Partial Answers to the Chapter End Problems

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Fundamental Finite Element Analysis and Applications

with Computations Using *Mathematica* and Matlab

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CHAPTER ONE

Finite Element Method
The Big Picture

1.1

$$\begin{pmatrix} 3.63562 & -2.1 & 1.29281 \\ -2.1 & 2.8 & -0.7 \\ 1.29281 & -0.7 & 2.23562 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} 70.7107 \\ 0 \\ 70.7107 \end{pmatrix}$$

1.2

$$10^7 \begin{pmatrix} 1.19505 & 0.357143 & -0.81044 & 0.302198 & -0.384615 & -0.659341 \\ 0.357143 & 0.659341 & 0.412088 & 0.43956 & -0.769231 & -1.0989 \\ -0.81044 & 0.412088 & 1.96429 & -1.07143 & -1.15385 & 0.659341 \\ 0.302198 & 0.43956 & -1.07143 & 2.85714 & 0.769231 & -3.2967 \\ -0.384615 & -0.769231 & -1.15385 & 0.769231 & 1.53846 & 0 \\ -0.659341 & -1.0989 & 0.659341 & -3.2967 & 0 & 4.3956 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} -30. \\ 45. \\ 0. \\ 0. \\ -30. \\ 45. \end{pmatrix}$$

1.3

$$\begin{pmatrix} 1.53743 & -0.18 & 0.104965 \\ -0.18 & 0.666 & -0.486 \\ 0.104965 & -0.486 & 1.84343 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} 46.7966 \\ 0 \\ 46.7966 \end{pmatrix}$$

1.4

$$\begin{pmatrix} 0.04 & -0.04 & 0 \\ -0.04 & 0.53 & 0.23 \\ 0 & 0.23 & 0.49 \end{pmatrix} \begin{pmatrix} T_1 \\ T_3 \\ T_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 50.4 \\ 50.4 \end{pmatrix}$$

1.5

$$\begin{pmatrix} 0.04 & 0.02 & -0.06 \\ 0.02 & 0.02 & -0.04 \\ -0.06 & -0.04 & 0.1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_4 \\ T_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

1.6

$$\begin{pmatrix} 200.625 & -0.4 & 99.775 \\ -0.4 & 0.4 & 0 \\ 99.775 & 0 & 200.225 \end{pmatrix}$$

{4500, 2000, 6500}

1.7

$$\begin{pmatrix} 75600. & 100800. & 0 & 0 & 0 & 0 & -75600. & -100800. \\ 100800. & 134400. & 0 & 0 & 0 & 0 & -100800. & -134400. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 262500. & 0 & 0 & 0 & -262500. \\ 0 & 0 & 0 & 0 & 350000. & 0 & -350000. & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -75600. & -100800. & 0 & 0 & -350000. & 0 & 425600. & 100800. \\ -100800. & -134400. & 0 & -262500. & 0 & 0 & 100800. & 396900. \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 14142.1 \\ -24142.1 \end{pmatrix}$$

1.8

$$\boldsymbol{K} = \begin{pmatrix} 63993.2 & 79991.4 & 0 & 0 & 0 & 0 & -63993.2 & -79991.4 \\ 79991.4 & 99989.3 & 0 & 0 & 0 & 0 & -79991.4 & -99989.3 \\ 0 & 0 & 0. & 0. & 0 & 0 & 0. & 0. \\ 0 & 0 & 0. & 210000. & 0 & 0 & 0. & -210000. \\ 0 & 0 & 0 & 0 & 239682. & -59920.6 & -239682. & 59920.6 \\ 0 & 0 & 0 & 0 & -59920.6 & 14980.1 & 59920.6 & -14980.1 \\ -63993.2 & -79991.4 & 0. & 0. & -239682. & 59920.6 & 303675. & 20070.9 \\ -79991.4 & -99989.3 & 0. & -210000. & 59920.6 & -14980.1 & 20070.9 & 324969. \end{pmatrix}$$

$$\boldsymbol{R}^{\mathrm{T}} = (\, 0. \quad 0. \quad 0. \quad 0. \quad 0. \quad 0. \quad 14142.1 \quad -24142.1 \,)$$

1.9

$$\begin{pmatrix} 246 & 198 & 0 & 0 & 0 & 0 & 221 & 179 & 0 & 0 & 188 & 194 \\ 198 & 273 & 0 & 0 & 0 & 0 & 182 & 242 & 0 & 0 & 221 & 186 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 221 & 182 & 0 & 0 & 0 & 0 & 215 & 167 & 0 & 0 & 198 & 185 \\ 179 & 242 & 0 & 0 & 0 & 0 & 167 & 243 & 0 & 0 & 225 & 183 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 188 & 221 & 0 & 0 & 0 & 0 & 198 & 225 & 0 & 0 & 266 & 180 \\ 194 & 186 & 0 & 0 & 0 & 0 & 185 & 183 & 0 & 0 & 180 & 201 \end{pmatrix} . \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \\ d_9 \\ d_{10} \\ d_{11} \\ d_{12} \end{pmatrix} == \begin{pmatrix} 10 \\ 6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 9 \\ 10 \\ 0 \\ 0 \\ 8 \\ 7 \end{pmatrix}$$

1.10

$$\boldsymbol{K} = \begin{pmatrix} 0.08 & -0.06 & -0.04 & 0.02 \\ -0.06 & 0.1 & 0 & -0.04 \\ -0.04 & 0 & 0.53 & 0.23 \\ 0.02 & -0.04 & 0.23 & 0.51 \end{pmatrix}$$

$$\boldsymbol{R}^{\mathrm{T}} = (\, 0 \quad 0 \quad 50.4 \quad 50.4 \,)$$

1.11

	u	v
1	0	0
2	0	0
3	0	0
4	0.0506837	−0.0736988

1.12

Nodal solution

	x−coord	y−coord	u	v
1	0.	0.	−0.1	0.2
2	400.	0.	0	0
3	0.	600.	0	0
4	400.	500.	0.0809795	−0.0423694

1.13

$$\begin{pmatrix} 182342 & 1479 & 239 \\ 1479 & 277735 & -1056 \\ 239 & -1056 & 33134 \end{pmatrix} . \begin{pmatrix} d_3 \\ d_5 \\ d_6 \end{pmatrix} = \begin{pmatrix} -2064 \\ 4224 \\ -5643 \end{pmatrix}$$

$\{-0.0112155, 0.014623, -0.169762\}$

1.14

$$\begin{pmatrix} 12560. & -1920. & -5000. & 0 & -2560. \\ -1920. & 8106.67 & 0 & 0 & 1920. \\ -5000. & 0 & 10000. & 0 & 0 \\ 0 & 0 & 0 & 6666.67 & 0 \\ -2560. & 1920. & 0 & 0 & 10120. \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_6 \end{pmatrix} = \begin{pmatrix} -19200. \\ -5600. \\ 0 \\ 46666.7 \\ 0 \end{pmatrix}$$

$\{u_2 = -2.21254,\ v_2 = -1.13317,\ u_3 = -1.10627,\ v_3 = 7.,\ u_6 = -0.344704\}$

1.15

	T(x,y)	∂T/∂x	∂T/∂y
1	$-2.94118\,x + 2.94118\,y + 99.4118$	−2.94118	2.94118
2	$-2.94118\,x + 2.94118\,y + 99.4118$	−2.94118	2.94118

1.16

Axial stress, $\sigma = \mathrm{E}\epsilon = -11.9905$ Axial force $= \sigma\mathrm{A} = -5995.25$

Axial stress, $\sigma = \mathrm{E}\epsilon = -35.4786$ Axial force $= \sigma\mathrm{A} = -17739.3$

Axial stress, $\sigma = \mathrm{E}\epsilon = -38.6919$ Axial force $= \sigma\mathrm{A} = -19345.9$

1.17

Nodal solution

	x−coord	y−coord	u	v
1	0.	0.	0	0
2	1.5	0.	0	-5.64054×10^{-6}
3	0.	1.5	0	0
4	1.5	1.	1.04785×10^{-6}	-5.64054×10^{-6}

Element solution

	Stress	Axial force
1	0.	0.
2	−25038.6	−20.0308
3	35136.4	28.1091
4	0.	0.

1.18

$$\frac{10x}{3} + 5y + 10$$

20

1.19

{−2.01064, −3.01064, 3.01064, 4.70213, −1.7234}

1.20

$$\left\{-\frac{4}{5}, \frac{3}{5}, -\frac{2}{5}, \frac{1}{5}\right\}$$

1.21

$$\left\{-\frac{29}{40}, -\frac{9}{20}, -\frac{7}{40}, -\frac{3}{40}, -\frac{1}{20}, -\frac{1}{40}\right\}$$

$$\left\{\frac{7}{40}, \frac{27}{20}, \frac{21}{40}, \frac{9}{40}, \frac{3}{20}, \frac{3}{40}\right\}$$

$$\left\{\frac{1}{8}, \frac{5}{4}, \frac{3}{8}, -\frac{1}{8}, -\frac{3}{4}, -\frac{11}{8}\right\}$$

1.22

$$\left\{\frac{6113}{4837}, -\frac{1224}{4837}, -\frac{1991}{4837}, \frac{4518}{4837}, \frac{14863}{4837}, \frac{62757}{9674}, -\frac{6693}{9674}, -\frac{3761}{691}, \frac{8139}{9674}, -\frac{923}{1382}\right\}$$

$$\left\{\frac{12709}{4837}, \frac{1304}{4837}, \frac{30416}{4837}, \frac{7706}{4837}, -\frac{7836}{4837}, -\frac{30439}{9674}, \frac{38239}{9674}, \frac{3090}{691}, -\frac{2601}{9674}, \frac{5039}{1382}\right\}$$

$$\left\{-\frac{7037}{4837}, \frac{776}{4837}, -\frac{2405}{4837}, \frac{15061}{4837}, \frac{1895}{4837}, -\frac{38649}{9674}, \frac{37059}{9674}, \frac{194}{691}, -\frac{8195}{9674}, \frac{2473}{1382}\right\}$$

1.23

$\{-2.01064, -3.01064, 3.01064, 4.70213, -1.7234\}$

1.24

$$\left\{-\frac{4}{5}, \frac{3}{5}, -\frac{2}{5}, \frac{1}{5}\right\}$$

1.25

$$\left\{-\frac{29}{40}, -\frac{9}{20}, -\frac{7}{40}, -\frac{3}{40}, -\frac{1}{20}, -\frac{1}{40}\right\}$$

1.26

$\{1.2638, -0.253049, -0.411619, 0.93405, 3.07277, 6.48718, -0.691854, -5.44284, 0.841327, -0.667873\}$

1.27

$\{-2.01064, -3.01064, 3.01064, 4.70213, -1.7234\}$

1.28

$$\left\{-\frac{4}{5}, \frac{3}{5}, -\frac{2}{5}, \frac{1}{5}\right\}$$

1.29

$$\left\{-\frac{29}{40}, -\frac{9}{20}, -\frac{7}{40}, -\frac{3}{40}, -\frac{1}{20}, -\frac{1}{40}\right\}$$

1.30

$\{1.2638, -0.253049, -0.411619, 0.93405, 3.07277, 6.48718, -0.691854, -5.44284, 0.841327, -0.667873\}$

1.31

$\{-0.725, -0.45, -0.175, -0.075, -0.05, -0.025\}$

1.32

$\{1.26032, -0.253115, -0.410117, 0.935126, 3.07278, 6.48661, -0.691353, -5.44312, 0.841222, -0.668436\}$

1.33

$\{d_1 = 1.53802,\ d_2 = -0.538023,\ d_3 = 1.61407,\ d_4 = 2.91635,\ d_5 = -1.44867,\ \lambda_1 = -2.15209\}$

1.34

$\{d_1 = -0.42562,\ d_2 = 0.85124,\ d_3 = 0.338843,$
$d_4 = 0.165289,\ d_5 = 0.157025,\ d_6 = 0.14876,\ \lambda_1 = 0.35124,\ \lambda_2 = 0.210744\}$

1.35

Penalty parameter, $\mu = 900000$.

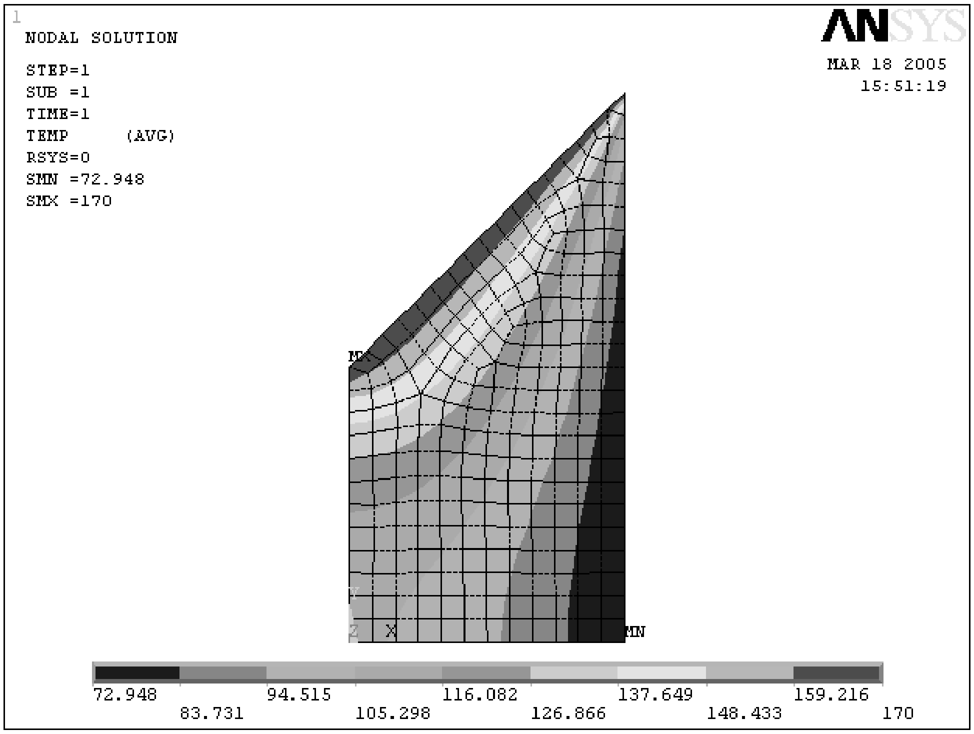
$\{d_1 = 1.53802,\ d_2 = -0.538024,\ d_3 = 1.61407,\ d_4 = 2.91635,\ d_5 = -1.44867\}$

1.36

Penalty parameter, $\mu = 300000$.

$\{d_1 = -0.425619,\ d_2 = 0.85124,\ d_3 = 0.338843,\ d_4 = 0.165289,\ d_5 = 0.157025,\ d_6 = 0.14876\}$

1.37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SUB =1
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SMX =170
ANSYS
MAR 18 2005
15:51:19
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MN
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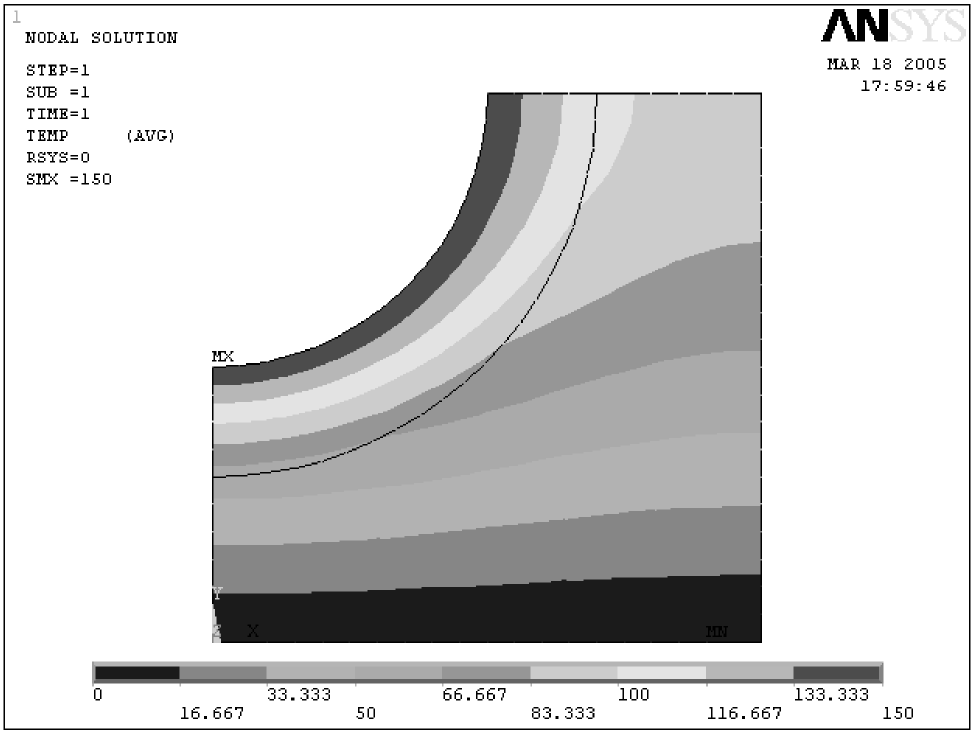
126.866
137.649
148.433
159.216
170

