication by d.

Solution. For the multiplication by d homomorphism $d : \mathbb{Z} \to d\mathbb{Z}$, the dualized homomorphism $d^* : \mathbb{Z}^* \to \mathbb{Z}^*$ is also multiplication by d for any group G where $\mathbb{Z}^* = \mathrm{Hom}(\mathbb{Z}, G)$. Then since $f_* : H_n(S^n;G) \to H_n(S^n;G)$ is multiplication by d, so is $f_\star : \mathrm{Hom}(H_n(S^n), G) \to \mathrm{Hom}(H_n(S^n), G)$. By the Universal Coefficient Theorem, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(S^n), G) & \longrightarrow & H^n(S^n;G) & \longrightarrow & \mathrm{Hom}(H_n(S^n), G) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f_\star} & & \\ 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(S^n), G) & \longrightarrow & H^n(S^n;G) & \longrightarrow & \mathrm{Hom}(H_n(S^n), G) & \longrightarrow & 0. \end{array}$$

Since $\mathrm{Ext}(H_{n-1}(S^n), G) = \mathrm{Ext}(0, G) = 0$, it follows that $H^n(S^n;G) \cong \mathrm{Hom}(H_n(S^n), G)$ and $f^* = f_\star$. That is, $f^* : H^n(S^n;G) \to H^n(S^n;G)$ is multiplication by d. ■

10. For the lens space $L_m(\ell_1, \dots, \ell_n)$ defined in Example 2.43, compute the cohomology groups using the cellular cochain complex and taking coefficients in $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_m$, and $\mathbb{Z}_p$ for p prime. Verify that the answers agree with those given by the universal coefficient theorem.

Solution. From Example 2.43 we have

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

To shorten syntax, we denote the lens space by X.

Case 1 G $= \mathbb{Z}$.
First we dualize with $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ to obtain the sequence

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

This follows from $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$ and the multiplication by 0 and m carrying through in the dualized maps. Then $H^i(X) = \ker_{i+1}/\mathrm{im}_i$ so that when i is even $H^i(X) = \mathbb{Z}/\{m\mathbb{Z}\} = \mathbb{Z}_m$. When i is odd, $H^i(X) = \{0\}/\{0\} = 0$. In the special case where $i = 0$ we have $H^0(X) = \mathbb{Z}/\{0\} = \mathbb{Z}$. The same is true for the special case $i = 2n-1$.

$$H^i(X) = \begin{cases} \mathbb{Z} & \text{for } i = 0, 2n-1 \\ \mathbb{Z}_m & \text{for } i \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Using the universal coefficient theorem, we have

$$0 \to \mathrm{Ext}(H_{i-1}(C), \mathbb{Z}) \to H^i(C;\mathbb{Z}) \to \mathrm{Hom}(H_i(C), \mathbb{Z}) \to 0.$$

When i is even we have

$$0 \to \mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}) \to H^i(C;\mathbb{Z}) \to \mathrm{Hom}(0, \mathbb{Z}) \to 0.$$

so that $H^i = \mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}) = \mathbb{Z}_m$. Similarly, when i is odd we have $H^i = \mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}) = 0$. In the special case where $i = 0$ we have $H^0(C, \mathbb{Z}) = \mathrm{Hom}(H_0(C), \mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$. When $i = 2n-1$ we have $H^{2n-1}(C, \mathbb{Z}) = \mathrm{Ext}(H_{2n-2}(C), \mathbb{Z}) \oplus \mathrm{Hom}(H_{2n-1}(C), \mathbb{Z}) = \mathrm{Ext}(0, \mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$. This verifies case 1 with $G = \mathbb{Z}$.

