HW3

1. $$\begin{cases} \dfrac{dx_0}{dt} = (2p_0 - 1)\, v_0 x_0 & \text{①} \\ \dfrac{dx_1}{dt} = 2(1-p_0)\, v_0 x_0 - d_1 x_1 \end{cases}$$

$p_0 = \dfrac{\tilde{p}}{1+\gamma x_1} \dfrac{\phi x_1}{1+\phi x_1}$ where $\tilde{p} = 0.8$ $d_1 = 1$ $v_0 = 1$ $\phi = 10$ $\gamma = 1$

$$p_0 = \frac{0.8}{1+x_1} \cdot \frac{10 x_1}{1+10x_1} = \frac{8x_1}{(1+x_1)(1+10x_1)}$$

Then $$\begin{cases} \dfrac{dx_0}{dt} = \left(\dfrac{16x_1}{(1+x_1)(1+10x_1)} - 1\right) \cdot x_0 = f \\ \dfrac{dx_1}{dt} = \left(2 - \dfrac{16x_1}{(1+x_1)(1+10x_1)}\right) \cdot x_0 - x_1 = g \end{cases}$$

$$J = \begin{pmatrix} \dfrac{\partial f}{\partial x_0} & \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial g}{\partial x_0} & \dfrac{\partial g}{\partial x_1} \end{pmatrix}$$

$$\frac{\partial f}{\partial x_0} = \left(\frac{16x_1}{(1+x_1)(1+10x_1)} - 1\right)$$

$$\frac{\partial g}{\partial x_0} = 2 - \frac{16x_1}{(1+x_1)(1+10x_1)}$$

$$\frac{\partial f}{\partial x_1} = \frac{16 - 160x_1^2}{(10x^2+11x+1)^2} \cdot x_0$$

$$\frac{\partial g}{\partial x_1} = \frac{160x_1^2 - 16}{(10x^2+11x+1)^2} \cdot x_0 \quad -1$$

Q3c. $y(x,t) = X(x)\,T(t)$

$$XT' = dX''T - kXT$$

$$\frac{T'}{T} = \frac{dX''}{X} - k = \lambda \Rightarrow \begin{cases} \frac{T'}{T} = \lambda \\ \frac{dX''}{X} = (\lambda+k) \end{cases}$$

Solve $X(x)$ $\quad dX'' = (\lambda+k)X$

$$\Rightarrow dX'' - (\lambda+k)X = 0.$$

$$X(x) = A \sin\sqrt{\frac{\lambda-k}{d}}\,x + B\cos\sqrt{\frac{\lambda-k}{d}}\,x$$

$$X'(x) = \sqrt{\frac{\lambda-k}{d}}\,A = 0 \Rightarrow A = 0.$$

$$X'(1) = -\sqrt{\frac{\lambda-k}{d}}\,B \sin\sqrt{\frac{\lambda-k}{d}} = 0 \Rightarrow \sin\sqrt{\frac{\lambda-k}{d}} = 0.$$

$$\Rightarrow \sqrt{\frac{\lambda-k}{d}} = n\pi$$

$$\Rightarrow \lambda = dn^2\pi^2 + k.$$

$$X(x) = B_n \cdot \cos \pi x, \quad n, 1, 2, 2.$$

Solve $T(t) = \frac{T'}{T} = \lambda \Rightarrow T' - \lambda T = 0 \Rightarrow T = e^{-\lambda t}$

$$= e^{-dn^2\pi^2 t - kt}$$

So $y(x,t) = \sum_{n=1}^{\infty} B_n \cos n\pi x \, e^{-dn^2\pi^2 t - kt}$

$$B_n = \frac{2}{\pi}\int_0^1 y_0 \cos(n\pi x)\,dx$$