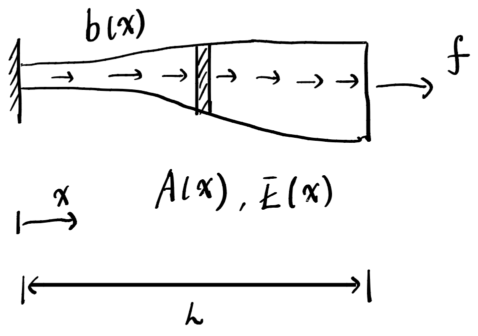
Lec 02 : 1D linear problem

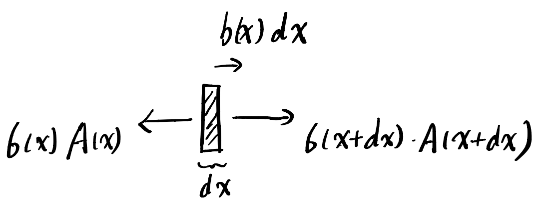
- Strong form
- Weak form
- Energy perspective
- Galerkin approximation
- Matrix form
- Element view
- Assembly
- Analysis

Strong form

A model problem:



* Equilibrium equation:



$$\sigma(x+dx)A(x+dx) - \sigma(x)A(x) + b(x)\,dx = 0$$

$$\Rightarrow \quad \frac{d\, A(x)\,\sigma(x)}{dx} + b(x) = 0$$

* Constitutive law: $\sigma(x) = E(x)\,\varepsilon(x)$ * Kinematic relation: $\varepsilon(x) = \frac{du}{dx}$

Therefore: Find $u: [0, L] \to \mathbb{R}$, s.t. $\frac{d}{dx}\left(AE\frac{du}{dx}\right) + b = 0$ (second-order linear elliptic PDE)

Boundary conditions: $u(0) = g$ (Dirichlet / Essential B.C.)

$$AE\frac{du}{dx}\Big|_{x=L} = f \quad \text{(Neumann / Natural B.C.)}$$

"PDEs are made by God, the boundary conditions by the Devil!" — Alan Turing

Weak form

If $R(x) := \frac{d}{dx}\left(AE\frac{du}{dx}\right) + b = 0$, then $\forall\, w(x) \in \underset{\text{a set of "smooth" functions}}{V}$, $\int_0^L R(x)\,w(x)\,dx = 0$.

$$\int_0^L \frac{d}{dx}\left(AE\frac{du}{dx}\right) w\,dx + \int_0^L b\,w\,dx = 0 \qquad \forall w \in V$$

w: test function
u: trial function

$$\Rightarrow \quad AE\frac{du}{dx}w\Big|_0^L - \int_0^L AE\frac{du}{dx}\frac{dw}{dx}\,dx + \int_0^L bw\,dx = 0 \quad \forall w \in V$$

Now, consider $V = \{ v \mid v \in \text{SMOOTH}([0,L]),\ v(0) = 0 \}$, we have

$$\underbrace{AE\frac{du}{dx}w\Big|_{x=L}}_{f\,w(L)} - \cancelto{0}{AE\frac{du}{dx}w\Big|_{x=0}} - \int_0^L AE\frac{du}{dx}\frac{dw}{dx}\,dx + \int_0^L bw\,dx = 0 \quad \forall w \in V$$

$$\Rightarrow \quad \int_0^L AE\frac{du}{dx}\frac{dw}{dx}\,dx = \int_0^L bw\,dx + f\,w(L) \quad \forall w \in V$$

The solution $u(x) \in S := \{ u(x) \mid u \in \text{SMOOTH}([0,L]),\ u(0) = g \}$

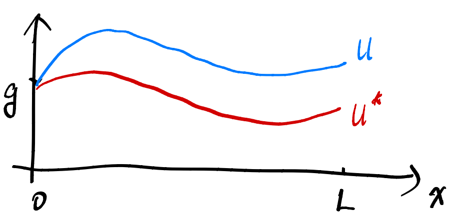
Let $a(w,u) = \int_0^L AE\frac{du}{dx}\frac{dw}{dx}\,dx$ and $F(w) = \int_0^L bw\,dx + f\,w(L) \Rightarrow a(w,u) = F(w)$

