Tuesday, January 18, 2022 6:22 PM

Ex: $f(x) = x^4$. Find the order of convergence if $x_k \to 0$ with fixed step steepest descent algorithm, $\alpha = \frac{1}{2}$.

Order of convergence: $\lim_{k\to\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|^p} = L$

$x^* = 0$ in this case, so

$$\lim_{k\to\infty} \frac{\|x_{k+1}\|}{\|x_k\|^p} = L$$

$$x_{k+1} = x_k - \alpha f'(x)$$
$$= x_k - \tfrac{1}{2}\cdot 4x^3 = x_k - 2x^3$$

$$\lim_{k\to\infty} \frac{\|x_k - 2x_k^3\|}{\|x_k\|^p}$$

If $x_k \to 0$

$$\lim_{x_k\to 0} \frac{|x_k - 2x_k^3|}{|x_k|^p}$$

If $p = 1$; This limit > 0, $< \infty$

$$\lim \frac{|x_k - 2x_k^3|}{|x_k|^3} = \frac{|1 - 2x_k^2|}{|x_k|^2} \approx \frac{1}{0} \to \infty$$

If you tried $p = 1$: $\lim_{x_k\to 0} \frac{|x_k - 2x_k^3|}{|x_k|} = \frac{|1 - 2x_k^2|}{1} = 1$

Def: Order of convergence: Given a sequence $x_k \to x^*$, order of convergence is $p \geq 1$ if

$$0 < \lim_{k\to\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|^p} < \infty$$

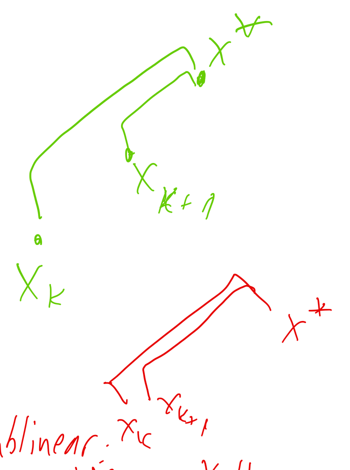
Linear: $p = 1$, $\lim_{k\to\infty} \frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|^p} = L < 1$

Sublinear: $p = 1, L = 1$

Superlinear: $p > 1$

$$\frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|}$$

(ratio ≤ 1



Sublinear: ratio $\frac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|}$ is 1

Ex: Suppose you want to perform one iteration of the function

$$f(x,y) = x^2 + 2y^2 - 2x + y + 1$$

from the current value of $x_k = (1,0)$. If we start at an initial value of $\bar{\alpha} = 1$, $\rho = 0.75$, how many times do we have to update the value of α to satisfy the Wolfe condition?

$$f_{k+1} \leq f_k - c_1 \alpha \nabla f_k^T \nabla f_k$$

with $c_1 = 1/4$?

$$f_k = f(x_k) = f(1,0) = 1 - 2 + 1 = 0$$

$$\nabla f = \begin{bmatrix} 2x - 2 \\ 4y + 1 \end{bmatrix} \qquad \nabla f(1,0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\nabla f(1,0)^T \nabla f(1,0) = 0^2 + 1^2 = 1$$

$$f_{k+1} = f(x_k - \alpha \nabla f_k)$$
$$= f\left(\begin{bmatrix} 1 - \alpha\cdot 0 \\ 0 - \alpha\cdot 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 1 \\ -\alpha \end{bmatrix}\right)$$

$$\text{LHS: } 1 + 2\alpha^2 - 2 - \alpha + 1 = 2\alpha^2 - \alpha \leq 0 - \tfrac{1}{4}\alpha$$
$$0 \leq \tfrac{3}{4}\alpha - 2\alpha^2$$
$$0 \leq \alpha\left(\tfrac{3}{4} - 2\alpha\right)$$
$$\alpha = 0, \quad \alpha = \tfrac{3}{8}$$

So if $\alpha < \frac{3}{8}$, then Wolfe I is satisfied.

To check steps of backtracking:

Multiply α by ρ at each step. $\rho = 0.75$

$\alpha = 1$ ~ 0 steps

$\alpha = 0.75$ ~ 1 step

$\alpha = (0.75)^2 = \frac{9}{16}$ ~ 2 steps

$\alpha = (0.75)^3 = \frac{27}{64}$ ~ 3 steps

$\alpha = (0.75)^4 \approx 0.316$ ~ 4 steps $< \frac{3}{8}$

Lecture: Steepest descent applied to

$$f(x) = \tfrac{1}{2}x^T Q x - b^T x$$

Obtained results (theorems for this case and the general case