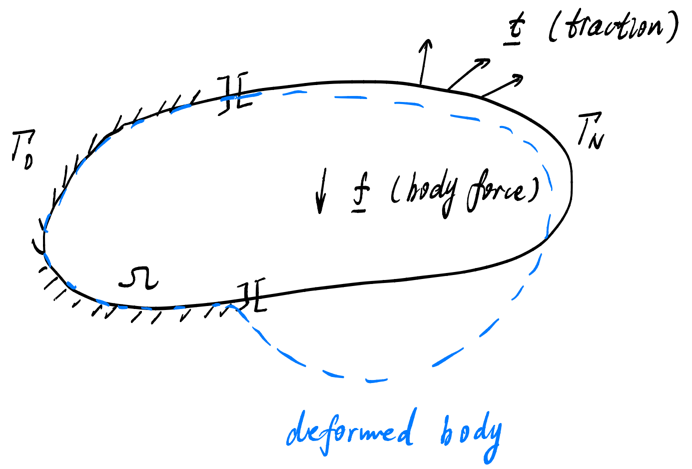
Lec 06: Linear elasticity; Spurious solutions

- Linear elasticity
- Incompressible elasticity (volume locking)
- Some other issues (hourglassing, shear locking)

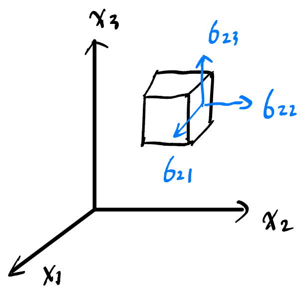
Linear elasticity

Strong form



$\Omega \subset \mathbb{R}^{n_{sd}}$

n_{sd}: spatial dimension (e.g., $n_{sd} = 3$)



$\underline{\underline{\sigma}}$: Cauchy stress tensor

Problem statement: Find displacement $\underline{u} : \Omega \to \mathbb{R}^{n_{sd}}$ such that

$$\begin{cases} \underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{f} = \underline{0} & \text{in } \Omega \quad \text{(equilibrium)} \\ \underline{u} = \underline{g} & \text{on } \Gamma_D \quad \text{(Dirichlet B.C.)} \\ \underline{n} \cdot \underline{\underline{\sigma}} = \underline{t} & \text{on } \Gamma_N \quad \text{(Neumann B.C.)} \end{cases} \quad \text{or} \quad \begin{cases} \sigma_{ij,i} + f_j = 0 & \text{in } \Omega \\ u_i = g_i & \text{on } \Gamma_D \\ n_i \sigma_{ij} = t_j & \text{on } \Gamma_N \end{cases}$$

Constitutive law: $\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}}$ (or $\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$)

Kinematic relation: $\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{u}\underline{\nabla} + \underline{\nabla}\underline{u})$ (or $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = u_{(i,j)}$)

Let us assume the body is <u>isotropic</u>, then

$$C_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda \delta_{ij}\delta_{kl} \Rightarrow \sigma_{ij} = \lambda u_{k,k} \delta_{ij} + 2\mu u_{(i,j)}$$

where λ and μ are the Lamé parameters, and μ is shear modulus often denoted by G. We can represent λ and μ using Young's modulus E and Poisson's ratio ν:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad , \quad \mu = G = \frac{E}{2(1+\nu)}$$

Weak form

Trial function space $S = \{ \underline{u} \mid \underline{u} \in [H^1(\Omega)]^{n_{sd}},\ \underline{u} = \underline{g} \text{ on } \Gamma_D \}$

This means $(u_1, u_2, u_3) \in H^1(\Omega) \times H^1(\Omega) \times H^1(\Omega) \quad \left(\underline{u}(\underline{x}) = \sum_{i=1}^{n_{sd}} u_i(\underline{x})\, \underline{e}_i \right)$

Test function space $V = \{ \underline{v} \mid \underline{v} \in [H^1(\Omega)]^{n_{sd}},\ \underline{v} = \underline{0} \text{ on } \Gamma_D \}$

Start with $\displaystyle \int_\Omega (\nabla \cdot \underline{\underline{\sigma}}) \cdot \underline{v} + \int_\Omega \underline{f} \cdot \underline{v} = \underline{0} \quad \forall\, \underline{v} \in V$

$$\Rightarrow \int_\Omega \sigma_{ij,i}\, v_j + \int_\Omega f_j v_j = 0 \Rightarrow \int_{\partial\Omega} n_i\, \sigma_{ij}\, v_j - \int_\Omega \sigma_{ij}\, v_{j,i} + \int_\Omega f_j v_j = 0$$

$$\Rightarrow \int_\Omega \sigma_{ij}\, v_{j,i} = \int_{\Gamma_N} t_j v_j + \int_\Omega f_j v_j \quad \text{or} \quad a(\underline{v}, \underline{u}) = f(\underline{v}) \quad \text{(weak form)}$$

