N1 Найти $\frac{dz}{dt}$, если $z = z(x,y)$, $x = x(t)$, $y = y(t)$

1) $z = x^2 + y^2 + xy$, $x = a\sin t$, $y = a\cos t$

$z = a^2 + \frac{a^2}{2}\sin 2t$, $\frac{dz}{dt} = \frac{a^2}{2}\cdot 2\cdot\cos 2t = a^2\cos 2t$

2) $z = x^2y^3u$, $x = t$, $y = t^2$, $u = \sin t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial z}{\partial u}\cdot\frac{du}{dt}$$

$$\frac{\partial z}{\partial x} = 2xy^3u = 2tt^6\sin t = 2t^7\sin t$$

$$\frac{\partial z}{\partial y} = 3x^2y^2u = 3t^2\cdot t^4\sin t = 3t^6\sin t$$

$$\frac{\partial z}{\partial u} = x^2y^3 = t^2\cdot t^6 = t^8$$

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2t, \quad \frac{du}{dt} = \cos t$$

$$\frac{dz}{dt} = 2t^7\sin t + 3t^6\sin t\cdot 2t + t^8\cos t = t^7(8\sin t + t\cos t)$$

N2 Для данных $z = f(x,y)$, $x(u,v)$, $y = y(u,v)$. Найти $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$ и dz:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

1) $z = x^3 + y^3$, где $x = uv$, $y = \frac{u}{v}$

$$\frac{\partial z}{\partial x} = 3x^2, \quad \frac{\partial z}{\partial y} = 3y^2, \quad \frac{\partial x}{\partial u} = v, \quad \frac{\partial y}{\partial u} = \frac{1}{v}, \quad \frac{\partial x}{\partial v} = u, \quad \frac{\partial y}{\partial v} = -\frac{u}{v^2}$$

$$\frac{\partial z}{\partial u} = 3(uv)^2\cdot v + 3\left(\frac{u}{v}\right)^2\cdot\frac{1}{v} = 3\left(u^2v^3 + \frac{u^2}{v^3}\right)$$

$$\frac{\partial z}{\partial v} = 3(uv)^2\cdot u + 3\left(\frac{u}{v}\right)^2\cdot\left(-\frac{u}{v^2}\right) = 3\left(u^3v^2 - \frac{u^3}{v^4}\right)$$

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}\cdot dv = 3\left(u^2v^3 + \frac{u^2}{v^3}\right)du + 3\left(u^3v^2 - \frac{u^3}{v^4}\right)dv$$

2) $z = \cos xy$, где $x = ue^{v}$, $y = v\ln u$

$\frac{\partial z}{\partial x} = -y\sin xy$, $\frac{\partial z}{\partial y} = -x\sin xy$

$\frac{\partial x}{\partial u} = e^{v}$, $\frac{\partial y}{\partial u} = \frac{v}{u}$, $\frac{\partial x}{\partial v} = ue^{v}$, $\frac{\partial y}{\partial v} = \ln u$

$$\frac{\partial z}{\partial u} = -v\ln u\sin(ue^{v}v\ln u)\cdot e^{v} - ue^{v}\sin(ue^{v}v\ln u)\cdot\frac{v}{u} =$$

$$= -\sin(vue^{v}\ln u)e^{v}\cdot v(\ln u + 1)$$

$$\frac{\partial z}{\partial v} = -v\ln u\sin(ue^{v}v\ln u)ue^{v} - ue^{v}\sin(ue^{v}v\ln u)\ln u =$$

$$= -\sin(vue^{v}\ln u)ue^{u}\ln u(v+1)$$

$$dz = -\sin(uve^{v}\ln u)ve^{v}(\ln u + 1)dv - \sin(uve^{v}\ln u)ue^{v}\ln u(v+1)du$$

N3 Найти производные $y'(x)$ неявных функций, заданных уравнениями:

$$xe^{2y} - y\ln x = 8$$

$$\frac{dy}{dx} = -\frac{F'_x(x,y)}{F'_y(x,y)} = -\frac{e^{2y} - \frac{y}{x}}{2xe^{2y} - \ln x}$$

N4 Составить уравнение касательной прямой и нормали к кривой $y = y(x)$, заданной уравнением $F(x,y) = 0$ в точке $M_0(x,y)$:

$$x^3y - y^3x = 6, \quad M_0(2,1)$$

$y - y_0 = f'(x_0)(x - x_0)$ – уравнение касательной

$y - y_0 = -\frac{1}{f'(x_0)}\cdot(x - x_0)$ – уравнение нормали

$3x^2y + x^3y' - 3y^2y'x - y^3 = 0$, $y' = \frac{y^3 - 3x^2y}{x^3 - 3y^2x}$

$y - 1 = \frac{1 - 3\cdot 4}{8 - 3\cdot 2}(x-2) = -\frac{11}{2}(x-2) \Rightarrow y = -\frac{11}{2}x - 10$ – уравн. касат.