Case 2 G $= \mathbb{Q}$.
First we dualize with $\mathrm{Hom}(\mathbb{Z}, \mathbb{Q})$ to obtain the sequence

$$0 \to \mathbb{Q} \xrightarrow{0} \mathbb{Q} \xrightarrow{m} \mathbb{Q} \to \cdots \to \mathbb{Q} \xrightarrow{m} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \to 0.$$

This follows from $\mathrm{Hom}(\mathbb{Z},\mathbb{Q}) = \mathbb{Q}$ and the multiplication by 0 and m carrying through in the dualized maps. Then $H^i(X) = \ker_{i+1}/\mathrm{im}_i$ so that when i is even $H^i(X) = \mathbb{Q}/\{m\mathbb{Q}\} = \mathbb{Q}/\mathbb{Q} = 0$. When i is odd, $H^i(X) = \{0\}/\{0\} = 0$. In the special case where $i = 0$ we have $H^0(X) = \mathbb{Q}/\{0\} = \mathbb{Q}$. The same is true for the special case $i = 2n-1$.

$$H^i(X) = \begin{cases} \mathbb{Q} & \text{for } i = 0, 2n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Using the universal coefficient theorem, we have

$$0 \to \mathrm{Ext}(H_{i-1}(C),\mathbb{Q}) \to H^i(C;\mathbb{Q}) \to \mathrm{Hom}(H_i(C),\mathbb{Q}) \to 0.$$

When i is even we have

$$0 \to \mathrm{Ext}(\mathbb{Z}_m,\mathbb{Q}) \to H^i(C;\mathbb{Q}) \to \mathrm{Hom}(0,\mathbb{Q}) \to 0.$$

so that $H^i = \mathrm{Ext}(\mathbb{Z}_m,\mathbb{Q}) = 0$. Similarly, when i is odd we have $H^i = \mathrm{Hom}(\mathbb{Z}_m,\mathbb{Q}) = 0$. In the special case where $i = 0$ we have $H^0(C,\mathbb{Q}) = \mathrm{Hom}(H_0(C),\mathbb{Q}) = \mathrm{Hom}(\mathbb{Z},\mathbb{Q}) = \mathbb{Q}$. When $i = 2n-1$ we have $H^{2n-1}(C,\mathbb{Q}) = \mathrm{Ext}(H_{2n-2}(C),\mathbb{Q}) \oplus \mathrm{Hom}(H_{2n-1}(C),\mathbb{Q}) = \mathrm{Ext}(0,\mathbb{Q}) \oplus \mathrm{Hom}(\mathbb{Z},\mathbb{Q}) = \mathbb{Q}$. This verifies case 2 with $G = \mathbb{Q}$.

Case 3 G $= \mathbb{Z}_\mathbf{m}$.
First we dualize with $\mathrm{Hom}(\mathbb{Z},\mathbb{Z}_m)$ to obtain the sequence

$$0 \to \mathbb{Z}_m \xrightarrow{0} \mathbb{Z}_m \xrightarrow{m} \mathbb{Z}_m \to \cdots \to \mathbb{Z}_m \xrightarrow{m} \mathbb{Z}_m \xrightarrow{0} \mathbb{Z}_m \to 0.$$

This follows from $\mathrm{Hom}(\mathbb{Z},\mathbb{Z}_m) = \mathbb{Z}_m$ and the multiplication by 0 and m carrying through in the dualized maps. Then $H^i(X) = \ker_{i+1}/\mathrm{im}_i$ so that when i is even $H^i(X) = \mathbb{Z}_m/\{m\mathbb{Z}_m\} = \mathbb{Z}_m/\{0\} = \mathbb{Z}_m$ since $m\mathbb{Z}_m$ takes each element to $0 \mod m$. For the same reason, when i is odd, $H^i(X) = Z_m/\{0\} = \mathbb{Z}_m$. In the special case where $i = 0$ we have $H^0(X) = \mathbb{Z}_m/\{0\} = \mathbb{Z}_m$. The same is true for the special case $i = 2n-1$.

$$H^i(X) = \begin{cases} \mathbb{Z}_m & \text{for } 0 \le i \le 2n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Using the universal coefficient theorem, we have

$$0 \to \mathrm{Ext}(H_{i-1}(C),\mathbb{Z}_m) \to H^i(C;\mathbb{Z}_m) \to \mathrm{Hom}(H_i(C),\mathbb{Z}_m) \to 0.$$