Let us work on $\displaystyle \int_\Omega \sigma_{ij}\, v_{j,i}$ a little further:

Note that $\underline{\underline{\sigma}}$ is a symmetric tensor, i.e., $\sigma_{ij} = \sigma_{ji}$, then

$$\sigma_{ij} v_{j,i} = \sigma_{ij} v_{(i,j)} = C_{ijkl}\, u_{(k,l)}\, v_{(i,j)} = \Big(\mu (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\, \delta_{ij}\delta_{kl} \Big)\, u_{(k,l)}\, v_{(i,j)}$$

$$= v_{(i,j)}\, \bar{C}_{ijkl}\, u_{(k,l)} + v_{i,i}\, \lambda\, u_{k,k} \quad , \text{ where } \bar{C}_{ijkl} = \mu (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Therefore $a(\underline{v}, \underline{u})$ is a symmetric bilinear form.

Also, $\sigma_{ij} v_{j,i} = v_{i,j} \bar{C}_{ijkl} u_{k,l} + v_{i,i} \lambda u_{k,k}$ due to symmetry of $\bar{C}_{ijkl}$.

Galerkin approximation

Trial function space $S^h = \left\{ \underline{u}^h \in S \,\middle|\, \underline{u}^h = \sum_{A=1}^{n_n} \underline{u}^A \phi^A(\underline{x}) \right\} \subset S$

Test function space $V^h = \left\{ \underline{v}^h \in V \,\middle|\, \underline{v}^h = \sum_{A=1}^{n_n} \underline{v}^A \phi^A(\underline{x}) \right\} \subset V$

Here, n_n : number of total nodes (# DoF $= n_n \times n_{sd}$)

ϕ^A : shape function (constructed just like before)

$\underline{u}^A = \sum_{i=1}^{n_{sd}} u_i^A \underline{e}_i$, u_i^A : unknown DoF

Galerkin problem statement: Find $u^h \in S^h$ s.t. $a(\underline{v}^h, \underline{u}^h) = F(\underline{v}^h)$ $\forall\, v^h \in V^h$

Matrix form

$$a\left(\sum_{A=1}^{n_n} \underline{v}^A \phi^A(\underline{x}), \sum_{A=1}^{n_n} \underline{u}^A \phi^A(\underline{x}) \right) = F\left(\sum_{A=1}^{n_n} \underline{v}^A \phi^A(\underline{x}) \right) \quad \forall\, v^h \in V^h$$

$$\Leftrightarrow \underbrace{\left\{ [\underline{v}^1]^T \, [\underline{v}^2]^T \cdots [\underline{v}^{n_n}]^T \right\}}_{\text{arbitrary}} \overbrace{\left(\begin{bmatrix} [\underline{\underline{K}}^{11}] & [\underline{\underline{K}}^{12}] & \cdots & [\underline{\underline{K}}^{1n_n}] \\ [\underline{\underline{K}}^{21}] & [\underline{\underline{K}}^{22}] & \cdots & [\underline{\underline{K}}^{2n_n}] \\ \vdots & & \ddots & \vdots \\ [\underline{\underline{K}}^{n_n1}] & [\underline{\underline{K}}^{n_n2}] & \cdots & [\underline{\underline{K}}^{n_nn_n}] \end{bmatrix} \begin{Bmatrix} [\underline{u}^1] \\ [\underline{u}^2] \\ \vdots \\ [\underline{u}^{n_n}] \end{Bmatrix} - \begin{Bmatrix} [\underline{F}^1] \\ [\underline{F}^2] \\ \vdots \\ [\underline{F}^{n_n}] \end{Bmatrix} \right)}^{\text{must be zero (matrix form)}} = 0$$

where $[\underline{v}^A]^T = [v_1^A, v_2^A]$, $[\underline{u}^B] = \begin{bmatrix} u_1^B \\ u_2^B \end{bmatrix}$, $\underset{1\times 2}{[\underline{v}^A]^T} \underset{2\times 2}{[\underline{\underline{K}}^{AB}]} \underset{2\times 1}{[\underline{u}^B]} = a(\underline{v}^A \phi^A(\underline{x}), \underline{u}^B \phi^B(\underline{x}))$