Energy perspective

The total potential energy is $\underset{\text{functional}}{\mathcal{E}[u]} = \int_0^L \frac{1}{2}\sigma\varepsilon \underbrace{A\,dx}_{dV} - f\cdot u(L) - \int b u\,dx$

$$\int_0^L \frac{AE}{2}\left(\frac{du}{dx}\right)^2 dx$$

The solution u^* is the one that minimizes $\mathcal{E}$.



u: A possible solution

u^*: The actual solution

$$u = u^* + \alpha w$$

Observations: $u \in S$, $u^* \in S$, $w \in V$, $\alpha \in \mathbb{R}$, $\mathcal{E}[u^*] \leq \mathcal{E}[u]$, $\forall w \in V$

To find u^*, take the functional differential and make it zero for all $w \in V$:

$$\delta\mathcal{E}[u;w] := \lim_{\alpha \to 0}\frac{\mathcal{E}[u+\alpha w] - \mathcal{E}[u]}{\alpha} = \frac{d}{d\alpha}\mathcal{E}[u+\alpha w]\Big|_{\alpha=0} = 0$$

$$\Rightarrow \quad \int_0^L AE\frac{du}{dx}\frac{dw}{dx}\,dx - f\cdot w(L) - \int bw\,dx = 0 \quad \forall w \in V.$$

This is just weak form!

Simple real space analogy

"Strong form": Find $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{R}^2$ s.t. $\begin{cases} u_1 - 1 = 0 \\ u_2 - 2 = 0 \end{cases}$

"Weak form": Find ... s.t. $\begin{cases} (u_1 - 1) v_1 = 0 \\ (u_2 - 2) v_2 = 0 \end{cases}$ $\forall \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$

"Energy perspective": Find ... s.t. $E(u_1, u_2) = \frac{1}{2}(u_1 - 1)^2 + \frac{1}{2}(u_2 - 2)^2$ is minimized

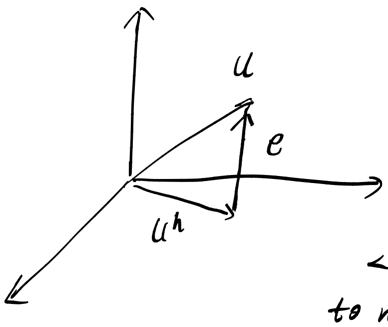
The analogy of the functional differential is $\dfrac{\partial E}{\partial (u_1, u_2)} \cdot (v_1, v_2) = (u_1 - 1) v_1 + (u_2 - 2) v_2 = 0$

Suggestion: If you can't figure out function space, think about their real space analogy.

Finite element roadmap

$$(S) \Longleftrightarrow (W) \Longrightarrow (G) \Longleftrightarrow (M)$$

strong form | weak form | Galerkin approx. | matrix form



u: exact sol.

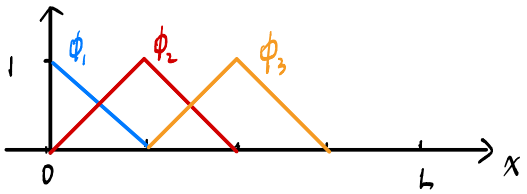
u^h: approx. sol.

$e = u - u^h$ error

$\langle e, w^h \rangle = 0$

to make e small.

Galerkin approximation



$$S^h := \left\{ u^h(x) \,\middle|\, u^h(x) = \sum_{i=1}^{N} u_i \phi_i(x),\ u(0) = g \right\}$$

$$V^h := \left\{ w^h(x) \,\middle|\, w^h(x) = \sum_{i=1}^{N} w_i \phi_i(x),\ w(0) = 0 \right\}$$

Note that $S^h \subset S$, $V^h \subset V$. Both S^h and V^h are finite-dimensional spaces.

For convenience, let trial solution $u^h(x) = g^h(x) + v^h(x)$, $u^h(x) \in S^h$

$g^h(x)$ is given, $g^h(x) = g\,\phi_1(x)$, and $v^h(x) \in V^h$.

