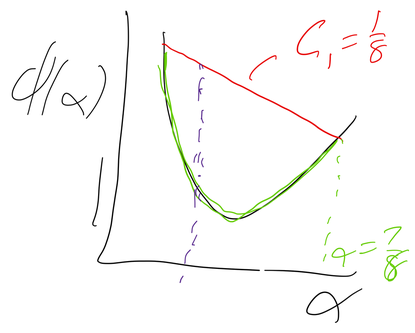
Thursday, January 13, 2022 6:00 PM

- Open note on canvas
- Available starting 6:30
  6:30 - 6:55PM

Ex: Consider minimizing the function $f(x,y) = (x-1)^2 + (y-2)^2$ by constant step size steepest descent method.

Find a step size α which satisfies Wolfe conditions (for any point) with $c_1 = \frac{1}{8}$, $c_2 = \frac{1}{2}$.

$\vec{x}_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix}$



I: $f_{k+1} \le f_k + c_1 \alpha p_k^T \nabla f_k$

II: $p_k^T \nabla f_{k+1} \ge c_2 p_k^T \nabla f_k$ $\quad p_k = -\nabla f_k$

$f_{k+1} = f(\vec{x}_k + \alpha p_k) = f(\vec{x}_k - \alpha \nabla f_k)$

$\nabla f(\vec{x}_k) = \begin{bmatrix} 2(x_k - 1) \\ 2(y_k - 2) \end{bmatrix}$

$f\left(\begin{bmatrix} x_k \\ y_k \end{bmatrix} - \alpha \begin{bmatrix} 2(x_k-1) \\ 2(y_k-2) \end{bmatrix}\right)$

$f\left(\begin{bmatrix} x_k - \alpha 2(x_k-1) \\ y_k - \alpha 2(y_k-2) \end{bmatrix}\right)$

$\left(x_k - 2\alpha(x_k-1) - 1\right)^2 + \left(y_k - 2\alpha(y_k-2) - 2\right)^2$

LHS

$\nabla f^T \nabla f$

$\left((x_k - 1) - 2\alpha(x_k - 1)\right)^2$

RHS: $f_k + c_1 \alpha p_k^T \nabla f_k$ or $f_k - c_1 \alpha \|\nabla f\|^2$

$= f\left(\begin{bmatrix} x_k \\ y_k \end{bmatrix}\right) - c_1 \alpha \left[(2(x_k-1))^2 + (2(y_k-2))^2\right]$

$= (x_k - 1)^2 + (y_k - 2)^2 - c_1 \alpha \left[4(x_k-1)^2 + 4(y_k-2)^2\right]$

Factor:

$= (x_k - 1)^2 (1 - 4c_1\alpha) + (y_k - 2)^2 (1 - 4c_1\alpha)$

LHS: $\left(x_k - 1 - 2\alpha(x_k - 1)\right)^2 + \cdots$

$= (x_k - 1)^2 (1 - 2\alpha)^2 + (y_k - 2)^2 (1 - 2\alpha)^2 \le (x_k - 1)^2 (1 - 4c_1\alpha) + (y_k - 2)^2 (1 - 4c_1\alpha)$

$(1 - 2\alpha)^2 \le 1 - 4c_1\alpha$

Given: $c_1 = \frac{1}{8}$.

$(1 - 2\alpha)^2 \le 1 - \frac{1}{2}\alpha$

$1 - 4\alpha + 4\alpha^2 \le 1 - \frac{1}{2}\alpha$

$0 \le 4\alpha - 4\alpha^2 - \frac{1}{2}\alpha$ $\quad \alpha > 0$

$0 \le \alpha\left(4 - 4\alpha - \frac{1}{2}\right)$

$\alpha > 0$ $\quad 4 - 4\alpha - \frac{1}{2} \ge 0$

$0 \le \frac{7}{2} - 4\alpha$

$4\alpha \le \frac{7}{2}$, $\alpha \le \frac{7}{8}$

If $c_1 = \frac{1}{8}$, then Wolfe I says $\alpha \le \frac{7}{8}$.

II: $p_k^T \nabla f_{k+1} \ge c_2 p_k^T \nabla f_k$ ensuring the slope isn't so steep

$-\nabla f_k^T \nabla f_{k+1} \ge -c_2 \nabla f_k^T \nabla f_k$

$\nabla f(x_k) = \begin{bmatrix} 2(x_k - 1) \\ 2(y_k - 2) \end{bmatrix}$

$\nabla f(\vec{x}_k - \alpha \nabla f_k) = \nabla f\left(\begin{bmatrix} x_k - 2\alpha(x_k-1) \\ y_k - 2\alpha(y_k-2) \end{bmatrix}\right)$

$\alpha \in \left(\frac{1}{4}, \frac{7}{8}\right)$ Choose any α in that interval.

Plug in.

Maybe it'll show up in a HW

Lemma: If $0 < c_1 < c_2 < 1$, then will exist intervals of α satisfying Wolfe conditions.