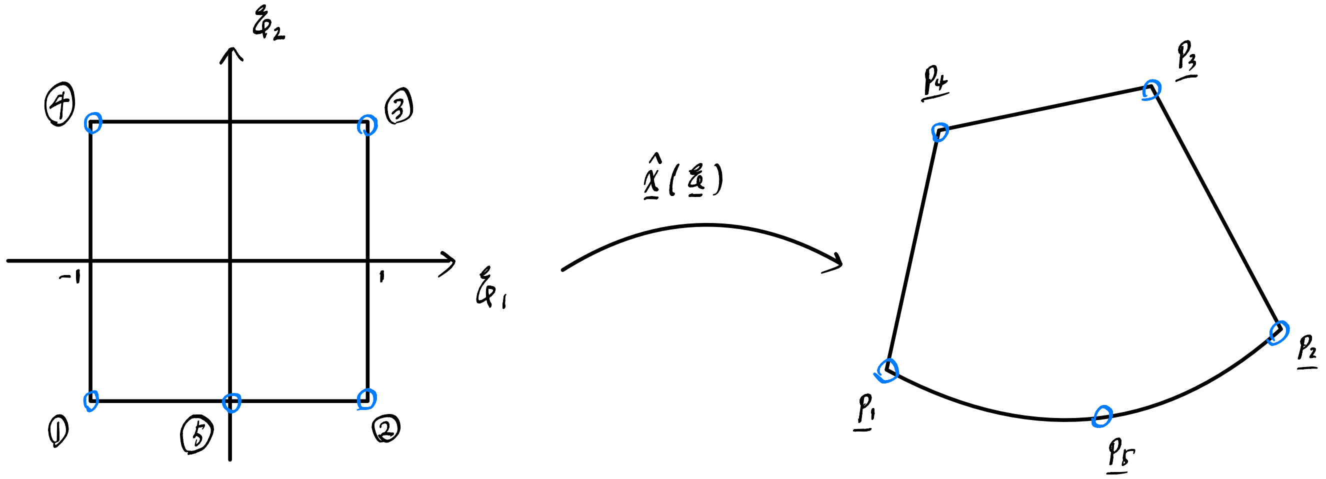
Lec 04: 2D linear problem

- Element technology
- Numerical integration
- Constraints

Element technology

Designing a customized element



$\hat{\Omega} = [-1,1] \times [-1,1]$

$$\hat{\underline{x}}(\underline{\xi}) = \sum_{i=1}^{5} \underline{p}_i \, \hat{\phi}_i(\underline{\xi})$$

Goal: Design shape functions $\{\hat{\phi}_i\}_{i=1}^{5}$ for the 5-node element s.t. Kronecker-δ and partition of unity properties are satisfied.

* Let $\hat{\phi}_5(\xi_1, \xi_2) = \ell_2^2(\xi_1)\,\ell_1^1(\xi_2) = \frac{1}{2}(1+\xi_1)(1-\xi_1)(1-\xi_2)$

Note that $\hat{\phi}_5(-1,-1) = \hat{\phi}_5(1,-1) = \hat{\phi}_5(1,1) = \hat{\phi}_5(-1,1) = 0$, and $\hat{\phi}_5(0,-1) = 1$. ✓

* Let $\hat{\phi}_1(\xi_1, \xi_2) = \ell_1^1(\xi_1)\,\ell_1^1(\xi_2) = \frac{1}{4}(1-\xi_1)(1-\xi_2)$, notice that $\hat{\phi}_1(0,-1) = \frac{1}{2} \neq 0$, not good.

Let $\hat{\phi}_1(\xi_1, \xi_2) = \ell_1^1(\xi_1)\,\ell_1^1(\xi_2) - \frac{1}{2}\hat{\phi}_5(\xi_1, \xi_2) = \frac{1}{4}(1-\xi_1)(1-\xi_2) - \frac{1}{4}(1+\xi_1)(1-\xi_1)(1-\xi_2)$. ✓

* Let $\hat{\phi}_2(\xi_1, \xi_2) = \ell_2^1(\xi_1)\,\ell_1^1(\xi_2) - \frac{1}{2}\hat{\phi}_5(\xi_1, \xi_2)$. ✓

