Homework II Solutions
of
Advanced Mechanical Vibrations class.

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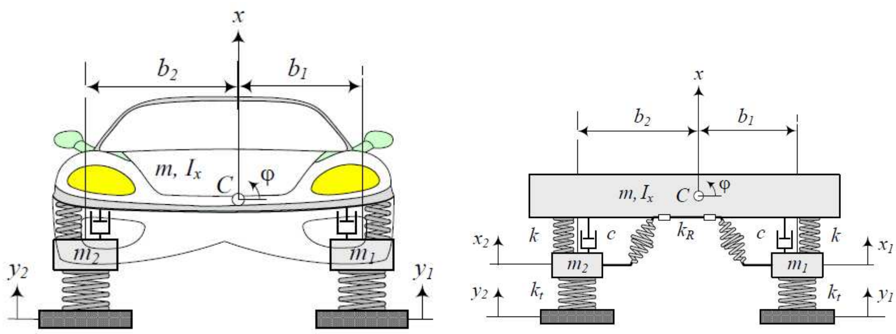
Abstract

Includes the solutions for the first homework problems of *Advanced Mechanical Engineering* class given at Bursa Technical University, in fall semester 2013.

Written by LaTeX using TeXstudio...

Problem 1

Half-car model of a vehicle is given in the fig. 1. The physical properties are as follows: $m = 420kg$, $m1 = m2 = 53kg$, $I_x = 820kgm^2$, $b_1 = 0.7m$, $b_2 = 0.75m$, $k = 10000Nm^{-1}$, $k_t = 200000Nm^{-1}$, $kR = 25000Nm^{-1}$, $k_4 = 1972900Nm^{-1}$, $c = 200Nsm^{-1}$. The vehicle is subject to base motions y_1 and y_2.



(a) whole car body (b) equivalent model

Figure 1: problem 1

Mass, damping and stiffness matrices of the car model are

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & I_x & 0 & 0 \\ 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 2c & cb_1 - cb_2 & -c & -c \\ cb_1 - cb_2 & cb_1^2 + cb_2^2 & -cb_1 & cb_2 \\ -c & -cb_1 & c & 0 \\ -c & cb_2 & 0 & c \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 2k & kb_1 - kb_2 & -k & -k \\ kb_1 - kb_2 & kb_1^2 + kb_2^2 & -kb_1 & kb_2 \\ -k & -kb_1 & k + k_t & 0 \\ -k & kb_2 & 0 & k + k_t \end{bmatrix}$$

Question 1

1. Find the undamped natural frequencies and vibration modes by
   (a) Solving characteristic polynomial.
   (b) Matrix iteration method.

2. Plot the vibration modes, comment them and make conclusions.
3. Compute modal mass and stiffness values.
4. Show that vibration modes are orthogonal to each other with respect to mass and stiffness matrices.

Solution 1

Solving characteristic polynomial.

Undamped system characteristic polynominal is found from the equation

$$\det\left(-\lambda^2\mathbf{M}+\mathbf{K}\right)=0$$

which is found

$$9.67\cdot 10^{8}\,\lambda^{8}-7.75\cdot 10^{12}\,\lambda^{6}+1.59\cdot 10^{16}\,\lambda^{4}-1.35\cdot 10^{18}\,\lambda^{2}+2.94\cdot 10^{19}=0$$

where the solution of natural frequencies are found as

$$\boldsymbol{\Lambda}=\begin{bmatrix}62.964\\62.951\\6.749\\6.517\end{bmatrix}\text{radian/s}$$

Ith mode shape of the system can be found by substituting λ_i on the equation

$$\left(-\lambda^2\mathbf{M}+\mathbf{K}\right)\mathbf{u}=\mathbf{0}$$

that is found

$$\begin{aligned}
500.0\,u_2+10000.0\,u_3+10000.0\,u_4+u_1\left(420.0\,\lambda_i^2-20000.0\right)&=0\\
500.0\,u_1+7000.0\,u_3-7500.0\,u_4+u_2\left(820.0\,\lambda_i^2-35525.0\right)&=0\\
10000.0\,u_1+7000.0\,u_2+u_3\left(53.0\,\lambda_i^2-210000.0\right)&=0\\
10000.0\,u_1-7500.0\,u_2+u_4\left(53.0\,\lambda_i^2-210000.0\right)&=0
\end{aligned}$$

with assuming $u_1=1$ and then normalizing the mode shapes, mode shapes are are found as

