

Notes on Support Vector Machines

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1 Question Formulation

Support Vector Machines aim to solve this problem: Given training data

$$D = (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$$

We want to find a hyperplane which separates different types of training data.

Any plane can be described with such an equation:

$$w^\top x + b = 0$$

The distance between any sample x and this plane is:

$$r = \frac{|w^\top x + b|}{\|w\|}$$

Assume that the hyperplane correctly classifies every sample, that is to say

$$w^\top x_i + b \geq +1, \forall y_i = +1$$

and

$$w^\top x_i + b \leq -1, \forall y_i = -1$$

The samples that are the closest to the hyperplane are called **support vectors**. The distance between the positive and negative support vectors are called the **margin**. The margin is

$$\gamma = \frac{2}{\|w\|}$$

Our goal is to find the hyperplane that maximizes the margin. A mathematical denotation of our problem is

$$\begin{aligned} & \max_{w, b} \frac{2}{\|w\|} \\ & s.t. \ y_i(w^\top x_i + b) \geq 1, \forall i \end{aligned}$$

Apparently, to maximize the margin, we need to maximize $\|w\|^{-1}$, this is equivalent to minimizing $\|w\|^2$, therefore the above equation can be rewritten as

$$\begin{aligned} & \min_{w, b} \frac{1}{2} \|w\|^2 \\ & s.t. \ y_i(w^\top x_i + b) \geq 1, \forall i \end{aligned}$$

2 Derivation

We use the **Lagrange Multiplier Method** to solve this problem. Assigning a multiplier α_i to each constraint, we have

$$L(w, b, \alpha) = \frac{1}{2} \|w\|^2 + \sum_{i=1}^m \alpha_i (1 - y_i (w^T x_i + b))$$

Let the partial derivative of L w.r.t. w and b equal to zero, we have

$$w = \sum_{i=1}^m \alpha_i x_i y_i$$

$$0 = \sum \alpha_i y_i$$

Therefore, the problem we need to solve can be reformulated as

$$\begin{aligned} \max_{\alpha} \quad & \sum_{i=1}^m \alpha_i - \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \alpha_i \alpha_j y_i y_j x_i^T x_j \\ \text{s.t.} \quad & \sum_{i=1}^m \alpha_i y_i = 0 \end{aligned}$$

2.1 Karush-Kuhn-Tucker Conditions

In order for the convex optimization problem to have a solution, the problem must satisfy the **Karush-Kuhn-Tucker Conditions** (KKT Conditions). The KKT conditions for the SVM problem are

$$\alpha_i \geq 0$$

$$y_i f(x_i) - 1 \geq 0$$

$$\alpha_i (y_i f(x_i) - 1) = 0$$

The KKT conditions are satisfied because for any sample (x_i, y_i) , either $\alpha_i = 0$ or $y_i f(x_i) = 1$. If $\alpha_i = 0$, then the sample is not on a support vector. If $\alpha_i > 0$, then we have $y_i f(x_i) = 1$. The sample is on a support vector. This property reveals the essence of SVM: the resulting hyperplane only depends on the support vectors.

3 Kernel Function

In the previous sections, we assume that we can find such a hyperplane that correctly classifies every node in the sample space. However this is not always the case. We use a function ϕ to embed our input feature vector x into a higher

dimension.

Let $\phi(x)$ denote the embedded vector. The SVM problem becomes

$$\min_{w,b} \frac{1}{2} \|w\|^2$$

$$s.t. \ y_i(w^\top \phi(x_i) + b) \geq 1, \forall i$$

and

$$\begin{aligned} \max_{\alpha} \sum_{i=1}^m \alpha_i - \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \alpha_i \alpha_j y_i y_j \phi(x_i)^\top \phi(x_j) \\ s.t. \sum_{i=1}^m \alpha_i y_i = 0 \end{aligned}$$

It is computationally expensive to compute $\phi(x_i)^\top \phi(x_j)$. Therefore we use a function to compute it.

$$\kappa(x_i, x_j) = \phi(x_i)^\top \phi(x_j)$$

The function κ is called the **kernel function**. Popular choices of kernel functions are

1. $\kappa(x_i, x_j) = x_i^\top x_j$
2. $\kappa(x_i, x_j) = (x_i^\top x_j)^d$
3. $\kappa(x_i, x_j) = \exp(-\frac{\|x_i - x_j\|^2}{2\sigma^2})$
4. $\kappa(x_i, x_j) = \exp(-\frac{\|x_i - x_j\|}{\sigma})$
5. $\kappa(x_i, x_j) = \tanh(\beta x_i^\top x_j + \theta)$