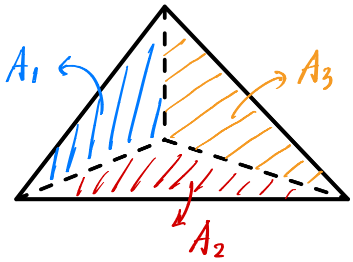
* Let $\hat{\phi}_3(\xi_1, \xi_2) = \ell_2^1(\xi_1)\,\ell_2^1(\xi_2)$ ✓ and $\hat{\phi}_4(\xi_1, \xi_2) = \ell_1^1(\xi_1)\,\ell_2^1(\xi_2)$ ✓

Verify that $\sum_{i=1}^{5} \hat{\phi}_i = 1$ and Kronecker-δ holds. More reading: Hughes book Section 3.7

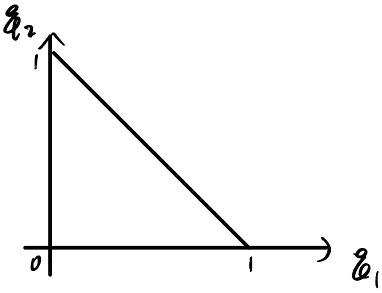
Shape functions for triangular elements

Area coordinate

$$r = \frac{A_1}{A}, \quad s = \frac{A_2}{A}, \quad t = \frac{A_3}{A}$$



Consider parametric domain $\hat{\Omega}$:



We have a map from (ξ_1, ξ_2) to (r, s, t):

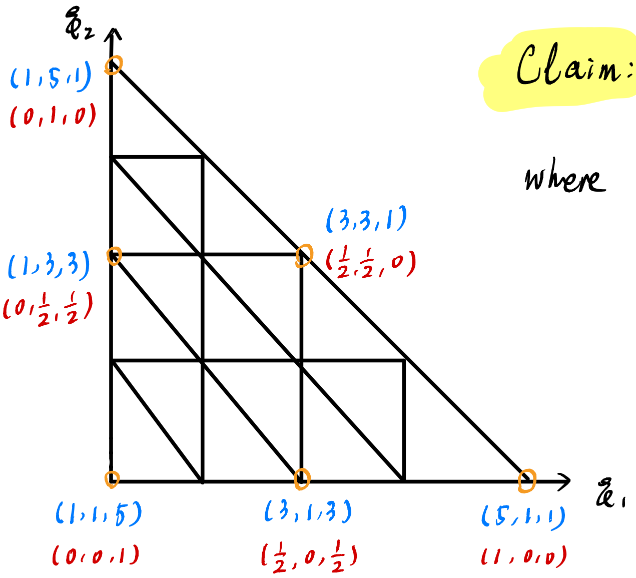
$$r = \xi_1$$

$$s = \xi_2$$

$$t = 1 - \xi_1 - \xi_2$$

$$\hat{\Omega} = \{ \underline{\xi} : \xi_1 \geq 0, \ \xi_2 \geq 0, \ \xi_1 + \xi_2 \leq 1 \}$$

Goal: Design shape functions $\{ \hat{\phi}_i(\underline{\xi}) \}$ with the help of area coordinate (r, s, t).



(I, J, K)

$(\bar{r}_I, \bar{s}_J, \bar{t}_K)$

Claim: $\hat{\phi}_i(\underline{\xi}) = T_I(r(\underline{\xi}))\, T_J(s(\underline{\xi}))\, T_K(t(\underline{\xi}))$

where

$$T_I(r) = \begin{cases} l_I^{I-1}(r) & \text{if } I \neq 1 \\ 1 & \text{if } I = 1 \end{cases}$$

Similar definitions for $T_J(s)$ and $T_K(t)$.

$$\left(l_I^{I-1}(r) = \frac{(r - \bar{r}_1)(r - \bar{r}_2) \cdots (r - \bar{r}_{I-1})}{(\bar{r}_I - \bar{r}_1)(\bar{r}_I - \bar{r}_2) \cdots (\bar{r}_I - \bar{r}_{I-1})} \right)$$

p: order (Here, $p = 4$)

