202C-hw2

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Question 1

Solution:

$$a) \quad x = \sqrt{z}, \quad \frac{dx}{dx} = \pm\frac{1}{2\sqrt{z}}$$
$$b) \quad x = log(z), \quad \frac{dx}{dz} = \frac{1}{z}$$
$$c) \quad x = \frac{1}{z}, \quad \frac{dx}{dz} = -\frac{1}{z^2}$$
$$d) \quad x = \frac{e^z}{1+e^z}, \quad \frac{dx}{dz} = \frac{e^z}{(1+e^z)^2}$$

Question 2

Solution:

$$x \sim Gamma(a, b)$$
$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-x\beta}, \quad x = \frac{1}{z}$$
$$Jacobian: \quad \left|\frac{dx}{dz}\right| = \left|-\frac{1}{z^2}\right| = z^{-2}$$
$$f(z) = \frac{b^a}{\Gamma(a)} z^{-a-1} e^{-\frac{b}{z}} \sim Inver.gam(a, b), \quad z > 0$$

Question 3

Solution:

a)

$$FIP = \frac{-\frac{d^2}{d\lambda^2} logp(\lambda)}{-\frac{d^2}{d\lambda^2}(logp(y|\lambda) + logp(\lambda))}$$
$$= \left[\frac{a-1}{a-1+\sum yi}\right]$$

b)

by weight average

$$
\begin{aligned}
E(y|\lambda) &= \frac{a+\sum yi}{b+n} \\
&= \frac{b}{b+n}\frac{a}{b} + \frac{n}{b+n}\overline{y}
\end{aligned}
$$

c)

From hw 1 we got:

$$
\begin{aligned}
f(y|a,b) &= \frac{b^a}{\Gamma(a)y!}\frac{\Gamma(y+a)}{(b+1)^{y+a}} \\
&= \frac{(y+a-1)!}{(a-1)!y!}\frac{b^a}{(b+1)^{y+a}} \\
&= \binom{y+a-1}{y} b^a (b+1)^{-y-a} \\
&= \binom{y+a-1}{y}\left(\frac{b}{1+b}\right)^a \left(\frac{1}{1+b}\right)^y
\end{aligned}
$$

$$
Which \quad f(y|a,b) \sim NB(a, \frac{b}{1+b})
$$

$$
\begin{aligned}
prior\ data\ mean &= \frac{a}{b} \\
prior\ sample\ size &= b\ (from\ weight\ average)
\end{aligned}
$$

Question 4

Solution:

$$
\begin{aligned}
y = e^{-zi} I_{0<zi<1}, \quad \left|\frac{dy}{dz}\right| &= \left|-e^{-zi}\right| = e^{-zi} I_{0<zi<1} \\
f(zi|\theta) &\propto \theta exp(-zi(\theta-1))e^{-zi} I_{0<zi<1} \\
&= \theta exp(-zi\theta) I_{0<zi<1} \\
&= \frac{\theta}{\Gamma(1)} zi^{1-1} e^{-zi(\theta)} I_{0<zi<1} \\
&= \theta e^{-zi(\theta)} \sim Gamma(1,\theta), 0<zi<1
\end{aligned}
$$

Also because alpha = 1,

$$
\theta e^{-zi(\theta)} \sim exp(\theta), 0<zi<1
$$

Question 5

Solution:

a)

$$Lf(y|\theta) = \theta^n e^{\theta - 1 log(\sum yi)} I_{0<yi<1}$$
$$f(\theta|z) \propto \theta^{a-1} e^{-\theta b}$$
$$f(\theta|y) \propto \theta^{a+n-1} e^{-\theta(b - \sum log(yi)} \sim gamma(a+n, b - \sum log(yi))$$

From posterior distribution:

$$mean = \frac{a+n}{b - \sum log(yi)}$$
$$variance = \frac{a+n}{(b - \sum log(yi))^2}$$
$$mode = \frac{a+n-1}{b - \sum log(yi)}$$

b)

$$FIP = \frac{-\frac{d^2}{d\theta^2} logp(\theta)}{-\frac{d^2}{d\theta^2}(logp(y|\theta) + logp(\theta))}$$
$$evaluated\ at\ mode = \frac{a-1}{a+n-1}$$

c)

$$FIP = \left[\frac{-\frac{d^2}{d\theta^2} logp(\theta)}{-\frac{d^2}{d\theta^2}(logp(y|\theta) + logp(\theta))}\right], is\ better$$

Easier to compute by using this formula

d)

From wikipedia, Pareto distribution and Zeta distribution.