for λ_1

$$\mathbf{u}_1=\begin{bmatrix}0.00859\\0.00011\\0.97766\\0.36874\end{bmatrix}$$

for λ_2

$$\mathbf{u}_2=\begin{bmatrix}-0.00015\\0.0031884\\-0.19903\\0.92811\end{bmatrix}$$

for λ_3

$$\mathbf{u}_3 = \begin{bmatrix} -0.69795 \\ -0.71612 \\ 0.04038 \\ 0.04902 \end{bmatrix}$$

for λ_4

$$\mathbf{u}_4 = \begin{bmatrix} -0.71609 \\ 0.6979 \\ 0.05428 \\ -0.01575 \end{bmatrix}$$

Plot the vibration modes, comment them and make conclusions.

Plot of vibration modes is given in fig. 2. What can be seen from the figure is; the dominant term in the first and second mode are displacements of the tires. These modes have two highest natural frequencies. In the third and fourth mode the dominant variables are displacement and the rotation of the car body. These modes have the lowest two natural frequencies.

It is concluded that, as expected, low frequency modes exist just because of the contribution of the body of the car and high frequency modes exist because of the contribution of the tires.

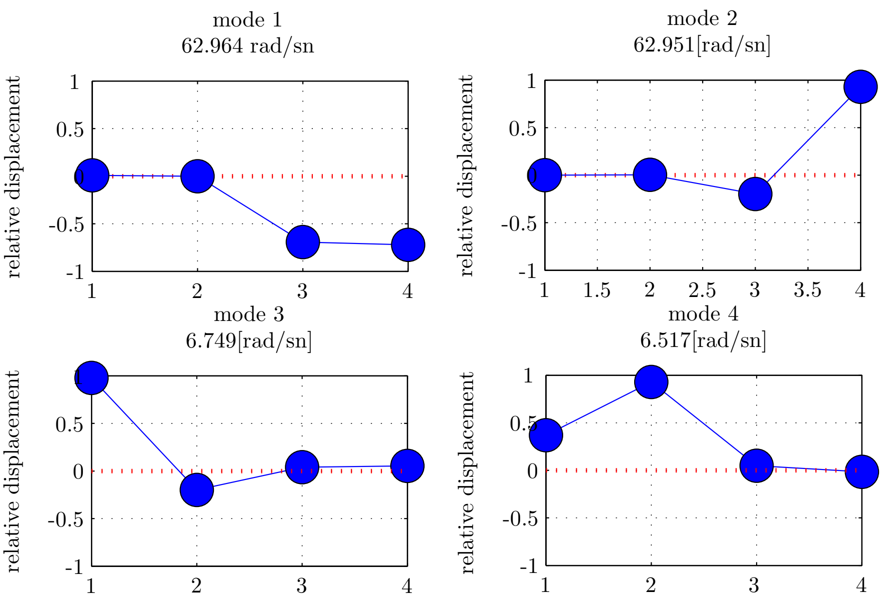


Figure 2: Vibration modes

Compute modal mass and stiffness matrices.

Modal mass and stiffness matrices are found from the relations

$$\mathbf{m} = \mathbf{\Phi}^T \mathbf{M} \mathbf{\Phi}$$
$$\mathbf{k} = \mathbf{\Phi}^T \mathbf{K} \mathbf{\Phi}$$

where $\mathbf{\Phi}$ is the eigen modes matrix. If we calculate them it is found that

$$\mathbf{m} = \begin{pmatrix} 53.0 & 0 & 0 & 0 \\ 0 & 53.0 & 0 & 0 \\ 0 & 0 & 434.0 & 0 \\ 0 & 0 & 0 & 764.0 \end{pmatrix}$$

and

$$\mathbf{k} = \begin{pmatrix} 2.1 \cdot 10^5 & 0 & 0 & 0 \\ 0 & 2.1 \cdot 10^5 & 0 & 0 \\ 0 & 0 & 1.98 \cdot 10^4 & 0 \\ 0 & 0 & 0 & 3.24 \cdot 10^4 \end{pmatrix}$$

Show that vibration modes are orthogonal to each other with respect to mass and stiffness matrices.

Zero bidiagonal elements in modal and stiffness matrices mean that the modes are orthogonal to each other with respect to mass and stiffness matrices. It is already showed that they are equal to zero in the previous answer.

