ECE 602 Assignment Solutions

Group Members	Student no
1 Meng Tianao	20810553
2 Ebere Thomas	20868189

Exercise 1:

② Let x and y be the sublevel set point.

Then $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$

$$f(x) \leq \alpha$$

$$f(y) \leq \alpha$$

$$f(\lambda x + (1-\lambda)y) \leq \lambda\alpha + (1-\lambda)\alpha$$

$$f(\lambda x + (1-\lambda)y) \leq \alpha$$

This means $\lambda x + (1-\lambda)y$ form a convex domain which makes $x, y \in S_\alpha(f)$ are convex

(b) $\{x \mid \|x-a\|_2 \le \theta \|x-b\|_2\}$

Expanding $\|x-a\|_2 \le \theta \|x-b\|_2$

$$(x-a)^T(x-a) \le \theta^2 (x-b)^T(x-b)$$

$$x^Tx - x^Ta - a^Tx + a^Ta \le \theta^2 (x^Tx - x^Tb - b^Tx + b^Tb)$$

$$x^Tx - x^Ta - a^Tx + a^Ta - \theta^2 (x^Tx - x^Tb - b^Tx + b^Tb) \le 0$$

$$(1-\theta^2)x^Tx - 2(a-\theta^2 b)^T x + a^Ta - \theta^2 b^Tb \le 0$$

$$f' = 2(1-\theta^2)x - 2(a-\theta^2 b)$$

$$f'' = 2(1-\theta^2) \ge 0$$

if the Hessian ≥ 0 then it is a convex set.

This means that the set of points are convex.

(c) Lets assume that a line passes through an arbitrary point c_1, c_2, c_3 ---- $c_k \in R^n$

Point c_1, c_2, c_3 ----- $c_k \in R^n$ and $\theta_1 c_1 + \theta_2 c_2 + \theta_3 c_3$ ----- $\theta_k c_k \in R^n$ lie on a line which is convex.

This means c_1, c_2, c_3 ---- $c_k \in R^n$ is a convex set.

So set $C = x \in R^n$ is a convex set.

(d)

$$\frac{df(x)}{dx} = \left(\sum_{i=1}^{n} x_i^p\right)^{(1-p)/p} x^{p-1} = \left(\frac{f(x)}{x_i}\right)^{1-p}$$

~~D~~ifferentiating twice we have:

$$\frac{d^2 f(x)}{dx_i dx_j} = \frac{1-p}{x_i}\left(\frac{f(x)}{x_i}\right)^{-p}\left(\frac{f(x)}{x_j}\right)^{1-p}$$

$$= \frac{1-p}{f(x)}\left(\frac{f(x)^2}{x_i x_j}\right)^{1-p}$$

Applying the cauchy-schwarz inequality $a^T b \leq \|a\|_2 \|b\|_2$

$a_i = \left(\frac{f(x)}{x_i}\right)^{-p/2}$, $b_i = y_i \left(\frac{f(x)}{x_i}\right)^{1-p/2}$ and

knowing that $\sum_i a_i^2 = 1$

Since $p < 1$ this means that $\left(\sum_{i=1}^{n} x_i^p\right)^{1/p}$,

$p < 1$ is not convex of all

(e)

$$f(x) = g(h(x)),$$

$$f''(x) = g''(h(x))[h'(x)]^2 + g'(h(x))h''(x)$$

From the question h is convex, g is convex and monotonically increasing

$f''(x)$ satisfies convexity if:

(i) $g''(\cdot)$ is convex, $h(\cdot)$ is convex and $g(\cdot)$ increasing

(ii) $g''(\cdot)$ is convex, $h(\cdot)$ is concave and $g(\cdot)$ decreasing

This satisfies that $f(x)$ is a convex function.

(4) dom $f = R^n_{++}$, which is convex.

if f is convex, we need to prove that

for all $x, y \in$ dom f, and θ with $0 \le \theta \le 1$

we have $f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$

However when $p<1$, $f(x)$ is not a norm

for example $n=1$, $p=0.5$..

$f(x) = x^2$ where $x \geq 0$.

the $f(x+y) = x^2 + y^2 + 2xy \geq x^2 + y^2$.

$\therefore$ when $p<1$ it is not a norm.

$\therefore$ we can not show $f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$
using triangle inequality

$\therefore$ $f(x)$ is not a convex function

Exercise 2.

(a) It is not a norm.

Because $f(X)$ is to get the total variance of X.
not only $X=0$, $f(X)=0$.

For example, when $m=n=3$, $X=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, which is not 0, but $f(X)=0$

thus $f(X)$ is not a norm.

(b) It is not a convex function

And it is not strictly convex.

(c) We can redefine, dom f : $\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & & & \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{bmatrix}$

$$f(x) = \sum_{i=1}^{n}\sum_{j=1}^{m} \sqrt{(x_{i+1,j} - x_{i,j})^2 + (x_{i,j+1} - x_{i,j})^2}$$

$$G(X) = \nabla f(X) = \begin{bmatrix} \frac{\partial f}{\partial x_{11}} & \cdots & \frac{\partial f}{x_{1n}} \\ \frac{\partial f}{\partial x_{21}} & \cdots & \frac{\partial f}{\partial x_{2n}} \\ \vdots & & \vdots \\ \frac{\partial f}{\partial_{n1}} & \cdots & \frac{\partial f}{\partial_{nm}} \end{bmatrix}$$

$$\frac{\partial f(x)}{\partial x_{ab}} = \underbrace{\frac{\partial}{\partial x_{ac}} \sum_{i=1}^{n}\sum_{j=1}^{m} \sqrt{(x_{i+1,j} - x_{i,j})^2 + (x_{i,j+1} - x_{i,j})^2}}_{\text{if } i \neq a,\ j \neq b:\ 0}$$

else : $-2x_{a,b} - 2x_{a+1,b} - 2x_{a,b} - 2x_{a,b+1}$

$$= \frac{4x_{a,b} - 2(x_{a+1,b} + x_{a,b+1})}{\sqrt{(x_{a+1,b} - x_{a,b})^2 + (x_{a,b+1} - x_{a,b})^2}}$$

$$\therefore \nabla f(x) = \frac{2D_c X + 2XD_r^T}{\sqrt{(D_c X)^2 + (XD_r^T)^2}}$$

At certain points like when there exists $x_{a+1,b} = x_{a,b+1} = x_{a,b}$, does not exsist $\nabla f(x)$

Exercise 3

see in attached file.