$$\Rightarrow a(\underline{v}^A \phi^A(\underline{x}), \underline{u}^B \phi^B(\underline{x})) = \int_\Omega v_i^A \frac{\partial \phi^A}{\partial x_j} \bar{C}_{ijkl} \frac{\partial \phi^B}{\partial x_l} u_k^B + v_i^A \frac{\partial \phi^A}{\partial x_i} \lambda \frac{\partial \phi^B}{\partial x_k} u_k^B$$

$$\Rightarrow K_{ik}^{AB} = \int_\Omega \frac{\partial \phi^A}{\partial x_j} \bar{C}_{ijkl} \frac{\partial \phi^B}{\partial x_l} + \frac{\partial \phi^A}{\partial x_i} \lambda \frac{\partial \phi^B}{\partial x_k}$$

(Can you find this integral ?)

$$\text{Also, } \overset{1\times 2}{[\underline{v}^A]^T} \overset{2\times 1}{[\underline{F}^A]} = F(\underline{v}^A \phi^A(x)) = \int_{\Gamma^N} t_j v_j^A \phi^A + \int_\Omega f_j v_j^A \phi^A$$

$$\Rightarrow \bar{F}_i^A = \int_{\Gamma^N} t_i \phi^A + \int_\Omega f_i \phi^A$$

After imposing Dirichlet B.C., you may solve for $\begin{Bmatrix} [\underline{u}^1] \\ [\underline{u}^2] \\ \vdots \\ [u^{n_n}] \end{Bmatrix}$.

Incompressible elasticity

Problem statement: Find displacement $\underline{u}(\underline{x})$ and pressure $p(\underline{x})$ such that

$$\begin{cases} \underline{\nabla}\cdot\underline{\underline{\sigma}} + \underline{f} = \underline{0} \text{ in } \Omega \text{ (equilibrium)} \\ \underline{\nabla}\cdot\underline{u} + p/\lambda = 0 \text{ in } \Omega \text{ (constraint)} \\ \underline{u} = \underline{g} \text{ on } \Gamma_D \text{ (Dirichlet B.C.)} \\ \underline{n}\cdot\underline{\underline{\sigma}} = \underline{t} \text{ on } \Gamma_N \text{ (Neumann B.C.)} \end{cases} \quad \text{or} \quad \begin{cases} \sigma_{ij,i} + f_j = 0 \text{ in } \Omega \\ u_{i,i} + p/\lambda = 0 \text{ in } \Omega \\ u_i = g_i \text{ on } \Gamma_D \\ n_i \sigma_{ij} = t_j \text{ on } \Gamma_N \end{cases}$$

Note: $\sigma_{ij} = \lambda u_{k,k} \delta_{ij} + 2\mu u_{(i,j)} = -p\delta_{ij} + 2\mu u_{(i,j)}$

This problem statement works for

① Incompressible solid: $\lambda \to \infty$ $(\underline{\nabla}\cdot\underline{u} = 0)$

② Compressible solid: λ is finite $(\underline{\nabla}\cdot\underline{u} + p/\lambda = 0)$

This (u, p) formulation will be the same as the (u) only formulation.

The weak form is to find $\underline{u} \in S$ and $p \in \mathcal{L}^2(\Omega)$ such that

$$\int_\Omega v_{i,j} \bar{C}_{ijkl} u_{k,l} - \int_\Omega v_{i,i} p - \int_\Omega q (u_{i,i} + p/\lambda) = \int_{\Gamma_N} t_i v_i + \int_\Omega f_i v_i$$

for $\forall \underline{v} \in V$ and $q \in \mathcal{L}^2(\Omega)$.

