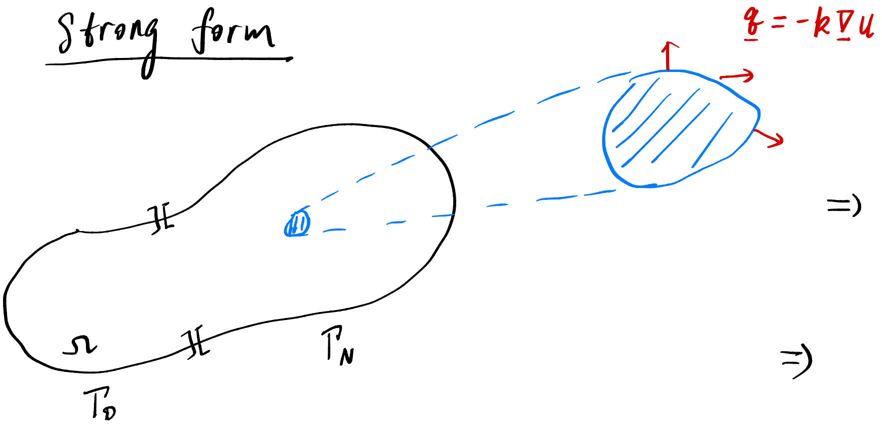
Lec 03 : 2D linear problem

- 2D steady state heat problem
- Isoparametric mapping
- Element technology

2D steady state heat problem

Strong form



$$\overbrace{-\int_S \underline{q}\cdot\underline{n}\,ds}^{\text{heat into system}} + \overbrace{\int_{\Delta V} f\,dx}^{\text{heat source}} = 0$$

$$\Rightarrow \int_S k\underline{\nabla}u\cdot\underline{n}\,ds + \int_{\Delta V} f\,dx = 0$$

$$\Rightarrow \int_{\Delta V} k\,\underline{\nabla}\cdot(\underline{\nabla}u) + f = 0 \quad \forall\, \Delta V$$

$$\Rightarrow -k\Delta u = f$$

$(\Gamma_D \cup \Gamma_N = \partial\Omega \text{ and } \Gamma_D \cap \Gamma_N = \emptyset)$

Heat problem: $-k\Delta u = f \quad \forall\, \underline{x} \in \Omega$

$u = g$ on Γ_D (Dirichlet B.C.)

$-k\underline{\nabla}u\cdot\underline{n} = h$ on Γ_N (Neumann B.C.)

(You cannot impose both conditions simultaneously!)

Weak form

Multiply both sides with a test function v:

$$\int_\Omega -k\Delta u\, v = -\int_\Omega \underline{\nabla}\cdot(k\underline{\nabla}u\, v) + \int_\Omega k\underline{\nabla}u\cdot\underline{\nabla}v = \int_\Gamma -k\underline{\nabla}u\cdot\underline{n}\, v + \int_\Omega k\underline{\nabla}u\cdot\underline{\nabla}v = \int_\Omega f v$$

trial function $u \in S = \{u : u \in H^1(\Omega),\ u = g \text{ on } \Gamma_D\}$

test function $v \in V = \{v : v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma_D\}$

($H^1(\Omega)$ is Sobolev space with order 1. Details will covered in next lecture)

Therefore, we have: Find $u \in S$ s.t. for all $v \in V$

$$\int_{\Gamma_N} -k\nabla u \cdot \underline{n}\, v + \int_{\Omega} k\nabla u \cdot \nabla v = \int_{\Omega} fv \Rightarrow \underbrace{\int_{\Omega} k\nabla u \cdot \nabla v}_{a(v,u)} = \underbrace{-\int_{\Gamma_N} hv + \int_{\Omega} fv}_{F(v)}$$