When i is even we have

$$0 \to \mathrm{Ext}(\mathbb{Z}_m,\mathbb{Z}_m) \to H^i(C;\mathbb{Z}_m) \to \mathrm{Hom}(0,\mathbb{Z}_m) \to 0.$$

so that $H^i = \mathrm{Ext}(\mathbb{Z}_m,\mathbb{Z}_m) = \mathbb{Z}_m$. Similarly, when i is odd we have $H^i = \mathrm{Hom}(\mathbb{Z}_m,\mathbb{Z}_m) = \mathbb{Z}_m$. In the special case where $i = 0$ we have $H^0(C,\mathbb{Z}_m) = \mathrm{Hom}(H_0(C),\mathbb{Z}_m) = \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_m) = \mathbb{Z}_m$. When $i = 2n-1$ we have $H^{2n-1}(C,\mathbb{Z}_m) = \mathrm{Ext}(H_{2n-2}(C),\mathbb{Z}_m) \oplus \mathrm{Hom}(H_{2n-1}(C),\mathbb{Z}_m) = \mathrm{Ext}(0,\mathbb{Z}_m) \oplus \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_m) = \mathbb{Z}_m$. This verifies case 3 with $G = \mathbb{Z}_m$.

Case 4 G $= \mathbb{Z}_\mathbf{p}$ **for p prime**.
First we dualize with $\mathrm{Hom}(\mathbb{Z},\mathbb{Z}_p)$ to obtain the sequence

$$0 \to \mathbb{Z}_p \xrightarrow{0} \mathbb{Z}_p \xrightarrow{m} \mathbb{Z}_p \to \cdots \to \mathbb{Z}_p \xrightarrow{m} \mathbb{Z}_p \xrightarrow{0} \mathbb{Z}_p \to 0.$$

This follows from $\mathrm{Hom}(\mathbb{Z},\mathbb{Z}_p) = \mathbb{Z}_p$ and the multiplication by 0 and m carrying through in the dualized maps. Now we separate into two cases:

Case 4 (a) $\mathbf{p} \nmid \mathbf{m}$.
If $p \nmid m$ then $\gcd(p,m) = 1$ and multiplication by m is an isomorphism. Then $H^i(X) = \ker_{i+1}/\mathrm{im}_i$ so that when i is even $H^i(X) = \mathbb{Z}_p/\{\mathbb{Z}_p\} = 0$. When i is odd, $H^i(X) = \{0\}/\{0\} = 0$. In the special case where $i = 0$ we have $H^0(X) = \mathbb{Z}_p/\{0\} = \mathbb{Z}_p$. The same is true for the special case $i = 2n-1$.

$$H^i(X) = \begin{cases} \mathbb{Z}_p & \text{for } i = 0, 2n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Using the universal coefficient theorem, we have

$$0 \to \operatorname{Ext}(H_{i-1}(C), \mathbb{Z}_p) \to H^i(C; \mathbb{Z}_p) \to \operatorname{Hom}(H_i(C), \mathbb{Z}_p) \to 0.$$

When i is even we have

$$0 \to \operatorname{Ext}(\mathbb{Z}_m, \mathbb{Z}_p) \to H^i(C; \mathbb{Z}_p) \to \operatorname{Hom}(0, \mathbb{Z}_p) \to 0.$$

so that $H^i = \operatorname{Ext}(\mathbb{Z}_m, \mathbb{Z}_p) = 0$ since $\gcd(p, m) = 1$. Similarly, when i is odd we have $H^i = \operatorname{Hom}(\mathbb{Z}_m, \mathbb{Z}_p) = 0$ since $\gcd(p, m) = 1$. In the special case where $i = 0$ we have $H^0(C, \mathbb{Z}_p) = \operatorname{Hom}(H_0(C), \mathbb{Z}_p) = \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}_p) = \mathbb{Z}_p$. When $i = 2n-1$ we have $H^{2n-1}(C, \mathbb{Z}_p) = \operatorname{Ext}(H_{2n-2}(C), \mathbb{Z}_p) \oplus \operatorname{Hom}(H_{2n-1}(C), \mathbb{Z}_p) = \operatorname{Ext}(0, \mathbb{Z}_p) \oplus \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}_p) = \mathbb{Z}_p$. This verifies case 4 (a) with $G = \mathbb{Z}_p$ for p prime and $p \nmid m$.

