Partial Answers to the Chapter End Problems

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Advanced Topics in Finite Element Analysis of Structures

with Computations Using *Mathematica* and Matlab

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CHAPTER ONE

Essential Background

1.1

$$w(1) + \int_0^1 (-w\,(u\,x^4 + x + u') - u'\,w')\,\mathrm{d}x = 0$$

1.2

$$u(x) = \frac{1}{17}\,(15\,x - 16)$$

1.3

$$\begin{pmatrix} \frac{2\,k\,L\,\pi\left(\frac{r_2^2}{2} - \frac{r_1^2}{2}\right)}{(r_1 - r_2)^2} & \frac{2\,k\,L\,\pi\left(\frac{r_2^2}{2} - \frac{r_1^2}{2}\right)}{(r_1 - r_2)\,(r_2 - r_1)} \\ \frac{2\,k\,L\,\pi\left(\frac{r_2^2}{2} - \frac{r_1^2}{2}\right)}{(r_1 - r_2)\,(r_2 - r_1)} & \frac{2\,k\,L\,\pi\left(\frac{r_2^2}{2} - \frac{r_1^2}{2}\right)}{(r_2 - r_1)^2} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}\,L\,\pi\,Q\,(r_1 - r_2)\,(2\,r_1 + r_2) \\ -\frac{1}{3}\,L\,\pi\,Q\,(r_1 - r_2)\,(r_1 + 2\,r_2) \end{pmatrix}$$

1.4

$$\left\{50,\ \frac{478}{23},\ \frac{1222}{23},\ 10\right\}$$

Exact solution: $u(x) = -\frac{10\,(1-5\,e^{100}+4\,e^{x})}{-1+e^{100}}$

1.5

Using two linear elements

	Range	Solution
1	$1 \le x \le 1.5$	$2.89749 - 0.897488\,x$
2	$1.5 \le x \le 2$	$2.10902 - 0.371845\,x$

Using 1 quadratic element

	Range	Solution
1	$1 \le x \le 2$	$0.534517\,x^2 - 2.25381\,x + 3.71929$

1.6

$I \approx 2.06155$

1.7

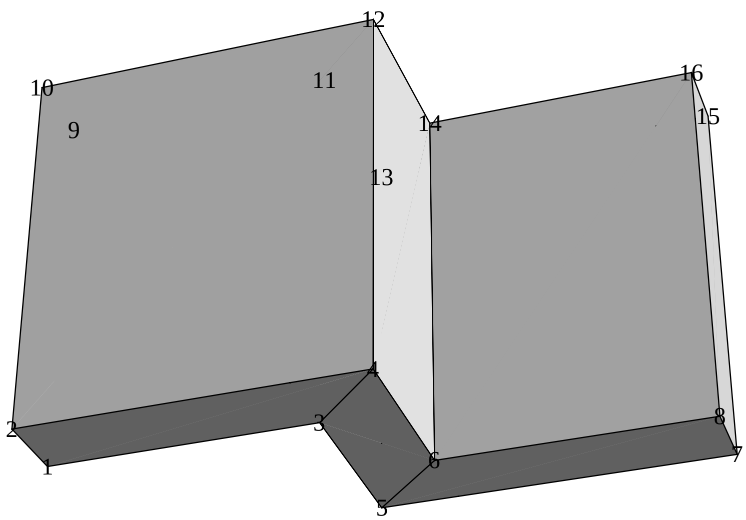
$\partial N_2/\partial \mathrm{x} = -\dfrac{s+2t-1}{2(s-t-4)}$; At s=0 and t=0: $\partial N_2/\partial \mathrm{x} = -\dfrac{1}{8}$

CHAPTER TWO

Analysis of Elastic Solids

2.1

Solution with three eight-node solid elements



	u	v	w
1	0	0	0
2	0	0	0
3	−0.0281233	−0.148096	−0.00123542
4	0.0357204	−0.194093	−0.00175711
5	−0.186379	−0.140973	−0.0018374
6	−0.117813	−0.216582	−0.00169579
7	−0.196634	−0.489654	−0.000698367
8	−0.115447	−0.490419	−0.000714844
9	0	0	0
10	0	0	0
11	−0.0281233	−0.148096	0.00123542
12	0.0357204	−0.194093	0.00175711
13	−0.186379	−0.140973	0.0018374
14	−0.117813	−0.216582	0.00169579
15	−0.196634	−0.489654	0.000698367
16	−0.115447	−0.490419	0.000714844

Ansys plane strain model

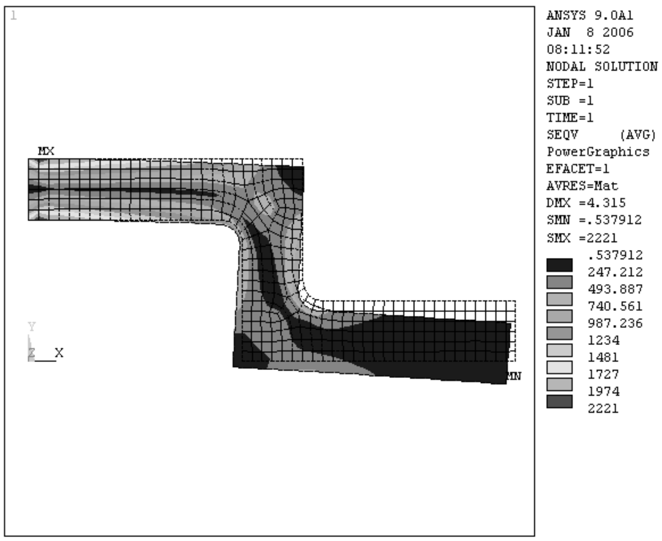
The planar model includes fillets to realistically capture stress concentrations at corners.

Deflections

MAXIMUM ABSOLUTE VALUES

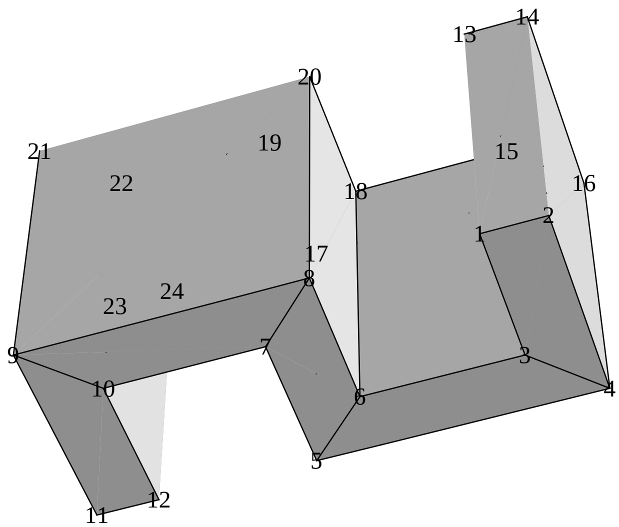
NODE	3	29	0	1
VALUE	-1.6737	-3.9774	0.0000	4.3149

Von Mises stresses



2.2

Solution with five eight-node solid elements



	u	v	w
1	0.029312	−0.0408152	−0.000102666
2	0.0248685	−0.040961	−0.00030369
3	0.0281897	−0.0329743	0.00171985
4	0.0310753	−0.0325326	0.00160147
5	0.0163573	−0.0308656	0.00104154
6	0.0198839	−0.0237311	0.00138904
7	0.0122978	−0.00959103	0.00142794
8	0.0159712	−0.0025363	0.00162177
9	0	0	0
10	0.00298233	−0.00104209	0.00131326
11	0	0	0
12	0.000264388	0.000237822	0.000397299
13	0.029312	−0.0408152	0.000102666
14	0.0248685	−0.040961	0.00030369
15	0.0281897	−0.0329743	−0.00171985
16	0.0310753	−0.0325326	−0.00160147
17	0.0163573	−0.0308656	−0.00104154
18	0.0198839	−0.0237311	−0.00138904
19	0.0122978	−0.00959103	−0.00142794
20	0.0159712	−0.0025363	−0.00162177
21	0	0	0

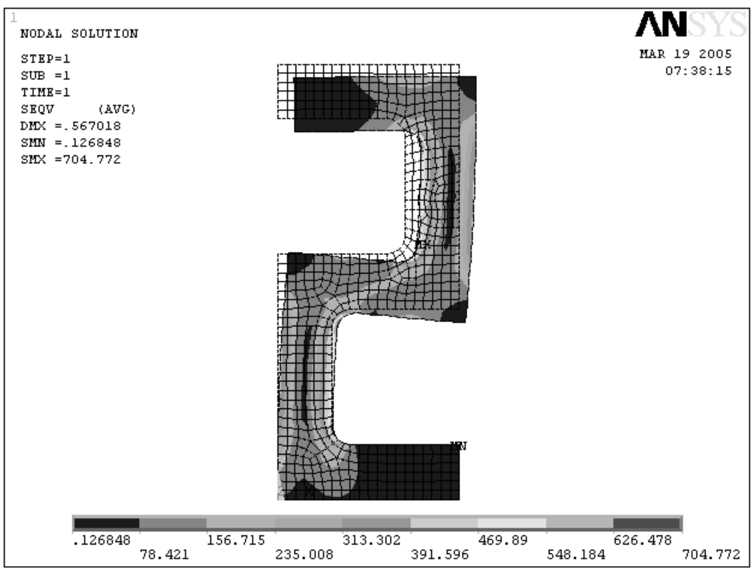
22	0.00298233	−0.00104209	−0.00131326
23	0	0	0
24	0.000264388	0.000237822	−0.000397299

