1 Lecture 7

1.1 Overview of This Lecture

In the previous lecture we defined topological space $(X, \mathcal{J})$. To understand it, you may want to see some examples. Have a look at definition 2.1, followed by many EXAMPLES in chapter 3. See also the topology of R and Topological Space in wikipedia.

Note that open set is merely an element of the set $\mathcal{J}$, and neighborhood, closed set are something we haven't defined yet. And we will not. Instead, we invite you to define them in the exercises (1.2.1, 1.2.2, and 1.2.3).

We introduce many new concepts in this lecture. It is recommended to read the textbook or wikipedia to understand them.

1.2 Proof of Things

Exercise 1.2.1 (definition of closed set)**.** An element O of $\mathcal{J}$ is called an open set of X. So what's the corresponding closed set of X, in terms of O? Is your definition for closed set well-defined?

Exercise 1.2.2 (definition of neighborhood)**.** The definition of neighborhood in metric space (X, d) is as follows: A subset N of X is called a neighborhood of a if there is a δ such that $B(a; \delta) \subset N$. That is, the neighborhood of a contains an open ball of a. But in topological space, we do not have open ball any more. We have only open set, which is an abstraction of open ball. Now give your definition of neighborhood in topological space (definition 2.2, chapter 3). Again, is your definition for neighborhood well-defined?

Exercise 1.2.3 (neighborhood and open set)**.** Let $(X, \mathcal{J})$ be a topological space. Prove it: a subset O of X is open if and only if O is a neighborhood of each of its points. (This is corollary 2.3 in chapter 3, and is what you did before in metric space.)

Definition 1.2.4 (subspace topology, definition 6.1, chapter 3)**.** Let $(X, \mathcal{J})$ be topological space and $Y \subset X$. Then $(Y, \mathcal{J}|_Y)$, where $\mathcal{J}|_Y = \{U | U = Y \cap O, O \in \mathcal{J}\}$, is a subspace topology. An element $U \in \mathcal{J}|_Y$ is an open set in Y, or *relatively open* in Y.

Remark 1.2.5 (remark for definition 1.2.4)**.** The main motivation to define *subspace topology* is as follows. Let $i : Y \to X$ be an inclusion map. We want to give Y a topology such that i is continuous.

Example 1.2.6 (example for definition 1.2.4)**.** Let $X = \mathbb{R}, Y = [0, 1)$ (with standard topology). Then $[0, 0.2)$, obviously not open in X, is open in $\mathcal{J}|_Y$.

Definition 1.2.7 (limit point)**.** Let $(X, \mathcal{J})$ be a topological space and $A \subset X$. We say $x \in X$ is a *limit point* of A if for each neighborhood N of x, we have $N/\{x\} \cap A \neq \emptyset$.

Remark 1.2.8 (remark for definition 1.2.7)**.** As you should verify, this definition of limit point in topological space is exactly the same as in metric space, with only one difference (what's the difference?). Also note that Mendelson devoloped theorems for topological space, without defining limit point. Here we take a different approach, and we will finally arrive at the same place as Mendelson.

Definition 1.2.9 (closure of a set)**.** Let A be a subset of a topological space. A point x is said to be in the *closure* of A if $x \in A$ or x is a limit point of A. The closure of A is denoted by $\bar{A}$.

Corollary 1.2.10 (corollary for definition 1.2.9)**.** *$A \subset \bar{A}$.*

Exercise 1.2.11 (exercise for definition 1.2.9)**.** Compare the definition here to the one in Mendelson (definition 4.3, chapter 2). Are the two definitions equivalent? Prove it!

Lemma 1.2.12 (lemma 4.3, chapter 3)**.** *Given a subset A of a topological space and a closed set F containing A, $\bar{A} \subset F$.*

proof skeleton. Recall that $\bar{A}$ contains all the points of A and all the limit points of A. Given $A \subset F$ and to prove $\bar{A} \subset F$, it is enough to show that all limit points of A are in F, where F is closed. (We have do something similar, remember it? Theorem 6.7 in chapter 2, and also in the lecture note 4. Check it!) In short, for each $x \in F^C \subset A^C$, that F^C is open means that F^C is a neighborhood of x, from which, given $F^C \subset A^C \iff F^C \cap A = \emptyset$, it follows that x is not a limit point of A. That is, for each $x \in F^C$, x is not a limit point of A. That is, all limit points of A are contained in F. We finished the proof. $\square$

Lemma 1.2.13 (lemma 4.4, chapter 3)**.** *Given a subset A of a topological space and a point $x \notin \bar{A}$, then $x \notin F$ for some closed set F containing A.*

proof skeleton. Lemma 1.2.12 gives you a closed set F containing A, while this lemma, in contrast, requires you to find a closed set F, which contains A. Think it: if $x \notin \bar{A}$, that is, $x \notin A$ and x is not a limit point of A, what will happen? This will lead you to the desired closed set F. □

Theorem 1.2.14 (theorem 4.5, chapter 3)**.** *Given a subset A of a topological space, $\bar{A} = \cap_{a \in I} F_a$ where $\{F_a\}_{a \in I}$ is the family of all closed sets containing A.*

proof skeleton. Immediate from lemma 1.2.12 and lemma 1.2.13, given that you understand the lemmas and know how to prove equality of two sets. □

Corollary 1.2.15 (corollary from theorem 1.2.14)**.** *$\bar{A}$ is closed.*

Theorem 1.2.16 (theorem 4.6, chapter 3)**.** *A is closed if and only if $A = \bar{A}$.*

proof skeleton. If $A = \bar{A}$, then A is closed since $\bar{A}$ is closed. If A is closed, then from lemma 1.2.12 and $A \subset \bar{A}$ we have $A = \bar{A}$. □

Definition 1.2.17 (Interior of A)**.** The interior of A, denoted by $int(A)$, is the largest open set contained in A.

Definition 1.2.18 (Boundary of A)**.** The boundary of A, denoted by ∂A, is defined as

$$\partial A = \bar{A} \cap (int(A))^C .$$

Remark 1.2.19. The concept of interior and boundary is heavily used in *Convex Geometry*, while closure oftentimes appears in the context of *Algebraic Geometry*.

Definition 1.2.20 (dense)**.** $A \subset X$, A is dense in X if $\bar{A} = X$.

Example 1.2.21. $\mathbb{Q}$ is dense in $\mathbb{R}$ since $\bar{\mathbb{Q}} = \mathbb{R}$.

Definition 1.2.22 (Hausdorff Space, definition 3.3, chapter 3)**.** A topological space $(X, \mathcal{J})$ is called a *Hausdorff space* or is said to satisfy the *Hausdorff axiom*, if for each pair a, b of distinct points of X, there are neighborhoods N and M of a and b respectively, such that $N \cap M = \emptyset$. *Hausdorff space* $(X, \mathcal{J})$ is also called *separable space*.

Example 1.2.23. $\mathbb{R}^n$ with standard topology is Hausdorff.

Example 1.2.24. R with Zariski topology is not seperable.

Definition 1.2.25 (Irreducible Space)**.** If X is not the union of two proper closed sets, i.e., there are not $Y_1, Y_2 \subsetneq X$, which are closed, such that $X = Y_1 \cup Y_2$.

Example 1.2.26. $\mathbb{R}$ is reducible.

Remark 1.2.27. If X is Hausdorff then X is reducible.

Theorem 1.2.28. *If X is irreducible, O is open in X, then O is irreducible and dense.*

Proof. You may want to prove it before the next lecture. □

1.3 Further Reading

3.1-3.6 in Mendelson.