Problem statement: Find $u^h = g^h + v^h \in S^h$, s.t. $a(w^h, u^h) = F(w^h)$, $\forall w^h \in V^h$

$\Rightarrow a(w^h, g^h + v^h) = a(w^h, v^h) + a(w^h, g^h) = F(w^h) \quad \forall w^h \in V^h$

$$\Rightarrow \quad a(w^h, v^h) = \tilde{F}(w^h) := F(w^h) - a(w^h, g^h) \quad \forall w^h \in V^h$$

Note that $w^h, v^h \in V^h$ the same space. (If different spaces, "Petrov-Galerkin".)

Matrix form

$$a(w^h, v^h) = a\left(\sum_{i=1}^{N} w_i \phi_i(x), \sum_{i=1}^{N} v_i \phi_i(x)\right) = \tilde{F}(w^h) = \tilde{F}\left(\sum_{i=1}^{N} w_i \phi_i(x)\right) \quad \forall w^h \in V^h$$

$$\Leftrightarrow \quad \begin{aligned} w_1 \, a\left(\phi_1(x), \sum_{i=1}^{N} v_i \phi_i(x)\right) &= w_1 \tilde{F}(\phi_1(x)) \quad \forall w_1 \in \mathbb{R} \\ w_2 \, a\left(\phi_1(x), \sum_{i=1}^{N} v_i \phi_i(x)\right) &= w_2 \tilde{F}(\phi_1(x)) \quad \forall w_2 \in \mathbb{R} \\ &\vdots \\ w_N \, a\left(\phi_1(x), \sum_{i=1}^{N} v_i \phi_i(x)\right) &= w_N \tilde{F}(\phi_1(x)) \quad \forall w_N \in \mathbb{R} \end{aligned}$$

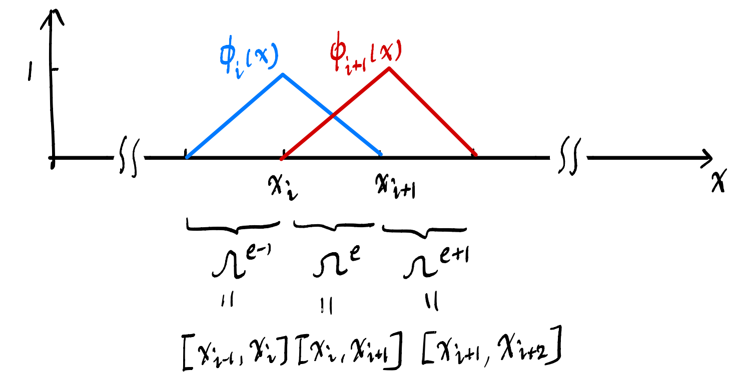
$$\Rightarrow \quad \begin{aligned} \sum_{i=1}^{N} a(\phi_1, \phi_i) v_i &= \tilde{F}(\phi_1) \\ \sum_{i=1}^{N} a(\phi_2, \phi_i) v_i &= \tilde{F}(\phi_2) \\ &\vdots \\ \sum_{i=1}^{N} a(\phi_N, \phi_i) v_i &= \tilde{F}(\phi_N) \end{aligned} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} a(\phi_1, \phi_1) & a(\phi_1, \phi_2) & \cdots & a(\phi_1, \phi_N) \\ a(\phi_2, \phi_1) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a(\phi_N, \phi_1) & \cdots & & a(\phi_N, \phi_N) \end{bmatrix}}_{[K]} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix}}_{[v]} = \underbrace{\begin{bmatrix} \tilde{F}(\phi_1) \\ \tilde{F}(\phi_2) \\ \vdots \\ \tilde{F}(\phi_N) \end{bmatrix}}_{[F]}$$

Just a reminder: $a(\phi_i, \phi_j) = \int_0^L AE \frac{d\phi_i}{dx} \frac{d\phi_j}{dx}$, $\quad \tilde{F}(\phi_i) = \int_0^L b \, \phi_i \, dx + f \cdot \phi_i(L) - \int_0^L AE \frac{d\phi_i}{dx} \frac{d\phi_1}{dx} g \, dx$