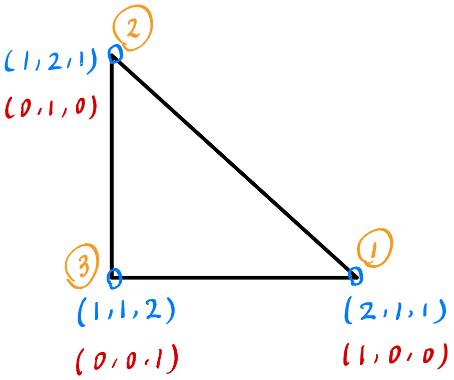
DoFs: $\frac{1}{2}(p+1)(p+2)$

I, J, K: node index

$(I = 1, 2, \cdots, p+1. \ J = 1, 2, \cdots, p+1. \ K = 1, 2, \cdots, p+1. \ I + J + K \leq p+3)$

$\bar{r}_I = \frac{I-1}{p}, \ \bar{s}_J = \frac{J-1}{p}, \ \bar{t}_K = \frac{K-1}{p}$ (knot coordinate)

Ex. Shape functions for 3-node linear triangle:

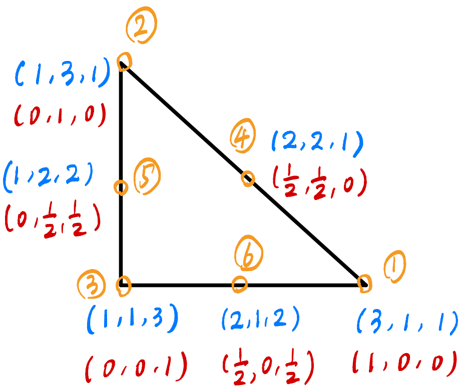


$$\hat{\phi}_{1}(\underline{\xi}) = T_2(r)\,T_1(s)\,T_1(t) = \ell_2^1(r)\cdot 1\cdot 1 = \frac{r-\bar{r}_1}{\bar{r}_2-\bar{r}_1} = r$$

$$\hat{\phi}_{2}(\underline{\xi}) = T_1(r)\,T_2(s)\,T_1(t) = 1\cdot \ell_2^1(s)\cdot 1 = \frac{s-\bar{s}_1}{\bar{s}_2-\bar{s}_1} = s$$

$$\hat{\phi}_{3}(\underline{\xi}) = T_1(r)\,T_1(s)\,T_2(t) = 1\cdot 1\cdot \ell_2^1(t) = \frac{t-\bar{t}_1}{\bar{t}_2-\bar{t}_1} = t$$

Ex. Shape functions for 6-node triangle:



$$\hat{\phi}_{1}(\underline{\xi}) = T_3(r)\,T_1(s)\,T_1(t) = \ell_3^2(r)\cdot 1\cdot 1 = \frac{(r-\bar{r}_1)(r-\bar{r}_2)}{(\bar{r}_3-\bar{r}_1)(\bar{r}_3-\bar{r}_2)}$$

$$= \frac{(r-0)(r-\frac{1}{2})}{(1-0)(1-\frac{1}{2})} = 2r(r-\tfrac{1}{2}) = r(2r-1)$$

$$\hat{\phi}_{5}(\underline{\xi}) = T_1(r)\,T_2(s)\,T_2(t) = 1\cdot \ell_2^1(s)\,\ell_2^1(t) = \frac{(s-\bar{s}_1)}{(\bar{s}_2-\bar{s}_1)}\cdot\frac{(t-\bar{t}_1)}{(\bar{t}_2-\bar{t}_1)}$$

$$= \frac{s-0}{\frac{1}{2}-0}\cdot\frac{t-0}{\frac{1}{2}-0} = 4st$$

Numerical integration

Motivation: Let $f: \Omega^e \subset \mathbb{R}^d \to \mathbb{R}$ be a given function. We are interested in computing

$$\int_{\Omega^e} f(\underline{x})\,d\underline{x}$$

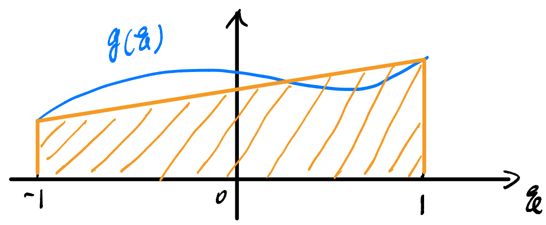
Recall $K^e_{\alpha\beta} = \int_{\Omega^e} k\,\nabla\phi_\alpha(\underline{x})\cdot\nabla\phi_\beta(\underline{x})\,d\underline{x} = \int_{\hat{\Omega}} k\left(\nabla_{\underline{\xi}}\hat{\phi}_\alpha(\underline{\xi})\cdot \underline{\underline{J}}_{\hat{x}}^{-1}(\underline{\xi})\right)\cdot\left(\nabla_{\underline{\xi}}\hat{\phi}_\beta(\underline{\xi})\cdot \underline{\underline{J}}_{\hat{x}}^{-1}(\underline{\xi})\right)|j(\underline{\xi})|\,d\underline{\xi}$