1.38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SUB =1
TIME=1
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SMX =150
ANSYS
MAR 18 2005
17:59:46
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MN
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X
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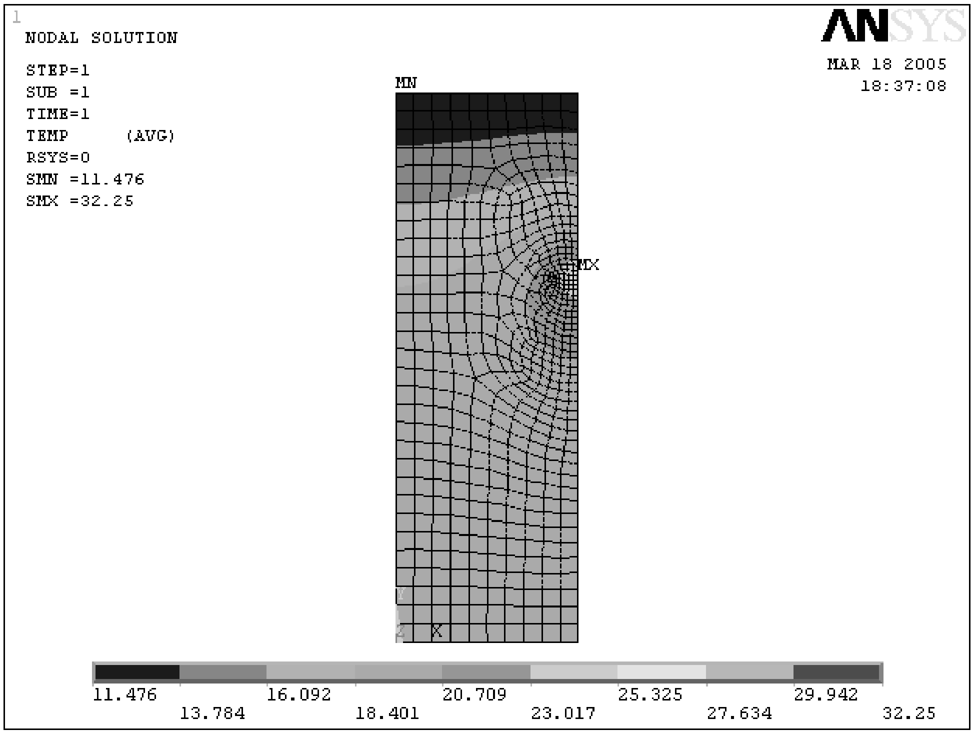
66.667
83.333
100
116.667
133.333
150

1.39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NODAL SOLUTION
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SUB =1
TIME=1
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SMX =32.25
ANSYS
MAR 18 2005
18:37:08
MN
MX
Y
Z X
11.476
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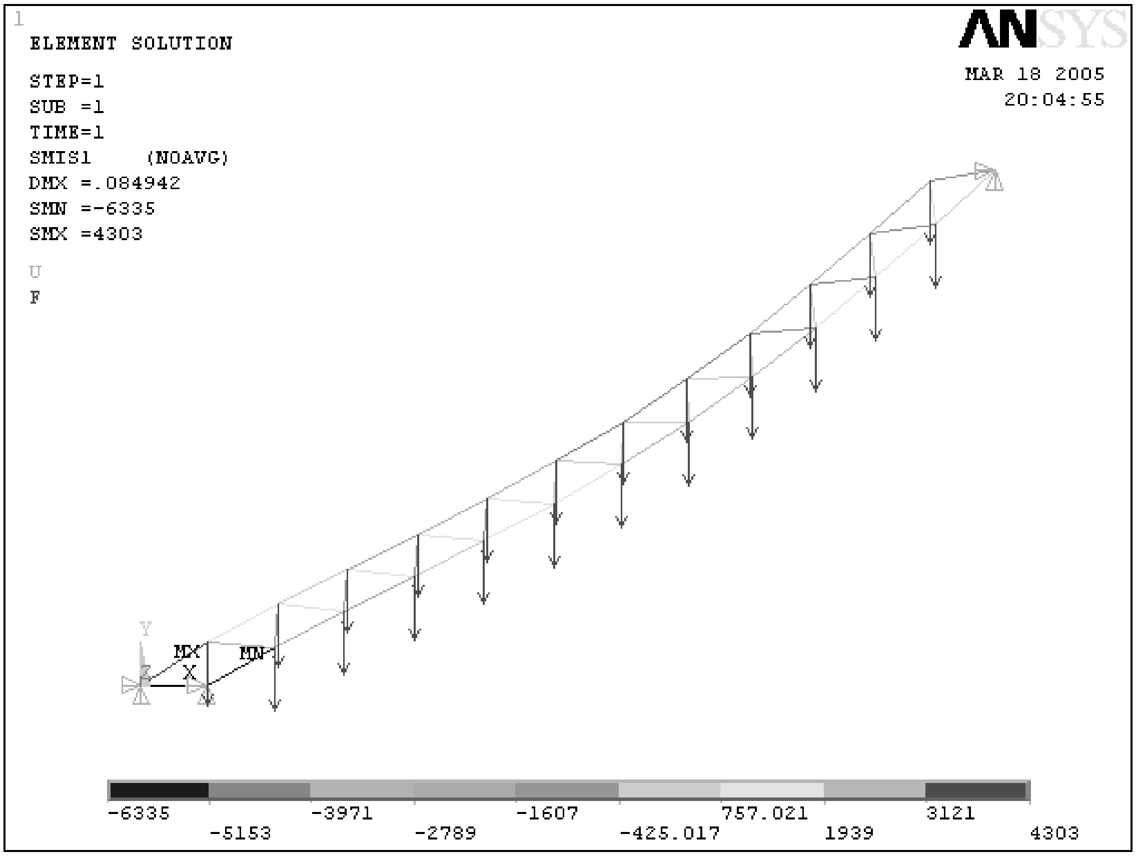
16.092
18.401
20.709
23.017
25.325
27.634
29.942
32.25

1.40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ELEMENT SOLUTION
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TIME=1
SMIS1 (NOAVG)
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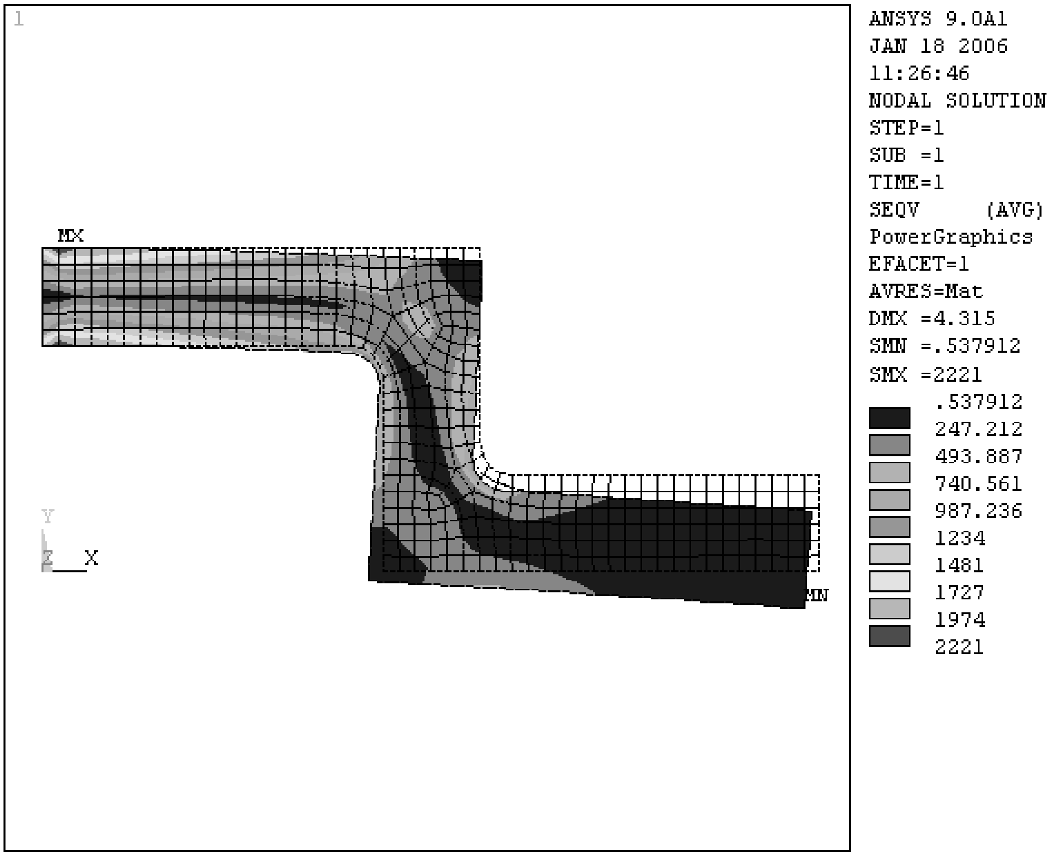
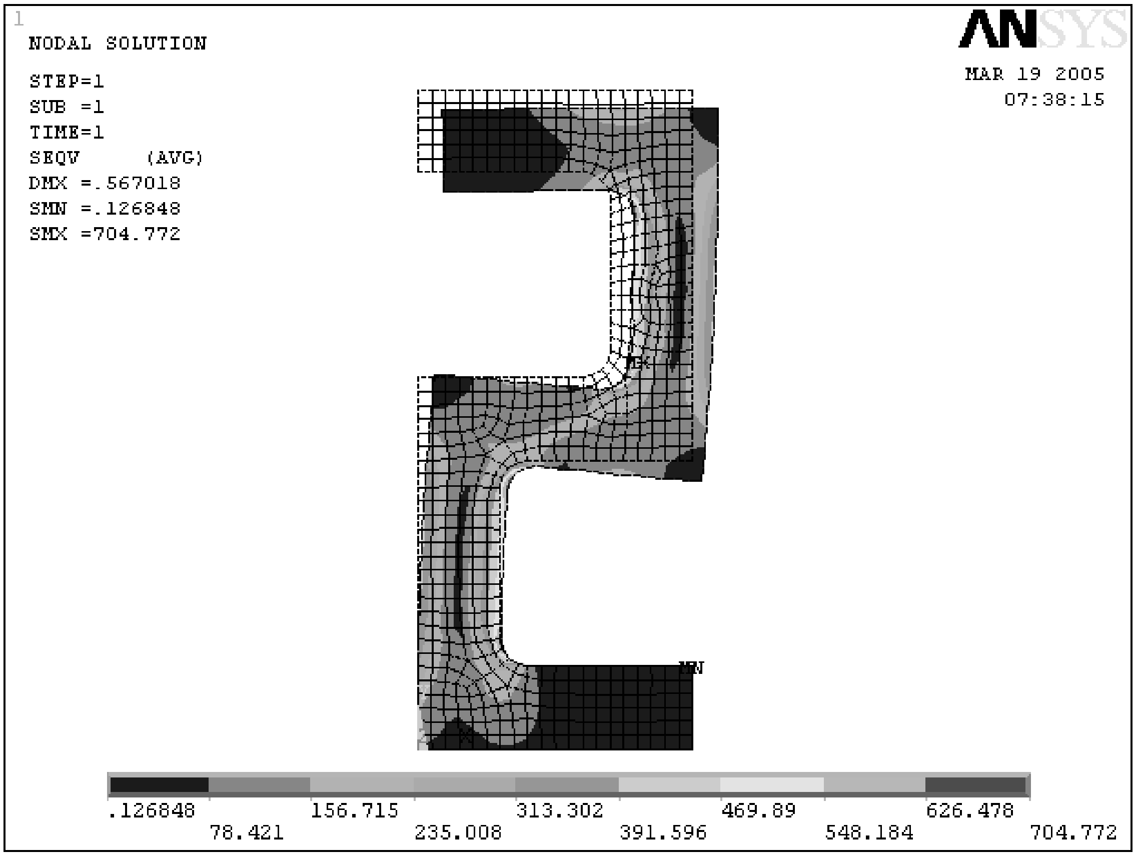
SMN =-6335
SMX =4303
U
F
ANSYS
MAR 18 2005
20:04:55
Y
MX
MN
X
Z
-6335
-5153
-3971
-2789
-1607
-425.017
757.021
1939
3121
4303

1.41


1
MX
MN
Y
Z X
ANSYS 9.0A1
JAN 18 2006
11:26:46
NODAL SOLUTION
STEP=1
SUB =1
TIME=1
SEQV (AVG)
PowerGraphics
EFACET=1
AVRES=Mat
DMX =4.315
SMN =.537912
SMX =2221
.537912
247.212
493.887
740.561
987.236
1234
1481
1727
1974
2221

1.42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NODAL SOLUTION
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SEQV (AVG)
DMX =.567018
SMN =.126848
SMX =704.772
ANSYS
MAR 19 2005
07:38:15
MX
MN
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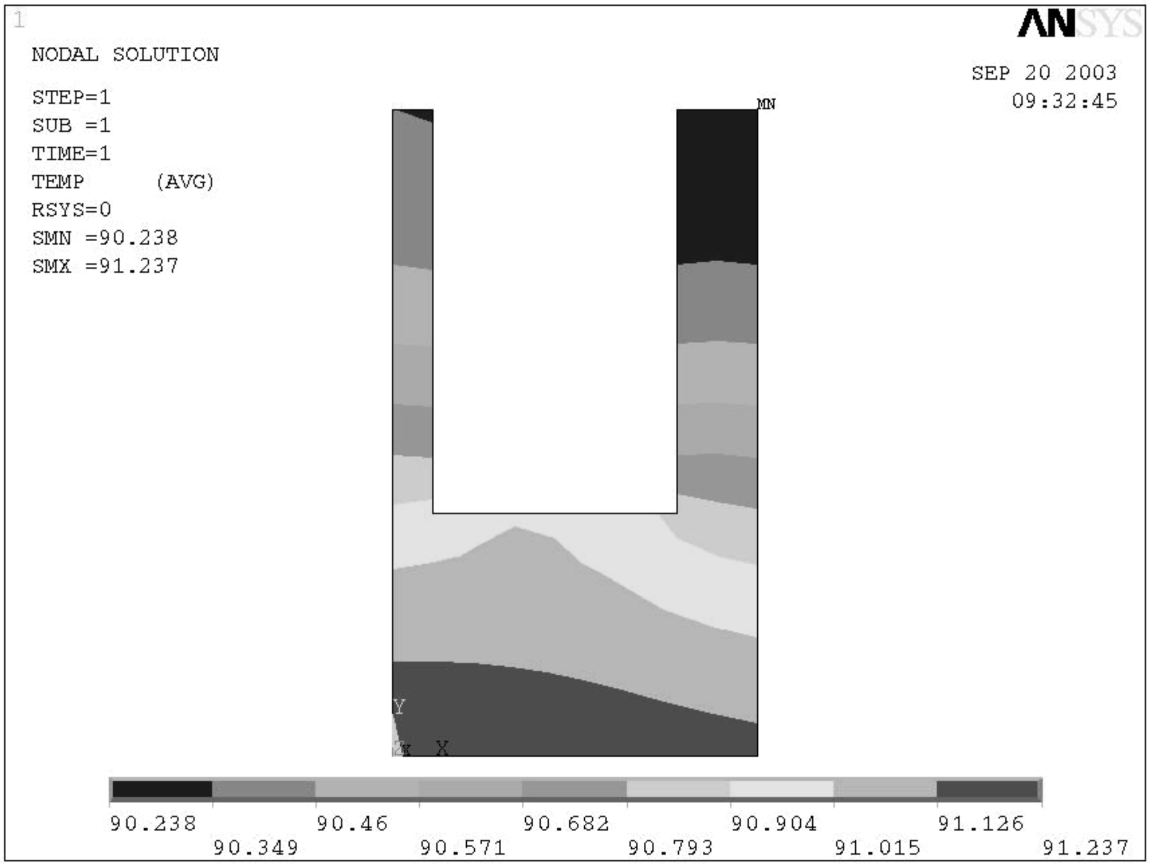
235.008
313.302
391.596
469.89
548.184
626.478
704.772

1.43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SMX =91.237
SEP 20 2003
09:32:45
MN
Y
X
90.238
90.349
90.46
90.571
90.682
90.793
90.904
91.015
91.126
91.237

CHAPTER TWO

Mathematical Foundation of the Finite Element Method

2.1

$$u(x) = \frac{(2\,\mathrm{EA}\,g + L\,q\,(L-x))\,x}{2\,\mathrm{EA}\,L}$$

2.2

$$T(x) = \frac{(k + h\,(L-x))\,T_0 + h\,x\,T_\infty}{k + h\,L}$$

2.3

$$T(r) = 20\,r^2 - 200\,r + 580$$

2.4

$$u(x) = \frac{3\,x}{2} + 2$$

2.5

$$u(x) = -\frac{240\,(x-1)\,x}{\pi\,(10+\pi^2)}$$

2.6

$$u(x) = \frac{2}{7}\,(4\,x-1)$$

2.7

$$u(x) = -1.65881 + 3.3853\,x$$

2.8

$$u(x) = \frac{1}{30}\left(17\,x^2 + 24\,x - 4\right)$$

2.9

$$u(x) = -\frac{5}{18}\,(x-1)\,x$$

2.10

$$u(x) = \frac{1}{128}\left(30\,x^2 + 71\,x - 59\right)$$

2.11

$$u(x) = \frac{3\,x}{2} - 2$$

2.12

$$u(x) = \frac{(2\,\mathrm{EA}\,g + L\,q\,(L-x))\,x}{2\,\mathrm{EA}\,L}$$

2.13

$$u(x) = \frac{3\,c\,L^2\,(L-x)\,x}{20\,\mathrm{EA}}$$

2.14

$$u(x) = \frac{x\,(\pi\,(61200 - 263\,x) + 64000\,(3\,x + 4))}{1378000000\,\pi}$$

2.15

$$-w''(x_0)\,u^{(3)}(x_0) + w''(x_\ell)\,u^{(3)}(x_\ell) + w'(x_0)\,u^{(4)}(x_0) - w'(x_\ell)\,u^{(4)}(x_\ell) - \\ w(x_0)\left(u^{(3)}(x_0) + u^{(5)}(x_0)\right) + w(x_\ell)\left(u^{(3)}(x_\ell) + u^{(5)}(x_\ell)\right) + \int_{x_0}^{x_\ell}(w\,x - u^{(3)}\left(w' + w^{(3)}\right))\,\mathrm{d}x = 0$$

2.16

$$\mathrm{EJ}_w\left(w'(0)\,\phi''(0) - w'(L)\,\phi''(L) - w(0)\,\phi^{(3)}(0) + w(L)\,\phi^{(3)}(L)\right) + \int_0^L((\mathrm{EJ}_w\,w'' - w\,\mathrm{GJ}_0)\,\phi'' - t\,w)\,\mathrm{d}x = 0$$

2.17

$$p(x) = \frac{7\,(1015625\,x^3 - 19125000\,x^2 + 88000000\,x + 1055502)}{502620}$$

2.18

$$u(x) = \frac{(2\,\mathrm{EA}\,g + L\,q\,(L-x))\,x}{2\,\mathrm{EA}\,L}$$

2.19

$$\mathrm{u[x]} \to 0.000154286\,x - 0.000257143\,x^2$$

2.20

$$u = \frac{12\,x^2}{86125\,\pi} - \frac{263\,x^2}{1378000000} + \frac{16\,x}{86125\,\pi} + \frac{153\,x}{3445000}$$

2.21

$$V(z) = -0.199251\,z^4 + 0.67403\,z^3 - 1.37394\,z^2 + 1.14916\,z + 3.75$$

2.22

$$t(z) = -0.0000379346z^4 + 0.00175396z^3 - 0.0233431z^2 - 0.0228071z + 2.81$$

2.23

$$f(x) = = -0.3x^3 + 0.575x^2 + 0.1$$

2.24

$$f(x) = = 0.368x^3 - 1.08x^2 + 1$$

2.25

$$A(x) = 10\left(-\frac{2x^6}{512578125} + \frac{17x^5}{15187500} - \frac{223x^4}{1822500} + \frac{253x^3}{40500} - \frac{5897x^2}{40500} + \frac{286x}{225} + 8\right)$$

2.26

	Range	ϵ	σ	F
1	$0 \le x \le 400$	0.0008125	162.5	32500.
2	$400 \le x \le 800$	−0.0001875	−37.5	−7500.