Ansys plane stress model

Deflections

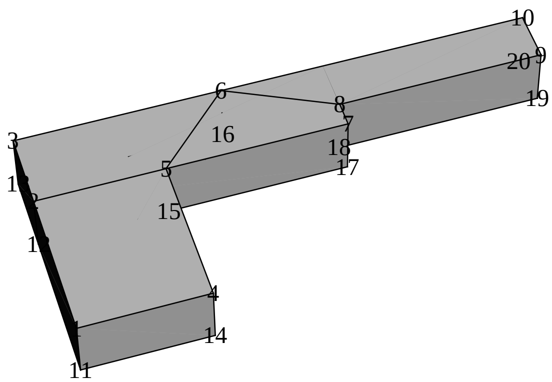
MAXIMUM ABSOLUTE VALUES				
NODE	162	86	0	145
VALUE	0.44392	-0.36244	0.0000	0.56712

Von Mises stresses



2.3

Solution with four eight-node solid elements



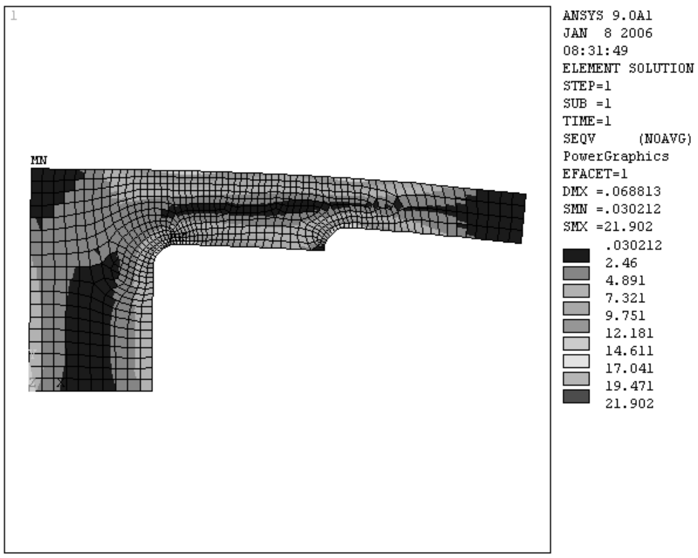
	u	v	w
1	0	0	0
2	0.00117475	0.00092064	-0.00010388
3	0.00261739	0.00124918	-0.0000660266
4	0	0	0
5	0.00121133	-0.00106776	0.000112101
6	0.00310604	-0.00308718	-9.57036×10^{-6}
7	0.000413212	-0.00664865	0.0000283445
8	0.00137316	-0.00682205	0.0000227296
9	0.00145549	-0.0145256	-7.66489×10^{-6}
10	0.00341233	-0.0145225	-6.92322×10^{-6}
11	0	0	0
12	0.00117475	0.00092064	0.00010388
13	0.00261739	0.00124918	0.0000660266
14	0	0	0
15	0.00121133	-0.00106776	-0.000112101
16	0.00310604	-0.00308718	9.57036×10^{-6}
17	0.000413212	-0.00664865	-0.0000283445
18	0.00137316	-0.00682205	-0.0000227296
19	0.00145549	-0.0145256	7.66489×10^{-6}
20	0.00341233	-0.0145225	6.92322×10^{-6}

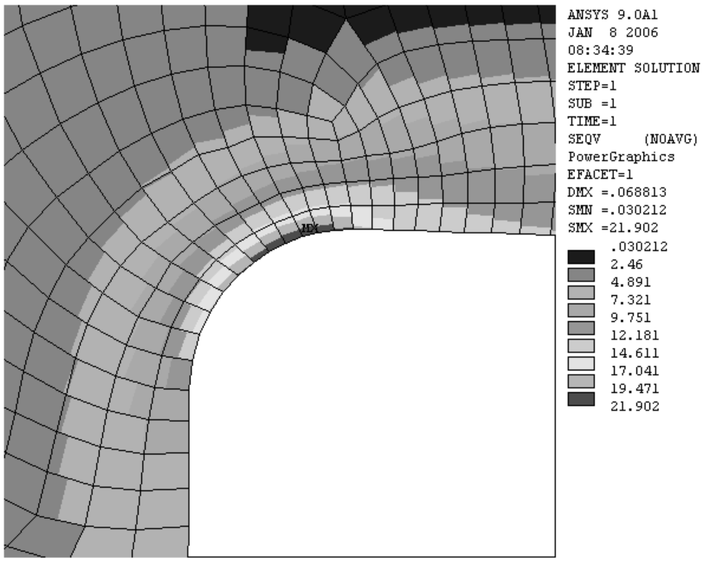
Ansys plane stress model

Deflections

```
MAXIMUM ABSOLUTE VALUES
NODE          98           98            0           98
VALUE   0.12303E-01-0.67705E-01  0.0000     0.68813E-01
```

Von Mises stresses

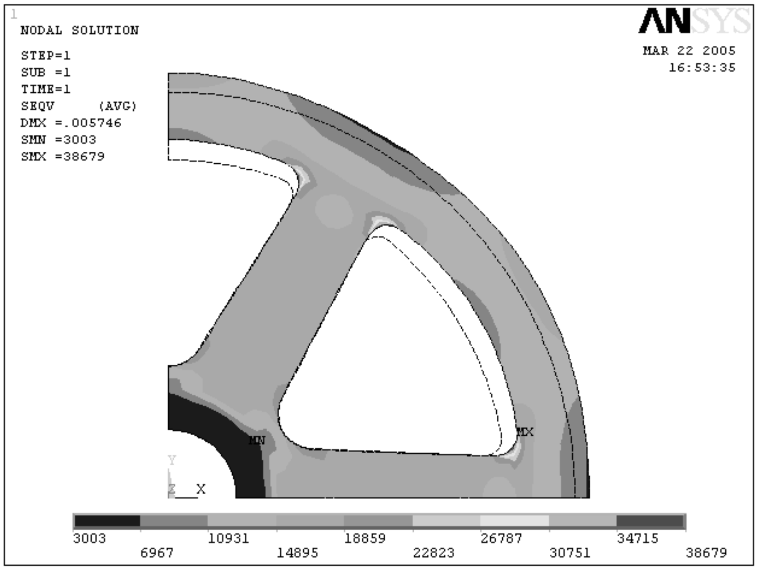




2.4

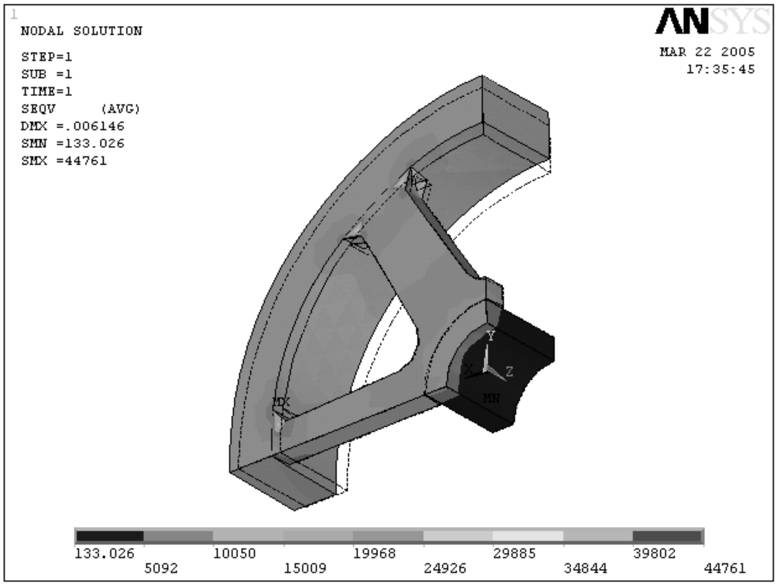
Ansys plane stress model

Von Mises stresses



Ansys solid model

Von Mises stresses



2.5

Ansys plane42 element: Rigid body patch test-Rotation

Nodal displacements with a rigid-body rotation of 60°.

$$\begin{pmatrix} 2.59808 & -1.5 \\ 1.16506 & -2.98205 \\ -0.267949 & -4.4641 \\ 0.933013 & 0.616025 \\ 0.366025 & -1.36603 \\ -1.13397 & -3.9641 \\ -0.732051 & 2.73205 \\ -1.36603 & -0.366025 \\ -2. & -3.4641 \end{pmatrix}$$

Specified displacements at exterior nodes

	u	v
1	2.59808	−1.5
2	1.16506	−2.98205
3	−0.267949	−4.4641
4	0.933013	0.616025
6	−1.13397	−3.9641
7	−0.732051	2.73205
8	−1.36603	−0.366025
9	−2.	−3.4641