Question 2

Suppose that while moving with a speed $\mathrm{V} = 60km/h$ the right tire passes over a half sine wave-like bump on the road as shown below, where $\mathrm{a} = 30cm$ and $\mathrm{b} = 5cm$. Assuming zero initial conditions plot the 5 seconds responses of the coordinates x_1 and x_2.

Solution 2

This problem can be solved numerically integrating the equations of motion. Before we do that, applied force on the second tire should be calculated. The force on the tire depends on the displacement of the road. Therefore, displacement profile with respect to time is

$$\begin{aligned} \mathrm{T} &= 0.072\frac{\mathrm{a}}{\mathrm{V}} \quad \text{(seconds)} \\ y_2 &= \left\{ \begin{array}{ccc} \mathrm{b}\sin\left(\frac{2\pi}{\mathrm{T}}t\right) & , & 0 \leq t \leq \frac{\mathrm{T}}{2} \\ 0 & , & \frac{\mathrm{T}}{2} < t \end{array} \right. \end{aligned}$$

with the found displacement of the road, complete force vector of the model is found as

$$\mathrm{F} = \begin{bmatrix} 0 \\ 0 \\ y_1 k_t \\ y_2 k_t \end{bmatrix}$$

With this calculated road input force, a numerical solution to the equations of motion of the car is done by a matlab function. This function is given in listing 2 and it calculates required derivatives of the system by using order reducing method. The derivatives can be integrated by a numerical method, for example Runge-Kutta. For the integration of derivatives `Ode45` function of matlab is used in which integration is done with Runge-Kutta45 method. Integration is completed with the command in listing 1.

Listing 1: integration commands

```
options = odeset('MaxStep',1e-3)
[t y]=ode45(@odefunc,[0 5],[0 0 0 0 0 0 0 0],options);
```

Listing 2: odefunc.m

```
function ydot=odefunc(t,y)
global i y2val tval;
kt=200000;
a=30/100;
b=5/100;
V=60000/3600;
halfperiod=a/V;
y1=0;
y2=b*sin(pi*t/halfperiod)*(t<halfperiod);
i=i+1;
y2val(i)=y2*kt;
tval(i)=t;
F1=[0; 0 ; y1*kt;y2*kt];
M=[420 0 0 0;0 820 0 0;0 0 53 0;0 0 0 53;];
K=[20000 -500 -10000 -10000;-500 35525 -7000 7500;-10000 -7000 ...
    210000 0;-10000 7500 0 210000;];
C=[400 -10 -200 -200;-10 210.5 -140 150;-200 -140 200 0;-200 150 0 ...
    200;];
disp=y(1:4);
vel=y(5:8);
ydot(1:4)=vel';
ydot(5:8)=((inv(M))*(-C*vel-K*disp+F1))';
end
```

Using listing 1 with listing 2 the resulting time dependent behaviors of the displacements of the right and the left tires are given in fig. 3.

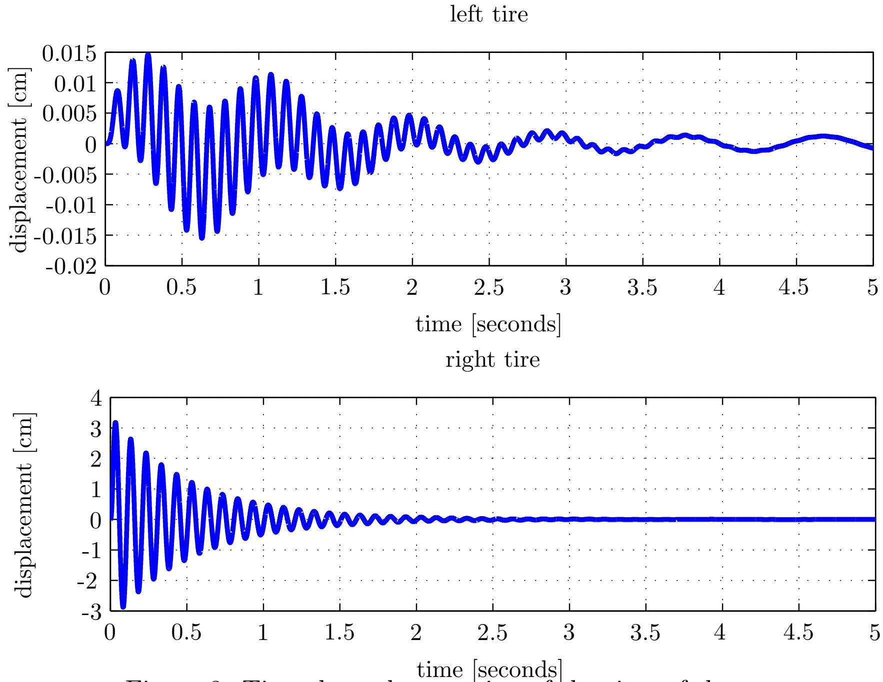


Figure 3: Time dependent motion of the tires of the car.