2.27

	Range	ϵ	σ	F
1	$0 \le x \le 300$	$\frac{1}{6000}$	$\frac{100}{3}$	$\frac{20000}{3}$
2	$300 \le x \le 900$	$-\frac{1}{12000}$	$-\frac{50}{3}$	$-\frac{10000}{3}$

2.28

	Range	Solution
1	$0 \le x \le 120$	$\frac{99x}{1093750}$
2	$120 \le x \le 240$	$\frac{171x}{2187500} + \frac{162}{109375}$
3	$240 \le x \le 360$	$\frac{9x}{218750} + \frac{162}{15625}$
4	$360 \le x \le 480$	$\frac{4212}{109375} - \frac{81x}{2187500}$
5	$480 \le x \le 600$	$\frac{324}{3125} - \frac{27x}{156250}$

2.29

	Range	Solution
1	$0 \le x \le \frac{100}{3}$	$\frac{16x}{3125\pi} + \frac{61x}{1500000}$
2	$\frac{100}{3} \le x \le \frac{200}{3}$	$\frac{16x}{1625\pi} + \frac{23x}{780000} - \frac{256}{1625\pi} + \frac{109}{292500}$
3	$\frac{200}{3} \le x \le 100$	$\frac{16x}{625\pi} + \frac{x}{60000} - \frac{5888}{4875\pi} + \frac{359}{292500}$

2.30

$$\frac{AE}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} L(2q_1 + q_2) \\ \frac{1}{6} L(q_1 + 2q_2) \end{pmatrix} + \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

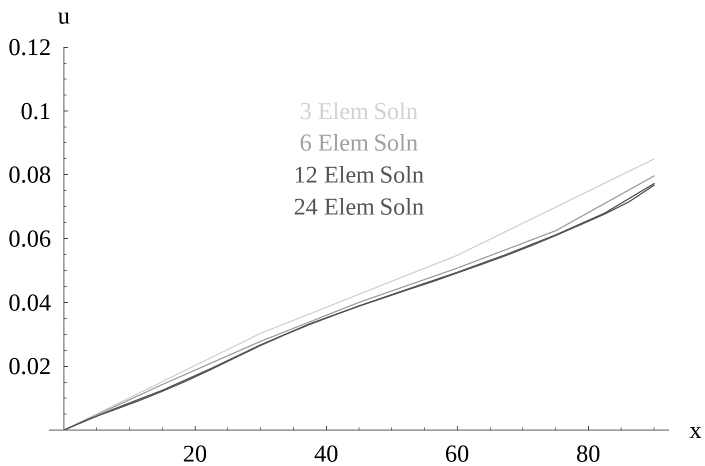
2.31

$$\begin{pmatrix} \frac{(A_l+A_r)E}{2\ell} & -\frac{(A_l+A_r)E}{2\ell} \\ -\frac{(A_l+A_r)E}{2\ell} & \frac{(A_l+A_r)E}{2\ell} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{6}(2q_l + q_r)\ell \\ \frac{1}{6}(q_l + 2q_r)\ell \end{pmatrix} + \begin{pmatrix} F_l \\ F_r \end{pmatrix}$$

2.32

Three element solution

$$\begin{pmatrix} 0 & 0 \\ 30. & 0.0302521 \\ 60. & 0.0547419 \\ 90. & 0.084994 \end{pmatrix}$$


u
0.12
0.1
0.08
0.06
0.04
0.02
3 Elem Soln
6 Elem Soln
12 Elem Soln
24 Elem Soln
20
40
60
80
x

CHAPTER THREE

One Dimensional Boundary Value Problem

3.1

$k(x) = \frac{1}{2}x^2; \quad p(x) = -x; \quad q(x) = -4$

NBC at $x = 1$:

$\alpha = 0; \quad \beta = -1$

EBC at $x = 3$: $u(3) = 1$

3.2

$\{0 \le x \le 0.0095,\ 1140.49 - 47.6108\,x\}$

$\{0.0095 \leq x \leq 0.2095,\ 1181.91 - 4408.41\,x\}$

$\{0.2095 \leq x \leq 0.21575,\ 268.326 - 47.6108\,x\}$

3.3

Four linear elements

	Range	Temp.	Heat loss
1	$0 \leq x \leq 0.00625$	$95. - 63.0279\,x$	0.0342645
2	$0.00625 \leq x \leq 0.0125$	$94.8901 - 45.446\,x$	0.0340981
3	$0.0125 \leq x \leq 0.01875$	$94.6712 - 27.9358\,x$	0.0339855
4	$0.01875 \leq x \leq 0.025$	$94.3437 - 10.4697\,x$	0.0339266

Total heat loss → 0.136275

Two quadratic elements

	Range	Temp.	Heat loss
1	$0 \leq x \leq 0.0125$	$1406.41\,x^2 - 71.8102\,x + 95.$	0.0683536
2	$0.0125 \leq x \leq 0.025$	$1397.14\,x^2 - 71.5933\,x + 94.9987$	0.0679033

Total heat loss → 0.136257

3.4

Three linear element solution

	Range	Temp.	Heat loss
1	$0 \leq x \leq \frac{1}{60}$	$17488.7\,x + 500$	−254.26
2	$\frac{1}{60} \leq x \leq \frac{1}{30}$	$4679.71\,x + 713.484$	−69.5234
3	$\frac{1}{30} \leq x \leq \frac{1}{20}$	$1076.66\,x + 833.586$	−21.5537

Total heat loss → −345.338

One quadratic element

	Range	Temp.	Heat loss
1	$0 \leq x \leq \frac{1}{20}$	$-245752.\,x^2 + 19686.3\,x + 500$	−337.908

Total heat loss → −337.908

3.5

Two linear element solution

	Range	Solution
1	$0 \le x \le \frac{3}{10}$	$\frac{x}{10500}$
2	$\frac{3}{10} \le x \le \frac{3}{5}$	$\frac{1}{17500} - \frac{x}{10500}$

Two quadratic element solution

	Range	Solution
1	$0 \le x \le \frac{3}{10}$	$-\frac{x^2}{13200} - \frac{x}{13200}$
2	$\frac{3}{10} \le x \le \frac{3}{5}$	$\frac{13x^2}{92400} - \frac{13x}{462000} - \frac{13}{385000}$

3.6

	Range	ϵ	σ	F
1	$0 \le x \le 0.3$	0.0000101024	707168.	1060.75
2	$0.3 \le x \le 0.6$	-0.000450934	-3.15654×10^7	-18939.2
3	$0.6 \le x \le 0.9$	0.000286541	2.00579×10^7	21060.8
4	$0.9 \le x \le 1.2$	0.000154291	1.08004×10^7	21060.8

3.7

1 element solution

	Range	Solution
1	$0 \le x \le \frac{4}{5}$	$0.000022956\,x$

2 element solution

	Range	Solution
1	$0 \le x \le \frac{2}{5}$	$0.0000315645\,x$
2	$\frac{2}{5} \le x \le \frac{4}{5}$	$0.0000143475\,x + 6.88679 \times 10^{-6}$

3 element solution

	Range	Solution
1	$0 \le x \le \frac{4}{15}$	$0.0000331586\,x$
2	$\frac{4}{15} \le x \le \frac{8}{15}$	$0.0000255066\,x + 2.04053\times10^{-6}$
3	$\frac{8}{15} \le x \le \frac{4}{5}$	$0.0000102027\,x + 0.0000102027$

3.8

	Range	Solution
1	$0 \le x \le 4.$	$277.778\,x + 14.7$
2	$4. \le x \le 8.$	$2236.92 - 277.778\,x$

3.9

	Range	Solution
1	$1 \le x \le 2$	$\frac{36x}{55} - \frac{287}{55}$
2	$2 \le x \le 3$	$\frac{51x}{55} - \frac{317}{55}$
3	$3 \le x \le 4$	$\frac{219x}{55} - \frac{821}{55}$

3.10

2 element solution

	Range	Solution
1	$1 \le x \le 2$	$2.37147 - 1.37147\,x$
2	$2 \le x \le 3$	$2.1976 - 1.28454\,x$

{Which[$1 \le x \le 2$, $2.37147 - 1.37147\,x$, $2 \le x \le 3$, $2.1976 - 1.28454\,x$]}

Solution convergence

Which[$1 \le x \le 3$, $2.32941 - 1.32941\,x$]

Which[$1 \le x \le 2$, $2.37147 - 1.37147\,x$, $2 \le x \le 3$, $2.1976 - 1.28454\,x$]

Which[$1 \le x \le \frac{5}{3}$, $2.40859 - 1.40859\,x$, $\frac{5}{3} \le x \le \frac{7}{3}$, $2.21073 - 1.28987\,x$, $\frac{7}{3} \le x \le 3$, $2.19739 - 1.28416\,x$]

Which[$1 \le x \le \frac{3}{2}$, $2.42941 - 1.42941\,x$, $\frac{3}{2} \le x \le 2$, $2.26582 - 1.32035\,x$, $2 \le x \le \frac{5}{2}$, $2.1703 - 1.27259\,x$, $\frac{5}{2} \le x \le 3$, 2

Which[$1 \le x \le \frac{7}{5}$, $2.44164 - 1.44164\,x$, $\frac{7}{5} \le x \le \frac{9}{5}$, $2.3109 - 1.34825\,x$, $\frac{9}{5} \le x \le \frac{11}{5}$, $2.20016 - 1.28673\,x$, $\frac{11}{5} \le x \le$

Solution using *Mathematica* function NDSolve

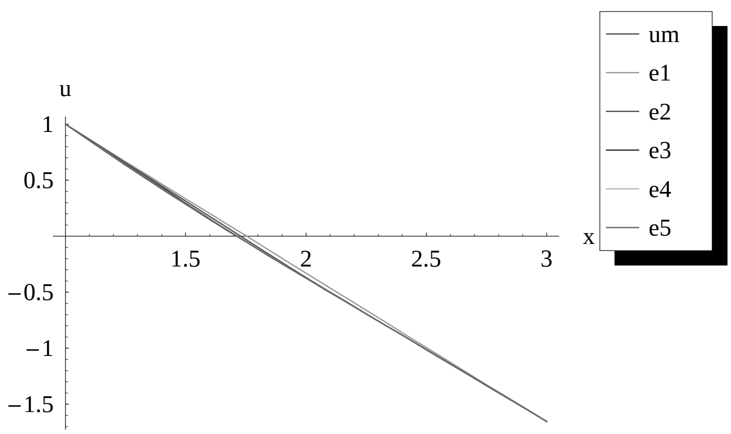
```
soln = NDSolve[{x^2 y''[x] + 2 x y'[x] - x y[x] + 4 == 0, y[1] == 1, y'[3] == 2 + 2 y[3]}, y, {x, 1, 3}];
um = y[x] /. soln[[1]]
```

InterpolatingFunction[(1. 3.), <>][x]

Comparison of different solutions

```
colors = Join[{{RGBColor[1, 0, 0]}},
Table[{RGBColor[Random[], Random[], Random[]]}, {Length[ua]}]];
```

```
Plot[Evaluate[Join[{um},ua]],{x,1,3},
AxesLabel->{" x", "u"},
PlotStyle->colors,
PlotLegend->{"um","e1", "e2", "e3", "e4", "e5"},
LegendPosition->{1,0}];
```



3.11

10 element solution

Solution summary

	Range	Solution
1	$0 \leq x \leq \frac{1}{10}$	$3.10287\,x$
2	$\frac{1}{10} \leq x \leq \frac{1}{5}$	$2.79914\,x + 0.030373$
3	$\frac{1}{5} \leq x \leq \frac{3}{10}$	$2.22141\,x + 0.145919$
4	$\frac{3}{10} \leq x \leq \frac{2}{5}$	$1.42623\,x + 0.384472$
5	$\frac{2}{5} \leq x \leq \frac{1}{2}$	$0.491446\,x + 0.758386$
6	$\frac{1}{2} \leq x \leq \frac{3}{5}$	$1.24983 - 0.491446\,x$
7	$\frac{3}{5} \leq x \leq \frac{7}{10}$	$1.8107 - 1.42623\,x$
8	$\frac{7}{10} \leq x \leq \frac{4}{5}$	$2.36733 - 2.22141\,x$
9	$\frac{4}{5} \leq x \leq \frac{9}{10}$	$2.82951 - 2.79914\,x$
10	$\frac{9}{10} \leq x \leq 1$	$3.10287 - 3.10287\,x$

3.12

2 element solution

	Range	Solution
1	$0.785398 \leq \theta \leq 1.1781$	$2.08239 - 1.37814\,\theta$
2	$1.1781 \leq \theta \leq 1.5708$	$1.83522 - 1.16833\,\theta$

CHAPTER FOUR

Trusses, Beams, and Frames

4.1

	x−coord	y−coord	u	v
1	0	72	0	−0.00791478
2	0	0	0	0
3	108	0	0.0361918	0

4.2

	x−coord	y−coord	u	v
1	0	72	0	0.0488567
2	0	0	0	0
3	108	0	0.109927	0.5

4.3

	x−coord	y−coord	u	v
1	0	0	0	−0.209497
2	−4000.	3000.	0	0
3	0	3000.	0	0
4	4000.	3000.	0	0

4.4

	x−coord	y−coord	u	v
1	0	0	0	0
2	96	0	0.0238171	0.0312798
3	0	72	0	0
4	96	72	0	0

4.5

	x−coord	y−coord	u	v
1	0	0	0.25	0
2	96	0	0.199743	0.0729465
3	0	72	0	0
4	96	72	0	0

4.6

	x−coord	y−coord	u	v
1	0	0	0	0
2	8000.	0	0	0
3	0	6000.	0.738808	0.0661619
4	8000.	6000.	0.85643	−0.0766952

4.7

	x−coord	y−coord	z−coord	u	v	w
1	0.	0.	10000.	2.08356	0	−0.222246
2	0.	8000.	0.	0	0	0
3	6928.2	−4000.	0.	0	0	0
4	−6928.2	−4000.	0.	0	0	0

4.8

	x−coord	y−coord	z−coord	u	v	w
1	0.	4000.	0.	−0.396825	−0.14881	−6.5448
2	−3000.	2000.	5000.	2.74335	5.00783	−1.70566
3	−3000.	0.	0.	0	0	0
4	3000.	0.	0.	0	0	0
5	0.	0.	5000.	0	0	0

4.9

	x−coord	y−coord	u	v
1	0.	0.	0	0
2	10.	0.	−0.015161	0

4.10

	x−coord	y−coord	u	v
1	0.	72.	0	−0.00960326
2	0.	0.	0	0
3	108.	0.	−0.0216073	0

4.11

	x−coord	y−coord	u	v
1	0	0	0	0
2	96	0	0.11641	−0.0208035
3	0	72	0	0
4	96	72	0	0

4.12

	x−coord	y−coord	z−coord	u	v	w
1	0.	4000.	0.	0	0	−8.74
2	−3000.	2000.	5000.	0	0	0
3	−3000.	0.	0.	0	0	0
4	3000.	0.	0.	0	0	0
5	0.	0.	5000.	0	0	0