Galerkin approximation

trial function $u^h \in S^h = \{ u^h \in H^1(\Omega) : u^h = \sum_{i=1}^{N} u_i \phi_i(\underline{x}),\ u^h = g \text{ on } \Gamma_D \}$

test function $v^h \in V^h = \{ v^h \in H^1(\Omega) : v^h = \sum_{i=1}^{N} v_i \phi_i(\underline{x}),\ v^h = 0 \text{ on } \Gamma_D \}$

Observations: S and V are infinite-dimensional spaces, while $S^h \subset S$ and $V^h \subset V$ are finite-dimensional spaces. N: Total number of DoFs (Degrees of freedom)

$\phi_i(\underline{x})$: Shape function (We will see later how to construct shape functions)

Galerkin problem: Find $u^h \in S^h$ s.t. $a(v^h, u^h) = F(v^h)$ $\forall\, v^h \in V^h$

Matrix form

$$a(v^h, u^h) = a\Big(\sum_{i=1}^{N} v_i \phi_i(\underline{x}), \sum_{i=1}^{N} u_i \phi_i(\underline{x})\Big) = F\Big(\sum_{i=1}^{N} v_i \phi_i(\underline{x})\Big) \quad \forall\, v^h \in V^h$$

Since v^h is arbitrary, we let v^h be $\phi_1(\underline{x})$ and the following holds:

$$a\Big(\phi_1(\underline{x}), \sum_{i=1}^{N} u_i \phi_i(\underline{x})\Big) = F(\phi_1(\underline{x})) \qquad (Eq.1)$$

Similarly, we have:

$$a\Big(\phi_2(\underline{x}), \sum_{i=1}^{N} u_i \phi_i(\underline{x})\Big) = F(\phi_2(\underline{x})) \qquad (Eq.2)$$

$$\vdots$$

$$a\Big(\phi_N(\underline{x}), \sum_{i=1}^{N} u_i \phi_i(\underline{x})\Big) = F(\phi_N(\underline{x})) \qquad (Eq.N)$$

$$\Rightarrow \underset{N\times N}{[K]}\,\underset{N}{[U]} = \underset{N}{[F]}$$

$K_{ij} = a(\phi_i(\underline{x}), \phi_j(\underline{x}))$ Global stiffness matrix

$U_j = u_j$ Unknown DoF to be solved

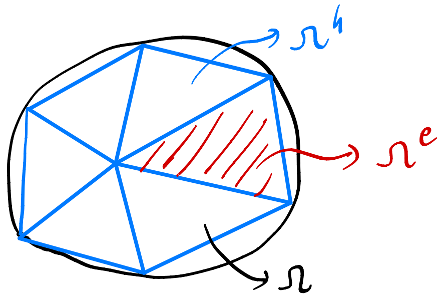
$F_i = F(\phi_i(\underline{x}))$ Right-hand-side vector

Remarks: ① Eq. 1 ~ Eq. N can be used to deduce back the Galerkin problem,

Therefore, Galerkin approx. $\Longleftrightarrow$ Matrix form

② Neumann B.C. is already reflected in the above matrix form, but Dirichlet B.C. needs to be imposed later.

③ **Claim:** For $\Omega \approx \Omega^h = \cup \Omega^e$, if we can evaluate $a(\phi_i, \phi_j)$ on any finite element cell Ω^e for any $i, j \in \{1, \cdots N\}$, we will be able to construct $[K]$.



Assembly procedure

for $e = 1, 2, \cdots, n_{el}$

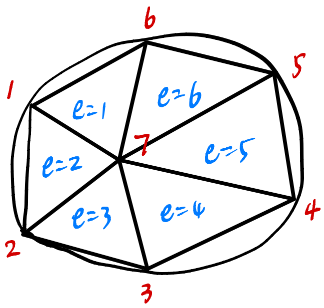
  for $\alpha, \beta = 1, 2, \cdots, n_{dof}$

    find $K^e_{\alpha\beta} = \int_{\Omega^e} k \nabla\phi_\alpha \cdot \nabla\phi_\beta$, do assembling: $K_{ij} = K_{ij} + K^e_{\alpha\beta}$