For Galerkin approximation, let $\underline{u}^h \in S^h$ and $\underline{v}^h \in V^h$ as usual.

$$p^h \in P^h = \{ p^h \in \mathcal{L}^2(\Omega) \mid p^h = \sum_{\tilde{A}=1}^{\tilde{n}_n} p^{\tilde{A}} \psi^{\tilde{A}}(\underline{x}) \}$$

$$q^h \in Q^h = \{ q^h \in \mathcal{L}^2(\Omega) \mid q^h = \sum_{\tilde{A}=1}^{\tilde{n}_n} q^{\tilde{A}} \psi^{\tilde{A}}(x) \}$$

(P^h and Q^h are the same)

Show (in homework) that the matrix form is:

$$\begin{array}{r} 2n_n \{ \\ \tilde{n}_n \{ \end{array} \begin{bmatrix} K & G \\ G^T & M \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix} \begin{array}{l} \} \, 2n_n \\ \} \, \tilde{n}_n \end{array}$$

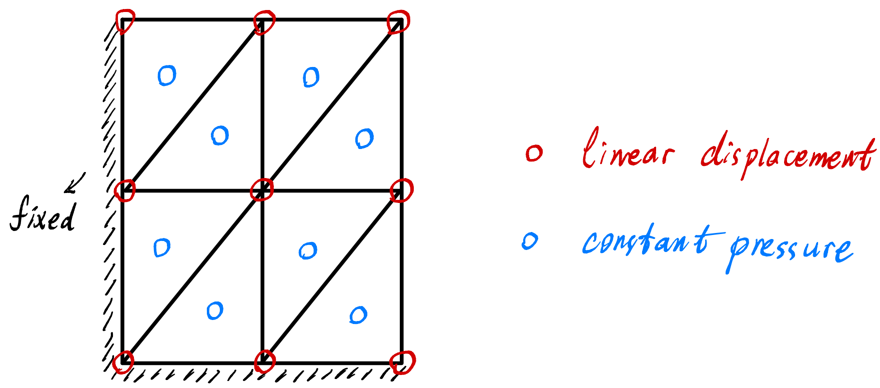
$$\begin{array}{cc} \underbrace{}_{2n_n} & \underbrace{}_{\tilde{n}_n} \end{array}$$

Find K_{ij}^{AB}, $G_i^{A\tilde{B}}$, $M^{\tilde{A}\tilde{B}}$, F_i^A

Volume locking

We will illustrate a <u>fundamental difficulty</u> using a simple example about incompressible solid. The example will show that you cannot arbitrarily choose approximate function spaces for u and p (S^h and P^h).

Consider a mesh:



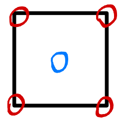
For incompressibility, we have $\lambda \to \infty$ and $\int_\Omega q^h \, \underline{\nabla} \cdot \underline{u}^h = \sum_{e=1}^{n_{el}} \int_{\Omega^e} q^h \, \underline{\nabla} \cdot \underline{u}^h = 0$,

where q^h is an arbitrary constant on each triangle.

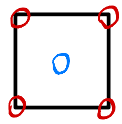
$\Rightarrow$ $\int_{\Omega^e} \underline{\nabla} \cdot \underline{u}^h = 0$ for any element Ω^e. This means the area of each triangle must NOT change.

In this example, it further implies that all displacement nodes o must be zero. We call this <u>volume locking</u> (or <u>mesh locking</u>).

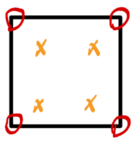
Remarks:

* Changing to a suitable pair of finite elements (e.g.,  bilinear displacements and constant pressure) can resolve the volume locking issue.

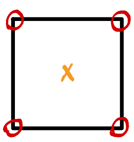
* A systematic study is beyound the scope of this course. Refer to the Babuška-Brezzi (or LBB) condition.

* In the nearly incompressible case (λ is finite and a very large number), the locking issue does NOT go away.
  - You may select a suitable scheme of finite elements (e.g., )
  - Alternatively, you may use the (u) only formulation and apply the "selective reduced integration" scheme:

$$K^{AB}_{ik} = \int_\Omega \frac{\partial \phi^A}{\partial x_j} \bar{C}_{ijkl} \frac{\partial \phi^B}{\partial x_l} + \int_\Omega \frac{\partial \phi^A}{\partial x_i} \lambda \frac{\partial \phi^B}{\partial x_k}$$



2×2 Gaussian quadrature



One-point Gaussian quadrature

  - It has been shown that the two strategies above are equivalent. See details in Fig 4.4.3 in Hughes book.

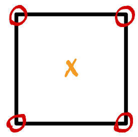
<u>Some other issues</u>

We will discuss some othe difficulties of the Galerkin FEM and possible remedies.

Hourglassing

Reduced integration may lead to spurious <u>zero-energy modes</u>.