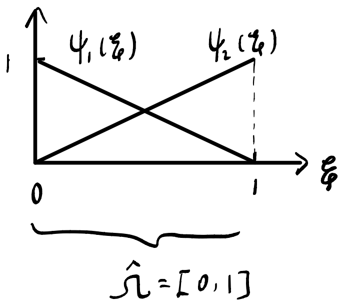
Dirichlet B.C. must be applied, i.e., $v_1 = 0$

$$\Rightarrow \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1N} \\ K_{21} & \ddots & & \vdots \\ \vdots & & & \vdots \\ K_{N1} & \cdots & & K_{NN} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} K_{21} & \cdots & K_{2N} \\ \vdots & \ddots & \vdots \\ K_{N1} & \cdots & K_{NN} \end{bmatrix} \begin{bmatrix} v_2 \\ \vdots \\ N_N \end{bmatrix} = \begin{bmatrix} F_2 \\ \vdots \\ F_N \end{bmatrix}$$

Element view



physical domain $\Omega = \bigcup_{e=1}^{n_{el}} \Omega^e$



$\psi_1(\xi) = 1 - \xi$

$\psi_2(\xi) = \xi$

(shape functions)

parametric/parent domain $\hat{\Omega}$

Instead of calculating something like " $\frac{d\phi_i}{dx}$ " over Ω, we will do it over $\hat{\Omega}$.

Let us focus on element domain $\Omega^e = [x_i, x_{i+1}]$, we define isoparametric mapping

$$\hat{x} : \hat{\Omega} \to \Omega^e, \quad \hat{x}(\xi) = x_i \psi_1(\xi) + x_{i+1} \psi_2(\xi), \text{ so } \hat{x}(0) = x_i, \ \hat{x}(1) = x_{i+1}$$

$$\xi \longmapsto x \qquad \text{(linear interpolation)}$$

$$= x_i (1 - \xi) + x_{i+1} \cdot \xi$$

Consider the inverse $\hat{\xi}(x) = \frac{x - x_i}{x_{i+1} - x_i}$

Then, $\phi_i(x)$ (over Ω^e) can be defined as $\phi_i(x) := \psi_1(\xi) = \psi_1(\hat{\xi}(x))$

$$\Rightarrow \phi_i(x) = 1 - \xi = 1 - \frac{x - x_i}{x_{i+1} - x_i} = \frac{x_{i+1} - x}{x_{i+1} - x_i} \quad \text{(This makes sense, } \phi_i(x_i) = 1, \ \phi_i(x_{i+1}) = 0\text{)}$$

Similarly, $\phi_{i+1}(x)$ (over Ω^e) can be found as $\phi_{i+1}(x) = \frac{x - x_i}{x_{i+1} - x_i}$.

Now, consider $a(\phi_i, \phi_{i+1})$ over Ω^e : $\int_{x_i}^{x_{i+1}} AE \frac{d\phi_i}{dx} \frac{d\phi_{i+1}}{dx} dx$

We can transform the integral to $\hat{\Omega}$

$$\int_{x_i}^{x_{i+1}} AE\,\frac{d\phi_i}{dx}\,\frac{d\phi_{i+1}}{dx}\,dx = \int_0^1 AE\,\frac{d\psi_1}{d\xi}\cdot\frac{d\xi}{dx}\cdot\frac{d\psi_2}{d\xi}\,\frac{d\xi}{dx}\cdot\frac{dx}{d\xi}\cdot d\xi \qquad \left(\xi \mapsto x\right)$$

$$= \int_0^1 \underset{\substack{\downarrow \\ \text{let } \hat{x}(\xi)=\frac{x_i+x_{i+1}}{2}}}{A(\hat{x}(\xi))\,E(\hat{x}(\xi))}\;\underset{\substack{\| \\ -1}}{\frac{d\psi_1}{d\xi}}\cdot\underset{\substack{\| \\ \frac{1}{x_{i+1}-x_i}}}{\left(\frac{d\hat{x}}{d\xi}\right)^{-1}}\cdot\underset{\substack{\| \\ 1}}{\frac{d\psi_2}{d\xi}}\cdot\underset{\substack{\| \\ \frac{1}{x_{i+1}-x_i}}}{\left(\frac{d\hat{x}}{d\xi}\right)^{-1}}\cdot\underset{\substack{\| \\ x_{i+1}-x_i}}{\left(\frac{d\hat{x}}{d\xi}\right)}\,d\xi \qquad \left(\hat{x}(\xi) \text{ is all you need}\right)$$

$$= -\frac{A^{i+1/2}\,E^{i+1/2}}{x_{i+1}-x_i}$$

Assembly

$\underset{N\times N}{[K]}\,\underset{N}{[v]} = \underset{N}{[F]}$, How to find K_{ij} in a computationally efficient way?