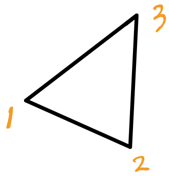
$K^e_{\alpha\beta}$: element stiffness matrix K_{ij}: global stiffness matrix

Note: A connectivity map is needed to map from (e, α) to i.
(or (e, β) to j)

Ex.



global DoF numbering



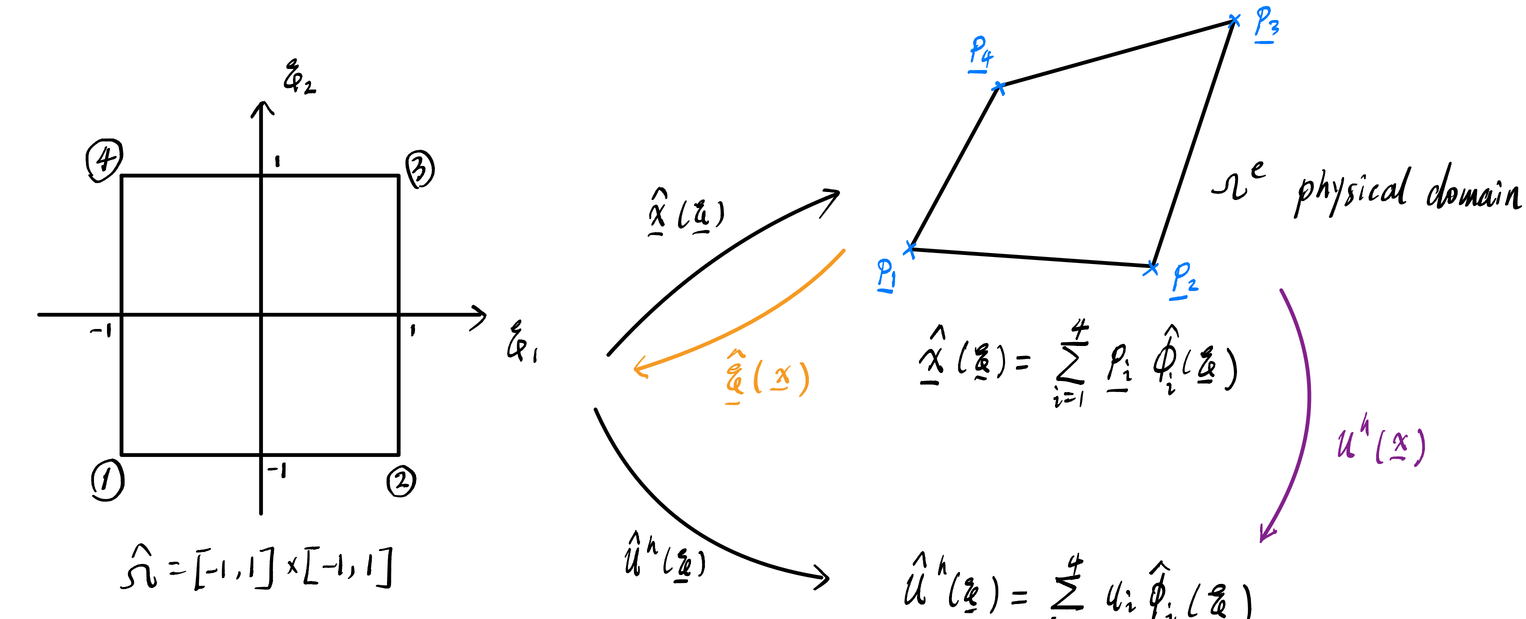
local DoF numbering

	$\alpha=1$	$\alpha=2$	$\alpha=3$
e=1	1	7	6
e=2	1	2	7
e=3	2	3	7
e=4	3	4	7
e=5	4	5	7
e=6	5	6	7

connectivity map

Next, we will show how to evaluate $\int_{\Omega^e} k\, \nabla\phi_\alpha \cdot \nabla\phi_\beta$.