Question 6

a)

from hw 1 Q2

$$p(\lambda|y) \propto \lambda^{\sum(yi)+a-1} e^{-\lambda(b+n)}, \lambda > 0$$

$$posterior\ follows\ gamma\ distribution\ \sim Gamma(\sum(yi) + a, b + n)$$

first normal approximation, by using mean and variance from posterior distribution.

$$mean = \frac{\sum(yi) + a}{b+n}, \quad variance = \frac{\sum(yi) + a}{(b+n)^2}$$

$$we\ have\ f(y|\lambda) \sim N(\frac{\sum(yi)+a}{b+n}, \frac{\sum(yi)+a}{(b+n)^2})$$

second normal approximation, by using fisher information and mode of lambda.

$$mode\ of\ \lambda = \frac{\sum(yi)+a-1}{b+n}$$
$$I = \frac{(b+n)^2}{\sum(yi)+a-1}$$
$$we\ have\ = f(y|\lambda) \sim N(\frac{\sum(yi)+a-1}{b+n}, \frac{\sum(yi)+a-1}{(b+n)^2})$$

b)

i: when the parameter from posterior distribution is large then we can say the normal approximation(s) to the gamma posterior are good.

ii: when the parameter from posterior distribution is close to 0 then we can say the normal approximation(s) to the gamma posterior are not good.

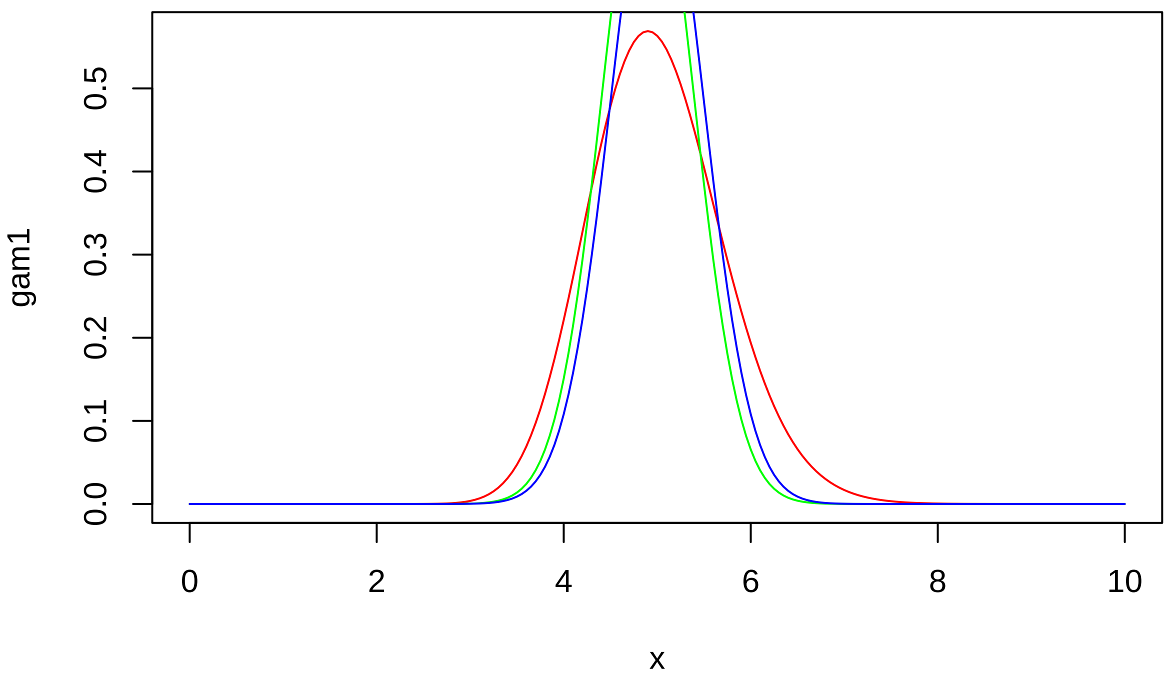
c)

