Quiz: Thursday
Last 20 mins of discussion
on canvas
Multiple choice / fill in the blank etc.
Covers material up to Lecture (with conditions)
Open Note

Line Search Algorithms:
Goal: Minimize f. Line search iteratively chooses x_k

$x_{k+1} = x_k + \alpha_k p_k$

α_k: step length, p_k determines the algorithm descent direction

Descent direction: $f(x_{k+1}) \le f(x_k)$
this is satisfied when $p_k^T \nabla f_k < 0$.

If $p_k^T \nabla f_k < 0$, then p_k is descent direction.

p_k vector, ∇f_k vector
$p_k^T \nabla f_k$ or $p_k \cdot \nabla f_k$
$\begin{bmatrix} a \\ b \end{bmatrix}$ $\begin{bmatrix} c \\ d \end{bmatrix}$

Usually $p_k = -B_k^{-1} \nabla f_k$, B_k is symmetric invertible

- Steepest descent: $B_k = I$, $p_k = -\nabla f_k$
- Newton's Method: $B_k = \nabla^2 f(x_k)$, $p_k = -\nabla^2 f(x_k)^{-1} \nabla f(x_k)$
- Quasi-Newton: B_k is an approximation of $\nabla^2 f(x_k)$

Ex: True/False: Given the objective func
$f(x,y) = x^2 - 2xy + 3x + 4y^2$
and initial guess $x_0 = (1,1)$, the direction $p_1 = \langle -1, 1 \rangle$ is a descent direction.
II: Find all minimizers of this problem.

- Verify $p_k^T \nabla f_k < 0$: $p_1^T \nabla f_1$

$p_1 = \langle -1, 1 \rangle$. $\nabla f_1 = \nabla f(x_1)$

$\nabla f = \begin{bmatrix} 2x - 2y + 3 \\ -2x + 8y \end{bmatrix}$

$\nabla f(1,1) = \begin{bmatrix} 2-2+3 \\ -2+8 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = -3 + 6 = 3$
$3 > 0$, p_1 isn't a descent direction

$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

- Finding minimizers,
First, set $\nabla f = 0$. Any solutions are stationary points.
Then compute $\nabla^2 f$, check if it's positive definite → minimizer
if it's not positive semidefinite, not a minimizer

$\nabla f = 0$: $\begin{bmatrix} 2x - 2y + 3 \\ -2x + 8y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2x - 2y + 3 = 0$
$-2x + 8y = 0$

$6y + 3 = 0$, $y = -\frac{1}{2}$

$-2x - 4 = 0$ $x = -2$.
$(-2, -\frac{1}{2})$ is stationary point.

$\nabla^2 f = \begin{bmatrix} 2 & -2 \\ -2 & 8 \end{bmatrix}$

1 method: Check that eigenvalues are positive.

$\det\left(\begin{bmatrix} 2-\lambda & -2 \\ -2 & 8-\lambda \end{bmatrix}\right) = 0$

$(2-\lambda)(8-\lambda) - 4 = 0$
$16 - 8\lambda - 2\lambda + \lambda^2 - 4 = 0$
$\lambda^2 - 10\lambda + 12 = 0$

$\lambda = \frac{10 \pm \sqrt{100 - 48}}{2}$ $10 - \sqrt{52}$
$\sqrt{100} - \sqrt{52} > 0$
$= \frac{10 \pm \sqrt{52}}{2}$ so $\lambda_1, \lambda_2 > 0$.
positive definite
$\Rightarrow (-2, -\frac{1}{2})$ is minimizer

2nd method: Thm: A symmetric diagonally dominant matrix with positive diagonal entries is positive definite.

Diagonally dominant: For every row of the matrix the magnitude of the diagonal entry of the row is $\ge$ the sum of the magnitudes of the non-diagonal entries in the same row.

$\begin{bmatrix} 2 & -2 \\ -2 & 8 \end{bmatrix}$
$|2| \ge |-2|$
$|8| \ge |-2|$
2, 8 > 0 so positive definite.

3x3: $\begin{bmatrix} 3 & -2 & \\ 1 & -3 & 2 \\ -1 & 2 & 4 \end{bmatrix}$
diagonally dominant b/c
row 1: $|3| \ge |-2| + |1| = 3$
row 2: $|-3| \ge |1| + |2| = 3$
row 3: $|4| \ge |-1| + |2| = 3$

Converse doesn't hold: If some diagonal entries aren't positive, $\not\Rightarrow$ not positive definite.

$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ $\lambda = -1, 2$