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MATH 105A **Midterm** Oct 27, 2021

1. (5pts) Consider the function $f(x) = x^2 + x - 5$. Let the initial value to be $p_0 = 0$. Use Newton's method ($p_{n+1} = p_n - f(p_n)/f'(p_n)$) to find the zero of f. Please calculate the number of first 2 steps of iteration: $\{p_1, p_2\}$.

$f'(x) = 2x + 1$

$$P_1 = P_0 - f(P_0)/f'(P_0) = 0 - \frac{(-5)}{1} = 5$$

$$P_2 = P_1 - \frac{f(P_1)}{f'(P_1)} = 5 - \frac{25}{11} = \frac{30}{11}$$

$\{P_1, P_2\} = \{5, \frac{30}{11}\}$

2. (5 pts) Suppose $x = 0.7127$ and $y = 0.7125$. Use 4-digit arithmetic with chopping to compute $9(x + y)$.

$x+y = 0.7127 + 0.7125 = 1.4252 \rightarrow 1.425$

$9 \cdot (x+y) = 9 \times 1.425 = 12.825 \rightarrow 12.82$

Thus $9(x+y) = 12.82$

3. (8 pts) For the following linear system

$$\begin{cases} x_2 - x_3 & = 4 \\ x_1 - x_2 + x_3 & = 6 \\ x_1 - x_3 & = 2 \end{cases}$$

(a) (4 pts) Use the Gaussian Elimination Algorithm to simplify the augmented matrix.

(b) (2 pts) Indicate all row interchanges in above elimination.

(c) (2 pts) Find x_1, x_2, x_3 using backward substitution.

(a) ca $\begin{pmatrix} 0 & 1 & -1 & 4 \\ 1 & -1 & 1 & 6 \\ 1 & 0 & -1 & 2 \end{pmatrix} \xrightarrow[\text{(b)}]{\text{Exchange row 2 with row 1}} \begin{pmatrix} 1 & -1 & 1 & 6 \\ 0 & 1 & -1 & 4 \\ 1 & 0 & -1 & 2 \end{pmatrix} \xrightarrow{E3-E_1} \begin{pmatrix} 1 & -1 & 1 & 6 \\ 0 & 1 & -1 & 4 \\ 0 & 1 & -2 & -4 \end{pmatrix}$

$\xrightarrow{E3-E2} \begin{pmatrix} 1 & -1 & 1 & 6 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -1 & -8 \end{pmatrix}$

(C) $X_3 = \frac{-8}{-1} = 8 \quad X_2 = \frac{4 + X_3}{1} = \frac{4+8}{1} = 12$

$X_1 = 6 + X_2 - X_3 = 6 + 12 - 8 = 10$

Thus $X_1 = 10 \quad X_2 = 12 \quad X_3 = 8$

4. (5 pts) Consider the following sigma notation

$$\sum_{i=1}^{n} b_i \sum_{j=1}^{i} (a_{ij} x_j - y_j)$$

How many multiplications are included? (Give your answer as a formula in n)
(Formular to be used: $\sum_{i=1}^{n} = n(n+1)/2$)

Consider multiplications. $\sum_{i=1}^{n} b_i \sum_{j=1}^{i} (a_{ij} x_j - y_j)$

For each j $(a_{ij} \cdot x_j - y_j)$ there are 1 multiplications

$\sum_{j=1}^{i} (a_{ij} x_j - y_j)$ means, there are $\sum_{j=1}^{i} 1 = i$ times

For each i, there is another multiplication added on

So for each i there is $i+1$ $\quad \sum_{i=1}^{n} i+1 = \boxed{\frac{n(n+1)}{2} + n}$

n

5. (7 pts)Given a sequence $\{p_{n+1} = p_n/2 + 2/p_n\}_{n=0}^{\infty}$ with $p_0 = 1$.

   (a) (2 pts) Verify $2p_n(p_{n+1} - 2) = (p_n - 2)^2$.

   (b) (2 pts) Given that p_n converges to 2, show that the sequence has a quadratic convergence order. (Hint: ~~use the format in (a))~~

   (c) (3 pts) Show that $|p_n - 2| \le 2^{-n}$. (Hint: use the format in (a))

(a) $P_{n+1} - 2 = \left(\frac{P_n}{2} + \frac{2}{P_n} - 2\right)$

Left hand side: LHS: $2P_n \cdot (P_{n+1} - 2) = P_n^2 + 4 - 4P_n = (P_n - 2)^2$

RHS $= (P_n - 2)^2$

Thus LHS = RHS

(b) $\lim\limits_{n\to\infty} \frac{|P_{n+1} - 2|}{|P_n - 2|^2} = \lim\limits_{n\to\infty} \frac{\left|\frac{(P_n-2)^2}{2P_n}\right|}{|P_n - 2|^2} = \lim\limits_{n\to\infty} \left|\frac{1}{2P_n}\right|$

Since P_n converges to 2 then $\lim\limits_{n\to\infty} P_n = 2$.

Thus $\lim\limits_{n\to\infty} \left|\frac{1}{2P_n}\right| = \frac{1}{4}$ which is a constant.

The sequence has a quadratic convergence order.

(c), $\cancel{(P_n - 2)^2 = \frac{2P_n(P_{n+1}-2)}{(P_n-2)}}$

Use Induction assume that $|P_n - 2| \le 2^{-n}$

Since $P_0 = 1$ $\quad P_1 = \frac{1}{2} + \frac{2}{1} = \frac{5}{2}$

$|P_0 - 2| = 1 \le 2^0 = 1$ The base case satisfy

Next Suppose $n = k$ $k \in \mathbb{Z}$ $|P_k - 2| \le 2^{-k}$

then $\left|\frac{2P_k(P_{k+1} - 2)}{(P_k - 2)}\right| \le 2^{-k}$

$|P_{k+1} - 2| \le 2^{-k} \cdot \frac{|P_k - 2|}{|2P_k|} = 2^{-(k+1)} \cdot \frac{|P_k - 2|}{|P_k|}$

Since $|P_k - 2| \le 2^{-k} < 1$ so $1 < P_k < 3$, so $|P_k - 2| < |P_k|$

Thus $\frac{|P_k - 2|}{|P_k|} < 1$

Therefore $|P_{k+1} - 2| \leq 2^{-(k+1)} \cdot \left|\frac{P_k - 2}{P_k}\right| < 2^{-(k+1)}$

Thus for $n = k+1$, the assumption also hold.

Thus. $|P_n - 2| \leq 2^{-n}$ holds for all $n = 0, 1, 2, \ldots$