Computed displacements

```
PRINT U    NODAL SOLUTION PER NODE

 ***** POST1 NODAL DEGREE OF FREEDOM LISTING *****

 LOAD STEP=     1  SUBSTEP=     1
  TIME=    1.0000      LOAD CASE=   0

 THE FOLLOWING DEGREE OF FREEDOM RESULTS ARE IN GLOBAL COORDINATES

   NODE      UX          UY          UZ          USUM
      1   2.5981     -1.5000      0.0000      3.0000
      2   1.1651     -2.9821      0.0000      3.2016
      3 -0.26795     -4.4641      0.0000      4.4721
      4  0.93301     0.61603      0.0000      1.1180
      5  0.36603     -1.3660      0.0000      1.4142
      6  -1.1340     -3.9641      0.0000      4.1231
      7 -0.73205      2.7321      0.0000      2.8284
      8  -1.3660    -0.36603      0.0000      1.4142
      9  -2.0000     -3.4641      0.0000      4.0000

MAXIMUM ABSOLUTE VALUES
NODE          1           3           0           3
VALUE    2.5981     -4.4641      0.0000      4.4721
```

```
PRINT S    ELEMENT SOLUTION PER ELEMENT


***** POST1 ELEMENT NODAL STRESS LISTING *****


 LOAD STEP=     1  SUBSTEP=     1

  TIME=    1.0000      LOAD CASE=   0
```

```
THE FOLLOWING X,Y,Z VALUES ARE IN GLOBAL COORDINATES


ELEMENT=       1        PLANE42
  NODE    SX          SY          SZ          SXY         SYZ         SXZ
     4  -714.29     -714.29      0.0000      0.95694E-13  0.0000      0.0000
     5  -714.29     -714.29      0.0000      0.17509E-12  0.0000      0.0000
     2  -714.29     -714.29      0.0000      0.21734E-13  0.0000      0.0000
     1  -714.29     -714.29      0.0000      0.23056E-12  0.0000      0.0000

ELEMENT=       2        PLANE42
  NODE    SX          SY          SZ          SXY         SYZ         SXZ
     5  -714.29     -714.29      0.0000      0.24772E-14  0.0000      0.0000
     6  -714.29     -714.29      0.0000     -0.19154E-13  0.0000      0.0000
     3  -714.29     -714.29      0.0000     -0.34503E-13  0.0000      0.0000
     2  -714.29     -714.29      0.0000      0.14726E-12  0.0000      0.0000

ELEMENT=       3        PLANE42
  NODE    SX          SY          SZ          SXY         SYZ         SXZ
     7  -714.29     -714.29      0.0000      0.16836E-12  0.0000      0.0000
     8  -714.29     -714.29      0.0000      0.15531E-12  0.0000      0.0000
     5  -714.29     -714.29      0.0000     -0.17370E-12  0.0000      0.0000
     4  -714.29     -714.29      0.0000      0.12758E-12  0.0000      0.0000

ELEMENT=       4        PLANE42
  NODE    SX          SY          SZ          SXY         SYZ         SXZ
     8  -714.29     -714.29      0.0000      0.71100E-13  0.0000      0.0000
```

```
9  -714.29     -714.29      0.0000    -0.25544E-12  0.0000      0.0000

6  -714.29     -714.29      0.0000    -0.11380E-12  0.0000      0.0000

5  -714.29     -714.29      0.0000    -0.10752E-12  0.0000      0.0000
```

The computed displacements at node 5 are as expected. However why do we have stresses in a rigid-body rotation? The answer will be clear after looking at different stress and strain measures discussed in Chapter 9. The given rotation obviously is large. For a proper analysis we must include large displacement effects.

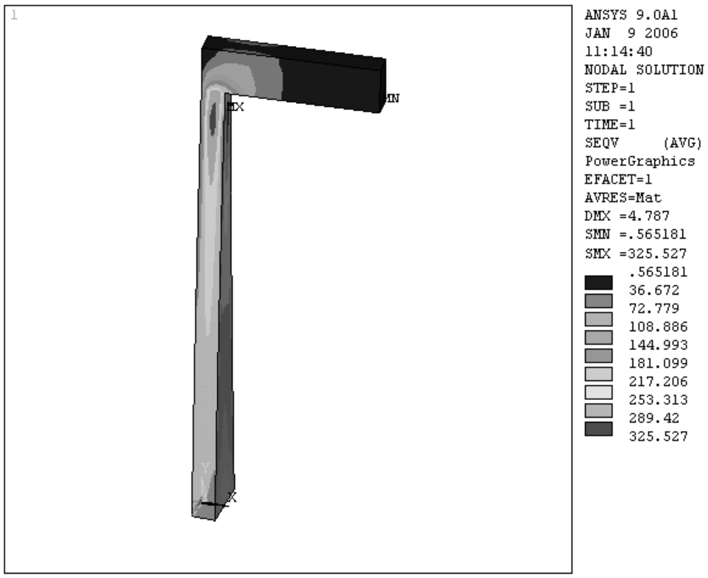
2.6

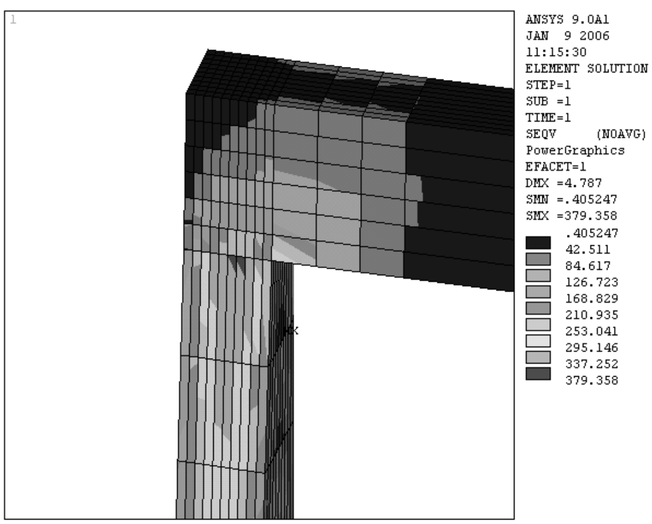
	x	y	u	v
1	0	0	0.1	0.2
2	20	0	4.1	0.2
3	0	10	0.1	−0.4
4	20	10	4.1	−0.4
5	4	3	0.9	0.02
6	14	4	2.9	−0.04
7	6	6	1.3	−0.16
8	14	7	2.9	−0.22

2.7 Programming Project

2.8 Programming Project

2.9





CHAPTER THREE

Solids of Revolution

3.1

	r	z	u	w
1	35	0	0.000531958	0
2	50	0	0.000420706	0
3	35	50	0.000535198	0
4	50	50	0.000453704	0

3.2

	r	z	u	w
1	10	0	0	0
2	20	0	0	0
3	10	40	−0.0188239	0.127113
4	20	40	0	0.122604
5	10	90	0.0410784	0.226363
6	20	90	0.0566833	0.216758

3.3

Nodal solution

	r	z	u	w
1	12	0	−0.0050849	0
2	20	0	−0.0221486	0
3	35	0	−0.0316438	0
4	12	20	−0.00612671	−0.021443
5	20	20	−0.0221733	−0.0198482
6	35	20	−0.03257	−0.0178926
7	12	40	−0.0100963	−0.0459928
8	20	40	−0.0258636	−0.0386396
9	35	40	−0.0370235	−0.0341886

3.4

	u	z
1	0.000517713	0
2	0.000426099	0
3	0.000517713	0
4	0.000426099	0

3.5

	u	z
1	0	0
2	0	0
3	−0.0203183	0.125564
4	0	0.122077
5	0.0332793	0.234792
6	0.0475977	0.221367

3.6

	u	z
1	−0.00524106	0
2	−0.0215315	0
3	−0.0319668	0
4	−0.00497884	−0.0211027
5	−0.0215631	−0.0197578
6	−0.0324841	−0.0174939
7	−0.0120671	−0.0505293
8	−0.0262301	−0.0389688
9	−0.0362975	−0.0334733

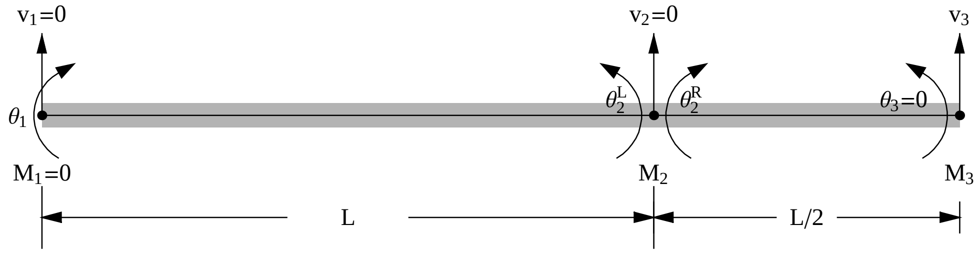
3.7 Programming Project

CHAPTER FOUR

Multi-Field Formulations for Beam Elements

4.1

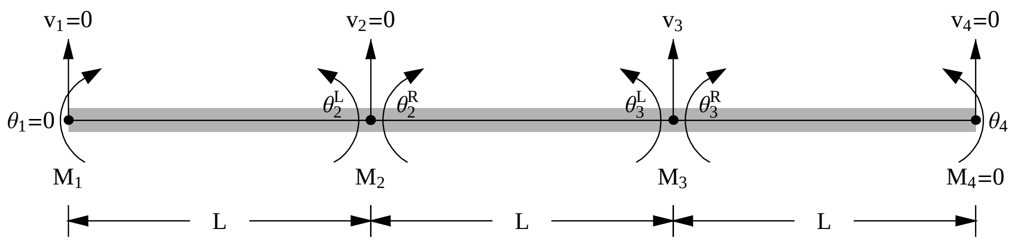
Using mixed-beam elements for EBT



$\{M_2=-360., M_3=840., R_1=-1.5, R_2=11.5, v_3=-0.178499, \theta_1=-0.000811359\}$

4.2

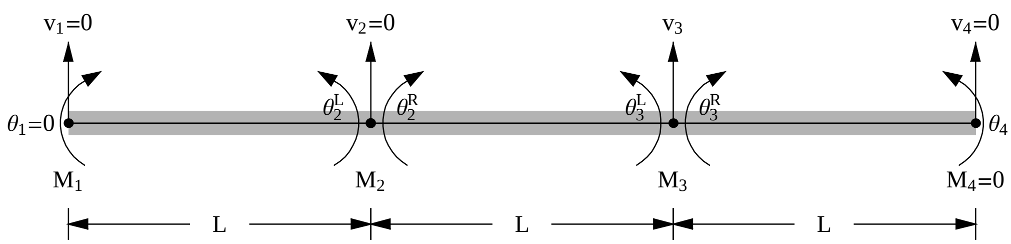
Using mixed-beam elements for EBT



$\{\{M_{1.} \to 818.182,\ M_{2.} \to -1636.36,\ M_{3.} \to 2181.82,$
$R_{1.} \to -20.4545,\ R_{2.} \to 52.2727,\ R_{4.} \to 18.1818,\ v_{3.} \to -0.834225,\ \theta_{4.} \to 0.0112299\}\}$

