ECE 602 hw2

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Exercise 1

(a)

(1) $I_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$

$f(x) = I_C(x)$

$f^*(x) = \sup_{x \in \mathrm{dom} f} (y^T x - f(x))$

$\therefore$ when $x \in C$, $f^*(x) = \sup_{x \in C} y^T x$

(2) $f(x) = \lambda \|\Sigma x\|_1$

$g(x) = \lambda \|x\|_1$

$\Downarrow$

$f(x) = g(\Sigma x)$

According to the basic properties

$f^*(y) = g^*(\Sigma^{-T} y)$

Assume $h(x) = \|x\|_1$

$g^*(y) = \lambda h^*(y/\lambda)$

$h^*(y) = \begin{cases} 0, & \|y\|_\infty \le 1 \\ \infty, & \text{otherwise} \end{cases}$

$\therefore g^*(y) = \begin{cases} 0, & \|y\|_\infty \le \lambda \\ \infty, & \text{otherwise.} \end{cases}$

$\therefore f^*(y) = \begin{cases} 0, & \|\Sigma^{-T} y\|_\infty \le \lambda \\ \infty, & \text{otherwise} \end{cases}$

(3) $f(x) = \lambda \|\Sigma x\|_2$

$g(x) = \lambda \|x\|_2$

$\Downarrow$

$f(x) = g(\Sigma x)$

According to the basic properties

$f^*(y) = g^*(\Sigma^{-T} y)$

Assume $h(x) = \|x\|_2$

$g^*(y) = \lambda h^*(y/\lambda)$

$$h^*(y) = \begin{cases} 0, & \|y\|_2 \le 1 \\ \infty, & \text{otherwise} \end{cases}$$

$$\therefore g^*(y) = \begin{cases} 0, & \|y\|_2 \le \lambda \\ \infty, & \text{otherwise.} \end{cases}$$

$$\therefore f^*(y) = \begin{cases} 0, & \|\Sigma^{-T} y\|_2 \le \lambda \\ \infty, & \text{otherwise} \end{cases}$$

(4) $f(x) = \frac{1}{2} x^T A x + b^T x + c$

$\because A \in S_{++}^n \quad \therefore f(x)$ is a convex function

$$f^*(y) = \sup_{x \in R^n} (y^T x - f(x))$$

$$= \sup_{x \in R^n} (y^T x - (\tfrac{1}{2} x^T A x + b^T x + c))$$

we assume $h(x) = y^T x - (\frac{1}{2} x^T A x + b^T x + c)$

$h'(x) = y^T - x^T A - b^T = 0. \Rightarrow x = A^{-1}(y - b)$

$\therefore f^*(y) = y^T A^{-1}(y-b) - \frac{1}{2}(y-b)^T (A^{-1})^T A A^{-1}(y-b) + b^T A^{-1}(y-b) + c)$

$\because A \in S_{++}^n$

$\therefore f^*(y) = y^T A^{-1}(y-b) - \frac{1}{2}(y-b)^T A^{-1}(y-b) - b^T A^{-1}(y-b) - c$

(5) $f(x) = -\sum_{i=1}^{n} \log x_i$

According to the basic properties.

若 $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$, $f^*(y_1, y_2) = f_1^*(y_1) + f_2^*(y_2)$

$\because f(x) = -\log x_1 - \cdots - \log x_n$ and $x \in R^{++}$

$\because (-\log x)^* = -1 - \log(-y)$

$\therefore f^*(y) = -1 - \log(y_1) - 1 - \log(y_2) - \cdots - 1 - \log(y_n)$

$= -(n + \log(y_1 \times y_2 \times \cdots \times y_n))$

(b)

(1) $\because \text{prox}_f(y) = \arg\min_x \left(\frac{1}{2}\|y-x\|_2^2 + f(x)\right)$

$\therefore \text{prox}_{I_C}(y) = \arg\min_x \left(\frac{1}{2}\|y-x\|_2^2 + I_C(x)\right)$

$= \arg\min_{x\in C} \frac{1}{2}\|y-x\|_2^2 = P_C(y)$

(2) Given $f(x) = \lambda\|\Sigma x\|_1$

assume $g(x) = \|\Sigma x\|_1$ $f(x) = \lambda g(x)$

$\text{Prox}_{\lambda g(x)}(v) = v - \lambda \,\text{proj}_{B\|\cdot\|_*}\left(\frac{v}{\lambda}\right)$

where $\text{proj}_{B\|\cdot\|_*}(\cdot)$ is the Orthogonal projection operator and $B_{\|\cdot\|_*}$ is the Norm unit Ball.

