Problem 6.

6.1

$$L(\mu|\bar{x}) = \prod_{i=1}^{n} (x_i|\mu) \propto \prod_{i=1}^{n} \exp\left(\frac{-(x_i-\mu)\Sigma^{-1}(x_i-\mu)}{2}\right)$$

$$= \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^{-1}\Sigma^{-1}(x_i-\mu)\right\}$$

① $\exp\{-\frac{n}{2}\mu^T\Sigma^{-1}\mu + n\bar{x}^{-1}\Sigma^{-1}\mu\}$, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$p(\mu|\eta, Z) \propto \exp\{-\frac{1}{2}\mu^T\Sigma^{-1}\mu + \eta^T Z^{-1}\mu\}$ ②

① × ② => $\exp\{-\frac{1}{2}\mu^T[n\Sigma^{-1}+Z^{-1}]\mu + (n\bar{x}^{-1}\Sigma^{-1}+\eta^T Z^{-1})\mu\}$

let $\Sigma=\delta^2$ $Z=z^2$ => $\mu \sim N(\eta, z^2)$ is conjugate for $x|\mu \sim N(\mu, \delta^2)$

6.2 From 6.1

we know ($p(\mu|x) \propto p(x|\mu)p(\mu)$)

$\mu|x_1,\dots,x_n \sim N(A,B)$, $B=[n\Sigma^{-1}+Z^{-1}]^{-1}$, $A = B[\Sigma^{-1}(n\bar{x}) + Z^{-1}\eta]$

Let $\Sigma=\delta^2$, $Z=z^2$, $\mu|x_1,\dots,x_n \sim N(a,b)$

=> $a = \frac{nz^2}{nz^2+\delta^2}\bar{x} + \frac{\delta^2}{nz^2+\delta^2}\eta$, $b = \frac{1}{\frac{n}{\delta^2}+\frac{1}{z^2}}$

=> we get, $E(\mu|x_1,\dots,x_n) = a = \frac{z^2}{z^2+\frac{\delta^2}{n}}\bar{x} + \frac{\frac{\delta^2}{n}}{z^2+\frac{\delta^2}{n}}\eta$

6.3 From 6.2, the mean of μ is the weighted sum of sample average and η.