The naive way: for $i=1,\dots,N$,

for $j=1,\dots,N$,

$$K_{ij} = a(\phi_i,\phi_j) = \sum_{e=1}^{n_{el}} \int_{\Omega^e} AE\,\frac{d\phi_i}{dx}\,\frac{d\phi_j}{dx}\,dx$$

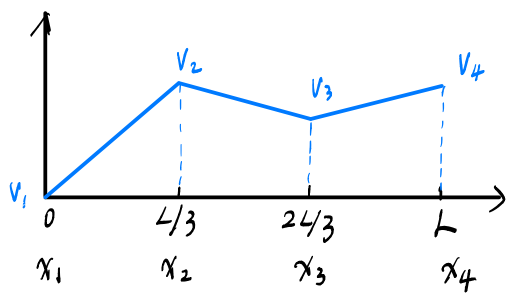
The better way: for $e=1,2,\dots,n_{el}$

find $K^e_{\alpha\beta} = \int_{\Omega^e} AE\,\frac{d\phi_\alpha}{dx}\,\frac{d\phi_\beta}{dx}\,dx$, <u>assemble</u> $K^e_{\alpha\beta}$ to K

$K^e_{\alpha\beta}$ ↑ element stiffness matrix

K ↑ global stiffness matrix

A complete example



Goal: Find $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ so that $u^h(x) = \sum_{i=1}^{4} v_i\,\phi_i(x) + g\,\phi_1(x)$

At element 1 ($\Omega^{e=1} = [x_1, x_2]$), $[K^e] = \dfrac{AE}{x_2 - x_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ (verify this by yourself)

At element 2 ($\Omega^{e=2} = [x_2, x_3]$), $[K^e] = \dfrac{AE}{x_3 - x_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

At element 3 ($\Omega^{e=3} = [x_3, x_4]$), $[K^e] = \dfrac{AE}{x_4 - x_3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

To assemble local $[K^e]$ to global $[K]$, we need <u>connectivity map</u>:

	$\alpha=1$	$\alpha=2$
$e=1$	1	2
$e=2$	2	3
$e=3$	3	4

Read the table as: "The second local DoF ($\alpha=2$) of element 3 ($e=3$) maps to global DoF #4."

(assume AE constant)

Therefore, we have

$$[K] = \frac{AE}{L/3} \begin{bmatrix} 1 & -1 & & \\ -1 & 1+1 & -1 & \\ & -1 & 1+1 & -1 \\ & & -1 & 1 \end{bmatrix} = \frac{AE}{L/3} \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix}$$

Also, $[F] = \begin{bmatrix} b\frac{L}{6} - \frac{AE}{L/3} g \\ b\frac{L}{3} + \frac{AE}{L/3} g \\ b\frac{L}{3} \\ b\frac{L}{6} + f \end{bmatrix}$. Apply Dirichlet B.C., $\dfrac{AE}{L/3} \begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} \frac{bL}{3} + \frac{AE}{L/3} g \\ \frac{bL}{3} \\ \frac{bL}{6} + f \end{bmatrix}$

$\Rightarrow$ solve for $\begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix}$ and $v_1 = 0$.

(Reminder: $\tilde{F}(\phi_i) = \int_0^L b\, \phi_i \, dx + f \cdot \phi_i(L) - \int_0^L AE \dfrac{d\phi_i}{dx} \dfrac{d\phi_1}{dx} g \, dx$)

<u>Analysis</u>

u^* exact sol.

u^h approx. sol.

$e = u - u^h$ error

$a(w^h, e) = a(w^h, u - u^h) = a(w^h, u) - a(w^h, u^h) = F(w^h) - F(w^h) = 0$

$\Rightarrow a(w^h, e) = 0 \quad \forall w^h \in V^h$ (Galerkin orthogonality)

(inner product)

Finite element gives the "best" solution in the assumed function space!