Math105LB final

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1 Introduction

In this final project, we will approximate the global surface temperature between 1922 and 1997 by using discrete least square method.
For the dataset, we consider $t_0, t_1, ..., t_n$ be a given set of time and $T_0, T_1, ..., T_n$ be the corresponding global surface temperature measure.
The method discrete least square will try to minimize the fost function $E = \sum_{i=0}^{n}(Pm(t_i) - T_i)^2$, where P_m is the approximate of the global surface temperature and can be written in the form of $P_m = a_m t^m + a_{m-1} t^{m-1} + ... + a_1 t + c_0$.
In this paper, several question will be addressed. First, I will show the algorithm for discrete least square approximation. Next, I will prove the basic algorithm method is correct, and show that the matrix that used in the algorithm is positive definite, which means it is invertible, so we can use the inverse matrix to get the coefficient for P_m directly.
In the second part of this paper, the code for this algorithm will be post, and it's outcome for degree(1,2,3,6) approximation by half test data will be pictured. The cost function will also be calculated both from test data and validation data for each degree approximation. The final goal for this project is to determine a best approximation model based on the error analysis.

2 Algorithm

Coefficient generater: INPUT, temperature data T, time data t, and degree m.
1. determine the number for input data n, and the degree for approximation m.
2. generate matrix X, which is

$$\begin{pmatrix} 1 & t_0 & ... & t_0^m \\ 1 & t_1 & ... & t_1^m \\ ... & ... & ... & ... \\ 1 & t_n & ... & t_n^m \end{pmatrix}$$

3. Generate Matrix M, which is $X^t X$, and generate $b = X^t y$.
4. Direct calculate $a = M^{-1} b$, which is the coefficient for approximate polynomial. Second is using the coefficient to get the approximate value by con-

structing the polynomial. Since the idea is easy, this function will be post in the following section.

3 Proof for basic theory

3.1 Proof A

Problem 3.1. Show that $a = (a_0, a_1, ..., a_m)^t$ satisfy $Ma = b$, where $M = X^tX$ and $b = X^ty$ with $X = \begin{pmatrix} 1 & t_0 & ... & t_0^m \\ 1 & t_1 & ... & t_1^m \\ ... & ... & ... & ... \\ 1 & t_n & ... & t_n^m \end{pmatrix}$
And $y = (T_0, T_1, ..., T_n)^t$

Proof. It is equivalent to show $Ma - b = 0$, thus we have $X^tXa - X^ty = 0 \leftrightarrow X^t(Xa - y)$. Notice that $Xa = \begin{pmatrix} Pm(t_0) \\ Pm(t_1) \\ ... \\ Pm(t_n) \end{pmatrix}$ then $Xa - y = \begin{pmatrix} Pm(t_0) - T_0 \\ Pm(t_1) - T_1 \\ ... \\ Pm(t_n) - T_n \end{pmatrix}$

Therefore,

$Ma - b = X^t(Xa - y) = \begin{pmatrix} \sum_{i=0}^{n} Pm(t_i) - T_i \\ \sum_{i=0}^{n} t_i^1(Pm(t_i) - T_i) \\ ... \\ \sum_{i=0}^{n} t_i^m(Pm(t_i) - T_i) \end{pmatrix}$

Which is equalvalent to $\frac{1}{2} \begin{pmatrix} \partial \sum_{i=0}^{n}(Pm(t_i) - T_i)^2/\partial a_0 \\ \partial \sum_{i=0}^{n}(Pm(t_i) - T_i)^2/\partial a_1 \\ ... \\ \partial \sum_{i=0}^{n}(Pm(t_i) - T_i)^2/\partial a_n \end{pmatrix}$

Since a is gained from by minimizing $\sum_{i=0}^{n}(Pm(t_i) - T_i)^2$, then for any i from 0 to n, the partial derivative should be 0, so we have $\partial(Pm(t_i) - T_i)^2/\partial a_i = 0$ thus we have $Ma = b$. □

3.2 Proof B

Problem 3.2. Next, show the matrix M is positive definite if $n \geq m$ and all t_i are distinct.

Proof. Notice that for any $c = (c_0, c_1, c_2, ..., c_m) \in \mathbb{R}^m$, $c^tMc = \sum_{i=0}^{n}(g(t_i))^2$, where $g(t) = c_mt^m + c_{m-1}t^{m-1} + ... + c_1t + c_0$. Since $g(t_i) \in \mathbb{R}$, then $g(t_i)^2 \geq 0$. Thus, $c^tMc = \sum_{i=0}^{n}(g(t_i))^2 \geq 0$.
Next, prove that $\sum_{i=0}^{n}(g(t_i))^2 \neq 0$. Suppose, to the contrary $\sum_{i=0}^{n}(g(t_i))^2 = 0$, then all $g(t_i) = 0$, then we have $g(t_i) = c_mt_i^m + c_{m-1}t_i^{m-1} + ... + c_1t_i + c_0 = g(t_j) = c_mt_j^m + c_{m-1}t_j^{m-1} + ... + c_1t_j + c_0$. If $t_i, t_j \neq 0$, then $t_i = \alpha t_j$ where $\alpha \neq 1$, then $g(t_i) - g(t_j) = 0 = c_m(\alpha^m - 1)t_j^m + ... + c_1(\alpha - 1)t_j$. Take $c = (0, 0, 0, ..., 0, 1, 1)$, then $(\alpha - 1)t_j = 0$, so $t_j = 0$, which contradict to assumption. So when $t \neq 0$,

```
function [a]=Least_square_appoximation(m,T,t)
n=length(t);
X=zeros(n,m+1);
X(:,1)=1;
for i=2:m+1
    X(:,i)=t.^(i-1);
end
M=X'*X;
b=X'*(T);
a=M^(-1)*b;
```