Isoparametric mapping



$$\hat{\Omega} = [-1,1]\times[-1,1]$$

parametric/parent domain

$$\hat{x}(\xi) = \sum_{i=1}^{4} p_i\, \hat{\phi}_i(\xi)$$

$$\hat{u}^h(\xi) = \sum_{i=1}^{4} u_i\, \hat{\phi}_i(\xi)$$

$$\hat{\phi}_1(\xi) = (\xi_1-1)(\xi_2-1)/4$$

$$\hat{\phi}_2(\xi) = -(\xi_1+1)(\xi_2-1)/4$$

$$\hat{\phi}_3(\xi) = (\xi_1+1)(\xi_2+1)/4$$

$$\hat{\phi}_4(\xi) = -(\xi_1-1)(\xi_2+1)/4$$

Remarks:

① Such element is called isoparametric element (both spatial coordinate $\hat{x}$ and field variable $\hat{u}^h$ are approximated using the same set of shape functions $\{\hat{\phi}_i\}$.

② It's possible to define the inverse map $\hat{\xi}(x)$. Then shape functions on physical domain can be defined as

$$\phi_i(x) := \hat{\phi}_i(\xi) = \hat{\phi}_i(\hat{\xi}(x)), \text{ and } u^h(x) = \sum_{i=1}^{4} u_i\, \phi_i(x).$$

Some useful math preliminaries:

Def. (Jacobian matrix) Consider a vector function $f: \mathbb{R}^n \to \mathbb{R}^m$ that inputs a vector and outputs another vector. Then the derivative of f is denoted as $J_f(x): \mathbb{R}^n \to \mathbb{R}^{m\times n}$. If we evaluate this derivative at a certain point $x \in \mathbb{R}^n$, we get the Jacobian matrix:

$$[J_f(x)] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

Def. (Jacobian determinant) If $m=n$, the determinant of Jacobian matrix is $j: \mathbb{R}^n \to \mathbb{R}$, $j(\underline{x}) = \det(\underline{\underline{J}}_f(\underline{x}))$, known as Jacobian determinant.

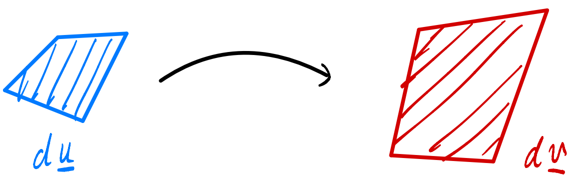
Note: In literature, "the Jacobian" can mean $\underline{\underline{J}}_f$ or j depending on the context.

<u>Proposition</u> As a consequence of the inverse function theorem that if $\hat{\underline{x}}: \hat{\Omega} \to \Omega^e$ is i) bijective (one-to-one and onto); ii) $\hat{\underline{x}} \in C^k$, $k \geq 1$; iii) $j(\underline{\xi}) > 0$ for all $\underline{\xi} \in \hat{\Omega}$, then the inverse mapping $\hat{\underline{\xi}} = \hat{\underline{x}}^{-1}: \Omega^e \to \hat{\Omega}$ exists and $\hat{\underline{\xi}} \in C^k$.