4.13

Spring forces

$$\left\{\frac{5P}{11}, -\frac{6P}{11}, -\frac{6P}{11}\right\}$$

4.14

Spring forces

$$\left\{\frac{2gk}{7}, \frac{2gk}{7}, \frac{2gk}{7}\right\}$$

4.15

v(L/4) = −0.143229 in;   M(L/4) = 520.833 k−in

4.16

	v	θ
1	0	0
2	$-\frac{10}{3}$	$-\frac{1}{20}$

4.17

Vertical displacement at the hinge = $-\frac{9}{22}$

4.18

Rotation of the right end = 0.00589169

4.19

Rotation of the right end = 0.00160714

4.20

Rotation of the right end = 0.00010582

4.21

The axial force in the spring = 27.7316

4.22

Vertical deflection at load location = -0.0499836

4.23

$v(L/4) = -0.143229$ in; $M(L/4) = 520.833$ k$-$in

4.24

Rotation of the right end = 0.00583333

4.25

Rotation of the right end = 0.00261905

4.26

$v(2L/3) = -\frac{1}{48}$ in; $M(2L/3) = \frac{5}{8}$ k$-$in

4.27

Rotation of the left end = 0.00675154

4.28

Rotation of the right end = -0.000843705

4.29

Vertical deflection at point load location = -54.5919 mm

4.30

Vertical deflection at mid-span = -0.0226372

4.31

Vertical deflection at mid-span = -0.000127098

4.32

Vertical deflection at point load location = -0.00152975

4.33

Rotation at right pin support = 0.002385

4.34

Vertical deflection at point where load P is applied = 0.000140669

4.35

Vertical deflection at free end = -5.27429

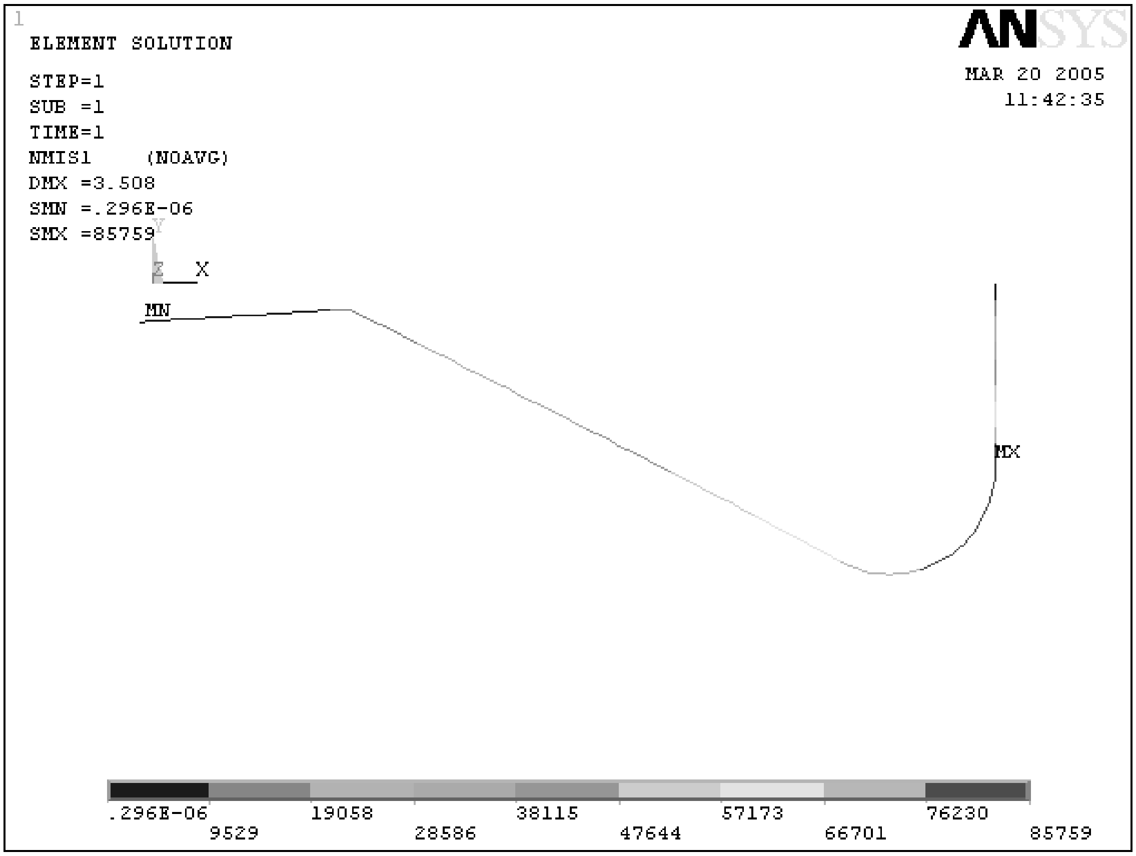
4.36

Vertical deflection at free end = -0.575948

4.37

Horizontal displacements at the top of the dam ≈ -0.0007

4.38



4.39

Vertical deflection at tip of the cantilever = −5.79544

4.40

Vertical deflection under load P = −16.0819

CHAPTER FIVE

Two Dimensional Elements

5.1

$k_x = k_y = -1; \quad P = 0; \quad Q = -2x - 2y + 4$

Essential: On side 3: $u = x^2$; On side 4: $u = y^2$

Natural: On side 1: $\alpha = 0$ and $\beta = -2 + 2x + x^3$

Natural: On side 2: $\alpha = 0$ and $\beta = 1 - 3y$

Weak form:

$$\int_{x=0}^{1} (2 - 2x - x^3)\, N_i \,\mathrm{dx} + \int_{y=0}^{1} -(1 - 3y)\, N_i \,\mathrm{dy} + \iint_A \left(\frac{\partial u}{\partial x} \frac{\partial N_i}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial N_i}{\partial y} + (-2x - 2y + 4)\, N_i \right) \mathrm{dA} = 0$$

5.2

Three parameter solution

$$\psi(x) = \frac{2199\,y\,x^2}{5438} - \frac{2199\,x^2}{10876} + \frac{1704\,y^2\,x}{2719} - \frac{13983\,y\,x}{27190} + \frac{5463\,x}{54380} - \frac{1704\,y^2}{2719} + \frac{1494\,y}{13595} + \frac{1383}{13595}$$

5.3

$$\boldsymbol{r}_{\mathrm{q}} = \begin{pmatrix} \frac{1}{9}\,a\,b\,(4\,Q_1 + 2\,Q_2 + Q_3 + 2\,Q_4) \\ \frac{1}{9}\,a\,b\,(2\,Q_1 + 4\,Q_2 + 2\,Q_3 + Q_4) \\ \frac{1}{9}\,a\,b\,(Q_1 + 2\,(Q_2 + 2\,Q_3 + Q_4)) \\ \frac{1}{9}\,a\,b\,(2\,Q_1 + Q_2 + 2\,Q_3 + 4\,Q_4) \end{pmatrix}$$

5.4

	x−coord	y−coord	u
1	0	0	$\frac{91}{43}$
2	2	0	1
3	2	1	1
4	0	1	$\frac{83}{43}$

5.5

Solution at element centroids

	x−coord	y−coord	ψ	$\partial\psi/\partial$x	$\partial\psi/\partial$y
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{23}{32}$	$-\frac{5}{8}$	$\frac{1}{8}$
2	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{13}{32}$	$-\frac{5}{8}$	$-\frac{5}{8}$

5.6

	x−coord	y−coord	T
1	0	0	300
2	2	0	300
3	2	1	345
4	0	1	75

5.7

Stresses at element centroid ($x = y = 0.25$)

τ_{yz} (N/cm^2)	τ_{xz} (N/cm^2)
700.	−700.

5.8

Stresses at element centroids

	τ_{yz} (MPa)	τ_{xz} (MPa)
1	2.23214	−2.23214
2	1.11607	−2.23214
3	2.23214	−1.11607

5.9

Cutoff frequencies

{3.87298, 3.4641, 3.4641, 1.73205, 0.}

5.10

Cutoff frequencies

{2.73861, 1.73205, 1.28512, 1.28512, 0.}

5.11

$$\text{Interpolation Functions} \to \begin{pmatrix} -\frac{9}{256}(x-2)\left(x-\frac{2}{3}\right)\left(x+\frac{2}{3}\right)(1-y) \\ \frac{27}{256}(x-2)\left(x-\frac{2}{3}\right)(x+2)(1-y) \\ -\frac{27}{256}(x-2)\left(x+\frac{2}{3}\right)(x+2)(1-y) \\ \frac{9}{256}\left(x-\frac{2}{3}\right)\left(x+\frac{2}{3}\right)(x+2)(1-y) \\ \frac{9}{256}\left(x-\frac{2}{3}\right)\left(x+\frac{2}{3}\right)(x+2)(y+1) \\ -\frac{27}{256}(x-2)\left(x+\frac{2}{3}\right)(x+2)(y+1) \\ \frac{27}{256}(x-2)\left(x-\frac{2}{3}\right)(x+2)(y+1) \\ -\frac{9}{256}(x-2)\left(x-\frac{2}{3}\right)\left(x+\frac{2}{3}\right)(y+1) \end{pmatrix}$$

5.12

Integral = 11979

5.13

Integral = 62.9638

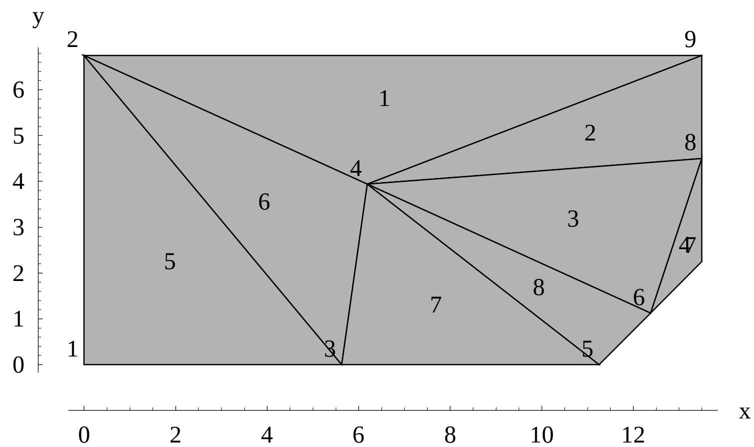
5.14

Integral = 215500

5.15

$\{\psi_5 = 295.891, \psi_8 = 290.897, \psi_{11} = 286.626\}$

5.16

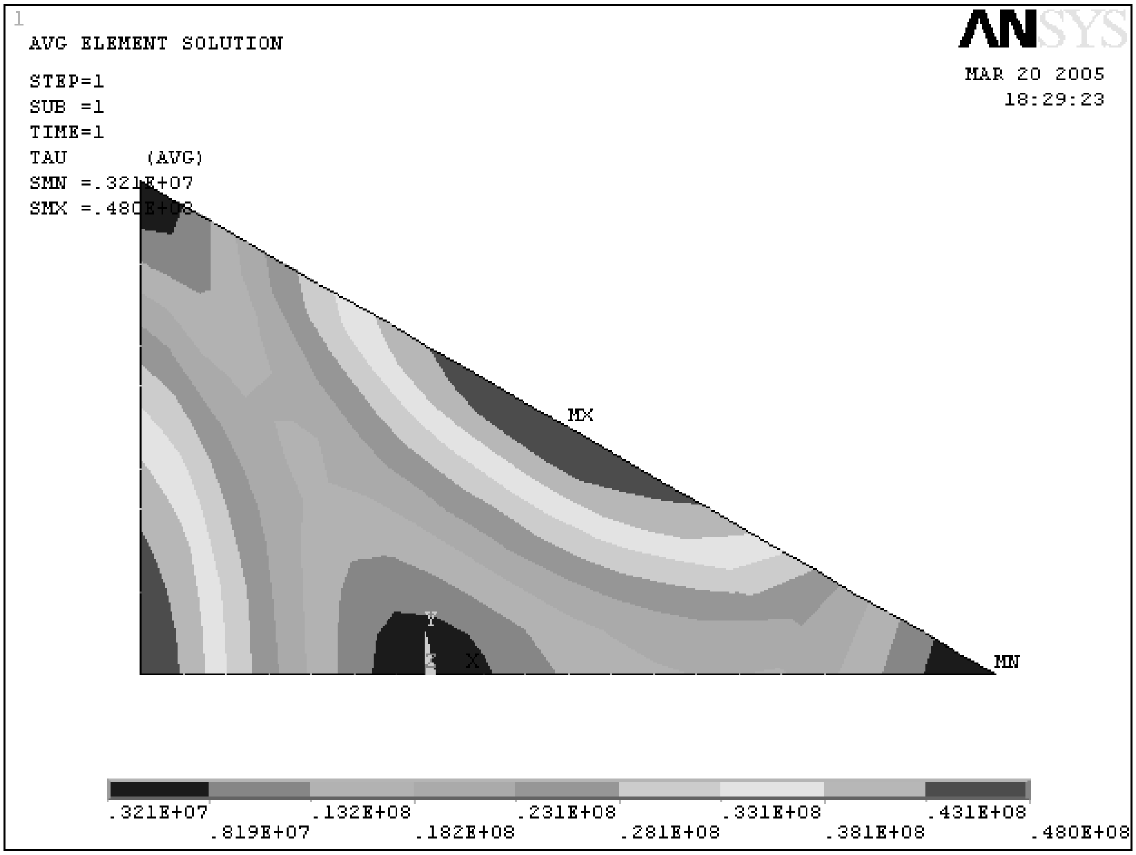


$\{\phi_1 = -98.3232, \phi_2 = -98.7015, \phi_3 = -58.6855, \phi_4 = -54.2459, \phi_5 = -22.0387, \phi_6 = -13.0548\}$

5.17

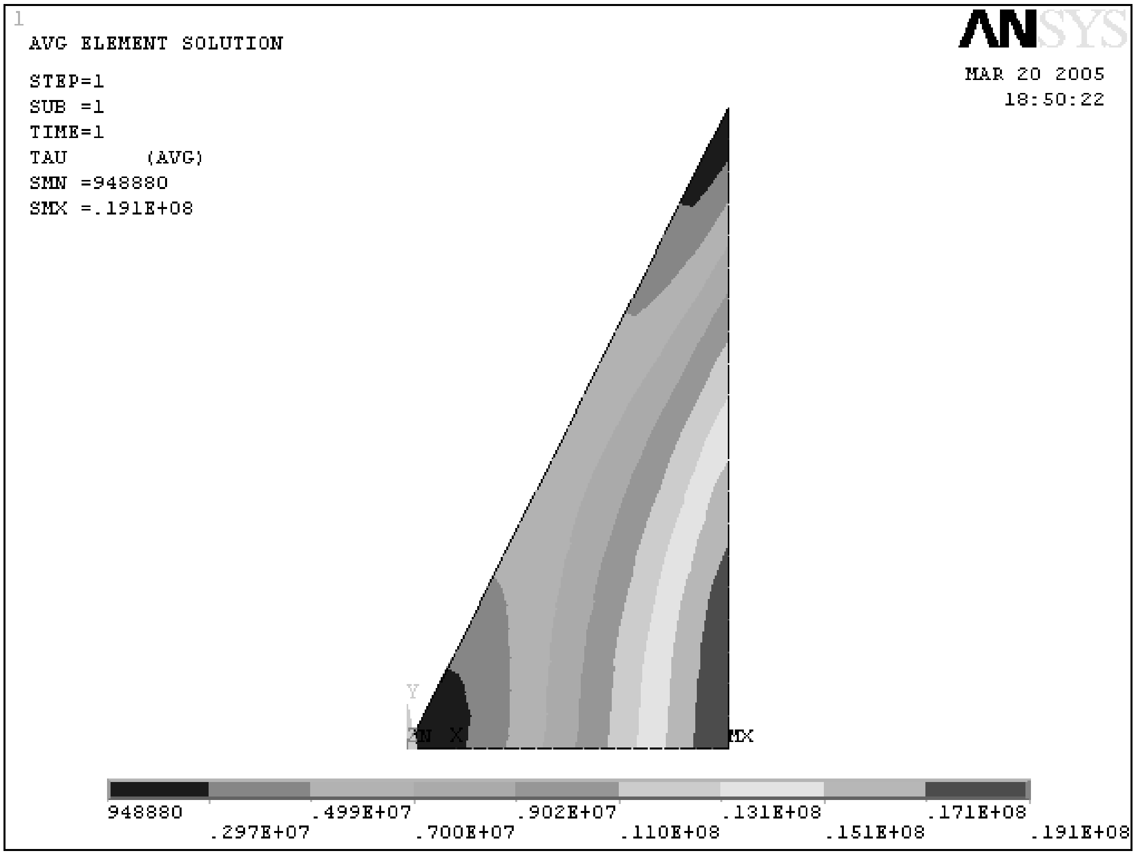
$\{\phi_4 = 0.000178326, \phi_6 = 0.000109739\}$

The maximum shear stress is 3.65 Pa.



5.18

$\{\phi_2 = 0.000142857,\ \phi_4 = 0.000247619,\ \phi_5 = 0.000180952\}$



5.19

Cutoff frequencies

{3.87298, 3.4641, 3.36855, 1.72577, 0.}

5.20

Cutoff frequencies

{2.68328, 1.71139, 1.3993, 1.29308, 0.}

CHAPTER SIX

Mapped Elements

6.1

$$\iint_{A_{xy}} (x - y)\, dA_{xy} = -\frac{23}{20}$$

6.2

$$\iint_{A_{xy}} (x\, y)\, dA_{xy} = \frac{151}{12}$$

6.3

$$\iint_{A_{xy}} (x^2\, y)\, dA_{xy} = \frac{911}{30} = 30.3667$$

6.4

Using 1×1 integration: $I = 18.$

Using 2×2 integration: $I = 12.5833$

6.5

Using 1×1 integration: $I = 22.75$

Using 2×2 integration: $I = 30.4167$

6.6

$$\text{detJ} = -2.27817\, t\, s^2 + 0.207107\, s^2 + 0.414214\, t\, s - 1.41421\, s - 6.02817\, t + 11.5992$$

6.7

$$\text{detJ} = \frac{3\, t^2\, s^2}{64} - \frac{7\, t\, s^2}{64} + \frac{3\, s^2}{32} + \frac{7\, t^2\, s}{64} - \frac{t\, s}{2} + \frac{35\, s}{64} + \frac{t^2}{32} - \frac{23\, t}{64} + \frac{49}{64}$$

6.8

$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{x} = \{0.0534091,\ 0.112652,\ -0.0471737,\ -0.0866226,\ -0.0322651\}$$
$$\partial \boldsymbol{N}^{\mathrm{T}}/\partial \mathrm{y} = \{-0.0582398,\ 0.113808,\ 0.0512185,\ -0.017848,\ -0.0889388\}$$

6.9

$$\text{detJ} = \frac{3\, s}{4} + \frac{t}{4} + 3$$

$I = = 28.2667$

From the exact value we can see that the 2×2 Gauss quadrature gives a reasonable approximation to the integral.