Case 4 (b) $\mathbf{p \mid m}$.

If $p \mid m$ then multiplication by m is the zero map and the cohomology group is the same as in case 3 with $G = \mathbb{Z}_m$.

$$H^i(X) = \begin{cases} \mathbb{Z}_p & \text{for } 0 \le i \le 2n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Using the universal coefficient theorem, we have

$$0 \to \operatorname{Ext}(H_{i-1}(C), \mathbb{Z}_p) \to H^i(C; \mathbb{Z}_p) \to \operatorname{Hom}(H_i(C), \mathbb{Z}_p) \to 0.$$

When i is even we have

$$0 \to \operatorname{Ext}(\mathbb{Z}_m, \mathbb{Z}_p) \to H^i(C; \mathbb{Z}_p) \to \operatorname{Hom}(0, \mathbb{Z}_p) \to 0.$$

so that $H^i = \operatorname{Ext}(\mathbb{Z}_m, \mathbb{Z}_p) = \mathbb{Z}_p$ since $\gcd(p, m) = p$. Similarly, when i is odd we have $H^i = \operatorname{Hom}(\mathbb{Z}_m, \mathbb{Z}_p) = \mathbb{Z}_p$ since $\gcd(p, m) = p$. The two special cases are the same as in case (a). This verifies case 4 (b) with $G = \mathbb{Z}_p$ for p prime and $p \mid m$. ■

3.2 Cup Product

3.3 Poincaré Duality

3.3.16. Show that $(\alpha \frown \varphi) \frown \psi = \alpha \frown (\varphi \smile \psi)$ for all $\alpha \in C_k(X; R)$, $\varphi \in C^l(X; R)$, and $\psi \in C^m(X; R)$. Deduce that cap product makes $H_*(X; R)$ a right $H^*(X; R)$-module.

Solution. On the right, we have

$$\begin{aligned} \alpha \frown (\varphi \smile \psi) &= (\varphi \smile \psi)(\alpha|_{[v_0, \dots, v_{\ell+m}]}) \cdot \alpha|_{[v_{\ell+m}, \dots, v_k]} \\ &= \varphi\left((\alpha|_{[v_0, \dots, v_{\ell+m}]})|_{[v_0, \dots, v_\ell]}\right) \cdot \psi\left((\alpha|_{[v_0, \dots, v_{\ell+m}]})|_{[v_\ell, \dots, v_m]}\right) \cdot \alpha|_{[v_{\ell+m}, \dots, v_k]} \\ &= \varphi(\alpha|_{[v_0, \dots, v_\ell]}) \cdot \psi(\alpha|_{[v_\ell, \dots, v_{\ell+m}]}) \cdot \alpha|_{[v_{\ell+m}, \dots, v_k]} \end{aligned}$$

where the final result is an element in $C_k \frown (C^\ell \smile C^m) \to C_k \frown C^{\ell+m} \to C_{k-(\ell+m)} = C_{k-\ell-m}$. On the left, we have

$$\begin{aligned} (\alpha \frown \varphi) \frown \psi &= \varphi(\alpha|_{[v_0, \dots, v_\ell]}) \cdot \alpha|_{[v_\ell, \dots, v_k]} \frown \psi \\ &= \varphi(\alpha|_{[v_0, \dots, v_\ell]}) \cdot \psi\left((\alpha|_{[v_\ell, \dots, v_k]})|_{[v_0, \dots, v_m]}\right) \cdot (\alpha|_{[v_\ell, \dots, v_k]})|_{[v_m, \dots, v_{k-\ell}]} \\ &= \varphi(\alpha|_{[v_0, \dots, v_\ell]}) \cdot \psi(\alpha|_{[v_\ell, \dots, v_{\ell+m}]}) \cdot \alpha|_{[v_{\ell+m}, \dots, v_k]} \end{aligned}$$