Consider 1D case: $\int_{-1}^{1} g(\xi)\,d\xi$, this can be approximated by numerical integration:

$$\int_{-1}^{1} g(\xi)\,d\xi = \sum_{l=1}^{n_{int}} g(\tilde{\xi}_l)\,w_l + R \approx \sum_{l=1}^{n_{int}} g(\tilde{\xi}_l)\,w_l \qquad \text{Quadrature formula}$$

Ex. Trapezoidal rule

$$\int_{-1}^{1} g(\xi)\,d\xi \approx g(-1) + g(1)$$



$n_{int} = 2$, $\tilde{\xi}_1 = -1$, $\tilde{\xi}_2 = 1$, $W_1 = 1$, $W_2 = 1$

$$R = -\frac{2}{3}\frac{d^2 g}{d\xi^2}(\bar{\xi}) \text{ at some } \bar{\xi}$$

Ex. Simpson's rule

$$\int_{-1}^{1} g(\xi)\,d\xi \approx \frac{1}{3}g(-1) + \frac{4}{3}g(0) + \frac{1}{3}g(1)$$

$n_{int} = 3$, $\tilde{\xi}_1 = -1$, $\tilde{\xi}_2 = 0$, $\tilde{\xi}_3 = 1$, $W_1 = \frac{1}{3}$, $W_2 = \frac{4}{3}$, $W_3 = \frac{1}{3}$

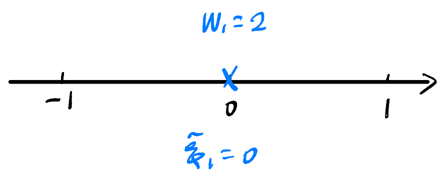
$$R = -\frac{g^{(4)}(\bar{\xi})}{90} \text{ at some } \bar{\xi}$$

We want to save computational cost: The fewer the integration points, the less the cost.

Gaussian quadrature

1. $n_{int} = 1$

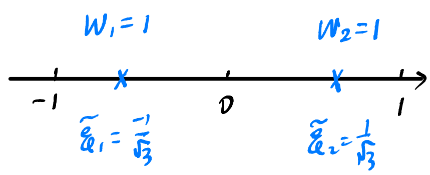
$$\int_{-1}^{1} g(\xi)\,d\xi \approx 2g(0)$$



$\tilde{\xi}_1 = 0$, $W_1 = 2$, $R = \frac{g^{(2)}(\bar{\xi})}{3}$

2. $n_{int} = 2$

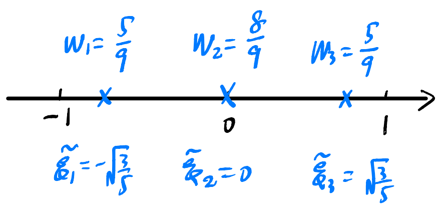
$$\int_{-1}^{1} g(\xi)\,d\xi \approx g\left(\frac{-1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right)$$



$\tilde{\xi}_1 = \frac{-1}{\sqrt{3}}$, $\tilde{\xi}_2 = \frac{1}{\sqrt{3}}$, $W_1 = 1$, $W_2 = 1$, $R = \frac{g^{(4)}(\bar{\xi})}{135}$

3. $n_{int} = 3$

$$\int_{-1}^{1} g(\xi)\,d\xi \approx \frac{5}{9}g\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}g(0) + \frac{5}{9}g\left(\sqrt{\frac{3}{5}}\right)$$



$\tilde{\xi}_1 = -\sqrt{\frac{3}{5}}$, $\tilde{\xi}_2 = 0$, $\tilde{\xi}_3 = \sqrt{\frac{3}{5}}$, $W_1 = \frac{5}{9}$, $W_2 = \frac{8}{9}$, $W_3 = \frac{5}{9}$, $R = \frac{g^{(6)}(\bar{\xi})}{15750}$

<u>Proposition</u> Gaussian quadrature scheme with n_{int} integration points can <u>exactly</u> integrate polynomial function $g(\xi)$ of order p, if $2n_{int} \geq p+1$.