6.10

$$\text{detJ} = \frac{s}{2} + \frac{3\, t}{4} + \frac{7}{4}$$

At node 3, s = t = 1; $\partial N_3/\partial \mathrm{y} = -\frac{1}{3}$

Using 1×1 Gauss quadrature: $\iint_A 64\, N_2\, N_3\, \mathrm{dA} = 28$

Using 2 point Gauss quadrature: $\int_c 4\, N_{3c}\, \mathrm{dc} = 8$

6.11

T at element center = -1

$\partial \mathrm{T} / \partial \mathrm{x}$ at node 5 = 1.21286

Finite Element Computations Over Mapped Elements

6.12

$$\boldsymbol{k}_{\mathrm{k}} = \begin{pmatrix} 0.755569 & 0.0477476 & -0.56552 & -0.237797 \\ 0.0477476 & 0.524955 & -0.465856 & -0.106847 \\ -0.56552 & -0.465856 & 1.05965 & -0.0282722 \\ -0.237797 & -0.106847 & -0.0282722 & 0.372916 \end{pmatrix}$$

6.13

$$\boldsymbol{k}_{\alpha} = \begin{pmatrix} -0.537852 & -0.238165 & 0.180604 & 0 & 0 & 0 & 0 & 0 \\ -0.238165 & -2.87502 & -0.484252 & 0 & 0 & 0 & 0 & 0 \\ 0.180604 & -0.484252 & -0.906983 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

6.14

$\{T_2 = 32.1398,\ T_3 = 30.0087\}$

6.15

$\{T_1 = 20.8893,\ T_2 = 21.5601,\ T_3 = 21.5124,\ T_4 = 140.972,\ T_8 = 155.004\}$

CHAPTER SEVEN

Analysis of Elastic Solids

7.1

Principal stresses = $\{121.698,\ 15.445,\ -77.143\}$ $\tau_{\max} = 99.4205$

Effective stress = 172.337 Factor of safety = 1.45065

7.2

Principal stresses = $\{74.6835,\ -10.1031,\ -124.58\}$ $\tau_{\max} = 99.632$

Effective stress = 173.205 Factor of safety = 1.1547

7.3

Stresses = $\{-p,\ -p,\ -p,\ 0,\ 0,\ 0\}$

7.4

Component stresses = {78.2807, 0., 0., 41.4982, 0., 0.}

$\sigma_e = 106.274$ MPa Factor of safety = $\sigma_f/\sigma_e = 2.82289$

7.5

Solution at element centers

	Coord	Disp	Stresses	Principal stresses	Effective Stress
1	10 $\frac{20}{3}$	−0.013 −0.0557143	−100. −30. 0 −150. 0 0	89.0292 0 −219.029	274.591

7.6

$$\left\{u_3 = 0,\ v_3 = \frac{11}{400}\right\}$$

7.7

Nodal solution

	x	y	u	v
1	0	0	0	0
2	50	10	0.140936	0.00489362
3	50	20	0.133596	0.0269149
4	0	20	0	0

	Coord	Disp	Stresses	Principal stresses	Effective Stress
1	$\frac{50}{3}$ 10	0.0469787 0.00163121	−13.1751 −164.953 0 2.63502 0 0	0. −13.1294 −164.998	158.841
2	$\frac{100}{3}$ $\frac{50}{3}$	0.0915106 0.0106028	26.3502 1.05401 0 −5.27005 0 0	27.4043 4.44089×10^{-16} 0.	27.4043

7.8

Nodal solution

	x	y	u	v
1	35.	0.	0.000412196	0
2	50.	0.	0.000352237	0
3	0.	50.	0	0.000352237
4	0.	35.	0	0.000412196

Solution at selected points on elements

	Coord	Disp	Stresses	Principal stresses	Effective Stress
1	21.25 21.25	0.000191108 0.000191108	0.960784 0.960784 0.576471 −0.999276 0 0	1.96006 0.576471 −0.0384919	1.77295

7.9

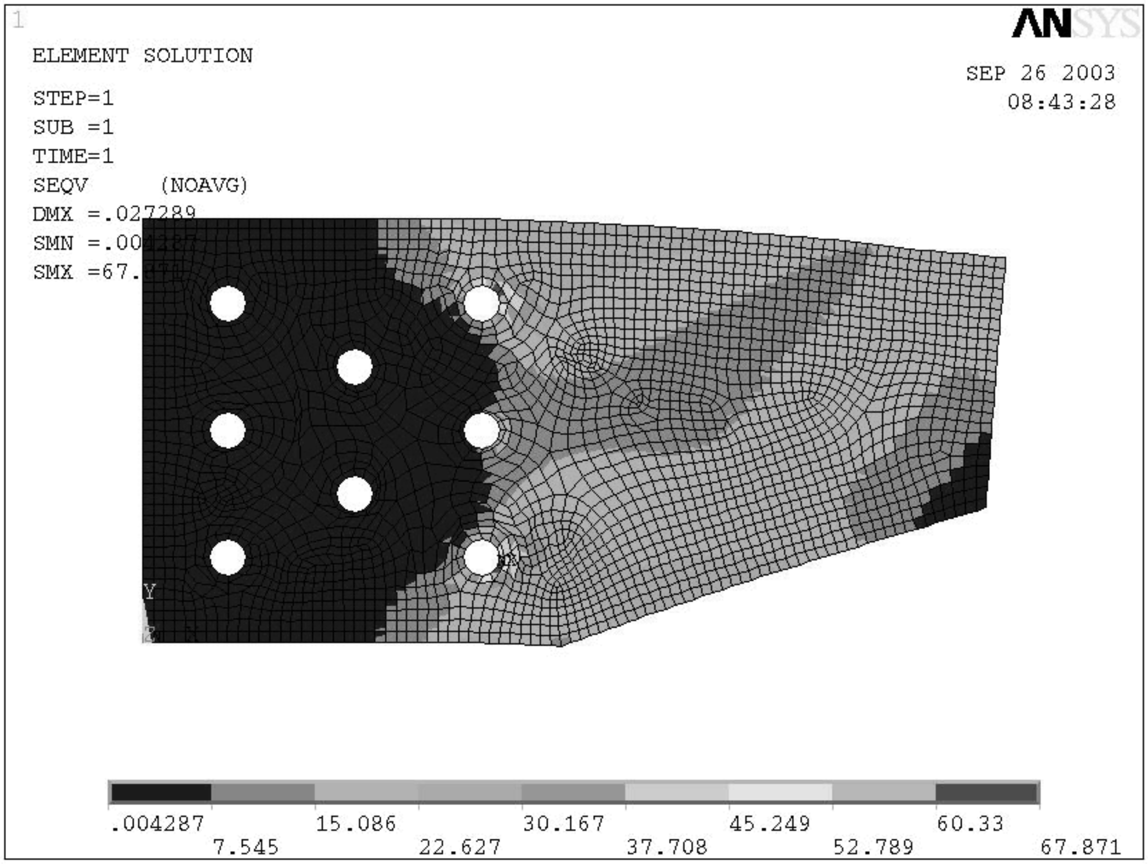
Nodal solution

	x	y	u	v
1	0.	48.	0	0.0361081
2	33.9411	33.9411	0.0257251	0.0257251
3	48.	0.	0.0361081	0
4	54.	0.	0.0337736	0
5	60.	0.	0.0322063	0
6	42.4264	42.4264	0.0229371	0.0229371
7	0.	60.	0	0.0322063
8	0.	54.	0	0.0337736

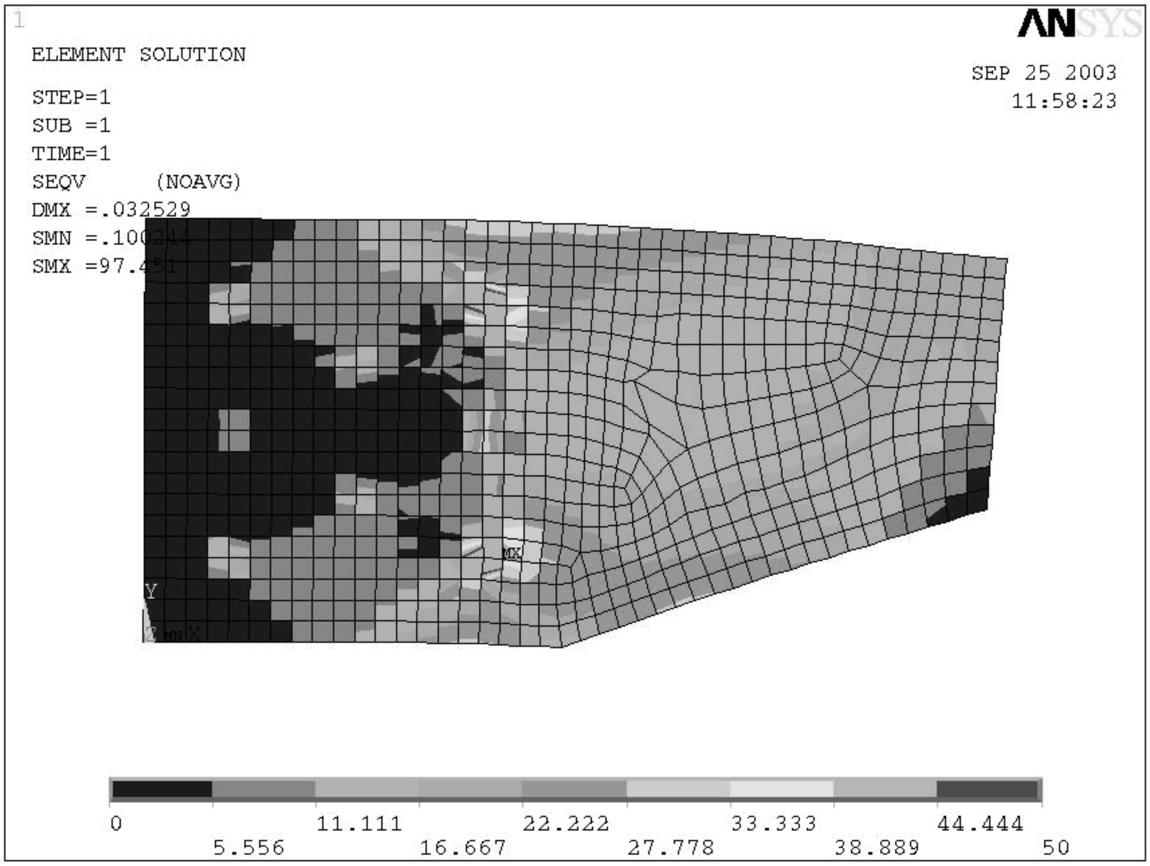
Solution at selected points on elements

	Coord	Disp	Stresses	Principal stresses	Effective Stress
1	38.1838 38.1838	0.0241393 0.0241393	8.56336 8.56336 5.13802 −11.0078 0 0	19.5712 5.13802 −2.44445	19.3713

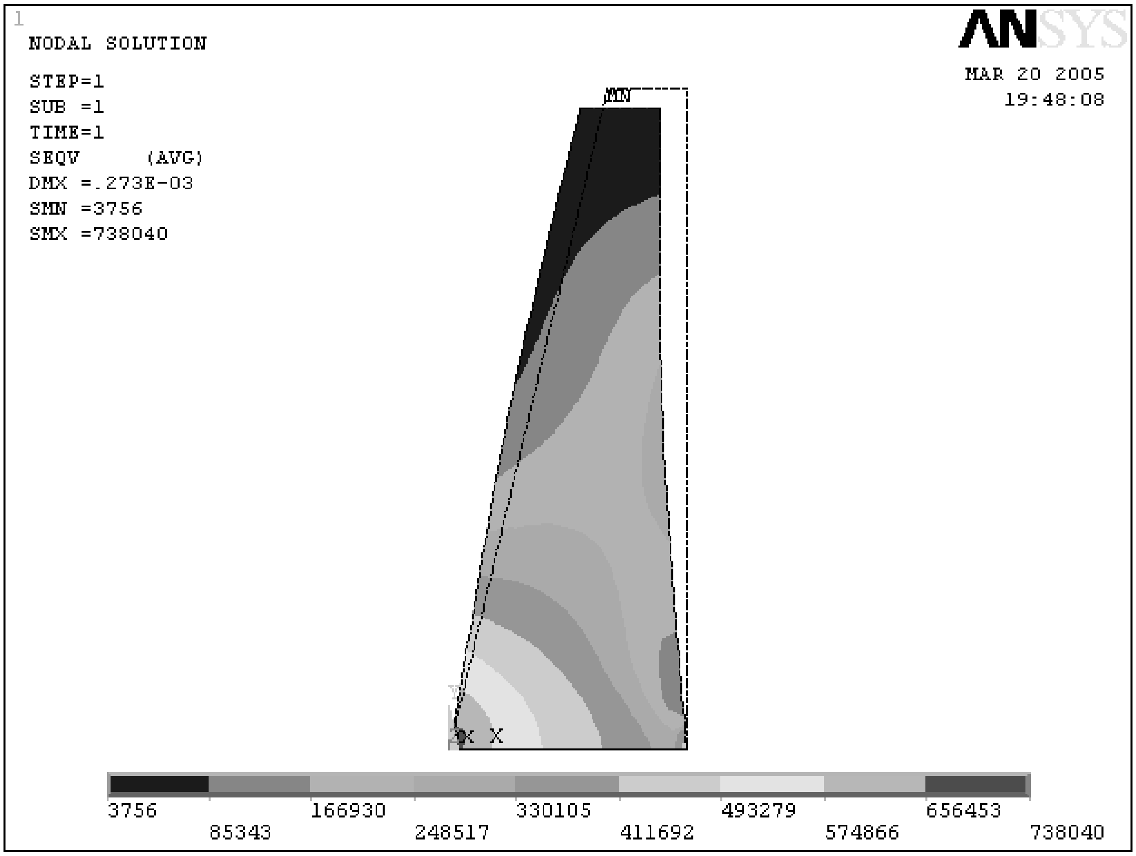
7.10 Stress Analysis of a Bolted Bracket



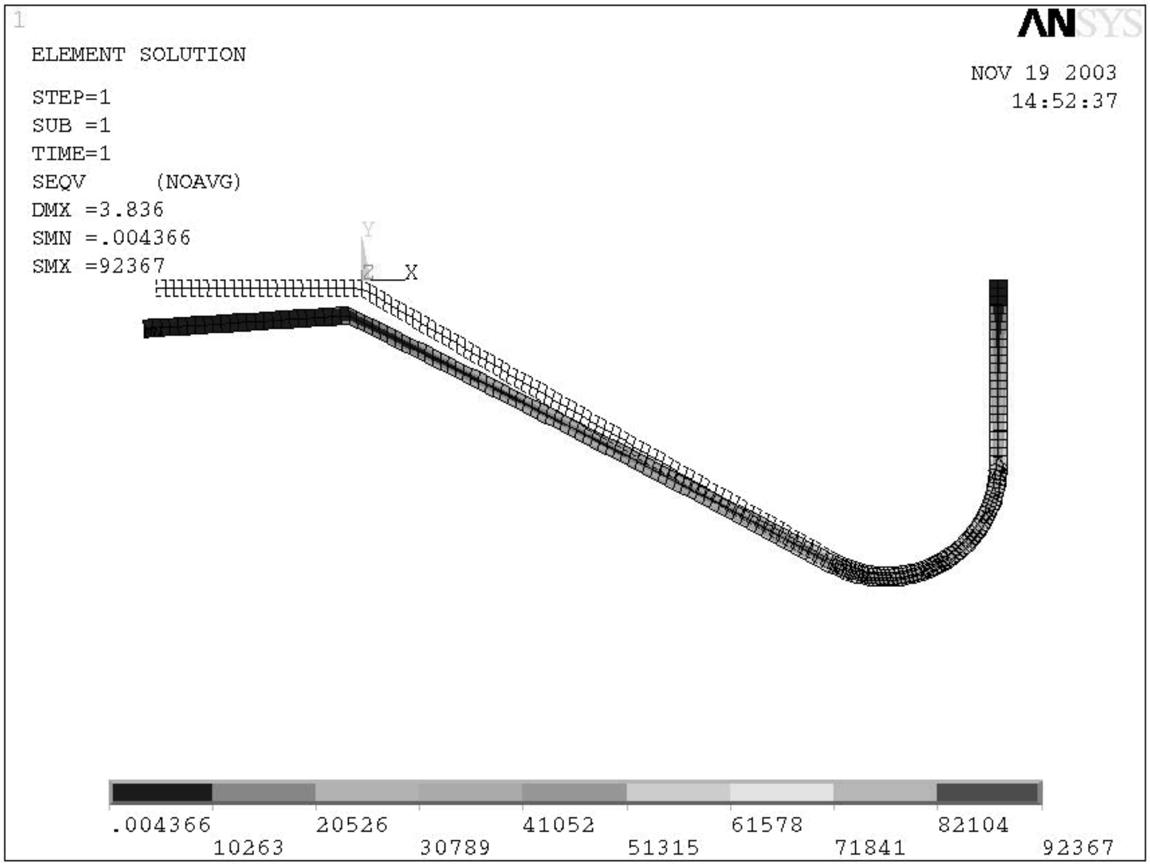
Model with no holes


1
ANSYS
ELEMENT SOLUTION
SEP 25 2003
11:58:23
STEP=1
SUB =1
TIME=1
SEQV (NOAVG)
DMX =.032529
SMN =.100244
SMX =97.451
MX
Y
Z
MN
X
0
5.556
11.111
16.667
22.222
27.778
33.333
38.889
44.444
50

