Уравнение Бернулли.

$y' - \dfrac{2y}{x} = -x^2y^2$

$z = \dfrac{1}{y}$

$dy = -y^2dz$

$z' + \dfrac{2}{x}z = x^2$

$\dfrac{dz}{z} = -\dfrac{2dx}{x}$

$z = \dfrac{C(x)}{x^2}$

$\dfrac{C'(x)x^2 - 2xC(x)}{x^4} + \dfrac{2C(x)}{x^3} = x^2$

$C'(x) = x^4,\ C(x) = \dfrac{x^5 + C_2}{5}$

$z = \dfrac{x^5 + C_2}{5x^2}$

$y = \dfrac{1}{z} = \dfrac{5x^2}{x^5 + C_2}$

Уравнение Эйлера.

$x^3y''' - x^2y'' + 2xy' - 2y = x^3$

$\lambda(\lambda-1)(\lambda-2) - \lambda(\lambda-1) + 2\lambda - 2 = 0$

$(\lambda-1)(\lambda^2 - 3\lambda + 2) = 0,\ \lambda_1 = \lambda_2 = 1,\ \lambda_3 = 2$

$y_0 = (C_1 + C_2t)e^t + C_3e^{2t}$

$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0\ \Rightarrow\ y_t''' - 4y_t'' + 5y_t' - 2y = x^3 = e^{3t}$

$3 \neq \lambda_1, \lambda_2, \lambda_3\ \Rightarrow\ y_1 = \alpha e^{3t},\ \alpha = \dfrac{1}{4}$

$y = y_0 + y_1 = (C_1 + C_2t)e^t + C_3e^{2t} + \dfrac{1}{4}e^{3t} = (C_1 + C_2\ln x)x + C^3x^2 + \dfrac{1}{4}x^3\ (x > 0)$

Уравнение, не разрешенное относительно производной.

$y = x + y' - \ln y'$

$p = y'$

$y = x + p - \ln p$

$dy = dx + dp - \frac{dp}{p}$

$pdx = dx + dp - \frac{dp}{p}$

$(p-1)dx = \dfrac{p-1}{p}dp$

а) $p \neq 1$

$dx = \dfrac{dp}{p}$

$x = \ln p + C_1$

$y = p + C_2$

$p = e^{x+C_1}$

$y = e^{x+C_1} + C_2$

б) $p = 1$

$y = x + 1$

Разложение по начальным условиям.

$$\begin{cases} y' = y + xe^y \sim y + x\left[1 + y + \dfrac{y^2}{2!} + \dfrac{y^3}{3!} + \ldots\right] \\ y(0) = 0 \end{cases}$$

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$

$y(0) = 0\ \Rightarrow\ a_0 = 0$

$\dfrac{x}{2}y^2 = \dfrac{x}{2}\left[a_1^2x^2 + 2a_1a_2x^3 + \bar{o}(x^3)\right]$

$\dfrac{x}{3!}y^3 = \dfrac{x}{3!}\left[a_1^3x^3 + \bar{o}(x^3)\right]$

$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + x + a_1x^2 + a_2x^3 + a_3x^4 + \dfrac{a_1^2x^3}{2} + \dfrac{2a_1a_2x^4}{2} + \dfrac{a_1^3x^4}{3!}$

$x^0\colon a_1 = 0$

$x^1\colon 2a_2 = a_1 + 1 \;\Rightarrow\; a_2 = \dfrac{1}{2}$

$x^2\colon 3a_3 = a_2 + a_1 \;\Rightarrow\; a_3 = \dfrac{1}{6}$

$x^3\colon 4a_4 = a_3 + \dfrac{a_1^2}{2} + a_2 = \dfrac{1}{6} + \dfrac{1}{2} = \dfrac{4}{6} \;\Rightarrow\; a_4 = \dfrac{1}{6}$

$y = \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{6} + \dots$

Задача на собственные функции.

$$\begin{cases} y'' = \lambda y \\ y(0) = 0 \\ y'(l) = 0 \end{cases}$$

$p^2 = \lambda$

1. $\lambda = 0$

   $y = C_1 + C_2 x$

   $y' = C_2$

   $\begin{cases} y(0) = 0 \;\Rightarrow\; C_1 = 0 \\ y'(l) = 0 \;\Rightarrow\; C_2 = 0 \end{cases} \;\Rightarrow\; y \equiv 0$

2. $\lambda > 0$

   $\mu = \sqrt{\lambda},\; p = \pm\mu$

   $y = C_1 e^{\mu x} + C_2 e^{-\mu x}$

   $y' = \mu C_1 e^{\mu x} - \mu C_2 e^{-\mu x}$

   $y(0) = 0 \;\Rightarrow\; C_1 + C_2 = 0 \;\Rightarrow\; C_1 = -C_2$

   $y'(l) = 0 \;\Rightarrow\; \mu C_1 e^{\mu l} + \mu C_1 e^{-\mu l} = 0 \;\Rightarrow\; \mu C_1 (e^{\mu l} + e^{-\mu l}) = 0 \;\Rightarrow\; C_1 = 0 \;\Rightarrow\; y \equiv 0$

3. $\lambda < 0$

   $\mu = \sqrt{-\lambda},\; p = \pm i\mu$

   $y = C_1 \cos\mu x + C_2 \sin\mu x$

   $y' = -\mu C_1 \sin\mu x + \mu C_2 \cos\mu x$

   $y(0) = 0 \;\Rightarrow\; C_1 = 0$

   $y'(l) = 0 \;\Rightarrow\; \mu C_2 \cos\mu l = 0$

   $\mu l = \dfrac{\pi}{2} + \pi n = \pi\left(\dfrac{1+2n}{\pi}\right)$

   $$\begin{cases} \lambda_k = -\dfrac{\pi^2(1+2k)^2}{4l^2} \\ y_k = \cos\dfrac{\pi(1+2k)}{2l} x \end{cases}$$