where the final result is an element in $(C_k \frown C^\ell) \frown C^m \to C_{k-\ell} \frown C^m \to C_{k-\ell-m}$. So both sides are the same and equality holds. Note that $(\alpha|_{[v_\ell, \dots, v_k]})|_{[v_0, \dots, v_m]} = \alpha|_{[v_\ell, \dots, v_{\ell+m}]}$ since we are taking $m - 0 = m$ vertices from ℓ to k. And similarly, $(\alpha|_{[v_\ell, \dots, v_k]})|_{[v_m, \dots, v_{k-\ell}]} = \alpha|_{[v_{\ell+m}, \dots, v_k]}$ since we are taking $k - (\ell + m)$ vertices.

Now we show that cap product makes $H_*(X;R)$ a right $H^*(X;R)$-module (see reference for conditions). First, for $m_1, m_2 \in H_*(X;R)$ and $r \in H^*(X;R)$, we have

$$(m_1 + m_2) \frown r = r \frown m_1 + r \frown m_2$$

because $\frown$ is an R-bilinear homomorphism. Next, for $m \in H_*(X;R)$ and $r_1, r_2 \in H^*(X;R)$, we have

$$m \frown (r_1 + r_2) = (r_1 + r_2)(m) \cdot m = r_1(m) \cdot m + r_2(m) \cdot m = m \frown r_1 + m \frown r_2.$$

The final two conditions are the identity $m \frown \mathbb{1} = \mathbb{1}(m) \cdot m = m$ and associativity, which was proven in the first part of this exercise. So $\frown$ makes $H_*(X;R)$ a right $H^*(X;R)$-module. ■

References: 1.

Additional Topics

3.A. Universal Coefficients for Homology

3.B. The General Künneth Formula

3.C. H–Spaces and Hopf Algebras

3.D. The Cohomology of SO(n)

3.E. Bockstein Homomorphisms

3.F. Limits and Ext

3.G. Transfer Homomorphisms

No exercises in this subsection.

3.H. Local Coefficients

4 Homotopy Theory

4.1 Homotopy Groups

1. Suppose a sum $f +' g$ pf maps $f, g : (I^n, \partial I^n) \to (X, x_0)$ is defined using a coordinate of I^n other than the first coordinate as in the usual sum $f + g$. Verify the formula $(f+g) +' (h+k) = (f +' h) + (g +' k)$, and deduce that $f +' k \simeq f + k$ so the two sums agree on $\pi_n(X, x_0)$, and also that $g +' h \simeq h + g$ so the addition is abelian.

Solution. Recall the usual definition for addition is

$$(f+g)(\mathbf{s}) = \begin{cases} f(2s_1, s_2 \dots s_n) & \text{for } s_1 \in [0, \frac{1}{2}] \\ g(2s_1 - 1, s_2, \dots s_n) & \text{for } s_1 \in [\frac{1}{2}, 1]. \end{cases}$$

Without loss of generality, assume addition uses the second coordinate:

$$(f+g)(\mathbf{s}) = \begin{cases} f(s_1, 2s_2, \dots s_n) & \text{for } s_2 \in [0, \frac{1}{2}] \\ g(s_1, 2s_2 - 1, \dots s_n) & \text{for } s_2 \in [\frac{1}{2}, 1]. \end{cases}$$