Example when postrior are good, I chose sum of yi = 10, alpha = 40, beta = 10

```
x <- seq(0, 10, by = 0.05)

gam1 <- dgamma(x, shape=50, rate = 10)
nor1 <- dnorm(x,mean = 49/10, 49/10^2)
nor12 <- dnorm(x,mean = 50/10, 50/10^2)

plot(x, gam1, type = "l", col = "red")
lines(x, nor1, type = "l", col = "green")
lines(x, nor12, type = "l", col = "blue")
```

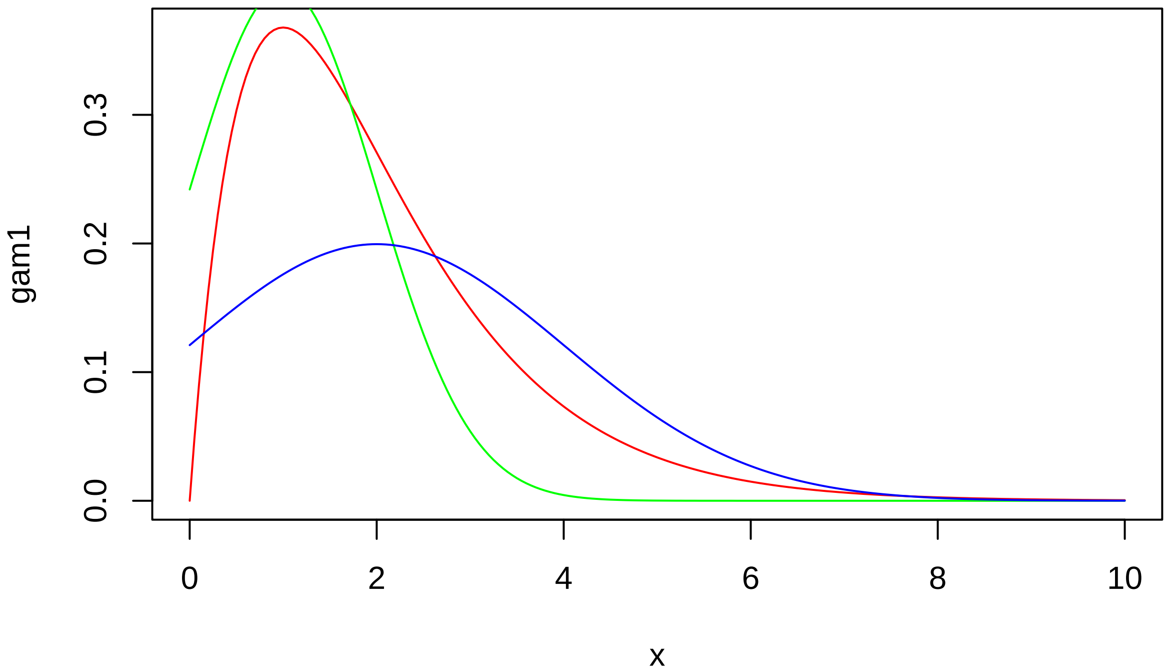


Example when posterior are not good, I chose sum of yi = 10, alpha = 2, beta = 1

```r
x <- seq(0, 10, by = 0.05)

gam1 <- dgamma(x, shape=2, rate = 1)
nor1 <- dnorm(x,mean = 1/1, 1/1^2)
nor12 <- dnorm(x,mean = 2/1, 2/1^2)

plot(x, gam1, type = "l", col = "red")
lines(x, nor1, type = "l", col = "green")
lines(x, nor12, type = "l", col = "blue")
```



d)

When alpha is large we get good normal approximation, when alpha is small we get poor normal approximation.