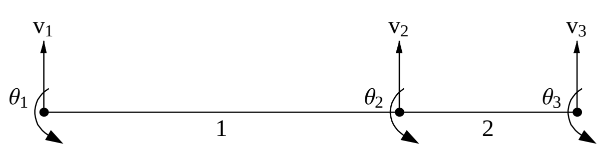
4.3

Using mixed-beam elements for EBT



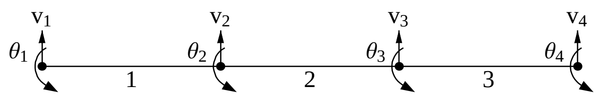
$\{\{M_{1.} \to 327.273,\ M_{2.} \to -654.545,\ M_{3.} \to 872.727,$
$R_{1.} \to -8.18182,\ R_{2.} \to 30.9091,\ R_{4.} \to 17.2727,\ v_{3.} \to -0.33369,\ \theta_{4.} \to 0.00449198\}\}$

4.4



$\{\theta_1 = 0.0000102496,\ \theta_2 = -0.0000206382,\ \mathrm{v}_3 = -0.00415919\}$

4.5

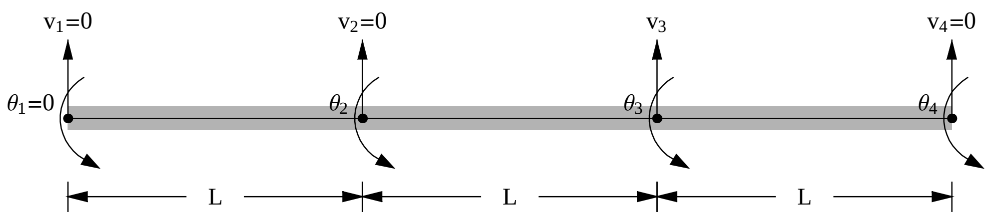


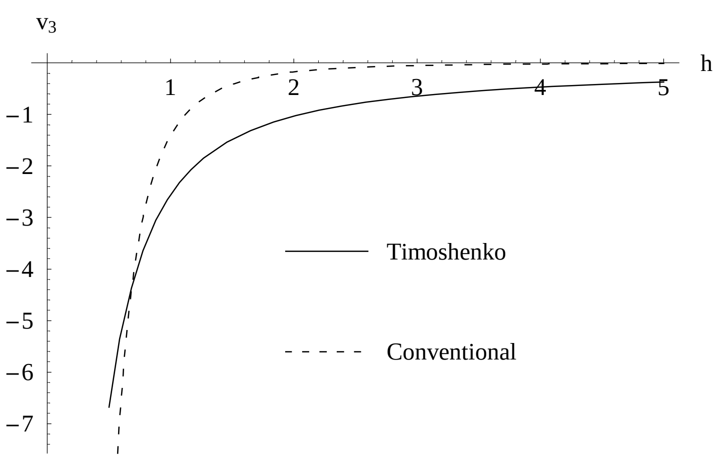
$\{\theta_2 = -0.000118835,\ v_3 = -0.0206492,\ \theta_3 = -0.0000384008,\ \theta_4 = 0.000275354\}$

4.6

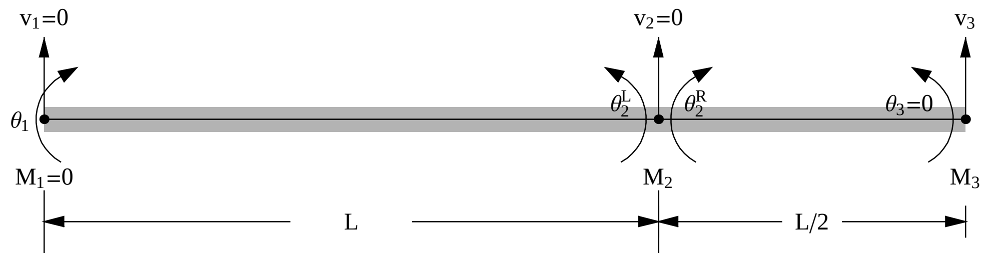
$\{\theta_2 = -0.0000475338,\ v_3 = -0.00825968,\ \theta_3 = -0.0000153603,\ \theta_4 = 0.000110142\}$

4.7



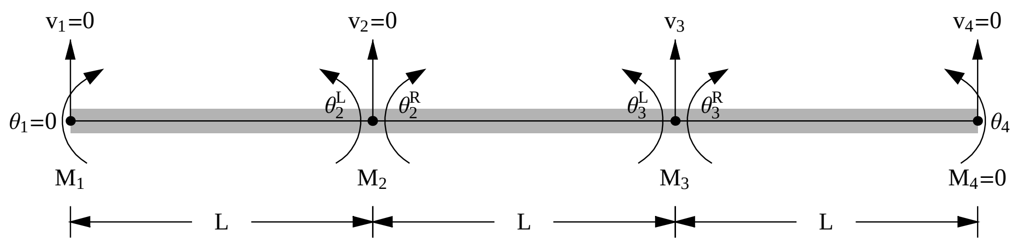


4.8



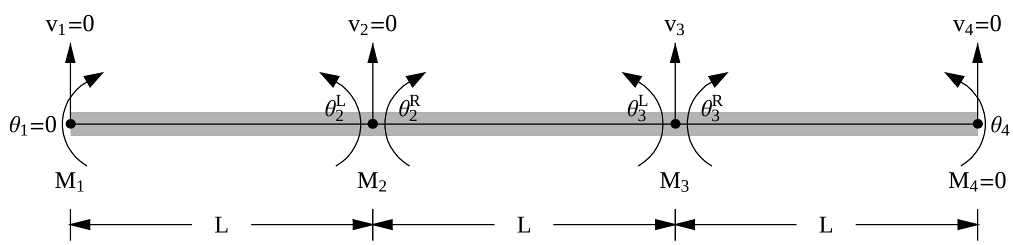
$\{M_2=-360.,\ M_3=840.,\ R_1=-1.5,\ R_2=11.5,\ v_3=-0.178499,\ \theta_1=-0.000811359\}$

4.9



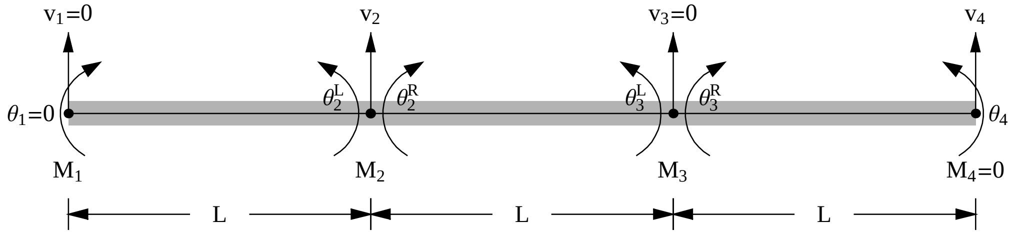
$\{\{M_{2.} \to -1636.36,\ M_{3.} \to 2181.82,\ M_{1.} \to 818.182,$
$R_{1.} \to -20.4545,\ R_{2.} \to 52.2727,\ R_{4.} \to 18.1818,\ v_{3.} \to -0.834225,\ \theta_{4.} \to 0.0112299\}\}$

4.10



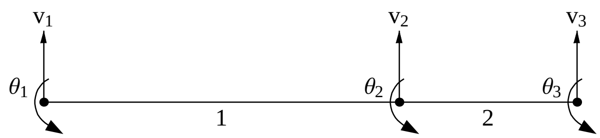
$\{\{M_{2.} \to -660.,\ M_{3.} \to 870.,\ M_{1.} \to 330.,\ R_{1.} \to -8.25,\ R_{2.} \to 31.,\ R_{4.} \to 17.25,\ v_{3.} \to -0.335294,\ \theta_{4.} \to 0.00452941\}\}$

4.11



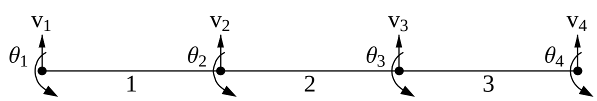
$\{M_3 = -30.,\ M_1 = -97.5,\ M_2 = 86.25,\ R_1 = 5.5625,\ R_3 = 4.9375,\ v_2 = -0.006,\ v_4 = 0.00413793,\ \theta_4 = 0.0000413793\}$

4.12



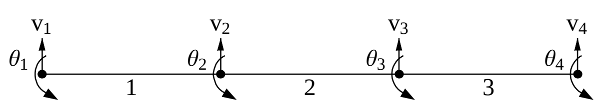
	v	θ
1	0	0.00134686
2	0	−0.00135497
3	−0.0842189	0

4.13



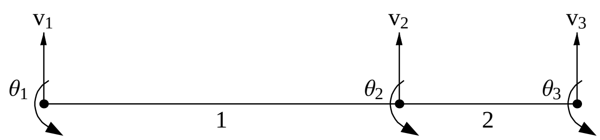
	v	θ
1	0	0
2	0	−0.000229926
3	−0.233016	−0.00344865
4	0	0.00724315

4.14



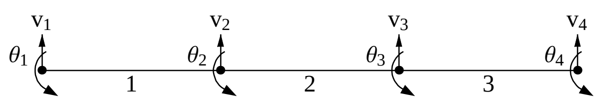
	v	θ
1	0	0
2	0	-0.0000919704
3	-0.0932063	-0.00137946
4	0	0.00289726

4.15



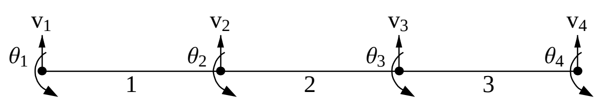
$\{\theta_1 = 0.000806982,\ \theta_2 = -0.00162491,\ v_3 = -0.181551\}$

4.16



$\{\theta_2 = -0.00487222,\ v_3 = -0.846617,\ \theta_3 = -0.00157443,\ \theta_4 = 0.0112895\}$

4.17



$\{\theta_2 = -0.00259852,\ v_3 = -0.403294,\ \theta_3 = -0.000839698,\ \theta_4 = 0.00602108\}$

CHAPTER FIVE

Multifield Formulations for Analysis of Elastic Solids

5.1

$$\begin{pmatrix} 0 & 0 & 0 & -A & \frac{2A}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{4A}{3} & 0 & 0 \\ 0 & 0 & 0 & A & \frac{2A}{3} & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & L & 0 \\ -\frac{2}{3} & \frac{4}{3} & -\frac{2}{3} & 0 & 0 & 0 & \frac{L}{3} \\ 0 & 0 & 0 & -L & 0 & L\,\mathrm{E} & 0 \\ 0 & 0 & 0 & 0 & -\frac{L}{3} & 0 & \frac{L\,\mathrm{E}}{3} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{Lq}{6} + F_L \\ \frac{2Lq}{3} \\ \frac{Lq}{6} + F_R \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

5.2

All are work modes.