$y - 1 = \frac{2}{11}\cdot(x-2) \Rightarrow y = \frac{2}{11}x + \frac{7}{11}$ – уравн. нормали

N5 Для данных функций найти требуемую частную производную или дифференциал

1) $z = \sin x \sin y$, $d^2 z$ - ?

$$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2\frac{\partial^2 z}{\partial x \partial y} \cdot dx \cdot dy + \frac{\partial^2 z}{\partial y^2} \cdot dy^2$$

$\frac{\partial z}{\partial x} = \sin y \cos x$, $\frac{\partial^2 z}{\partial x \partial y} = \cos x \cos y$

$\frac{\partial^2 z}{\partial x^2} = -\sin x \sin y$, $\frac{\partial^2 z}{\partial y^2} = -\sin x \sin y$

$$d^2 z = -\sin x \sin y \, dx^2 + 2\cos x \cos y \, dx dy - \sin x \sin y \, dy^2$$

2) $z = xy + \sin(x+y)$, $\frac{\partial^2 z}{\partial x^2}$ - ?

$\frac{\partial z}{\partial x} = y + \cos(x+y)$, $\frac{\partial^2 z}{\partial x^2} = -\sin(x+y)$

3) $z = \operatorname{arctg} \frac{x+y}{1-xy}$, $\frac{\partial^2 z}{\partial x \partial y}$ - ?

$$\frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} \cdot \left(\frac{1}{1-xy} - \frac{x+y}{(1-xy)^2}(-y)\right) = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} \cdot \left(\frac{1-xy+yx+y^2}{(1-xy)^2}\right) =$$

$$= \frac{1+y^2}{(1-xy)^2+(x+y)^2} = \frac{1+y^2}{1-2xy+x^2y^2+x^2+y^2+2xy} = \frac{1+y^2}{1+x^2+y^2+x^2y^2} =$$

$$= \frac{1+y^2}{1+x^2+y^2(1+x^2)} = \frac{1}{1+x^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 0$$

N6 Найти y', y'', y''' для неявной функции $y=y(x)$, заданной неявно уравнением $x^2-xy+2y^2+x-y=1$ при $x=0$, если $y(0)=1$.

$$2x-y-xy'+4yy'+1-y'=0$$

$$y'(-x+4y-1)+(2x-y+1)=0$$

$$y'=\frac{2x-y+1}{x-4y+1}, \quad y'(0)=\frac{-1+1}{-4+1}=0$$

$$y''(4y-x-1)+y'(4y'-1)+2-y'=0$$

$$y''=-\frac{4y'^2-y'+2-y'}{4y-x-1}=\frac{4y'^2-2y'+2}{x+1-4y}=\frac{2(2y'^2-y'+1)}{x+1-4y}$$

$$y''(0)=\frac{2\cdot(1)}{1-4}=-\frac{2}{3}$$

$$y'''(x+1-4y)+y''(1-4y')-2(4y'y''-y'')=0$$

$$y'''=\frac{4y''(4y'-1)}{x+1-4y}$$

$$y'''(0)=\frac{-\frac{8}{3}\cdot(-1)}{-3}=-\frac{8}{9}$$

N7 Для функции $z=\operatorname{arctg}\frac{y}{x}$ построить линии уровня и градиент. Сравнить их направление в точках $(1,1)$, $(1,-1)$.

$$\operatorname{grad} z=\frac{\partial z}{\partial x}\vec{i}+\frac{\partial z}{\partial y}\vec{j}$$