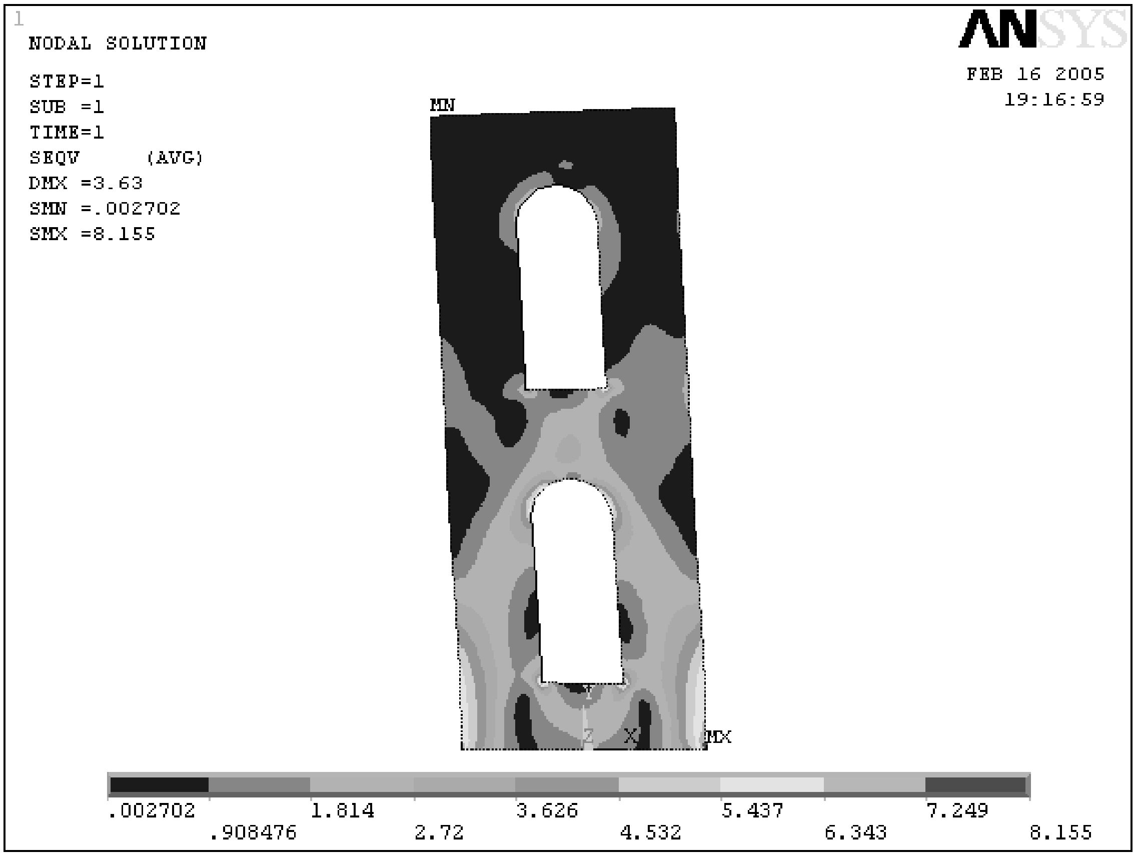
7.11 Stresses in a concrete dam



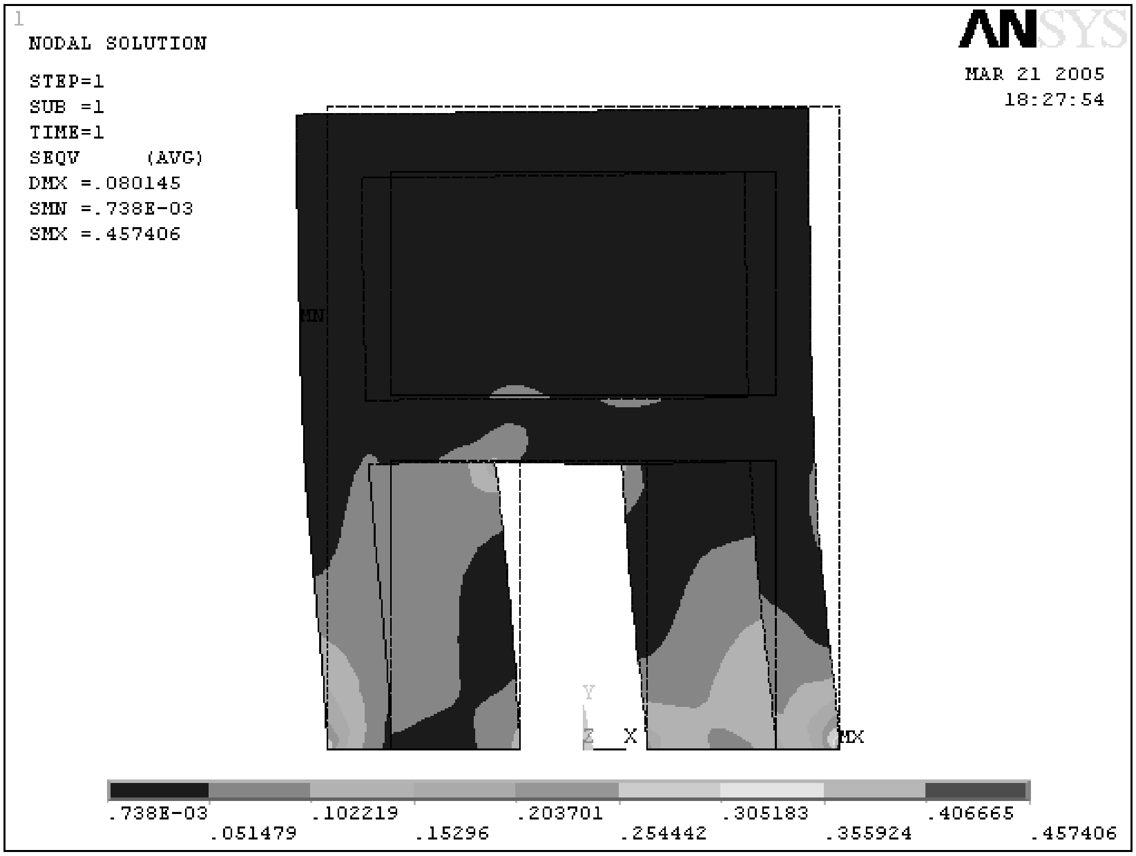
7.12


1
ANSYS
ELEMENT SOLUTION
STEP=1
SUB =1
TIME=1
SEQV (NOAVG)
DMX =3.836
SMN =.004366
SMX =92367
NOV 19 2003
14:52:37
Y
Z
X
.004366
10263
20526
30789
41052
51315
61578
71841
82104
92367

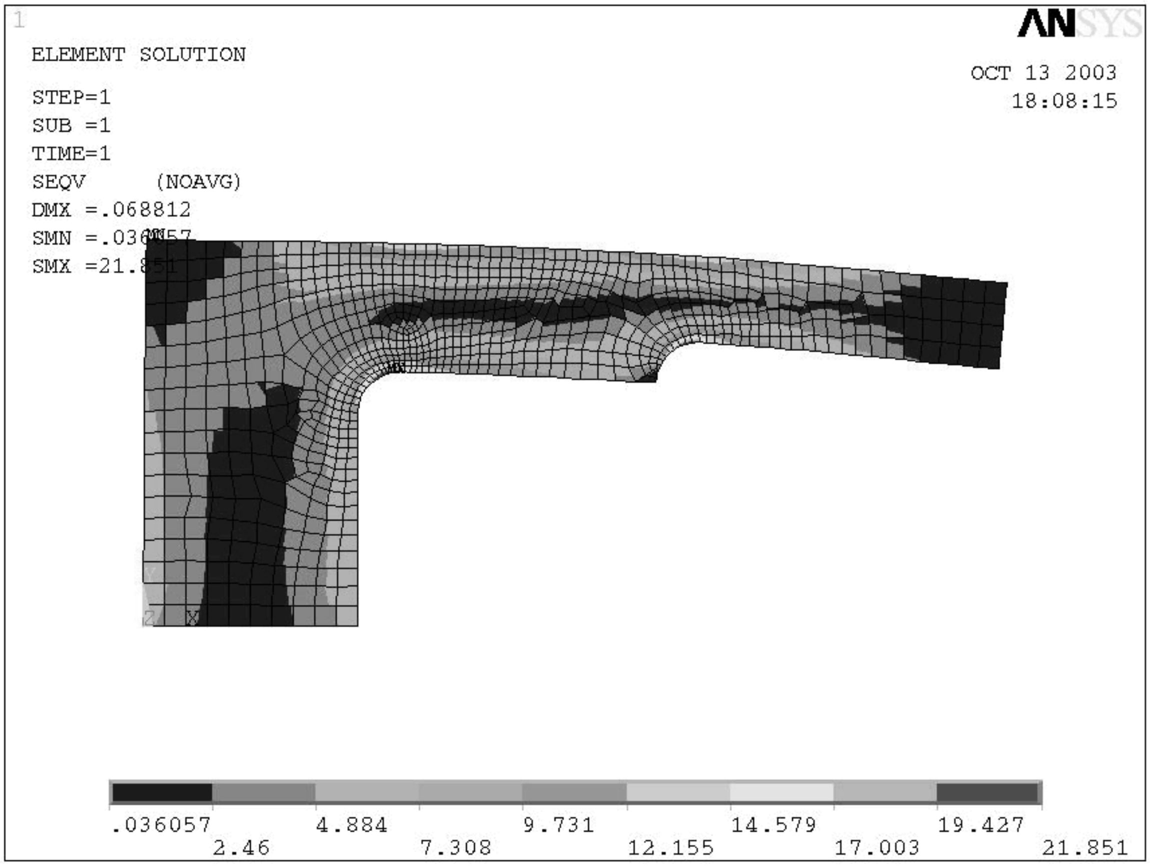
7.13


1
NODAL SOLUTION
STEP=1
SUB =1
TIME=1
SEQV (AVG)
DMX =3.63
SMN =.002702
SMX =8.155
ANSYS
FEB 16 2005
19:16:59
MN
MX
Y
Z
X
.002702
.908476
1.814
2.72
3.626
4.532
5.437
6.343
7.249
8.155

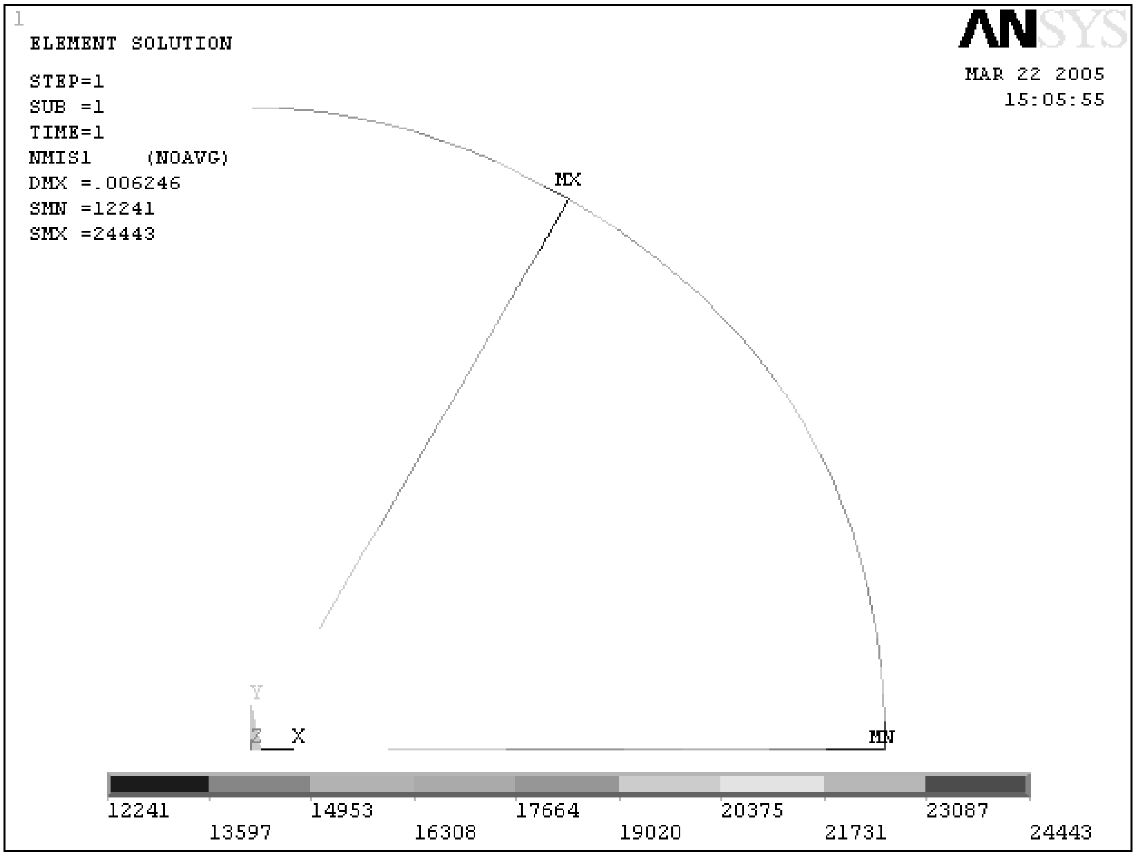
7.14

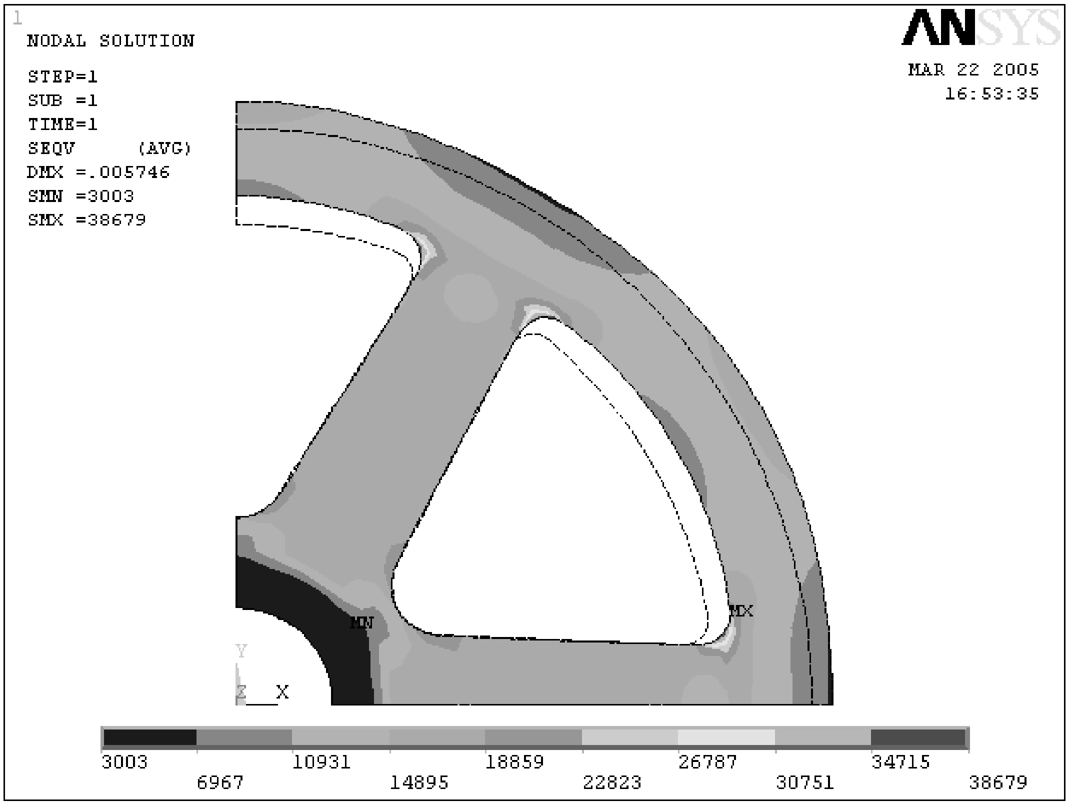

1
NODAL SOLUTION
STEP=1
SUB =1
TIME=1
SEQV (AVG)
DMX =.080145
SMN =.738E-03
SMX =.457406
ANSYS
MAR 21 2005
18:27:54
MN
MX
Y
Z X
.738E-03
.051479
.102219
.15296
.203701
.254442
.305183
.355924
.406665
.457406

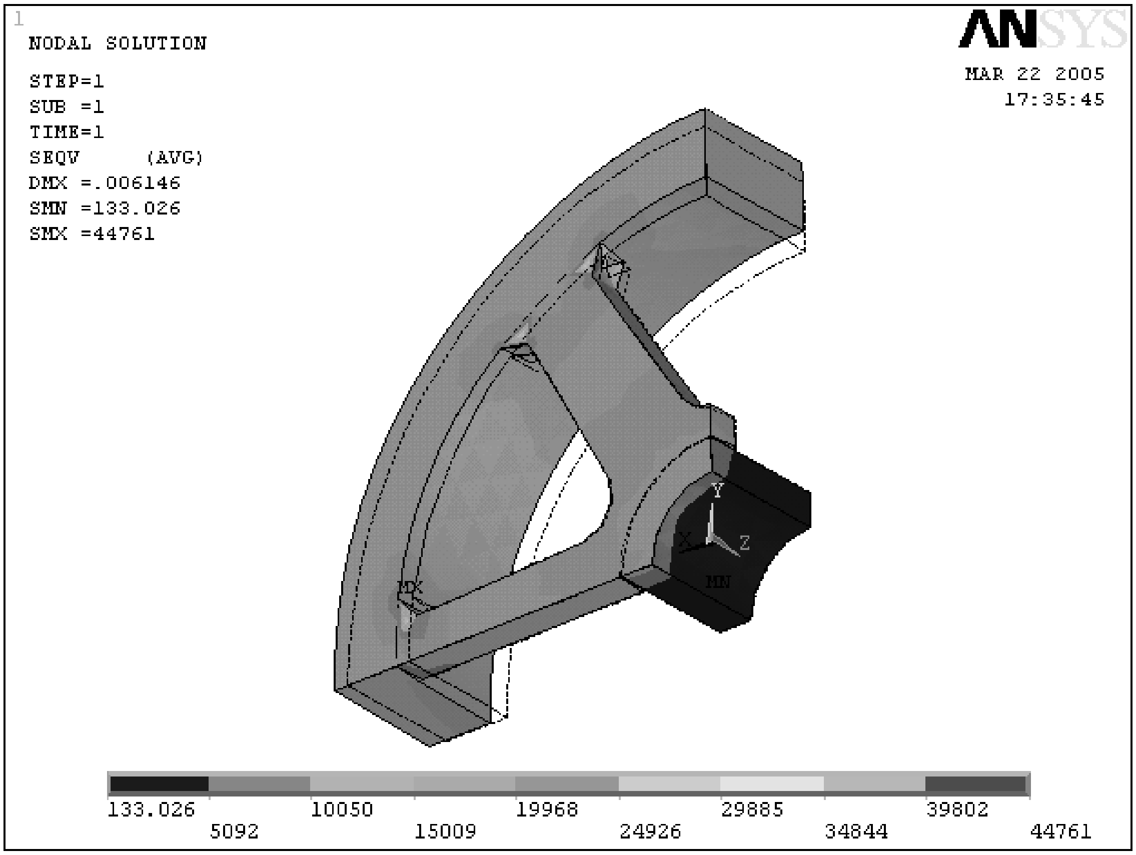
7.15 Stresses in an Aluminum machine part



7.16 Stresses in Spoked Flywheels




1
NODAL SOLUTION
STEP=1
SUB =1
TIME=1
SEQV (AVG)
DMX =.005746
SMN =3003
SMX =38679
ANSYS
MAR 22 2005
16:53:35
MN
MX
Y
Z X
3003
6967
10931
14895
18859
22823
26787
30751
34715
38679