On the left, we have

$$(f+g)(\mathbf{s}) = \begin{cases} f(2s_1, s_2 \dots s_n) & \text{for } s_1 \in [0, \frac{1}{2}] \\ g(2s_1 - 1, s_2, \dots s_n) & \text{for } s_1 \in [\frac{1}{2}, 1] \end{cases} \quad \text{and} \quad (h+k)(\mathbf{s}) = \begin{cases} h(2s_1, s_2 \dots s_n) & \text{for } s_1 \in [0, \frac{1}{2}] \\ k(2s_1 - 1, s_2, \dots s_n) & \text{for } s_1 \in [\frac{1}{2}, 1] \end{cases}$$

so that

$$((f+g) +' (h+k))(\mathbf{s}) = \begin{cases} f(2s_1, 2s_2, \dots s_n) & \text{for } s_1 \in [0, \frac{1}{2}], \text{ and } s_2 \in [0, \frac{1}{2}] \\ g(2s_1 - 1, 2s_2, \dots s_n) & \text{for } s_1 \in [\frac{1}{2}, 1], \text{ and } s_2 \in [0, \frac{1}{2}] \\ h(2s_1, 2s_2 - 1, \dots s_n) & \text{for } s_1 \in [0, \frac{1}{2}], \text{ and } s_2 \in [\frac{1}{2}, 1] \\ k(2s_1 - 1, 2s_2 - 1, \dots s_n) & \text{for } s_1 \in [\frac{1}{2}, 1], \text{ and } s_2 \in [\frac{1}{2}, 1]. \end{cases}$$

On the right, we have

$$(f +' h)(\mathbf{s}) = \begin{cases} f(s_1, 2s_2, \dots s_n) & \text{for } s_2 \in [0, \frac{1}{2}] \\ h(s_1, 2s_2 - 1, \dots s_n) & \text{for } s_2 \in [\frac{1}{2}, 1] \end{cases} \quad \text{and} \quad (g +' k)(\mathbf{s}) = \begin{cases} g(s_1, 2s_2, \dots s_n) & \text{for } s_2 \in [0, \frac{1}{2}] \\ k(s_1, 2s_2 - 1, \dots s_n) & \text{for } s_2 \in [\frac{1}{2}, 1] \end{cases}$$

so that

$$((f +' h) + (g +' k))(\mathbf{s}) = \begin{cases} f(2s_1, 2s_2, \dots s_n) & \text{for } s_2 \in [0, \frac{1}{2}], \text{ and } s_1 \in [0, \frac{1}{2}] \\ h(2s_1, 2s_2 - 1, \dots s_n) & \text{for } s_2 \in [\frac{1}{2}, 1], \text{ and } s_1 \in [0, \frac{1}{2}] \\ g(2s_1 - 1, 2s_2, \dots s_n) & \text{for } s_2 \in [0, \frac{1}{2}], \text{ and } s_1 \in [\frac{1}{2}, 1] \\ k(2s_1 - 1, 2s_2 - 1, \dots s_n) & \text{for } s_2 \in [\frac{1}{2}, 1], \text{ and } s_1 \in [\frac{1}{2}, 1]. \end{cases}$$

I.e. both sides agree and equality holds. If we take $g = h = 0$, then we get

$$(f+g) +' (h+k) = (f +' h) + (g +' k) \implies (f+0) +' (0+k) = (f +' 0) + (0 +' k) \implies f +' k = f + k$$

so that both additions agree on π_n. And taking $f = k = 0$ we have $g +' h = h + g$ so the addition is abelian. ∎

4.2 Elementary Methods of Calculation

4.3 Connections with Cohomology

Additional Topics

4.A. Basepoints and Homotopy

4.B. The Hopf Invariant

4.C. Minimal Cell Structures

No exercises in this subsection.

4.D. Cohomology of Fiber Bundles

4.E. The Brown Representability Theorem

No exercises in this subsection.

4.F. Spectra and Homology Theories

4.G. Gluing Constructions

4.H. Eckmann-Hilton Duality

4.I. Stable Splittings of Spaces

4.J. The Loopspace of a Suspension

4.K. The Dold-Thom Theorem

4.L. Steenrod Squares and Powers