Eigenvalues:{1.42857, 0.769231, 0.769231, 0.0925926, 0.0925926, 0, 0, 0}

5.3

$$\boldsymbol{P}^{\mathrm{T}} = \begin{pmatrix} 1 & s & t & 0 & 0 \\ 0 & 1 & s & t & 0 \\ 0 & 0 & 1 & s & t \end{pmatrix}$$

All are work modes.

Eigenvalues:{1.2171, 0.715005, 0.650081, 0.121892, 0.085197, 0, 0, 0}

5.4

All are work modes.

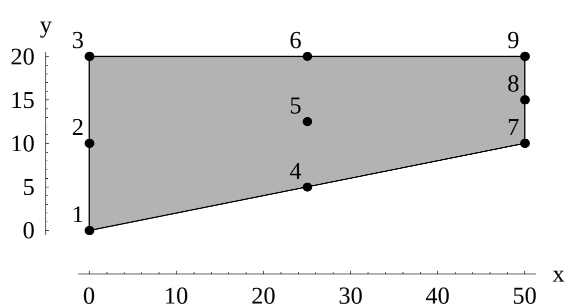
Eigenvalues:{0.769231, 0.769231, 0.434783, 0.0925926, 0.0925926, 0, 0, 0}

5.5

Nodal solution

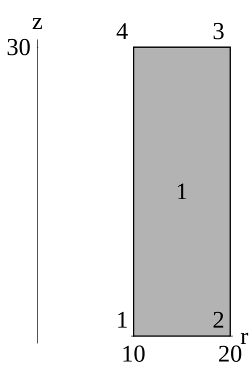
	x	y	u	v
1	0	0	0	0
2	50	10	0.202609	−0.0323944
3	50	20	0.205101	0.0523294
4	0	20	0	0

5.6



	u	v
1	0	0
2	0	0
3	0	0
4	0.101861	−0.0284674
5	0.104682	0.00699846
6	0.101556	0.0428647
7	0.191474	0.0198994
8	0.180643	0.017815
9	0.190381	0.0143666

5.7



	u	w
1	0	0
2	0	0
3	0.0000972836	−0.0000706801
4	0.000150769	-9.80215×10^{-6}

5.8 Programming Project

CHAPTER SIX

Plates and Shells

6.1

Maximum deflection = 0.0560523

6.2

	x−coord	y−coord	w	θ_x	θ_y
1	0	0	0	0	0
2	5	0	0	0.0738563	0
3	5	$\frac{5}{2}$	0.114294	0	0
4	0	$\frac{5}{2}$	0	0	−0.0623135

6.3

	x−coord	y−coord	w	θ_x	θ_y
1	0	0	0	0	0
2	10	5	0	−0.10891	0
3	0	5	0	0	0

6.4

Nodal solution

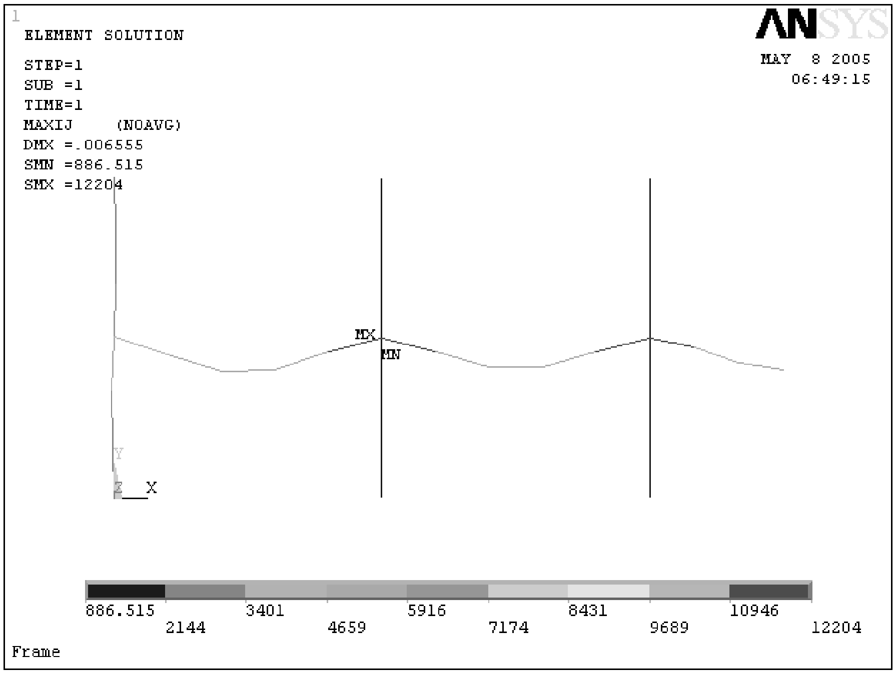
	x−coord	y−coord	w	θ_x	θ_y
1	0	0	0	0	0
2	5	0	0	0.0539835	0
3	5	$\frac{5}{2}$	0.0682754	0	0
4	0	$\frac{5}{2}$	0	0	−0.0270237

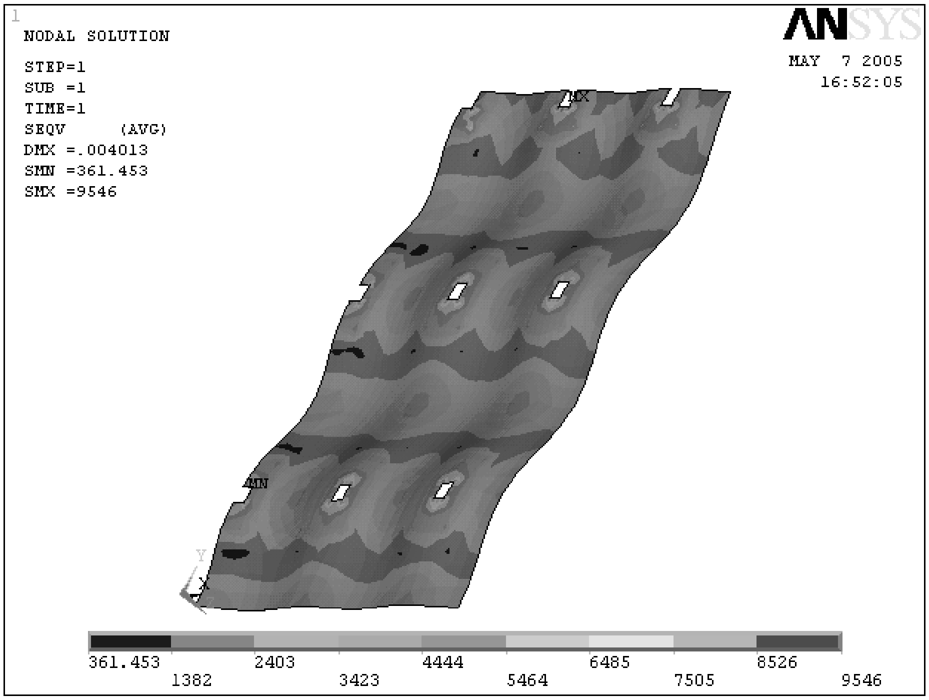
6.5

Prepare a short report on different plate and shell elements available in Ansys or any other commercial finite element software available. Look through the documentation and determine the theory on which each element is based.