<u>Proof</u> Let $g(\xi) = a_0 + a_1\xi + a_2\xi^2 + \cdots + a_p\xi^p$

$$\int_{-1}^{1} g(\xi)\, d\xi = \int_{-1}^{1} a_0 d\xi + \int_{-1}^{1} a_1\xi + \cdots + \int_{-1}^{1} a_p\xi^p = 2a_0 + 0\cdot a_1 + \cdots = \underline{g}^T \underline{a}$$

where $\underline{g} = \begin{bmatrix} 2 \\ 0 \\ \vdots \end{bmatrix}$, $\underline{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_p \end{bmatrix}$

$$\int_{-1}^{1} g(\xi)\, d\xi \approx \sum_{\ell=1}^{n_{int}} g(\tilde{\xi}_\ell) W_\ell = \overbrace{[W_1\ W_2 \cdots W_{n_{int}}]}^{\underline{W}^T} \overbrace{\begin{bmatrix} g(\tilde{\xi}_1) \\ g(\tilde{\xi}_2) \\ \vdots \\ g(\tilde{\xi}_{n_{int}}) \end{bmatrix}}^{\underline{g}}$$

$$= \overbrace{[W_1\ W_2 \cdots W_{n_{int}}]}^{\underline{W}^T} \overbrace{\begin{bmatrix} 1 & \tilde{\xi}_1 & \tilde{\xi}_1^2 & \cdots & \tilde{\xi}_1^p \\ 1 & \tilde{\xi}_2 & \tilde{\xi}_2^2 & \cdots & \tilde{\xi}_2^p \\ & & \vdots & & \\ 1 & \tilde{\xi}_{n_{int}} & \tilde{\xi}_{n_{int}}^2 & \cdots & \tilde{\xi}_{n_{int}}^p \end{bmatrix}}^{\underline{\underline{M}}} \overbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_p \end{bmatrix}}^{\underline{a}} = \underline{W}^T \underline{\underline{M}}\, \underline{a}$$

We want $\underline{g}^T \underline{a} = \underline{W}^T \underline{\underline{M}}\, \underline{a}$ to hold for arbitrary $\underline{a}$. $\Rightarrow (\underline{g}^T - \underline{W}^T \underline{\underline{M}})\cdot \underline{a} = 0 \quad \forall\, \underline{a}$

$\Rightarrow \underline{g}^T - \underline{W}^T \underline{\underline{M}} = 0 \Rightarrow \underline{\underline{M}}^T \underline{W} = \underline{g}$ ($p+1$ equations to be satisfied by $2n_{int}$ parameters)

Note: The $2n_{int}$ parameters include $\{W_1, W_2 \cdots W_{n_{int}}\}$ and $\{\tilde{\xi}_1, \tilde{\xi}_2, \cdots, \tilde{\xi}_{n_{int}}\}$.

<u>Ex.</u> Show that the Gaussian quadrature scheme for $n_{int} = 2$ can exactly integrate $g(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3$.

<u>Sol.</u> ① Since $2n_{int} = 4 \geq 4 = p+1$, the argument holds.

② Go through the reasoning (see homework).

Gaussian quadrature in several dimensions

Construct by employing 1D Gaussian rules on each coordinate seperately:

$$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\,d\xi\,d\eta \approx \int_{-1}^{1}\left(\sum_{\ell^{(1)}=1}^{n_{int}^{(1)}} g(\tilde{\xi}^{(1)}_{\ell^{(1)}},\eta)\,W^{(1)}_{\ell^{(1)}}\right)d\eta \approx \sum_{\ell^{(1)}=1}^{n_{int}^{(1)}}\sum_{\ell^{(2)}=1}^{n_{int}^{(2)}} g(\tilde{\xi}^{(1)}_{\ell^{(1)}},\tilde{\eta}^{(2)}_{\ell^{(2)}})\,W^{(1)}_{\ell^{(1)}}W^{(2)}_{\ell^{(2)}}$$