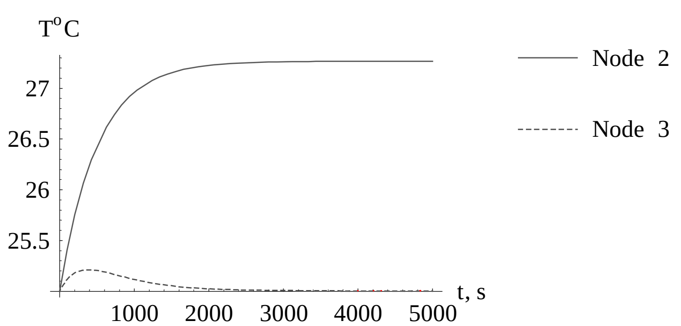

1
NODAL SOLUTION
STEP=1
SUB =1
TIME=1
SEQV (AVG)
DMX =.006146
SMN =133.026
SMX =44761
ANSYS
MAR 22 2005
17:35:45
Y
X
Z
MN
MX
133.026
5092
10050
15009
19968
24926
29885
34844
39802
44761

CHAPTER EIGHT

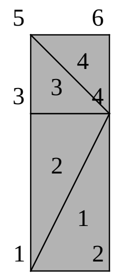
Transient Problems

8.1

$$\begin{pmatrix} 6777 & 2259 \\ 2259 & 4518 \end{pmatrix}\begin{pmatrix} \dot{T}_2 \\ \dot{T}_3 \end{pmatrix} + \begin{pmatrix} 15.035 & 7.5 \\ 7.5 & 15.0175 \end{pmatrix}\begin{pmatrix} T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} 597.5 \\ 580. \end{pmatrix}$$

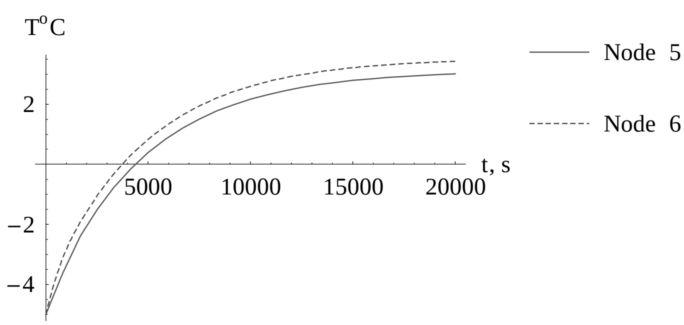


8.2



$$\begin{pmatrix} 201.449 & 50.3623 & 50.3623 & 100.725 & 0 & 0 \\ 50.3623 & 100.725 & 0 & 50.3623 & 0 & 0 \\ 50.3623 & 0 & 151.087 & 75.5435 & 25.1812 & 0 \\ 100.725 & 50.3623 & 75.5435 & 302.174 & 50.3623 & 25.1812 \\ 0 & 0 & 25.1812 & 50.3623 & 100.725 & 25.1812 \\ 0 & 0 & 0 & 25.1812 & 25.1812 & 50.3623 \end{pmatrix} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \end{pmatrix} +$$

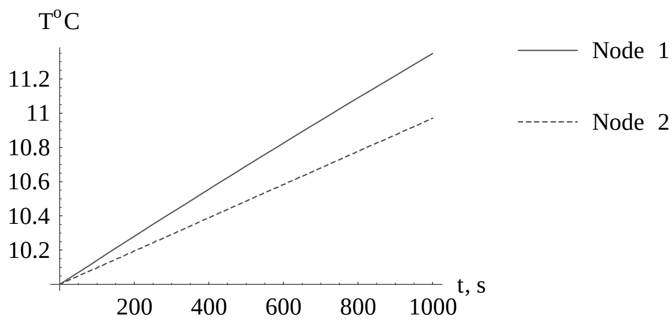
$$\begin{pmatrix} 1.3075 & -1.046 & -0.2615 & 0 & 0 & 0 \\ -1.046 & 1.3075 & 0 & -0.2615 & 0 & 0 \\ -0.2615 & 0 & 2.3535 & -1.569 & -0.523 & 0 \\ 0 & -0.2615 & -1.569 & 2.3535 & 0 & -0.523 \\ 0 & 0 & -0.523 & 0 & 1.246 & -0.423 \\ 0 & 0 & 0 & -0.523 & -0.423 & 1.246 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \\ -\frac{3}{2} \\ -\frac{3}{2} \end{pmatrix}$$



8.3

The final global system of equations after adjusting for essential boundary conditions is as follows

$$\begin{pmatrix} 26685.8 & 13342.9 & 0 \\ 13342.9 & 71876.5 & 7816.09 \\ 0 & 7816.09 & 15632.2 \end{pmatrix} \begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \end{pmatrix} + \begin{pmatrix} 2.55681 & 0.581682 & 0 \\ 0.581682 & 5.05772 & 0.774077 \\ 0 & 0.774077 & 2.56443 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} 82.2986 \\ 198.719 \\ 132.3 \end{pmatrix}$$



8.4

Natural frequencies and mode shapes

	ω (rad/sec)	f (Hz)	T (sec)	Mode shape
1	4196.21	667.848	0.00149735	0.676257 0.757172
2	12309.9	1959.19	0.000510416	−0.827969 1.30859

8.5

Natural frequencies and mode shapes

{2527.97, 3319.11}

$$\begin{pmatrix} -0.52637 & 0.850256 \\ 0.88527 & 0.465078 \end{pmatrix}$$

8.6

Natural frequencies and mode shapes

{97.3496, 125.354}

$$\begin{pmatrix} 0.755295 & 0.655385 \\ -0.655385 & 0.755295 \end{pmatrix}$$

8.7

Natural frequencies and mode shapes

{136.503, 408.009, 433.192, 533.73}

$$\begin{pmatrix} 0.703631 & 0.0700205 & 0.703631 & -0.0700205 \\ 0.10039 & 0.699944 & -0.10039 & 0.699944 \\ 0.0634883 & -0.704251 & 0.0634883 & 0.704251 \\ 0.701213 & -0.0911096 & -0.701213 & -0.0911096 \end{pmatrix}$$

8.8

Natural frequencies and mode shapes

$\{5.50805\times10^{-6}, 167.146, 223.775, 358.339, 532.145, 1225.18\}$

$$\begin{pmatrix} -4.62267\times10^{-17} & -1.28255\times10^{-15} & -0.596285 & 2.48247\times10^{-15} & 0.745356 & -0.298142 \\ 0.738959 & 0.0277773 & 0.0479423 & -0.037125 & 0.626771 & -0.238012 \\ -0.0647607 & 0.809102 & 0.0486193 & -0.114284 & 0.534295 & -0.200661 \\ 0.031967 & 0.0422761 & -0.0528745 & 0.66381 & -0.659161 & 0.345337 \\ 0.0311645 & 0.0252551 & -0.0357075 & -0.669731 & -0.69779 & -0.248326 \\ 0.013388 & 0.00594057 & -0.023626 & -0.412237 & 0.102114 & 0.904909 \end{pmatrix}$$

8.9

Natural frequencies and mode shapes

{78.5105, 711.616, 1395.17, 3261.03}

$$\begin{pmatrix} -0.334142 & -0.881305 & -1.3366\times10^{-15} & 0.334142 \\ 0.57735 & -1.36539\times10^{-16} & -0.57735 & 0.57735 \\ -0.706047 & 0.0547294 & -1.59796\times10^{-16} & 0.706047 \\ 0.57735 & 3.6484\times10^{-17} & 0.57735 & 0.57735 \end{pmatrix}$$

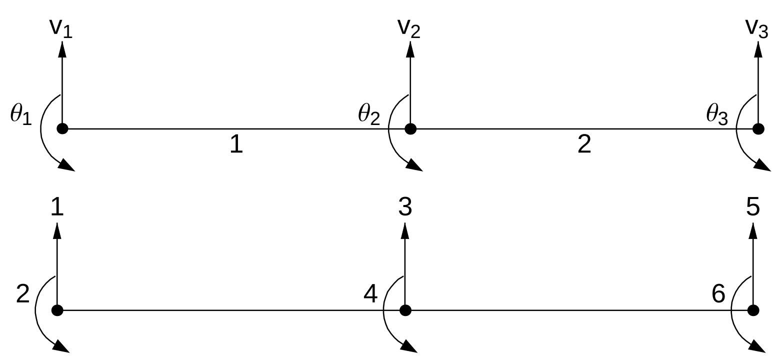
8.10

Natural frequencies and mode shapes

{805.267, 882.587, 1195.83}

$$\begin{pmatrix} -0.873687 & -0.332045 & -0.355552 \\ -0.421046 & 0.647647 & 0.635038 \\ -0.0956541 & -0.286418 & 0.953318 \end{pmatrix}$$

8.11



Global matrices after incorporating essential boundary conditions

$$\boldsymbol{m} = \begin{pmatrix} 0.0167143 & 0 & -0.104464 \\ 0 & 9.64286 & -3.61607 \\ -0.104464 & -3.61607 & 4.82143 \end{pmatrix}$$

$$\boldsymbol{k} = \begin{pmatrix} 213.333 & 0 & 8000. \\ 0 & 1.6\times10^{6} & 400000. \\ 8000. & 400000. & 800000. \end{pmatrix}$$

$$\boldsymbol{r}^{\mathrm{T}} = (\,-60. \quad 1500. \quad 0\,)$$

Time history of nodal solution

Time	DOF	Disp	Vel	Acc
0	1	0	0	0
	2	0	0	0
	3	0	0	0.
	4	0	0	0.
	5	0	0	0
	6	0	0	0.
0.001	1	0	0	0
	2	0	0	0
	3	−0.000127972	−0.255943	−511.887
	4	6.52715×10^{-6}	0.0130543	26.1086
	5	0	0	0
	6	1.95908×10^{-6}	0.00391816	7.83632
0.002	1	0	0	0
	2	0	0	0
	3	−0.000777531	−1.04318	−1062.58
	4	0.0000371082	0.0481078	43.9984
	5	0	0	0
	6	9.48963×10^{-6}	0.011143	6.61327
0.003	1	0	0	0
	2	0	0	0
	3	−0.00249963	−2.40103	−1653.13
	4	0.000108476	0.0946276	49.0412
	5	0	0	0
	6	0.0000204648	0.0108074	−7.28446
0.004	1	0	0	0
	2	0	0	0
	3	−0.00587142	−4.34255	−2229.91
	4	0.00022586	0.14014	41.9841
	5	0	0	0
	6	0.0000220795	−0.00757793	−29.4861
0.005	1	0	0	0
	2	0	0	0
	3	−0.011446	−6.80669	−2698.37
	4	0.000383511	0.175162	28.0586
	5	0	0	0
	6	-5.21514×10^{-6}	−0.0470114	−49.3807
0.006	1	0	0	0
	2	0	0	0
	3	−0.0196676	−9.63648	−2961.21
	4	0.000569168	0.196153	13.9237
	5	0	0	0
	6	−0.0000783954	−0.0993492	−55.295
0.007	1	0	0	0
	2	0	0	0
	3	−0.0307849	−12.598	−2961.81
	4	0.000769818	0.205148	4.06751
	5	0	0	0
	6	−0.000201679	−0.147219	−40.4444

0.008	1	0	0	0
	2	0	0	0
	3	−0.044801	−15.4343	−2710.73
	4	0.000975622	0.206459	−1.44542
	5	0	0	0
	6	−0.000360654	−0.17073	−6.57864
0.009	1	0	0	0
	2	0	0	0
	3	−0.0614833	−17.9303	−2281.44
	4	0.00118004	0.202381	−6.71073
	5	0	0	0
	6	−0.000523885	−0.15573	36.5787
0.01	1	0	0	0
	2	0	0	0
	3	−0.0804283	−19.9598	−1777.42
	4	0.00137641	0.190344	−17.3638
	5	0	0	0
	6	−0.000651513	−0.099526	75.8301

8.12

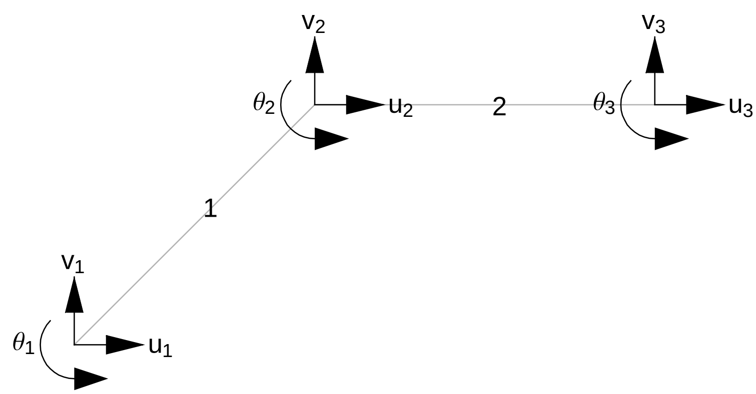


Figure 8.18. Two element model for the plane frame

Global matrices after incorporating essential boundary conditions

$$\boldsymbol{m} = \begin{pmatrix} 1.90286 & -0.102857 & -36.0018 & 0.797143 & 0.102857 & 21.2738 \\ -0.102857 & 1.90286 & 36.0018 & 0.102857 & 0.797143 & -21.2738 \\ -36.0018 & 36.0018 & 1666.29 & -21.2738 & 21.2738 & -1249.71 \\ 0.797143 & 0.102857 & -21.2738 & 3.70286 & -0.102857 & 36.0018 \\ 0.102857 & 0.797143 & 21.2738 & -0.102857 & 3.90857 & 14.9124 \\ 21.2738 & -21.2738 & -1249.71 & 36.0018 & 14.9124 & 3332.57 \end{pmatrix}$$

$$\boldsymbol{k} = \begin{pmatrix} 8.3642\times10^{6} & 8.30247\times10^{6} & -3.92837\times10^{6} & -8.3642\times10^{6} & -8.30247\times10^{6} & -3.92837\times10^{6} \\ 8.30247\times10^{6} & 8.3642\times10^{6} & 3.92837\times10^{6} & -8.30247\times10^{6} & -8.3642\times10^{6} & 3.92837\times10^{6} \\ -3.92837\times10^{6} & 3.92837\times10^{6} & 6.66667\times10^{8} & 3.92837\times10^{6} & -3.92837\times10^{6} & 3.33333\times10^{8} \\ -8.3642\times10^{6} & -8.30247\times10^{6} & 3.92837\times10^{6} & 2.50309\times10^{7} & 8.30247\times10^{6} & 3.92837\times10^{6} \\ -8.30247\times10^{6} & -8.3642\times10^{6} & -3.92837\times10^{6} & 8.30247\times10^{6} & 8.42593\times10^{6} & 1.62718\times10^{6} \\ -3.92837\times10^{6} & 3.92837\times10^{6} & 3.33333\times10^{8} & 3.92837\times10^{6} & 1.62718\times10^{6} & 1.33333\times10^{9} \end{pmatrix}$$

$$\boldsymbol{r}^{T} = (\, 12727.9 \quad -12727.9 \quad -540000. \quad 12727.9 \quad -12727.9 \quad 540000. \,)$$

Time history of nodal solution

Time	DOF	Disp	Vel	Acc
0	1	0	0	5174.34
	2	0	0	−4420.
	3	0	0	18.3648
	4	0	0	1446.04
	5	0	0	−2954.7
	6	0	0	105.277
	7	0	0	0
	8	0	0	0
	9	0	0	0
0.001	1	0.00183535	3.67069	2167.05
	2	−0.00216981	−4.33962	−4259.23
	3	-9.63473×10^{-6}	−0.0192695	−56.9037
	4	0.000414303	0.828607	211.178
	5	−0.00102521	−2.05042	−1146.14
	6	0.0000411559	0.0823118	59.3462
	7	0	0	0
	8	0	0	0
	9	0	0	0
0.002	1	0.00645218	5.56298	1617.53
	2	−0.00863759	−8.59594	−4253.42
	3	−0.0000545939	−0.0706489	−45.8551
	4	0.000864672	0.0721304	−1724.13
	5	−0.00344596	−2.79108	−335.182
	6	0.000145407	0.12619	28.4099
	7	0	0	0
	8	0	0	0
	9	0	0	0
0.003	1	0.0135154	8.56341	4383.34
	2	−0.0195489	−13.2267	−5008.12
	3	−0.000103311	−0.0267848	133.583
	4	0.000429732	−0.942011	−304.152
	5	−0.00660953	−3.53606	−1154.78
	6	0.000279593	0.142183	3.57705
	7	0	0	0
	8	0	0	0
	9	0	0	0
0.004	1	0.0243155	13.0368	4563.48
	2	−0.0351789	−18.0332	−4604.93
	3	−0.0000509029	0.1316	183.187
	4	−0.000119237	−0.155925	1876.33
	5	−0.010706	−4.65689	−1086.9
	6	0.000408729	0.116087	−55.7699
	7	0	0	0
	8	0	0	0
	9	0	0	0
0.005	1	0.0389857	16.3035	1969.91
	2	−0.0546126	−20.8342	−997.029
	3	0.000132756	0.235717	25.0461
	4	0.00014845	0.691299	−181.878
	5	−0.0156927	−5.31648	−232.274
	6	0.000486692	0.0398399	−96.7241
	7	0	0	0
	8	0	0	0
	9	0	0	0

	1	0.0555934	16.912	−752.874
	2	−0.0758254	−21.5914	−517.324
	3	0.000357364	0.213498	−69.4838
	4	0.000486564	−0.0150723	−1230.86
0.006	5	−0.0208822	−5.06259	740.058
	6	0.000482633	−0.0479584	−78.8725
	7	0	0	0
	8	0	0	0
	9	0	0	0
	1	0.0724475	16.7961	520.962
	2	−0.0979574	−22.6726	−1645.06
	3	0.000574048	0.21987	82.2277
	4	0.000169029	−0.619996	21.0157
0.007	5	−0.0258101	−4.79309	−201.057
	6	0.000416664	−0.0839785	6.83225
	7	0	0	0
	8	0	0	0
	9	0	0	0
	1	0.0897333	17.7755	1437.92
	2	−0.121091	−23.5948	−199.428
	3	0.000848287	0.328607	135.246
	4	−0.000245159	−0.208381	802.215
0.008	5	−0.0310133	−5.61342	−1439.62
	6	0.000353337	−0.0426759	75.773
	7	0	0	0
	8	0	0	0
	9	0	0	0
	1	0.106907	16.5714	−3846.17
	2	−0.143957	−22.1371	3114.79
	3	0.00117402	0.322869	−146.722
	4	−0.000147595	0.40351	421.567
0.009	5	−0.0369513	−6.26264	141.181
	6	0.000340708	0.0174181	44.4149
	7	0	0	0
	8	0	0	0
	9	0	0	0
	1	0.121003	11.6217	−6053.31
	2	−0.163878	−17.7038	5751.93
	3	0.00138287	0.0948185	−309.379
	4	0.0000894329	0.0705448	−1087.5
0.01	5	−0.0430209	−5.87648	631.145
	6	0.000378079	0.0573229	35.3947
	7	0	0	0
	8	0	0	0
	9	0	0	0