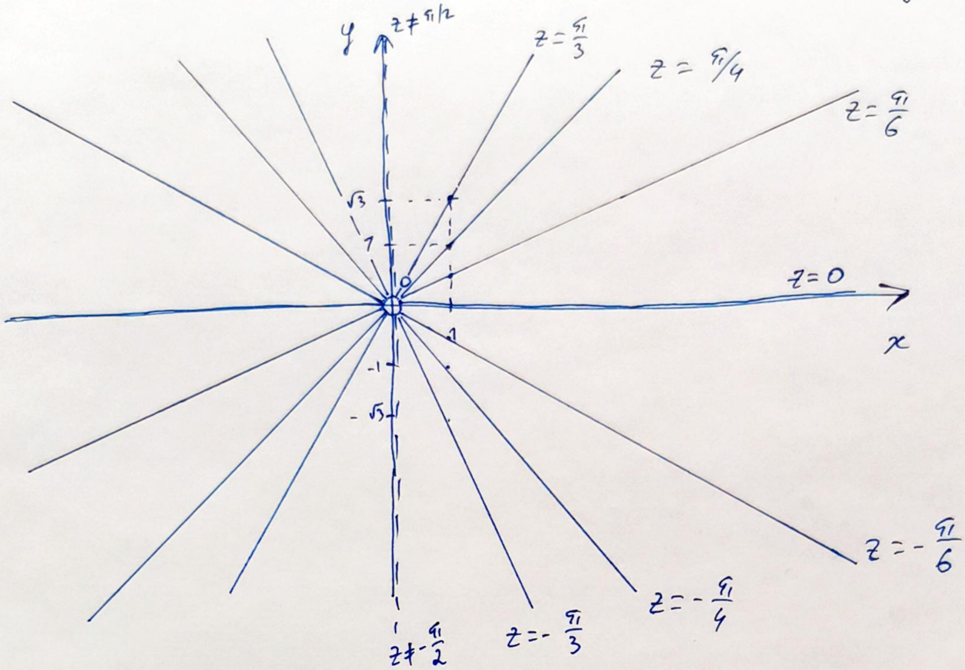
$$\frac{\partial z}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{-y}{x^2}=-\frac{y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}=\frac{x}{x^2+y^2}$$

$$z\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right), \quad x\neq 0$$

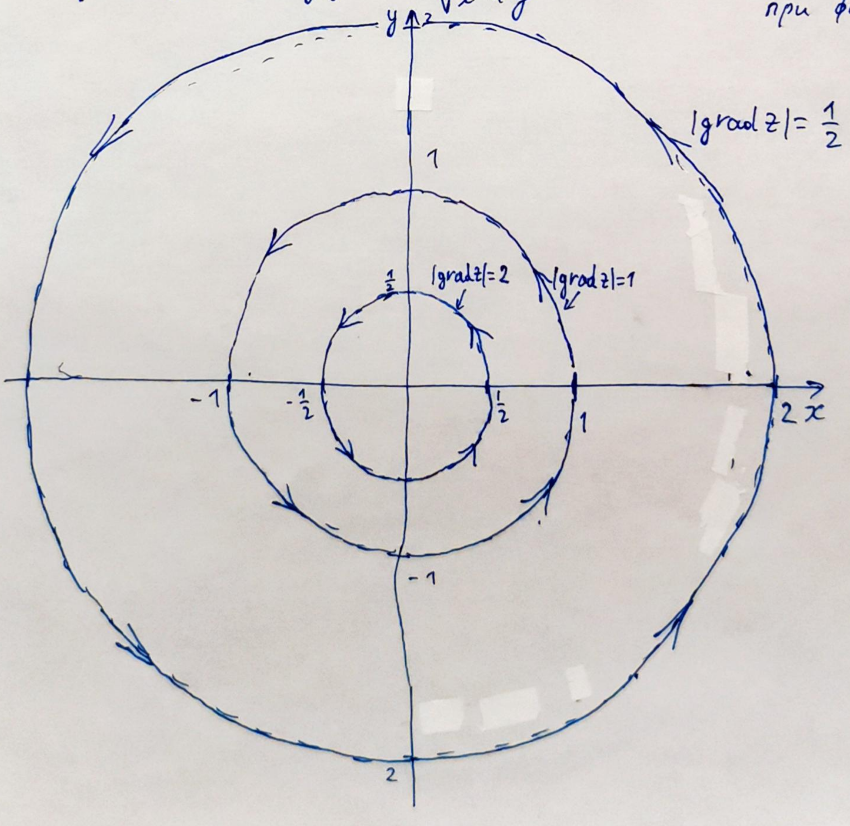
$$\lim_{\substack{x\to 0\\ y\to 0}} \left(\operatorname{arctg}\frac{y}{x}\right) = \left| \begin{matrix} x=\rho\cos\varphi \\ y=\rho\sin\varphi \\ \frac{y}{x} = \operatorname{tg}\varphi \end{matrix} \right| = \lim_{\substack{\rho\to 0\\ \varphi\to\alpha}} \operatorname{tg}\varphi = \operatorname{tg}\alpha \Rightarrow$$

предела не существует



$$|\operatorname{grad} z| = \sqrt{\frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2}} = \frac{1}{\sqrt{x^2+y^2}} = u \Rightarrow \frac{1}{u^2} = x^2+y^2$$

при фиксир. u уравн. окружн.



$\vec{a} = \text{grad}\, z \,(1,1) = -\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$

$\vec{b} = \text{grad}\, z \,(1,-1) = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$