Problem 2

In fig. 4, a beam with fixed one end, carrying a rigid tip mass M , and exhibits transverse vibrations is given. Physical properties are as follows: $L = 1m$, $E = 210GPa$, $\rho = 7800kgm^{-3}$, sizes of the rectangular cross section $a = 2cm$ $b = 1cm$, M is twice of the beam mass.

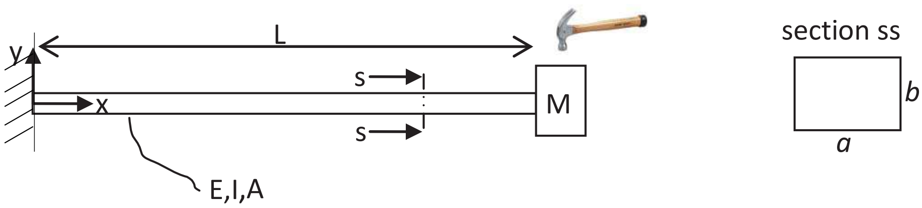


Figure 4: Problem 2

Question 1

Derive the equations of motion with suitable boundary conditions.

Solution 1

The transverse equation of motion for the beam is given by

$$EI\ \frac{\partial^4 w}{\partial x^4} + \rho A\ \frac{\partial^2 w}{\partial t^2} = 0$$

The boundary conditions for the problem are as follows:

1. zero displacement at $x = 0$:

$$w\,(0,t) = 0$$

2. zero slope of displacement at $x = 0$:

$$\frac{\partial w\,(0,t)}{\partial x} = 0$$

3. shear force acceleration of lumped mass equilibrium at $x = L$:

$$EI\ \frac{\partial w^3\,(L,t)}{\partial x^3} = M\frac{\partial w^2\,(L,t)}{\partial t^2}$$

4. zero moment at $x = L$:

$$EI\ \frac{\partial w^2\,(L,t)}{\partial x^2} = 0$$

Question 2

Find the first five natural frequencies (Hz) and vibration modes of the system. Plot the vibration modes in the same figure.

Solution 2

Using seperation of variables method, the displacements of the beam can be written as

$$u\left(x,t\right)=U\left(x\right)T\left(t\right) \tag{1}$$

using this assumption with the equations of motion we have

$$\begin{aligned} U^{IV}-\lambda^4 U &= 0 & (2)\\ \ddot{T}+\omega^2 T &= 0 & (3)\end{aligned}$$

Using these differential equations , the space dependent part of the solution of transverse vibration of beam is found as

$$X(x)=A\cosh\left(\lambda x\right)+B\sinh\left(\lambda x\right)+C\cos\left(\lambda x\right)+D\sin\left(\lambda x\right) \tag{4}$$

and time dependent part as

$$T(t)=\bar{A}\cos\left(\omega t\right)+\bar{B}\sin\left(\omega t\right)$$

where $\omega^2=\lambda^4\dfrac{EI}{\rho A}$. In these equations have 4 variables to determine in the space dependent part. Using boundary conditions, these variables can be determined. Using first two boundary conditions

$$U\left(0,t\right)=0 \quad ; \quad A+C=0 \quad \Longrightarrow \quad A=-C \tag{5}$$

$$U'\left(0,t\right)=0 \quad ; \quad B+D=0 \quad \Longrightarrow \quad B=-D \tag{6}$$