<u>Theorem</u> (Change of variables) Let $\Omega \subset \mathbb{R}^n$ be an open set, $\underline{\varphi}: \Omega \to \mathbb{R}^n$ be a vector-valued function, and $f: \underline{\varphi}(\Omega) \to \mathbb{R}$. Subject to some regular technical conditions:

$$\int_{\underline{\varphi}(\Omega)} f(\underline{v})\, d\underline{v} = \int_{\Omega} f(\underline{\varphi}(\underline{u}))\, |j(\underline{u})|\, d\underline{u} \qquad (\underline{v} = \underline{\varphi}(\underline{u}))$$

Note: $dv_1 dv_2 \cdots dv_n = |j(\underline{u})| = |\det(\underline{\underline{J}}_{\varphi}(\underline{u}))|\, du_1 du_2 \cdots du_n$



$|j(\underline{u})| = \dfrac{d\underline{v}}{d\underline{u}}$ is <u>the ratio of areas</u>

Now, let's go back to $K^e_{\alpha\beta} = \int_{\Omega^e} k\, \underline{\nabla}\phi_\alpha(\underline{x}) \cdot \underline{\nabla}\phi_\beta(\underline{x})\, d\underline{x}$, we will transform this integral to the parametric domain $\hat{\Omega}$:

(recall $\phi_\alpha(\underline{x}) = \hat{\phi}_\alpha(\hat{\underline{\xi}}(\underline{x}))$)

$$\int_{\Omega^e} k\, \underline{\nabla}\phi_\alpha(\underline{x}) \cdot \underline{\nabla}\phi_\beta(\underline{x})\, d\underline{x} = \int_{\Omega^e} k \frac{\partial \phi_\alpha}{\partial x_j} \frac{\partial \phi_\beta}{\partial x_j}\, d\underline{x} = \int_{\Omega^e} k \left(\frac{\partial \hat{\phi}_\alpha}{\partial \xi_i} \frac{\partial \hat{\xi}_i}{\partial x_j}\right)\left(\frac{\partial \hat{\phi}_\beta}{\partial \xi_i} \frac{\partial \hat{\xi}_i}{\partial x_j}\right) d\underline{x}$$

$$= \int_{\Omega^e} k \left(\begin{bmatrix} \frac{\partial \hat{\phi}_\alpha}{\partial \xi_1} & \frac{\partial \hat{\phi}_\alpha}{\partial \xi_2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial \hat{\xi}_1}{\partial x_1} & \frac{\partial \hat{\xi}_1}{\partial x_2} \\ \frac{\partial \hat{\xi}_2}{\partial x_1} & \frac{\partial \hat{\xi}_2}{\partial x_2} \end{bmatrix} \right) \cdot \left(\begin{bmatrix} \frac{\partial \hat{\phi}_\beta}{\partial \xi_1} & \frac{\partial \hat{\phi}_\beta}{\partial \xi_2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial \hat{\xi}_1}{\partial x_1} & \frac{\partial \hat{\xi}_1}{\partial x_2} \\ \frac{\partial \hat{\xi}_2}{\partial x_1} & \frac{\partial \hat{\xi}_2}{\partial x_2} \end{bmatrix} \right) dx_1 dx_2$$

$$= \int_{\hat{\Omega}} k \left(\left[\frac{\partial \hat{\phi}_\alpha}{\partial \xi_1} \;\; \frac{\partial \hat{\phi}_\alpha}{\partial \xi_2} \right] \cdot \underbrace{\begin{bmatrix} \frac{\partial \hat{x}_1}{\partial \xi_1} & \frac{\partial \hat{x}_1}{\partial \xi_2} \\ \frac{\partial \hat{x}_2}{\partial \xi_1} & \frac{\partial \hat{x}_2}{\partial \xi_2} \end{bmatrix}^{-1}}_{= \left[\underline{\underline{J}}_{\hat{x}}(\underline{\xi}) \right]^{-1}} \right) \cdot \left(\left[\frac{\partial \hat{\phi}_\beta}{\partial \xi_1} \;\; \frac{\partial \hat{\phi}_\beta}{\partial \xi_2} \right] \cdot \begin{bmatrix} \frac{\partial \hat{x}_1}{\partial \xi_1} & \frac{\partial \hat{x}_1}{\partial \xi_2} \\ \frac{\partial \hat{x}_2}{\partial \xi_1} & \frac{\partial \hat{x}_2}{\partial \xi_2} \end{bmatrix}^{-1} \right) \cdot \underbrace{|j(\underline{\xi})|}_{= \left| \det\left(\underline{\underline{J}}_{\hat{x}}(\underline{\xi}) \right) \right|} \, d\xi_1 \, d\xi_2$$