Gaussian rule (1) applied to ξ

Gaussian rule (2) applied to η

$$\Rightarrow \int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\,d\xi\,d\eta \approx \boxed{\sum_{\ell=1}^{n_{int}} g(\tilde{\xi}_\ell,\tilde{\eta}_\ell)\,W_\ell}\ ,\ \text{where}$$

$$n_{int} = n_{int}^{(1)}\,n_{int}^{(2)},\quad \tilde{\xi}_\ell = \tilde{\xi}^{(1)}_{\ell^{(1)}},\quad \tilde{\eta}_\ell = \tilde{\eta}^{(2)}_{\ell^{(2)}},\quad W_\ell = W^{(1)}_{\ell^{(1)}}W^{(2)}_{\ell^{(2)}}$$

Ex. If 1D one-point rule is used in each direction:

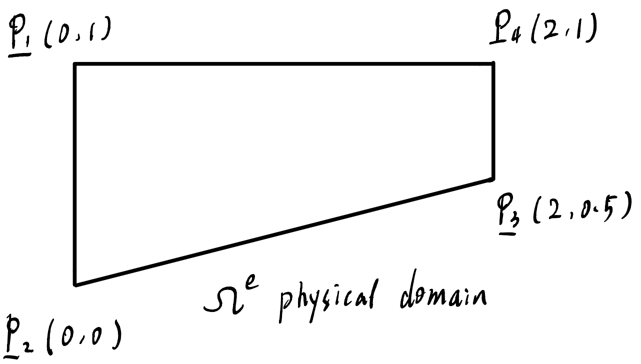
$$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\,d\xi\,d\eta \approx 4\,g(0,0)$$

Ex. If 1D two-point rule is used in each direction:

$$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\,d\xi\,d\eta \approx g\left(\tfrac{-1}{\sqrt3},\tfrac{-1}{\sqrt3}\right)+g\left(\tfrac{-1}{\sqrt3},\tfrac{1}{\sqrt3}\right)+g\left(\tfrac{1}{\sqrt3},\tfrac{-1}{\sqrt3}\right)+g\left(\tfrac{1}{\sqrt3},\tfrac{1}{\sqrt3}\right)$$

Question: Why can we "decouple" the rule in several dims to 1D respectively?

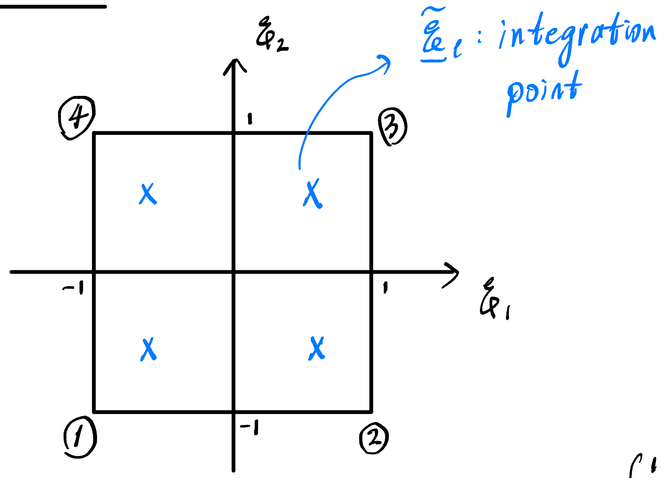
A practical example



Goal: Find $K^e_{\alpha\beta} = \int_{\Omega^e} k\,\nabla\phi_\alpha(\underline{x})\cdot\nabla\phi_\beta(\underline{x})\,d\underline{x}$

$k = 5$

Recall:



$\hat{\Omega} = [-1,1] \times [-1,1]$

parametric/parent domain

$$\hat{\phi}_1(\underline{\xi}) = (\xi_1 - 1)(\xi_2 - 1)/4$$
$$\hat{\phi}_2(\underline{\xi}) = -(\xi_1 + 1)(\xi_2 - 1)/4$$
$$\hat{\phi}_3(\underline{\xi}) = (\xi_1 + 1)(\xi_2 + 1)/4$$
$$\hat{\phi}_4(\underline{\xi}) = -(\xi_1 - 1)(\xi_2 + 1)/4$$

$$\int_{-1}^{1}\int_{-1}^{1} g(\underline{\xi})\, d\underline{\xi} \approx \sum_{l=1}^{N_{int}} g(\tilde{\underline{\xi}}_l) W_l$$