Using other boundary conditions

$$\begin{aligned} EIU''\left(L,t\right)=0 \quad ; \quad & \lambda^2\Big(A\cosh\left(\lambda L\right)+B\sinh\left(\lambda L\right)\\ & -C\cos\left(\lambda L\right)-D\sin\left(\lambda L\right)\Big)=0\\ EIU'''\left(L,t\right)-M\omega^2 U\left(L,t\right)=0 \quad ; \quad & EI\lambda^3\Big(A\sinh\left(\lambda L\right)+B\cosh\left(\lambda L\right)\\ & +C\sin\left(\lambda L\right)-D\cos\left(\lambda L\right)\Big)\\ & -M\omega^2\Big(A\cosh\left(\lambda L\right)+B\sinh\left(\lambda L\right)\\ & +C\cos\left(\lambda L\right)+D\sin\left(\lambda L\right)\Big)\end{aligned}$$

from the third boundary condition we have

$$B = -\frac{\cosh(\lambda L) + \cos(\lambda L)}{\sinh(\lambda L) + \sin(\lambda L)} A \tag{7}$$

using all relations in the fourth boundary condition it is found that

$$A\Bigg[\left(\sinh(L\,\lambda) - \sin(L\,\lambda) - \frac{\cos(L\,\lambda) + \cosh(L\,\lambda)}{\sin(L\,\lambda) + \sinh(L\,\lambda)}(\cos(L\,\lambda) + \cosh(L\,\lambda))\right) - \frac{M\,\lambda^4}{\rho\,A}\left(\cosh(L\,\lambda) - \cos(L\,\lambda) + \frac{\cos(L\,\lambda) + \cosh(L\,\lambda)}{\sin(L\,\lambda) + \sinh(L\,\lambda)}(\sin(L\,\lambda) - \sinh(L\,\lambda))\right)\Bigg] = 0$$

where for a non trivial solution $A \neq 0$ should be satisfied. Therefore, the expression inside the brackets must be zero. This results an equation in terms of λ where solution corresponds to the eigen frequencies of the beam.

Since it is impossible to obtain explicit expression for the λ, numerical solution is done. Therefore, the first five space frequencies are found as

$$\Lambda = \begin{bmatrix} 3.9256 \\ 7.0685 \\ 10.2101 \\ 13.3517 \\ 16.4933 \end{bmatrix} \text{rad/m}$$

The first five natural frequencies that are found as

$$\Omega = \begin{bmatrix} 726 \\ 2351 \\ 4906 \\ 8389 \\ 12801 \end{bmatrix} \text{Hz}$$

Mode shape that corresponds to an arbitrary λ is found from eq. (4) using the relations eqs. (5) to (7). The expression for the mode shapes for a specific λ is found as

$$U(x) = A\Bigg[\cosh(\lambda\,x) - \cos(\lambda\,x) + \frac{\sin(\lambda\,x)\ (\cos(L\,\lambda) + \cosh(L\,\lambda))}{\sin(L\,\lambda) + \sinh(L\,\lambda)} - \frac{\sinh(\lambda\,x)\ (\cos(L\,\lambda) + \cosh(L\,\lambda))}{\sin(L\,\lambda) + \sinh(L\,\lambda)}\Bigg]$$

The plot of the shape $U(x)$ is given in fig. 5.

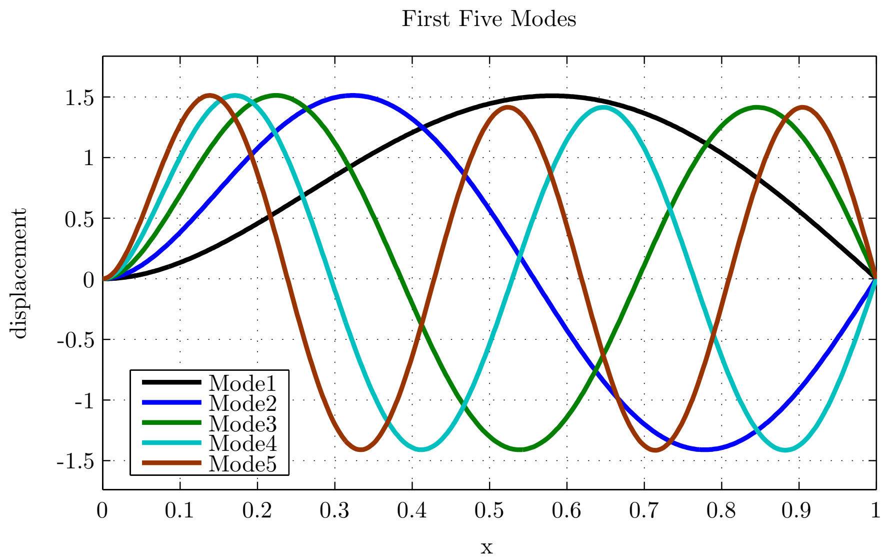


Figure 5: First five vibration modes of the beam.