Figure 1: code result

$\sum_{i=0}^{n}(g(t_i))^2 \neq 0$. When $t_i = 0$, then let $c = (1, 1, 1, ..., 1)$, $g(t) = c_0 = 1 \neq 0$.
Thus, in general, $\sum_{i=0}^{n}(g(t_i))^2 \neq 0$.
Therefore matrix M is positive definite. $\square$

4 Code, Graphing, and Error Results

4.1 Code

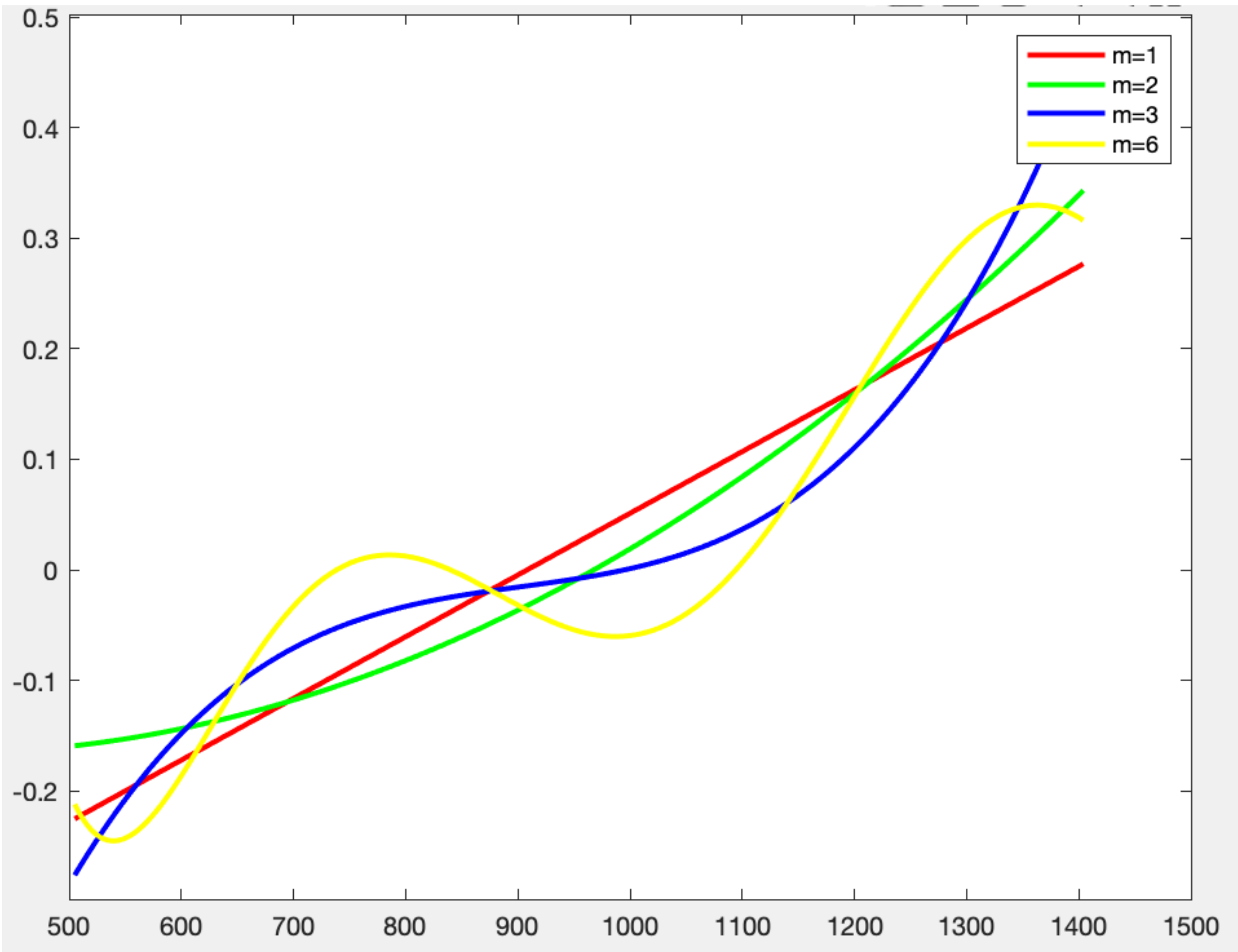

Figure 2: ploting result