The projection onto the L_∞ Unit Ball is given.

$$\text{Proj}_{B\|\cdot\|_\infty}(x) = \begin{cases} -1 & x_i < -1 \\ x_i & -1 \le x_i \le 1 \\ 1 & x_i > 1 \end{cases}$$

In summary

$\text{Prox}_{\lambda\|\cdot\|_*}(v) = v - \lambda \,\text{Proj}_{B\|\cdot\|_\infty}\left(\frac{v}{\lambda}\right)$

$$\text{Prox}_{f_i}(x) = \sigma_i x_i - \lambda \,\text{Proj}_{B\|\cdot\|_\infty}\left(\frac{\sigma_i x_i}{\lambda}\right) = \begin{cases} \sigma_i x_i - 1 & \text{计 } \frac{x_i}{\sigma_i} < -\lambda \\ \sigma_i x_i - \frac{x_i}{\sigma_i} & \text{计 } -\lambda \le \frac{x_i}{\sigma_i} \le \lambda \\ \sigma_i x_i + 1 & \text{计 } \frac{x_i}{\sigma_i} > \lambda \end{cases}$$

(3) Given $f(x) = \lambda \|\Sigma x\|_2$

assume $g(x) = \|\Sigma x\|_2$ $f(x) = \lambda g(x)$

$\text{Prox}_{\lambda g(x)}(v) = v - \lambda \, \text{proj}_{B\|\cdot\|_*}(\frac{v}{\lambda})$

where $\text{proj}_{B\|\cdot\|_*}(\cdot)$ is the Orthogonal projection operator

and $B_{\|\cdot\|_*}$ is the Norm unit Ball.

The projection onto the L_2 Unit Ball is given.

$$\text{Proj}_{B\|\cdot\|_2}(x) = \begin{cases} \frac{x}{\|x\|} & \text{if } \|x\|_2 > 1 \\ x & \text{if } \|x\|_2 \le 1 \end{cases}$$

In summary

$\text{Prox}_{\lambda\|\cdot\|_2}(v) = v - \lambda \, \text{Proj}_{B\|\cdot\|_2}(\frac{v}{\lambda})$

$$\text{Prox}_f(x) = \Sigma x - \lambda \, \text{Proj}_{B\|\cdot\|}\left(\frac{\Sigma^{-1} x}{\lambda}\right) = \begin{cases} \Sigma x - \frac{\lambda \Sigma^{-1} x}{\|\Sigma^{-1} x\|_2} & \text{if } \|\Sigma^{-1} x\|_2 > \lambda \\ \Sigma x & \text{if } \|\Sigma^{-1} x\|_2 \le \lambda \end{cases}$$

(4) $f(x) = \frac{1}{2} x^T A x + b^T x + c$

$$\text{prox}_f(y) = \arg\min_x \left(\frac{1}{2}\|y - x\|_2^2 + \frac{1}{2} x^T A x + b^T x + c \right)$$

$\because$ both $\frac{1}{2}\|y - x\|_2^2$ and $\frac{1}{2} x^T A x + b^T x + c$ are convex function

$\therefore$ the sum of them is convex.

assume $h(x) = \frac{1}{2}\|y - x\|_2^2 + \frac{1}{2} x^T A x + b^T x + c$

$\because \frac{1}{2}\|y - x\|_2^2 = \frac{1}{2}\|x - y\|_2^2 = \frac{1}{2}(x - y)^T \cdot (x - y) = \frac{1}{2}(x^T x - x^T y - y^T x + y^T y)$

$\therefore h'(x) = \frac{1}{2}(2x - 2y) + x^T A + b^T$

$= x^T + x^T A - y^T + b^T = 0$

$\Rightarrow x^* = (y - b)(I + A)^{-1}$

$\therefore \text{prox}_f(y) = (y - b)(I + A)^{-1}$

(5) $f(x) = -\sum_{i=1}^{n} \log x_i$

This is fully separable.