CHAPTER NINE

P—Formulation

9.1

```
p = {1/2 Sqrt[3/2] (s^2 - 1), 1/2 Sqrt[5/2] s (s^2 - 1), 1/8 Sqrt[7/2] (5 s^4 - 6 s^2 + 1)};
```

$$p = \left\{\frac{1}{2}\sqrt{\frac{3}{2}}\,(s^2-1),\ \frac{1}{2}\sqrt{\frac{5}{2}}\,s\,(s^2-1),\ \frac{1}{8}\sqrt{\frac{7}{2}}\,(5\,s^4-6\,s^2+1)\right\};$$

```
dp = D[p, s];

Table[Integrate[dp[[i]] dp[[j]], {s, -1, 1}], {i, 1, 3}, {j, 1, 3}]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

9.2

	Range	Solution
1	$0 \le x \le \frac{4}{5}$	$0.000022956\,x$

Solution with one p-mode

	Range	Solution
1	$0 \le x \le \frac{4}{5}$	$0.0000401729\,x - 0.0000215212\,x^2$

Solution with two p-modes

	Range	Solution
1	$0 \le x \le \frac{4}{5}$	$-0.0000179344\,x^3 + 0.\,x^2 + 0.000034434\,x$

9.3

Element equations:
$$\begin{pmatrix} -2.84764 & -2.86328 & -0.0224879 & -0.00684593 & 3.49721 & 0.00957872 & 0.0179 \\ -2.86328 & -2.85025 & -0.00684593 & -0.0198714 & 3.49881 & -0.00797645 & 0.0163 \\ -0.0224879 & -0.00684593 & 0.0224879 & 0.00684593 & 0.0179632 & -0.00957872 & -0.0179 \\ -0.00684593 & -0.0198714 & 0.00684593 & 0.0198714 & 0.016361 & 0.00797645 & -0.0163 \\ 3.49721 & 3.49881 & 0.0179632 & 0.016361 & -4.28417 & -0.00098119 & -0.0210 \\ 0.00957872 & -0.00797645 & -0.00957872 & 0.00797645 & -0.00098119 & 0.0107503 & 0.0009 \\ 0.0179632 & 0.016361 & -0.0179632 & -0.016361 & -0.0210192 & 0.00098119 & 0.0210 \\ -0.00957872 & 0.00797645 & 0.00957872 & -0.00797645 & 0.00098119 & -0.0107503 & -0.0009 \end{pmatrix}$$

9.4

Nodal solution summary

dof	x	y	Value
ϕ_1	0	0	−1.30795
ϕ_2	1	0	0
ϕ_3	0	1	−3.99197
ϕ_4	1	1	1
ϕ_5	0	2	0
ϕ_6	1	2	1

Element solution summary

	x	y	ϕ	$\partial\phi/\partial x$	$\partial\phi/\partial y$
1	0.5	0.5	−1.97741	2.83314	−0.790941
2	0.5	1.5	−1.98683	4.43836	2.53129

9.5

Nodal solution summary

dof	x	y	Value
u_1	0	0	0
u_2	0	$\frac{1}{2}$	0
u_3	0	1	0
u_4	1	0	0
u_5	1	$\frac{1}{2}$	0
u_6	1	1	0

Element solution summary

	x	y	u	$\partial u/\partial x$	$\partial u/\partial y$
1	0.5	0.25	0.0539941	0.	0.423281
2	0.5	0.75	0.54205	0.	1.52894

APPENDIX B

Variational Form for Boundary Value Problems

B.1

Compute variation of the following functionals

(a) $F[u, x] = \frac{u^2}{x^3} + e^x$

(b) $F[u', u, x] = \int_0^1 (x^2\, u'^2 + u^3 + x)\, \mathrm{dx}$

(c) $F[u, x] = u^2\, u'^2 + x^3$

(d) $F[u'', u', u, x] = \int_{-1}^1 (\frac{u''}{u} + u'\, x)\, \mathrm{dx}$

(a) $F[u, x] = \frac{u^2}{x^3} + e^x$ $\delta\mathrm{F} = \frac{2\,\mathrm{u}\delta\mathrm{u}}{x^3}$

(b) $F[u', u, x] = \int_0^1 (x^2\, u'^2 + u^3 + x)\, \mathrm{dx}$ $\delta\mathrm{F} = \int_0^1 (x^2\, 2\, u'\, \delta\mathrm{u}' + 3\, u^2\, \delta\mathrm{u})\, \mathrm{dx}$

(c) $F[u, x] = u^2\, u'^2 + x^3$ $\delta\mathrm{F} = 2\, u\, \delta\mathrm{u}\, u'^2 + u^2\, 2\, u'\, \delta\mathrm{u}'$

(d) $F[u'', u', u, x] = \int_{-1}^{1}(\frac{u''}{u} + u' x)\,\mathrm{dx} = \int_{-1}^{1}(u''\, u^{-1} + u'\, x)\,\mathrm{dx}$

$$\delta \mathrm{F} = \int_{-1}^{1}(\delta \mathrm{u}''\, u^{-1} - u''\, u^{-2}\, \delta \mathrm{u} + \delta \mathrm{u}'\, x)\,\mathrm{dx}$$

B.2

Determine an equivalent variational form for the following boundary value problem.

$$\frac{d^2 u}{\mathrm{dx}^2} + x^2 = 0; \quad 0 < x < 1$$

with the boundary conditions

$$u(0) = 1$$

$$u'(1) + 2\,u(1) = 1$$

The equivalent functional can be derived using the steps outlined in section as follows.

$$\int_0^1 \left(\frac{d^2 u}{\mathrm{dx}^2} + x^2\right) \delta \mathrm{u}\ \mathrm{dx} = 0$$

Integrating the first term by parts

$$\left[\frac{\mathrm{du}}{\mathrm{dx}}\, \delta \mathrm{u}\right]_0^1 + \int_0^1 \left(-\frac{\mathrm{du}}{\mathrm{dx}}\, \frac{d(\delta \mathrm{u})}{\mathrm{dx}} + x^2\, \delta \mathrm{u}\right)\ \mathrm{dx} = 0$$

$$u'(1)\, \delta \mathrm{u}(1) - u'(0)\, \delta \mathrm{u}(0) + \int_0^1 (-\frac{\mathrm{du}}{\mathrm{dx}}\, \delta(\frac{\mathrm{du}}{\mathrm{dx}}) + x^2\, \delta \mathrm{u})\ \mathrm{dx} = 0$$

Taking into consideration the natural boundary condition and requiring trial solutions to satisfy essential boundary condition

$$[1 - 2\,u(1)]\, \delta \mathrm{u}(1) + \int_0^1 (-\frac{\mathrm{du}}{\mathrm{dx}}\, \delta(\frac{\mathrm{du}}{\mathrm{dx}}) + x^2\, \delta \mathrm{u})\ \mathrm{dx} = 0$$

$$\delta[u(1) - u(1)^2] + \delta\left[\int_0^1 \left\{-\frac{1}{2}\,(\frac{\mathrm{du}}{\mathrm{dx}})^2 + x^2\, u\right\}\ \mathrm{dx}\right] = 0$$

Thus the appropriate functional is as follows.

$$I(u) = u(1) - u(1)^2 + \int_0^1 \left\{-\frac{1}{2}\,(\frac{\mathrm{du}}{\mathrm{dx}})^2 + x^2\, u\ \right\} \mathrm{dx}$$

B.3

Consider the following boundary value problem

$$u'' \sin(x) + u' \cos(x) + u \sin(x) = 0; \quad \pi/4 < x < \pi/2$$

$$u(\pi/4) = 1 \text{ and } u'(\pi/2) = 2$$

Derive an equivalent variational functional for the problem. Note that the differential equation can be written as follows.

$$\frac{d}{\mathrm{dx}}[u' \sin(x)] + u \sin(x) = 0$$

The equivalent variational functional is as follows.

$$\int_{\pi/4}^{\pi/2} \left(\frac{d}{dx} [u' \sin(x)] + u \sin(x) \right) \delta u \, dx = 0$$

Integration by parts

$$(\delta u \, u' \sin(x))_{x=\pi/2} - (\delta u \, u' \sin(x))_{x=\pi/4} + \int_{\pi/4}^{\pi/2} (-\delta u' \, u' \sin(x) + \delta u \, u \sin(x)) \, dx = 0$$

Incorporating $u'(\pi/2) = 2$ and requiring trial solutions to be such that $\delta u(\pi/4) = 0$, we have

$$2 \, \delta u(\pi/2) + \delta\left[\int_{\pi/4}^{\pi/2} (-1/2 \, u'^2 \sin(x) + 1/2 \, u^2 \sin(x)) \, dx\right] = 0$$

$$\delta\left[2 \, u(\pi/2) + \int_{\pi/4}^{\pi/2} (-1/2 \, u'^2 \sin(x) + 1/2 \, u^2 \sin(x)) \, dx\right] = 0$$

Thus the functional is as follows.

$$2 \, u(\pi/2) + \int_{\pi/4}^{\pi/2} (-1/2 \, u'^2 + 1/2 \, u^2) \sin(x) \, dx$$

B.4

Consider finite element solution of the following boundary value problem

$$-u'' + x = 0; \quad \pi/4 < x < \pi/2$$

$u(\pi/4) = 1$ and $u'(\pi/2) + 2 = 0$

Verify that the following is an appropriate functional for the problem.

$$I(u) = 2 \, u(\pi/2) + \int_{\pi/4}^{\pi/2} \left(\frac{1}{2} u'^2 + x \, u\right) dx$$

Taking variation of the functional

$$\delta I(u) = 2 \, \delta u(\pi/2) + \int_{\pi/4}^{\pi/2} (u' \, \delta u' + x \delta u) \, dx$$

Integrating the first term inside the integral by parts

$$\delta I(u) = 2 \, \delta u(\pi/2) + (u' \, \delta u)_{\pi/2} - (u' \, \delta u)_{\pi/4} + \int_{\pi/4}^{\pi/2} (-u'' \, \delta u + x \delta u) \, dx$$

Using the natural boundary condition

$$\delta I(u) = 2 \, \delta u(\pi/2) - 2 \, \delta u(\pi/2) - u'(\pi/4) \, \delta u(\pi/4) + \int_{\pi/4}^{\pi/2} (-u'' + x) \, \delta u \, dx$$

or $\quad \delta I(u) = -u'(\pi/4) \, \delta u(\pi/4) + \int_{\pi/4}^{\pi/2} (-u'' + x) \, \delta u \, dx$

For admissible trial solutions $\delta u(\pi/4)$ must be 0, therefore

$$\delta I(u) = \int_{\pi/4}^{\pi/2} (-u'' + x) \, \delta u \, dx.$$

Since δ u is arbitrary, for $\delta I(u)$ to be 0, we must have $-u'' + x = 0$ which is the governing differential equation. Thus the functional is appropriate for the given boundary value problem.

B.5

A functional is given as follows.

$$I(u) = \int_0^{\ell} \left(\tfrac{1}{2}\,\mathrm{EA}\,u'^2 - u \sin\left(\tfrac{\pi x}{\ell}\right)\right) \mathrm{dx} - \tfrac{1}{2}\,k\,u(\ell)^2 - \tfrac{1}{\pi}\,u(0)$$

Determine the corresponding boundary value problem.

The first variation of the variational functional is as follows.

$$\delta \mathrm{I} = \int_0^{\ell} \left(\mathrm{EA}\,u'\,\delta \mathrm{u}' - \delta \mathrm{u} \sin\left(\tfrac{\pi x}{\ell}\right)\right) \mathrm{d}x - k\,u(\ell)\,\delta \mathrm{u}(\ell) - \tfrac{1}{\pi}\,\delta \mathrm{u}(0)$$

Integrate the first term by parts

$$\delta \mathrm{I} = (\mathrm{EA}\,u'\,\Box u)_{x=\ell} - (\mathrm{EA}\,u'\,\Box u)_{x=0} + \int_0^{\ell} \left(-\tfrac{d}{\mathrm{dx}}\,(\mathrm{EA}\,u')\,\Box u - \sin\left(\tfrac{\pi x}{\ell}\right)\Box u\right) \mathrm{d}x - k\,u(\ell)\,\delta \mathrm{u}(\ell) - \tfrac{1}{\pi}\,\delta \mathrm{u}(0)$$

Combining the boundary terms

$$\delta \mathrm{I} = -\int_0^{\ell} \left(\tfrac{d}{\mathrm{dx}}\,(\mathrm{EA}\,u') + \sin\left(\tfrac{\pi x}{\ell}\right)\right)\Box u\, \mathrm{d}x + (\mathrm{EA}\,u'(\ell) - k\,u(\ell))\,\delta \mathrm{u}(\ell) + \left(-\mathrm{EA}\,u'(0) - \tfrac{1}{\pi}\right)\delta \mathrm{u}(0)$$

For $\delta \mathrm{I}$ to be zero the following conditions must be met

$$\tfrac{d}{\mathrm{dx}}\,(\mathrm{EA}\,u') + \sin\left(\tfrac{\pi x}{\ell}\right) = 0 \quad 0 < x < \ell$$

Either $\quad \mathrm{EA}\,u'(\ell) - k\,u(\ell) = 0 \quad$ or $\quad \delta \mathrm{u}(\ell) = 0$

Either $\quad -\mathrm{EA}\,u'(0) - \tfrac{1}{\pi} = 0 \quad$ or $\quad \delta \mathrm{u}(0) = 0$

Thus the equivalent BVP

$$\tfrac{d}{\mathrm{dx}}\,(\mathrm{EA}\,u') + \sin\left(\tfrac{\pi x}{\ell}\right) = 0 \quad 0 < x < \ell$$

$\mathrm{EA}\,u'(\ell) - k\,u(\ell) = 0 \quad$ or $\quad u(\ell) = \text{Specified value}$

$-\mathrm{EA}\,u'(0) - \tfrac{1}{\pi} = 0 \quad$ or $\quad u(0) = \text{Specified value}$

B.6

The potential energy for the problem of the torsion of a thin-walled section with warping restraint can be written as follows.

$$\Pi = \int_0^{L} \left(\tfrac{E J_w}{2}\,\phi''^2 + \tfrac{G J_0}{2}\,\phi'^2 - t(x)\,\phi\right) \mathrm{dx}$$

where E is the modulus of elasticity, G is shear modulus, J_w is the warping constant, J_0 is the torsional constant, t is thickness of the section, and ϕ is the angle through which a cross-section rotates. Determine the governing differential equation and appropriate boundary conditions.

The first variation of the variational functional is as follows.

$$\delta\Pi = \int_0^L \left(\frac{E\,J_w}{2}\,2\,\phi''\,\delta\phi'' + \frac{G\,J_0}{2}\,2\,\phi'\,\delta\phi' - t\,\delta\phi\right) dx = \int_0^L \left(E\,J_w\,\phi''\,\frac{d^2(\delta\phi)}{d\phi^2} + G\,J_0\,\phi'\,\frac{d(\delta\phi)}{d\phi} - t\,\delta\phi\right) dx$$

Integrate the first term by parts twice

$$\int_0^L \left(E\,J_w\,\phi''\,\frac{d^2(\delta\phi)}{d\phi^2}\right) dx = \left[E\,J_w\,\phi''\,\frac{d(\delta\phi)}{d\phi}\right]_0^L - \int_0^\ell \left(\frac{d(E\,J_w\,\phi'')}{d\phi}\,\frac{d(\delta\phi)}{d\phi}\right) dx$$

$$= \left[E\,J_w\,\phi''\,\frac{d(\delta\phi)}{d\phi}\right]_0^L - \left[\frac{d(E\,J_w\,\phi'')}{d\phi}\,\delta\phi\right]_0^L + \int_0^\ell \left(\frac{d^2(E\,J_w\,\phi'')}{d\phi^2}\,\delta\phi\right) dx$$

Integrate the second term by parts

$$\int_0^L \left(G\,J_0\,\phi'\,\frac{d(\delta\phi)}{d\phi}\right) dx = [G\,J_0\,\phi'\,\delta\phi]_0^L - \int_0^\ell \left(\frac{d(G\,J_0\,\phi')}{d\phi}\,\delta\phi\right) dx$$

Combining the terms

$$\delta\Pi = \left[E\,J_w\,\phi''\,\frac{d(\delta\phi)}{d\phi}\right]_0^L + \left[\left(-\frac{d(E\,J_w\,\phi'')}{d\phi} + G\,J_0\,\phi'\right)\delta\phi\right]_0^L + \int_0^\ell \left(\frac{d^2(E\,J_w\,\phi'')}{d\phi^2} - \frac{d(G\,J_0\,\phi')}{d\phi} - t\right)\delta\phi\,dx$$

For $\delta\Pi$ to be zero the following conditions must be met

$$\frac{d^2(E\,J_w\,\phi'')}{d\phi^2} - \frac{d(G\,J_0\,\phi')}{d\phi} - t = 0; \quad 0 < x < L$$

At $x = 0$ and L: either $E\,J_w\,\phi'' = 0$ or $\frac{d(\delta\phi)}{d\phi} = \delta\phi' = 0$

At $x = 0$ and L : Either $-\frac{d(E\,J_w\,\phi'')}{d\phi} + G\,J_0\,\phi' = 0$ or $\delta\phi = 0$

Thus the equivalent BVP

$$\frac{d^2(E\,J_w\,\phi'')}{d\phi^2} - \frac{d(G\,J_0\,\phi')}{d\phi} - t = 0; \quad 0 < x < L$$

At $x = 0$ and L: either $E\,J_w\,\phi'' = 0$ or $\phi' =$ specified

At $x = 0$ and L : Either $-\frac{d(E\,J_w\,\phi'')}{d\phi} + G\,J_0\,\phi' = 0$ or $\phi =$ specified