Home Assignment 3

ECE 602 - Introduction to Optimization

Due: April 24, 2020

Exercise 1

Derive the conjugates of the following functions.

(a) *Max function:* $f(x) = \max_{i=1,\cdots,n} x_i$ on $\mathbf{R}^n$.

(b) *Sum of largest elements:* $f(x) = \sum_{i=1}^{r} x_{[i]}$ on $\mathbf{R}^n$.

(c) *p-norm:* $f(x) = ||x||_p$ on $\mathbf{R}^n$.

Exercise 2

The relative entropy between two vectors $x, y \in \mathbf{R}^n_{++}$ is defined as

$$\sum_{k=1}^{n} x_k \log\left(\frac{x_k}{y_k}\right) .$$

This is a convex function, jointly in x and y. In the following problem we calculate the vector x that minimizes the relative entropy with a given vector y, subject to the following constraints on x:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \sum_{k=1}^{n} x_k \log\left(\frac{x_k}{y_k}\right) \\ \text{subject to} & Ax = b \\ & \mathbf{1}^T x = 1 \\ & x \geq \mathbf{0} \, . \end{array}$$

The given parameters are $y \in \mathbf{R}^n_{++}$, $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$. Note that $\mathbf{1}^T x = 1$ and $x \geq \mathbf{0}$ mean that x is a probability vector. Derive the Lagrange dual of this problem and simplify it to get

$$\underset{\nu}{\text{maximize}} \quad b^T \nu - \log \sum_{k=1}^{n} y_k e^{a_k^T \nu} ,$$

where a_k is the k-th column of A. Note that $\nu \in \mathbf{R}^m$ is the Lagrange multiplier associated with the constraint $Ax = b$.

Exercise 3

In class, we formulated the problem of finding the largest Euclidean ball that lies in a polyhedron described by linear inequalities,

$$\mathcal{P} = \{x \in \mathbf{R}^n \mid a_i^T x \leq b_i,\, i = 1, \cdots, m\},$$

as a linear program. What if we find the largest ball in the l_∞-norm, i.e.,

$$\mathcal{B} = \{x_c + ru \mid ||u||_\infty \leq 1\},$$

instead of the largest Euclidean ball? Can it still be formulated as a linear program?

Exercise 4

Consider the equality constrained least-squares problem

$$\begin{array}{ll} \text{minimize} & \|Ax - b\|_2^2 \\ \text{subject to} & Cx = h, \end{array}$$

where $A \in \mathbf{R}^{m\times n}$ with $\operatorname{rank} A = n$, and $C \in \mathbf{R}^{p\times n}$ with $\operatorname{rank} C = p$. Give the KKT conditions, and derive expressions for the primal solution x^* and the dual solution ν^*.