Recall $\underline{\hat{x}}(\underline{\xi}) = \sum_{i=1}^{4} \underline{p}_i \, \hat{\phi}_i(\underline{\xi})$, $\Rightarrow \underbrace{\left[\underline{\underline{J}}_{\hat{x}}(\underline{\xi}) \right]}_{2\times 2} = \sum_{i=1}^{4} \underbrace{\begin{bmatrix} p_{i1} \\ p_{i2} \end{bmatrix}}_{2\times 1} \cdot \underbrace{\left[\frac{\partial \hat{\phi}_i}{\partial \xi_1} \;\; \frac{\partial \hat{\phi}_i}{\partial \xi_2} \right]}_{1\times 2}$.

Observation: To compute $K^e_{\alpha\beta}$, we only need the shape function gradient $\underline{\nabla}_{\xi} \hat{\phi}_i(\underline{\xi})$!

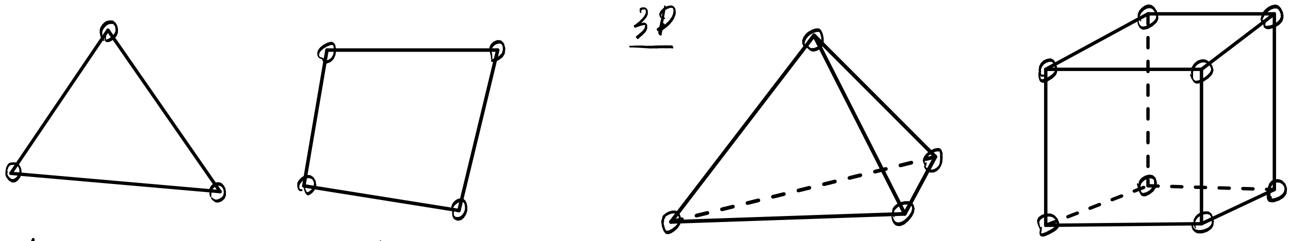
(or $\left[\underline{\nabla}_{\xi} \hat{\phi}_i(\underline{\xi}) \right]^T = \left[\frac{\partial \hat{\phi}_i}{\partial \xi_1} \;\; \frac{\partial \hat{\phi}_i}{\partial \xi_2} \right]$)

We need a systematic way to construct $\hat{\phi}_i(\underline{\xi})$ over $\hat{\Omega}$.

Element technology

Finite element types

2D   3D



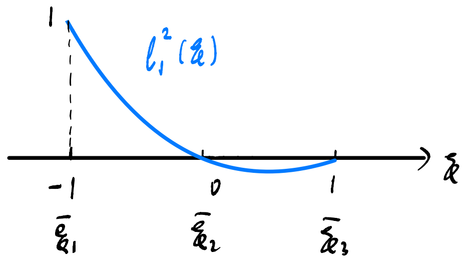
triangle   quadrilateral   tetrahedron   hexahedron

Lagrange polynomials

(n_n: number of nodes)

$$l_a^{n_n - 1}(\xi) = \frac{\prod_{b=1, b\neq a}^{n_n} (\xi - \bar{\xi}_b)}{\prod_{b=1, b\neq a}^{n_n} (\xi_a - \bar{\xi}_b)} = \frac{(\xi - \bar{\xi}_1)(\xi - \bar{\xi}_2)\cdots(\xi - \bar{\xi}_{a-1})(\xi - \bar{\xi}_{a+1})\cdots(\xi - \bar{\xi}_{n_n})}{(\bar{\xi}_a - \bar{\xi}_1)(\bar{\xi}_a - \bar{\xi}_2)\cdots(\bar{\xi}_a - \bar{\xi}_{a-1})(\bar{\xi}_a - \bar{\xi}_{a+1})\cdots(\bar{\xi}_a - \bar{\xi}_{n_n})}$$