Consider a 2D bilinear elasticity element, using one-point Gaussian quadrature for integration:



One-point Gaussian quadrature

Elastic energy can be defined as $E = \frac{1}{2} a(\underline{u}, \underline{u})$

reminder:

$$\underline{u}^h = \sum_{A=1}^{n_n} \underline{u}^A \phi^A(\underline{x})$$

$$u^h_{i,j} = \sum_{A=1}^{n_n} u_i^A \frac{\partial \phi^A}{\partial x_j}$$

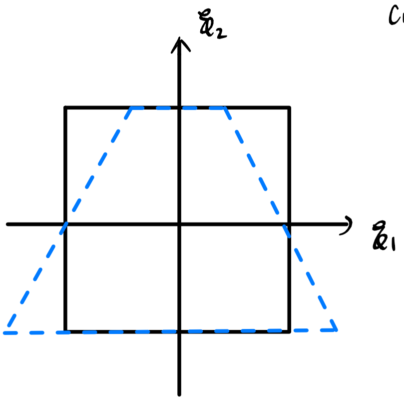
Then,

$$\frac{1}{2} a(\underline{u}^h, \underline{u}^h) = \frac{1}{2} \int_\Omega \left(u^h_{i,j} \bar{C}_{ijkl} u^h_{k,l} + u^h_{i,i} \lambda u^h_{k,k} \right) d\underline{x}$$

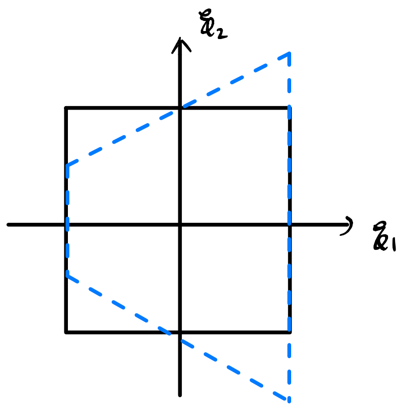
$$\approx \frac{1}{2} \left(u^h_{i,j} \bar{C}_{ijkl} u^h_{k,l} + u^h_{i,i} \lambda u^h_{k,k} \right) \Big|_{\underline{\xi}=\underline{0}} \cdot j(\underline{\xi}=\underline{0}) \cdot \underset{\substack{\uparrow \\ \text{weight}}}{4}$$

Hourglass mode I: $\underline{u}^h(\underline{\xi}) = \underset{\substack{\uparrow \\ \text{constant}}}{c}\, \xi_1 \xi_2\, \underline{e}_1$

Hourglass mode II: $\underline{u}^h(\underline{\xi}) = \underset{\substack{\uparrow \\ \text{constant}}}{c}\, \xi_1 \xi_2\, \underline{e}_2$



parametric domain
deformed shape



parametric domain
deformed shape

Note: $u^h_{i,j} = \frac{\partial u^h_i}{\partial x_j} = \frac{\partial u^h_i}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_j} \Big|_{\underline{\xi}=\underline{0}} = 0$ for $1 \le i, j \le n_{sd}$ for both mode I and II

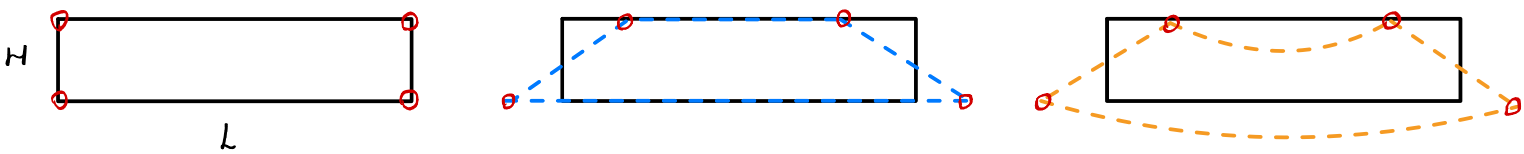
Therefore $E^h = \frac{1}{2} a(\underline{u}^h, \underline{u}^h) = 0$ for the two modes.

Why it is called hourglassing?  Such deformation creates no energy!

The element is considered as too "soft". But we still want to use one-point integration due to efficiency, see Hughes book Section 4.8 for the Kosloff-Frazier element as a solution.

Shear locking

Sometimes the standard Galerkin elements can be too "stiff". One such case occurs for bending of a typical 2D bilinear elasticity element



element (physical domain)

$\frac{L}{H} >> 1$ makes situation worse

element response

$\underline{u}^h(\underline{x}) = c\, x_1 x_2\, \underline{e}_1$

constant

correct response

bending-like

Solution to this challenge: Enrich the element with more modes (see Hughes book Section 4.7)