$(\mathrm{prox}_f(y))_i = \mathrm{prox}_{f_i}(y_i)$

$\mathrm{prox}_{f_i}(y_i) = \arg\min_{x_i} \left(\frac{1}{2}\|y_i - x_i\|_2^2 - \log x_i\right)$

where $\frac{1}{2}\|y_i - x_i\|_2^2 - \log x_i$ is a convex function

assume $h(x_i) = \frac{1}{2}\|y_i - x_i\|_2^2 - \log x_i$

$= \frac{1}{2} \cdot (y_i^2 - 2y_i x_i + x_i^2) - \log x_i$

$h'(x_i) = -y_i + x_i - \frac{1}{x_i} = 0$.

where $x_i > 0$

$\therefore \quad x_i^2 - x_i y_i - 1 = 0$.

$x_i^* = \dfrac{y_i \pm \sqrt{y_i^2 + 4}}{2}$

$\therefore \quad x_i^* = \dfrac{y_i + \sqrt{y_i^2 + 4}}{2}$

$(\mathrm{prox}_f(y))_i = \mathrm{prox}_{f_i}(y_i) = \dfrac{y_i + \sqrt{y_i^2 + 4}}{2}$

Exercise 2

(a) $(\mathrm{prox}_f(x))_i = \mathrm{prox}_{f_i}(x_i)$

(b) $\mathrm{prox}_f(y) = \arg\min_{x_1} \left(\frac{1}{2}\|y - x_1\|_2^2 + f(x_1) \right)$

$h(x) = \frac{1}{2}(y-x_1)^T(y-x_1) + f(x_1)$

$= \frac{1}{2}(y^Ty - y^Tx_1 - x_1^Ty + x_1^Tx_1) + f(x_1)$

$h'(x) = \frac{1}{2}(-2y + 2x_1) + f'(x_1)$

$= -y + x_1 + f'(x_1) = 0$

$\therefore \mathrm{prox}_f(y) = y - f'(x_1^*)$

when $x_2 = Wx_1$

$\mathrm{prox}_f(y) = \arg\min_{x_2} \left(\frac{1}{2}\|y - x_2\|_2^2 + f(x_2) \right)$

$g(x_1) = \frac{1}{2}(y - Wx_1)^T(y - Wx_1) + f(Wx_1)$

$= \frac{1}{2}(y^Ty - y^TWx_1 - x_1^TW^Ty + x_1x_1^T) + f(Wx_1)$

$g'(x_1) = -y^TW + x_1^T + (f'(Wx_1))^TW = 0$

$\mathrm{prox}_f(y) = W^Ty - W^Tf'(Wx_1^*)$

$\therefore \mathrm{prox}_f(Wx) = W^T \mathrm{prox}_f(x)$

when f is separable.

$(\mathrm{prox}_f(Wx))_i = \underbrace{W_i^T}_{W^T\ i\text{th row.}} \mathrm{prox}_{f_i}(x_i)$

(3)

Given $f(x) = \lambda \|Wx\|_1$

assume $g(x) = \|Wx\|_1$ $\quad f(x) = \lambda g(x)$

$$\text{Prox}_{\lambda g(x)}(v) = v - \lambda \, \text{proj}_{B\|\cdot\|_*}\left(\frac{v}{\lambda}\right)$$

where $\text{proj}_{B\|\cdot\|_*}(\cdot)$ is the orthogonal projection operator

and $B_{\|\cdot\|_*}$ is the Norm unit Ball.

The projection onto the L_∞ Unit Ball is given.

$$\text{Proj}_{B\|\cdot\|_\infty}(x) = \begin{cases} -1 & x_i < -1 \\ x_i & -1 \le x_i \le 1 \\ 1 & x_i > 1 \end{cases}$$

In summary

$$\text{Prox}_{\lambda\|\cdot\|_1}(v) = v - \lambda \, \text{Proj}_{B\|\cdot\|_\infty}\left(\frac{v}{\lambda}\right)$$

$\because$ W square and orthogonal $\therefore$ $\underset{i\text{th row}}{W_i^T} = W_i^{-1}$

$$\text{Prox}_{f_i}(x) = W_i x - \lambda \, \text{Proj}_{B\|\cdot\|_\infty}\left(\frac{x^T W_i^T}{\alpha}\right) = \begin{cases} W_i x - 1 & \text{if } x^T W_i^T < -\lambda \\ W_i x - x^T W_i^T & \text{if } -\lambda \le x^T W_i^T \le \lambda \\ W_i x + 1 & \text{if } x^T W_i^T > \lambda \end{cases}$$