$$K^e_{\alpha\beta} = \int_{\hat{\Omega}} k \left(\begin{bmatrix} \frac{\partial \hat{\phi}_\alpha}{\partial \xi_1} & \frac{\partial \hat{\phi}_\alpha}{\partial \xi_2} \end{bmatrix} \cdot \underbrace{\begin{bmatrix} \frac{\partial \hat{x}_1}{\partial \xi_1} & \frac{\partial \hat{x}_1}{\partial \xi_2} \\ \frac{\partial \hat{x}_2}{\partial \xi_1} & \frac{\partial \hat{x}_2}{\partial \xi_2} \end{bmatrix}^{-1}}_{= [\underline{\underline{J}}_{\hat{x}}(\underline{\xi})]^{-1}} \right) \cdot \left(\begin{bmatrix} \frac{\partial \hat{\phi}_\beta}{\partial \xi_1} & \frac{\partial \hat{\phi}_\beta}{\partial \xi_2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial \hat{x}_1}{\partial \xi_1} & \frac{\partial \hat{x}_1}{\partial \xi_2} \\ \frac{\partial \hat{x}_2}{\partial \xi_1} & \frac{\partial \hat{x}_2}{\partial \xi_2} \end{bmatrix}^{-1} \right) \cdot \underbrace{|j(\underline{\xi})|}_{= |\det(\underline{\underline{J}}_{\hat{x}}(\underline{\xi}))|} d\xi_1 d\xi_2$$

$$\hat{\underline{x}}(\underline{\xi}) = \sum_{i=1}^{4} \underline{p}_i \hat{\phi}_i(\underline{\xi}), \Rightarrow \underbrace{[\underline{\underline{J}}_{\hat{x}}(\underline{\xi})]}_{2\times 2} = \sum_{i=1}^{4} \underbrace{\begin{bmatrix} p_{i1} \\ p_{i2} \end{bmatrix}}_{2\times 1} \cdot \underbrace{\begin{bmatrix} \frac{\partial \hat{\phi}_i}{\partial \xi_1} & \frac{\partial \hat{\phi}_i}{\partial \xi_2} \end{bmatrix}}_{1\times 2}.$$

In matrix format:

$$\underbrace{[\underline{\underline{J}}_{\hat{x}}(\underline{\xi})]}_{2\times 2} = \underbrace{\begin{bmatrix} \underline{p}_1 & \underline{p}_2 & \underline{p}_3 & \underline{p}_4 \end{bmatrix}}_{2\times 4} \underbrace{\begin{bmatrix} \frac{\partial \hat{\phi}_1}{\partial \xi_1} & \frac{\partial \hat{\phi}_1}{\partial \xi_2} \\ \frac{\partial \hat{\phi}_2}{\partial \xi_1} & \frac{\partial \hat{\phi}_2}{\partial \xi_2} \\ \frac{\partial \hat{\phi}_3}{\partial \xi_1} & \frac{\partial \hat{\phi}_3}{\partial \xi_2} \\ \frac{\partial \hat{\phi}_4}{\partial \xi_1} & \frac{\partial \hat{\phi}_4}{\partial \xi_2} \end{bmatrix}}_{4\times 2} = \begin{bmatrix} 0 & 0 & 2 & 2 \\ 1 & 0 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4}(\xi_2 - 1) & \frac{1}{4}(\xi_1 - 1) \\ -\frac{1}{4}(\xi_2 - 1) & -\frac{1}{4}(\xi_1 + 1) \\ \frac{1}{4}(\xi_2 + 1) & \frac{1}{4}(\xi_1 + 1) \\ -\frac{1}{4}(\xi_2 + 1) & -\frac{1}{4}(\xi_1 - 1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ \frac{\xi_2 - 3}{8} & \frac{\xi_1 + 1}{8} \end{bmatrix} \Rightarrow j = \frac{3 - \xi_2}{8}, \quad [\underline{\underline{J}}_{\hat{x}}(\underline{\xi})]^{-1} = \begin{bmatrix} \frac{\xi_1 + 1}{-\xi_2 + 3} & \frac{8}{\xi_2 - 3} \\ 1 & 0 \end{bmatrix}$$