6.6 Computational project

Input data file


1
ELEMENT SOLUTION
STEP=1
SUB =1
TIME=1
MAXIJ (NOAVG)
DMX =.006555
SMN =886.515
SMX =12204
ANSYS
MAY 8 2005
06:49:15
MX
MN
Y
Z X
886.515
2144
3401
4659
5916
7174
8431
9689
10946
12204
Frame


1
NODAL SOLUTION
STEP=1
SUB =1
TIME=1
SEQV (AVG)
DMX =.004013
SMN =361.453
SMX =9546
ANSYS
MAY 7 2005
16:52:05
MX
MN
361.453
1382
2403
3423
4444
5464
6485
7505
8526
9546

CHAPTER SEVEN

Introduction to Nonlinear Problems

7.1

Quadratic solution $u(x) = 0.197847\,x^2 + 0.312499\,x + \frac{1}{27}$

Cubic solution $u(x) = 0.156708\,x^3 - 0.276866\,x^2 + 0.651649\,x + \frac{1}{27}$

7.2

Quadratic solution $u(x) = 0.36118\,x^2 - 1.20384\,x + 1$

Cubic solution $u(x) = -0.297702\,x^3 + 1.18883\,x^2 - 1.66833\,x + 1$

7.3

Quadratic solution $u(x) = 0.846013\,x^2 - 1.84489\,x + 1.49888$

Cubic solution $u(x) = 0.599143\,x^3 - 1.88008\,x^2 + 2.1394\,x - 0.358459$

7.4

Quadratic solution $u(x) = 1.81956\,x^2 - 5.62534\,x + 4.30579$

Cubic solution $u(x) = 1.04138\,x^3 - 2.85862\,x^2 + 1.11953\,x + 1.19771$

7.5

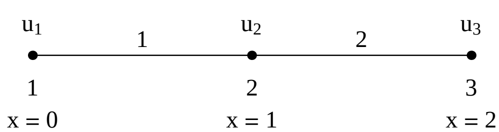
Quadratic solution $u(x) = 1.81962\,x^2 - 5.92742\,x + 4.6078$

Cubic solution $u(x) = 1.09736\,x^3 - 3.11873\,x^2 + 1.20061\,x + 1.32076$

7.6

$$\boldsymbol{k} = \begin{pmatrix} \frac{4\,\ell^2\,u_1^3 + 3\,\ell^2\,u_2\,u_1^2 + 2\left(\ell^2\,u_2^2 + 10\right)u_1 + u_2\left(\ell^2\,u_2^2 - 20\right)}{20\,\ell} \\ \frac{\ell^2\,u_1^3 + 2\,\ell^2\,u_2\,u_1^2 + \left(3\,\ell^2\,u_2^2 - 20\right)u_1 + 4\,u_2\left(\ell^2\,u_2^2 + 5\right)}{20\,\ell} \end{pmatrix}$$

$$\boldsymbol{r}_E = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

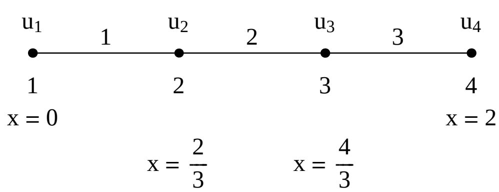


Total nodal values, $\boldsymbol{d}^{(4)} = \left\{\frac{1}{27},\ 0.539496,\ 1.42052\right\}$

7.7

$$\boldsymbol{k} = \begin{pmatrix} \frac{1}{5}\left(12\,\ell\,u_1^3 + 9\,\ell\,u_2\,u_1^2 + \left(6\,\ell\,u_2^2 + \frac{5}{\ell}\right)u_1 + 3\,\ell\,u_2^3 - \frac{5\,u_2}{\ell}\right) \\ \frac{3\,\ell\,u_1^3}{5} + \frac{6}{5}\,\ell\,u_2\,u_1^2 + \left(\frac{9\,\ell\,u_2^2}{5} - \frac{1}{\ell}\right)u_1 + \frac{12\,\ell\,u_2^3}{5} + \frac{u_2}{\ell} \end{pmatrix}$$

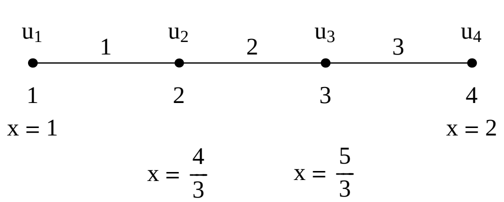
$$\boldsymbol{r}_E = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



Total nodal values, $\boldsymbol{d}^{(6)} = \left\{1,\ 0.294696,\ 0.152172,\ \frac{1}{27}\right\}$

7.8

$$
\begin{aligned}
\boldsymbol{k} = \Big\{ & \frac{1}{360\,\ell^3} \\
& \big(45\,\ell^2\,\big(3\,\ell\,(14\,\mathrm{x1}+15\,\ell)+6\,\mathrm{x1}\,(7\,\mathrm{x1}+8\,\ell)\,\mathrm{Log}[\mathrm{x1}]-6\,\mathrm{x1}\,(3\,\mathrm{x1}+4\,\ell)\,\mathrm{Log}[\mathrm{x1}]^2+4\,\mathrm{x1}\,(\mathrm{x1}+2\,\ell)\,\mathrm{Log}[\mathrm{x1}]^3-42 \\
& \quad (\mathrm{x1}+\ell)^2\,\mathrm{Log}[\mathrm{x1}+\ell]+18\,(\mathrm{x1}+\ell)^2\,\mathrm{Log}[\mathrm{x1}+\ell]^2-4\,(\mathrm{x1}+\ell)^2\,\mathrm{Log}[\mathrm{x1}+\ell]^3\big)+360\,\ell^2\,\mathrm{u}_1+576\,\ell^4\,\mathrm{u}_1^3- \\
& \quad 360\,\ell^2\,\mathrm{u}_2+20\,\ell\,\big(\ell\,\big(-132\,\mathrm{x1}^2-132\,\mathrm{x1}\,\ell+\ell\,(9+19\,\ell)\big)-6\,\mathrm{x1}^2\,(22\,\mathrm{x1}+27\,\ell)\,\mathrm{Log}[\mathrm{x1}]+18\,\mathrm{x1}^2\,(2\,\mathrm{x1}+3\,\ell) \\
& \quad \mathrm{Log}[\mathrm{x1}]^2+6\,(22\,\mathrm{x1}-5\,\ell)\,(\mathrm{x1}+\ell)^2\,\mathrm{Log}[\mathrm{x1}+\ell]-18\,(2\,\mathrm{x1}-\ell)\,(\mathrm{x1}+\ell)^2\,\mathrm{Log}[\mathrm{x1}+\ell]^2\big)\,\mathrm{u}_2+30 \\
& \quad \big(\ell\,\big(36\,\mathrm{x1}^3+30\,\mathrm{x1}^2\,\ell-12\,\mathrm{x1}\,\ell^2+7\,\ell^3\big)+12\,\mathrm{x1}^3\,(3\,\mathrm{x1}+4\,\ell)\,\mathrm{Log}[\mathrm{x1}]-12\,\big(3\,\mathrm{x1}^4+4\,\mathrm{x1}^3\,\ell+\ell^4\big)\,\mathrm{Log}[\mathrm{x1}+\ell]\big) \\
& \quad \mathrm{u}_2^2+144\,\ell^4\,\mathrm{u}_2^3+18\,\mathrm{u}_1^2\,\big(5\,\big(\ell\,\big(12\,\mathrm{x1}^3+42\,\mathrm{x1}^2\,\ell+52\,\mathrm{x1}\,\ell^2+25\,\ell^3\big)+ \\
& \quad 12\,\mathrm{x1}\,\big(\mathrm{x1}^3+4\,\mathrm{x1}^2\,\ell+6\,\mathrm{x1}\,\ell^2+4\,\ell^3\big)\,\mathrm{Log}[\mathrm{x1}]-12\,(\mathrm{x1}+\ell)^4\,\mathrm{Log}[\mathrm{x1}+\ell]\big)+24\,\ell^4\,\mathrm{u}_2\big)- \\
& \quad 4\,\mathrm{u}_1\,\big(-5\,\ell\,\big(\ell\,\big(132\,\mathrm{x1}^2+294\,\mathrm{x1}\,\ell+\ell\,(-9+170\,\ell)\big)+12\,\mathrm{x1}\,\big(11\,\mathrm{x1}^2+27\,\mathrm{x1}\,\ell+18\,\ell^2\big)\,\mathrm{Log}[\mathrm{x1}]- \\
& \quad 36\,\mathrm{x1}\,\big(\mathrm{x1}^2+3\,\mathrm{x1}\,\ell+3\,\ell^2\big)\,\mathrm{Log}[\mathrm{x1}]^2-132\,(\mathrm{x1}+\ell)^3\,\mathrm{Log}[\mathrm{x1}+\ell]+36\,(\mathrm{x1}+\ell)^3\,\mathrm{Log}[\mathrm{x1}+\ell]^2\big)+ \\
& \quad 15\,\big(\ell\,\big(36\,\mathrm{x1}^3+78\,\mathrm{x1}^2\,\ell+36\,\mathrm{x1}\,\ell^2-13\,\ell^3\big)+12\,\mathrm{x1}^2\,\big(3\,\mathrm{x1}^2+8\,\mathrm{x1}\,\ell+6\,\ell^2\big)\,\mathrm{Log}[\mathrm{x1}]- \\
& \quad 12\,(3\,\mathrm{x1}-\ell)\,(\mathrm{x1}+\ell)^3\,\mathrm{Log}[\mathrm{x1}+\ell]\big)\,\mathrm{u}_2-72\,\ell^4\,\mathrm{u}_2^2)\big), \\
& \frac{1}{360\,\ell^3}\big(-45\,\ell^2\,\big(3\,(14\,\mathrm{x1}-\ell)\,\ell+42\,\mathrm{x1}^2\,\mathrm{Log}[\mathrm{x1}]-18\,\mathrm{x1}^2\,\mathrm{Log}[\mathrm{x1}]^2+4\,\mathrm{x1}^2\,\mathrm{Log}[\mathrm{x1}]^3- \\
& \quad 6\,\big(7\,\mathrm{x1}^2+6\,\mathrm{x1}\,\ell-\ell^2\big)\,\mathrm{Log}[\mathrm{x1}+\ell]+6\,\big(3\,\mathrm{x1}^2+2\,\mathrm{x1}\,\ell-\ell^2\big)\,\mathrm{Log}[\mathrm{x1}+\ell]^2-4\,\big(\mathrm{x1}^2-\ell^2\big)\,\mathrm{Log}[\mathrm{x1}+\ell]^3\big)+ \\
& \quad 144\,\ell^4\,\mathrm{u}_1^3+20\,\ell\,\big(\ell\,\big(18+132\,\mathrm{x1}^2+9\,\ell-30\,\mathrm{x1}\,\ell+8\,\ell^2\big)+132\,\mathrm{x1}^3\,\mathrm{Log}[\mathrm{x1}]-36\,\mathrm{x1}^3\,\mathrm{Log}[\mathrm{x1}]^2- \\
& \quad 12\,\big(11\,\mathrm{x1}^3+6\,\mathrm{x1}^2\,\ell-3\,\mathrm{x1}\,\ell^2+2\,\ell^3\big)\,\mathrm{Log}[\mathrm{x1}+\ell]+36\,\big(\mathrm{x1}^3+\ell^3\big)\,\mathrm{Log}[\mathrm{x1}+\ell]^2\big)\,\mathrm{u}_2- \\
& \quad 90\,\big(\ell\,\big(12\,\mathrm{x1}^3-6\,\mathrm{x1}^2\,\ell+4\,\mathrm{x1}\,\ell^2-3\,\ell^3\big)+12\,\mathrm{x1}^4\,\mathrm{Log}[\mathrm{x1}]-12\,\big(\mathrm{x1}^4-\ell^4\big)\,\mathrm{Log}[\mathrm{x1}+\ell]\big)\,\mathrm{u}_2^2+ \\
& \quad 576\,\ell^4\,\mathrm{u}_2^3-6\,\mathrm{u}_1^2\,\big(5\,\big(\ell\,\big(36\,\mathrm{x1}^3+78\,\mathrm{x1}^2\,\ell+36\,\mathrm{x1}\,\ell^2-13\,\ell^3\big)+ \\
& \quad 12\,\mathrm{x1}^2\,\big(3\,\mathrm{x1}^2+8\,\mathrm{x1}\,\ell+6\,\ell^2\big)\,\mathrm{Log}[\mathrm{x1}]-12\,(3\,\mathrm{x1}-\ell)\,(\mathrm{x1}+\ell)^3\,\mathrm{Log}[\mathrm{x1}+\ell]\big)-48\,\ell^4\,\mathrm{u}_2\big)+ \\
& \quad 4\,\mathrm{u}_1\,\big(5\,\ell\,\big(\ell\,\big(-18-132\,\mathrm{x1}^2-9\,\ell-132\,\mathrm{x1}\,\ell+19\,\ell^2\big)-6\,\mathrm{x1}^2\,(22\,\mathrm{x1}+27\,\ell)\,\mathrm{Log}[\mathrm{x1}]+18\,\mathrm{x1}^2\,(2\,\mathrm{x1}+3\,\ell) \\
& \quad \mathrm{Log}[\mathrm{x1}]^2+6\,(22\,\mathrm{x1}-5\,\ell)\,(\mathrm{x1}+\ell)^2\,\mathrm{Log}[\mathrm{x1}+\ell]-18\,(2\,\mathrm{x1}-\ell)\,(\mathrm{x1}+\ell)^2\,\mathrm{Log}[\mathrm{x1}+\ell]^2\big)+ \\
& \quad 15\,\big(\ell\,\big(36\,\mathrm{x1}^3+30\,\mathrm{x1}^2\,\ell-12\,\mathrm{x1}\,\ell^2+7\,\ell^3\big)+12\,\mathrm{x1}^3\,(3\,\mathrm{x1}+4\,\ell)\,\mathrm{Log}[\mathrm{x1}]- \\
& \quad 12\,\big(3\,\mathrm{x1}^4+4\,\mathrm{x1}^3\,\ell+\ell^4\big)\,\mathrm{Log}[\mathrm{x1}+\ell]\big)\,\mathrm{u}_2+108\,\ell^4\,\mathrm{u}_2^2)\big)\Big\}; \qquad \boldsymbol{r}_{\mathrm{E}} = \{0,\ 0\}
\end{aligned}
$$