Ex.



$$l_1^2(\xi) = \frac{(\xi - \bar{\xi}_2)(\xi - \bar{\xi}_3)}{(\bar{\xi}_1 - \bar{\xi}_2)(\bar{\xi}_1 - \bar{\xi}_3)} = \frac{(\xi - 0)(\xi - 1)}{(-1-0)(-1-1)} = \frac{1}{2}\xi(\xi - 1)$$

Remarks: * The order p of the polynomial $\ell_a^{n_n-1}$ is $p = n_n - 1$, so we can also write ℓ_a^p.

* $\ell_a^p(\bar{\xi}_b) = \delta_{ab} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$ (Kronecker-δ property)

* $\sum_{a=1}^{n_n} \ell_a^{n_n-1}(\xi) = 1$ (partition of unity)

Proof (Patition of unity):

Define $p(\xi) = \sum_{a=1}^{n_n} \ell_a^{n_n-1}(\xi) - 1$, then $p(\xi)$ is a polynomial with order $n_n - 1$.

Also $p(\xi_a) = 0$ for $a = 1, 2, \cdots, n_n$. Then $p(\xi)$ has n_n roots. Therefore $p(\xi) = 0$.

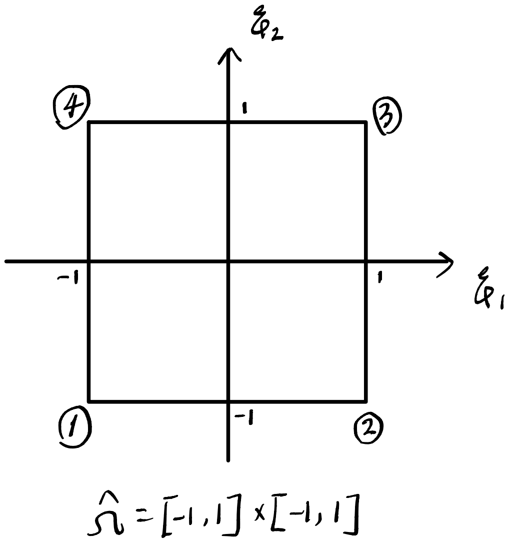
Shape function: We can define 1D shape functions with n_n nodes to be:

$$\hat{\phi}_a(\xi) = \ell_a^{n_n-1}(\xi), \quad a = 1, 2, \cdots, n_n$$

We can define 2D shape functions with $n_n \times n_n$ nodes to be:

$$\hat{\phi}_a(\underline{\xi}) = \hat{\phi}_a(\xi_1, \xi_2) = \ell_b^{n_n-1}(\xi_1) \cdot \ell_c^{n_n-1}(\xi_2).$$

Ex. The 2D shape functions for first-order quadrilateral element (Q4) are



$\hat{\Omega} = [-1,1] \times [-1,1]$

$$\hat{\phi}_1(\xi_1, \xi_2) = \ell_1^1(\xi_1)\, \ell_1^1(\xi_2) = (\xi_1 - 1)(\xi_2 - 1)/4$$

$$\hat{\phi}_2(\xi_1, \xi_2) = \ell_2^1(\xi_1)\, \ell_1^1(\xi_2) = -(\xi_1 + 1)(\xi_2 - 1)/4$$

$$\hat{\phi}_3(\xi_1, \xi_2) = \ell_2^1(\xi_1)\, \ell_2^1(\xi_2) = (\xi_1 + 1)(\xi_2 + 1)/4$$

$$\hat{\phi}_4(\xi_1, \xi_2) = \ell_1^1(\xi_1)\, \ell_2^1(\xi_2) = -(\xi_1 - 1)(\xi_2 + 1)/4$$