Consider $K_{12}^{e} = \int_{\Omega^e} k \nabla \phi_1(\underline{x}) \nabla \phi_2(\underline{x})\, d\underline{x}$

$$K_{12}^{e} = \int_{\hat{\Omega}} 5 \left(\left[\tfrac{1}{4}(\xi_2-1) \;\; \tfrac{1}{4}(\xi_1-1) \right] \begin{bmatrix} \frac{\xi_1+1}{-\xi_2+3} & \frac{8}{\xi_2-3} \\ 1 & 0 \end{bmatrix} \right) \cdot \left(\left[-\tfrac{1}{4}(\xi_2-1) \;\; -\tfrac{1}{4}(\xi_1+1) \right] \begin{bmatrix} \frac{\xi_1+1}{-\xi_2+3} & \frac{8}{\xi_2-3} \\ 1 & 0 \end{bmatrix} \right) \cdot \frac{3-\xi_2}{8} \cdot d\underline{\xi}$$

$$K_{12}^{e} \approx \sum_{l=1}^{4} g(\underline{\tilde{\xi}}_l) W_l = g(\tfrac{-1}{\sqrt3}, \tfrac{-1}{\sqrt3}) + g(\tfrac{-1}{\sqrt3}, \tfrac{1}{\sqrt3}) + g(\tfrac{1}{\sqrt3}, \tfrac{-1}{\sqrt3}) + g(\tfrac{1}{\sqrt3}, \tfrac{1}{\sqrt3})$$

Consider the contribution from quadrature point $(-\frac{1}{\sqrt3}, -\frac{1}{\sqrt3})$:

$$g(-\tfrac{1}{\sqrt3}, -\tfrac{1}{\sqrt3}) = 5 \left(\left[\tfrac{1}{4}(-\tfrac{1}{\sqrt3}-1) \;\; \tfrac{1}{4}(-\tfrac{1}{\sqrt3}-1) \right] \cdots = -1.68 \right.$$

Homework: Complete this calculation and find $K_{\alpha\beta}^{e}$ for $\alpha = 1, 2$ and $\beta = 1, 2$.

Constraints

Recall our 1D problem $[K][v] = [F]$, where

$$[K] = \frac{AE}{L/3} \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix}, \quad [v] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}, \quad [F] = \begin{bmatrix} b\frac{L}{6} - \frac{AE}{L/3} g \\ b\frac{L}{3} + \frac{AE}{L/3} g \\ b\frac{L}{3} \\ b\frac{L}{6} + f \end{bmatrix}$$

We want to impose $v_1 = 0$ (from Dirichlet B.C.) and solve for $[v]$.

For simplicity, assume $A=1, E=1, L=3, b=2, g=1, f=1$, so that

$$[K] = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix}, \quad [F] = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 2 \end{bmatrix}.$$

Previously, we considered reduced system:

$$[K_{red}] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad [F_{red}] = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}. \Rightarrow [v_{red}] = \begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 13 \end{bmatrix}.$$

Observations:

* $[K]$ is symmetric positive semidefinite ($[x]^T[K][x] \geq 0 \quad \forall [x] \in \mathbb{R}^N$, and if $[x] = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, $[x]^T[K][x] = 0$, physical interpretation: rigid body motion)

* $[K_{red}]$ is symmetric positive definite (it is invertible).

* Do we have other ways to impose the constraint?

Lagrange multiplier method

In the optimization context:

$$\min_{[v]} E([v])$$
$$\text{s.t. } g([v]) = 0$$

where $E([v]) = \frac{1}{2}[v]^T[K][v] - [v]^T[F]$ is the energy function to be minimized, $g([v]) = v_1 - 0 = 0$ is the constraint that must be satisfied.

The Lagrange multiplier method tries to find the stationary points of $\mathcal{L}$:

$$\mathcal{L}([v], \lambda) := E([v]) + \lambda g$$

that is:

$$\frac{\partial \mathcal{L}}{\partial [v]} = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

$$\Rightarrow \begin{cases} [K][v] - [F] + \lambda \dfrac{\partial g}{\partial [v]} = 0 \\ g([v]) = 0 \end{cases} \quad (N+1 \text{ equations}, N+1 \text{ variables})$$

For our problem:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 2 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 11 \\ 13 \\ 7 \end{bmatrix}$$