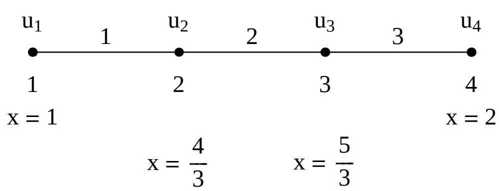


Total nodal values, $\boldsymbol{d}^{(5)} = \left\{\frac{1}{2},\ 0.565264,\ 0.75045,\ \frac{1}{2} + \log(2)\right\}$

7.9

$$\boldsymbol{k} = \begin{pmatrix} -\frac{(u_1-u_2)(2\ell u_1+\ell u_2+6)}{6\ell} \\ -\frac{(u_1-u_2)(\ell u_1+2\ell u_2-6)}{6\ell} \end{pmatrix}$$

$$\boldsymbol{r}_E = \begin{pmatrix} \frac{\ell^4}{20} + \frac{\mathrm{x1}\,\ell^3}{4} + \frac{\mathrm{x1}^2\,\ell^2}{2} + \frac{\mathrm{x1}^3\,\ell}{2} \\ \frac{\ell^4}{5} + \frac{3\,\mathrm{x1}\,\ell^3}{4} + \mathrm{x1}^2\,\ell^2 + \frac{\mathrm{x1}^3\,\ell}{2} \end{pmatrix}$$

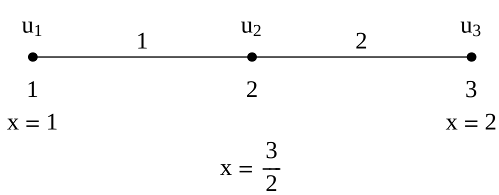


Total nodal values, $\boldsymbol{d}^{(3)} = \left\{\frac{1}{2},\ 0.0793862,\ -0.0534516,\ \frac{1}{3}\right\}$

7.10

$$\boldsymbol{k} = \begin{pmatrix} -\frac{(u_1-u_2)(2\ell u_1+\ell u_2+6)}{6\ell} \\ -\frac{(u_1-u_2)(\ell u_1+2\ell u_2-6)}{6\ell} \end{pmatrix}$$

$$\boldsymbol{r}_E = \begin{pmatrix} \frac{\ell^4}{20} + \frac{\mathrm{x1}\,\ell^3}{4} + \frac{\mathrm{x1}^2\,\ell^2}{2} + \frac{\mathrm{x1}^3\,\ell}{2} \\ \frac{\ell^4}{5} + \frac{3\,\mathrm{x1}\,\ell^3}{4} + \mathrm{x1}^2\,\ell^2 + \frac{\mathrm{x1}^3\,\ell}{2} \end{pmatrix}$$

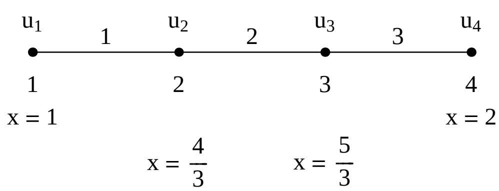


Total nodal values, $\boldsymbol{d}^{(3)} = \left\{\frac{1}{2}, -0.185885, 0.032486\right\}$

7.11

$$\boldsymbol{k} = \begin{pmatrix} -\frac{(u_1-u_2)(2\ell u_1+\ell u_2+6)}{6\ell} \\ -\frac{(u_1-u_2)(\ell u_1+2\ell u_2-6)}{6\ell} \end{pmatrix}$$

$$\boldsymbol{r}_E = \lambda \begin{pmatrix} \frac{\ell^4}{20} + \frac{\text{x1}\,\ell^3}{4} + \frac{\text{x1}^2\,\ell^2}{2} + \frac{\text{x1}^3\,\ell}{2} \\ \frac{\ell^4}{5} + \frac{3\,\text{x1}\,\ell^3}{4} + \text{x1}^2\,\ell^2 + \frac{\text{x1}^3\,\ell}{2} \end{pmatrix}$$

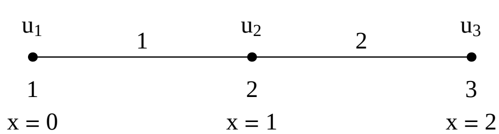


Nodal values, $\boldsymbol{d} = \left\{\frac{1}{2}, 0.0792459, -0.0534103, \frac{1}{3}\right\}$

7.12

$$\boldsymbol{k} = \begin{pmatrix} -\frac{(u_1-u_2)(2\ell u_1+\ell u_2-6)}{6\ell} \\ -\frac{(u_1-u_2)(\ell u_1+2\ell u_2+6)}{6\ell} \end{pmatrix}$$

$$\boldsymbol{r}_E = \lambda \begin{pmatrix} \frac{\ell^4}{20} + \frac{\text{x1}\,\ell^3}{4} + \frac{\text{x1}^2\,\ell^2}{2} + \frac{\text{x1}^3\,\ell}{2} \\ \frac{\ell^4}{5} + \frac{3\,\text{x1}\,\ell^3}{4} + \text{x1}^2\,\ell^2 + \frac{\text{x1}^3\,\ell}{2} \end{pmatrix}$$



Nodal values, d = $\left\{\frac{1}{2}, 1.12385, 0.944394\right\}$

7.13

$$\boldsymbol{k} = \begin{pmatrix} \frac{5 P \ell^2 \epsilon \sigma T_1^4 + 4 P \ell^2 \epsilon \sigma T_2 T_1^3 + 3 P \ell^2 \epsilon \sigma T_2^2 T_1^2 + 2 \left(P \ell^2 \epsilon \sigma T_2^3 + 15 A k\right) T_1 + P \ell^2 \epsilon \sigma T_2^4 - 30 A k T_2}{30 \ell} \\ \frac{P \ell^2 \epsilon \sigma T_1^4 + 2 P \ell^2 \epsilon \sigma T_2 T_1^3 + 3 P \ell^2 \epsilon \sigma T_2^2 T_1^2 + \left(4 P \ell^2 \epsilon \sigma T_2^3 - 30 A k\right) T_1 + 5 T_2 \left(P \ell^2 \epsilon \sigma T_2^3 + 6 A k\right)}{30 \ell} \end{pmatrix};$$

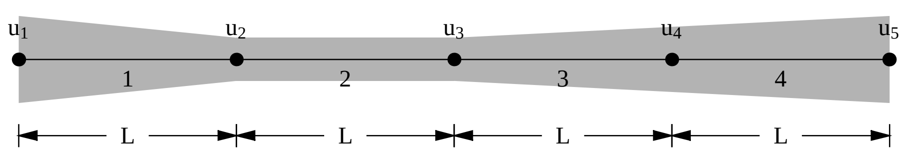
$$\boldsymbol{r}_E = \begin{pmatrix} \frac{1}{2} P \ell \epsilon \sigma T_\infty^4 + \frac{q \ell}{2} \\ \frac{1}{2} P \ell \epsilon \sigma T_\infty^4 + \frac{q \ell}{2} \end{pmatrix}$$

Total nodal values, $\boldsymbol{d}^{(5)}$ = {500, 339.574, 304.63}

CHAPTER EIGHT

Material Nonlinearity

8.1



Average area for each element

{1500, 600, 1050, 1950}

Total nodal values

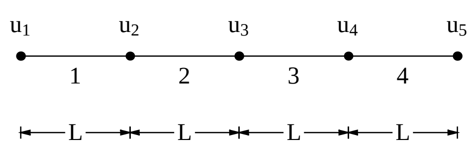
	u
1	0
2	0.108849
3	−1.24394
4	−0.435378
5	0

8.2

	u
1	0
2	0.179874
3	−1.65121
4	−0.577925
5	0

8.3

$$\boldsymbol{r}_q = \frac{1}{6}\,\rho\,A\,\omega^2 \begin{pmatrix} -2\,x_1^2 + x_1\,x_2 + x_2^2 \\ -\,x_1^2 - x_1\,x_2 + 2\,x_2^2 \end{pmatrix}$$



	u
1	0
2	0.0477439
3	0.0770413
4	0.0938676
5	0.100181

8.4

	u
1	0
2	0.0664385
3	0.0899684
4	0.106611
5	0.112924

8.5

	u	v
1	8.1821	−12.0004
2	0	0
3	0	0
4	0	0

8.6

	u	v
1	0	0
2	0.155035	0.297951
3	0	0
4	0	0

8.7

	u	v
1	0	0
2	0.154994	0.256663
3	0	0
4	0	0

8.8

Final $\beta = 0.444443 \Longrightarrow \sigma_C = \{200., -0.000204798, 0., 0., 0., 0.\}$ and $F = -0.000102399$

8.9 Programming Project

CHAPTER NINE

Geometric Nonlinearity

9.1

The second PK and Cauchy stress tensors

$$\begin{pmatrix} 119.082 & 0 \\ 0 & -268.645 \end{pmatrix}$$

$$\begin{pmatrix} 24.7298 & 166.59 \\ 166.59 & -167.631 \end{pmatrix}$$

9.2

The second PK and Cauchy stress tensors

Large strains

$$\begin{pmatrix} -46.5812 & 138.242 \\ 138.242 & 178.062 \end{pmatrix}$$

$$\begin{pmatrix} 111.676 & 232.978 \\ 232.978 & 228.165 \end{pmatrix}$$

Small strains. Displacement 1000 times less.

$$\begin{pmatrix} 0.0251358 & 0.227147 \\ 0.227147 & 0.252424 \end{pmatrix}$$

$$\begin{pmatrix} 0.0253574 & 0.227279 \\ 0.227279 & 0.252494 \end{pmatrix}$$

With the Kirchhoff material assumption and plane *stress* constitutive matrix we get the following stresses.

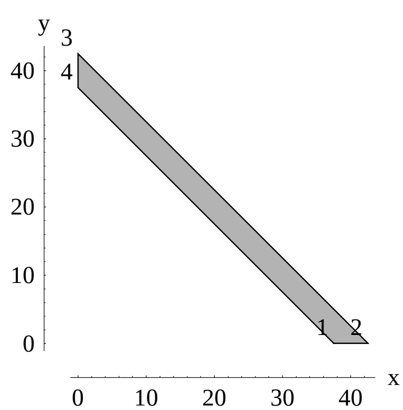
Large strains

{41.0354, 410.354, 227.273}

Small strains. Displacement 1000 times less.

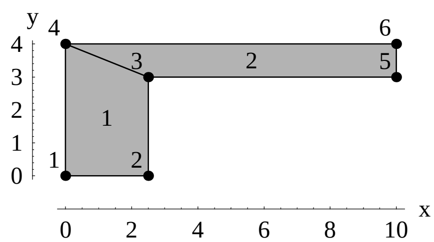
{0.0252683, 0.252683, 0.227273}

9.3



	u	v
1	0	0
2	0	0
3	0	−22.9665
4	0	−23.3059

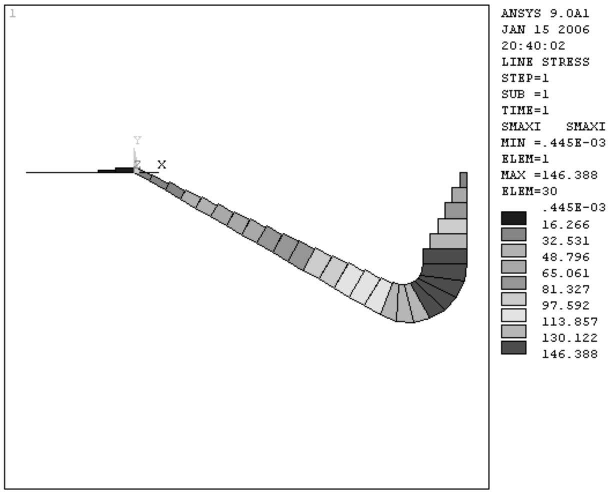
9.4



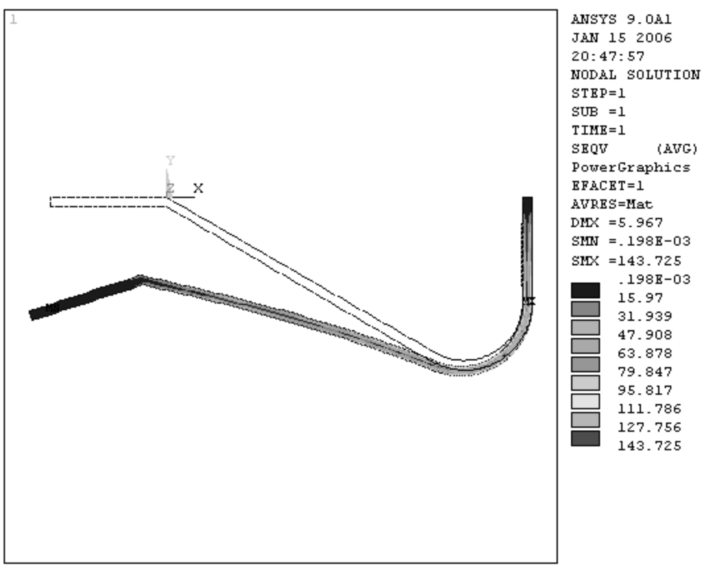
	u	v
1	0	0
2	0	0
3	0.00128751	−0.00103326
4	0.00193114	0.00109351
5	0.00151194	−0.00774094
6	0.00246313	−0.00773134

9.5

Maximum stress using beam model



vonMises stresses with a plane stress model



9.6

Buckling load (kN).

153600.

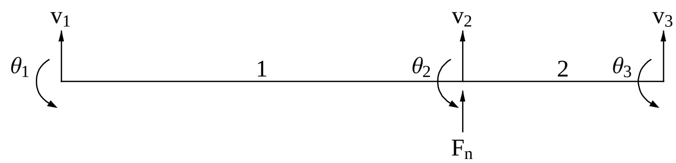
CHAPTER TEN

Contact Problems

10.1

$\{\{v_2 \to -2.73965,\ \theta_2 \to -0.0273965,\ v_4 \to -0.739552,\ \theta_4 \to -0.00739552,\ F_n \to 1.05181\}\}$

10.2



$\{\{\theta_1 \to -0.000870984,\ v_2 \to -3.75,\ \theta_2 \to -0.00107053,\ \theta_3 \to 0.00786584,\ F_n \to 15248.4\}\}$

10.3

Square of distance between node 5 and the target surface is

$(0.25 - 0.75\,a)^2 + 0.25$

At the minimum point

$\{\{a \to 0.333333\}\}$

The gap (signed distance) is

0.5

10.4

Square of distance between node 5 and the target surface is

$1250.\,a^2 + \left(14.6447\,a^2 + 20.\right)^2$

At the minimum point

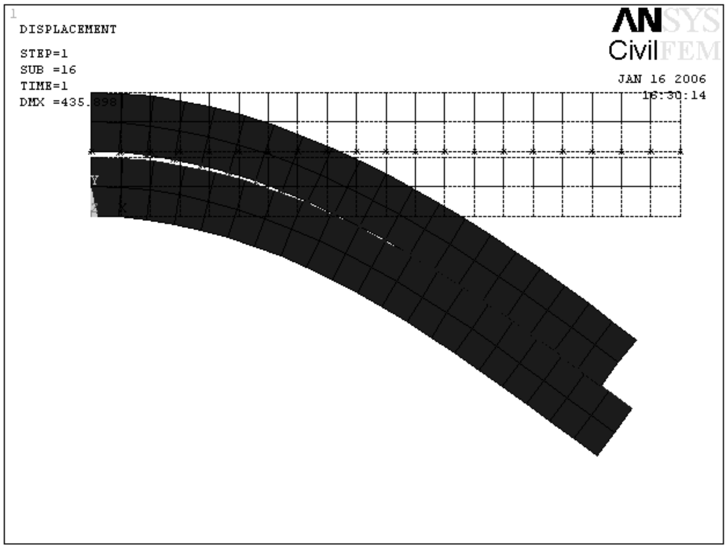
$\{\{a \to 0.\},\ \{a \to 0. - 2.06879\,i\},\ \{a \to 0. + 2.06879\,i\}\}$

The gap (signed distance) is

20.

10.5

Deformed shape



Von Mises Stresses