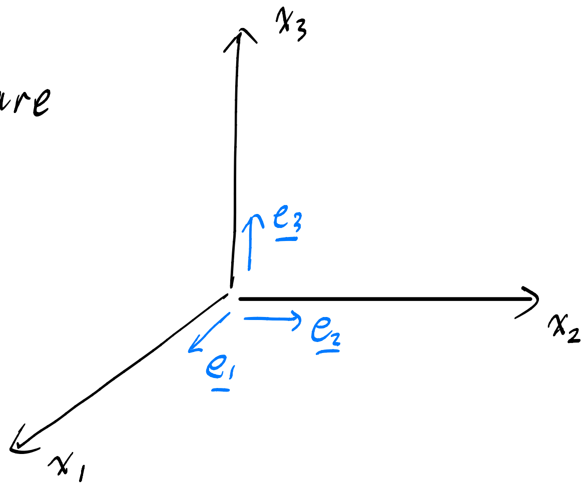
Lec 01 : Mathematical preliminaries

- Tensors
- Derivatives
- Integration theorems
- Classification of PDEs

Tensors

In Cartesian systems, the base vectors are



$$|\underline{e}_1| = \underline{e}_1 \cdot \underline{e}_1 = 1 \ , \ |\underline{e}_2| = |\underline{e}_3| = 1$$

$$\underline{e}_i \cdot \underline{e}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

A scalar (zeroth-order tensor) can be g.

A vector (first-order tensor) can be $\underline{u} = u_1 \underline{e}_1 + u_2 \underline{e}_2 + u_3 \underline{e}_3$

Then, $u_i = \underline{u} \cdot \underline{e}_i$. Also, $\underline{u} = \sum_{i=1}^{3} u_i \underline{e}_i = u_i \underline{e}_i$ (summation of dummy index)

$$\boxed{\text{Ex.} \quad \underline{u} \cdot \underline{e}_i = u_k \underline{e}_k \cdot \underline{e}_i = u_k \delta_{ik} = u_i}$$ (free index)

Def. A second order tensor $\underline{T}$ is a linear operator that acts on a vector $\underline{a}$ and returns another vector $\underline{b}$.

$$\underline{\underline{T}} : \underline{a} \longmapsto \underline{b}, \quad \underline{\underline{T}}(\alpha \underline{a}) = \alpha \underline{\underline{T}}(\underline{a}), \quad \underline{\underline{T}}(\underline{a}+\underline{b}) = \underline{\underline{T}}(\underline{a}) + \underline{\underline{T}}(\underline{b})$$

Tensor dot product with vector:

$$\underline{\underline{T}}(\underline{a}) = \underline{\underline{T}} \cdot \underline{a} = T_{ij}\, \underline{e}_i \otimes \underbrace{\underline{e}_j \cdot a_k \underline{e}_k}_{\delta_{jk}} = T_{ij}\, a_k\, \delta_{jk}\, \underline{e}_i = T_{ij}\, a_j\, \underline{e}_i$$

$\underline{\underline{T}}(\underline{a}) = \underline{b} = b_i \underline{e}_i$ We may write $T_{ij} a_j = b_i$

Dyadic product: $\underline{\underline{A}} = \underline{u} \otimes \underline{v}$, then $\underline{\underline{A}} \cdot \underline{a} = (\underline{u} \otimes \underline{v}) \cdot \underline{a} = \underline{u}\,(\underline{v} \cdot \underline{a})$

Ex. A projection tensor (along $\underline{n}$ direction) can be written as $\underline{\underline{T}} = \underline{n} \otimes \underline{n}$

Show that for any vector $\underline{u}$, $\underline{\underline{T}}(\underline{\underline{T}}(\underline{u})) = \underline{\underline{T}}(\underline{u})$.

Sol. $\underline{\underline{T}}(\underline{u}) = \underline{n} \otimes \underline{n} \cdot \underline{u} = (\underline{n} \cdot \underline{u})\, \underline{n}$

$$\underline{\underline{T}}(\underline{\underline{T}}(\underline{u})) = \underline{n} \otimes \underline{n} \cdot (\underline{n} \cdot \underline{u})\, \underline{n} = (\underline{n} \cdot \underline{u})\, \underline{n}$$

Tensor components:

If $\underline{\underline{T}} = T_{ij}\, \underline{e}_i \otimes \underline{e}_j$, then $[\underline{\underline{T}}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$.

If $\underline{a} = a_i \underline{e}_i$, then $[a] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

We can verify that $[\underline{\underline{T}} \cdot \underline{a}] = [\underline{\underline{T}}] \cdot [a]$ matrix-vector product

Tensor dot product with another tensor:

$$\underline{\underline{A}} \cdot \underline{\underline{B}} = A_{ij}\, \underline{e}_i \otimes \underline{e}_j \cdot B_{mk}\, \underline{e}_m \otimes \underline{e}_k = A_{ij} B_{mk}\, \delta_{mj}\, \underline{e}_i \otimes \underline{e}_k$$

(single dot product)

$$= A_{ij} B_{jk}\, \underline{e}_i \otimes \underline{e}_k$$

$$\underline{\underline{A}} : \underline{\underline{B}} = A_{ij}\, \underline{e}_i \otimes \underline{e}_j : B_{mk}\, \underline{e}_m \otimes \underline{e}_k = A_{ij} B_{mk}\, \delta_{im} \delta_{jk} = A_{ij} B_{ij}$$

(double dot product)

Fourth-order tensor (e.g., elastic modulus tensor $\underline{\underline{\underline{\underline{C}}}}$)

$$\underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}} = C_{ijkl}\, \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k \otimes \underline{e}_l : \varepsilon_{pq}\, \underline{e}_p \otimes \underline{e}_q = C_{ijkl}\, \varepsilon_{kl}$$

Identity tensor:

$\underline{\underline{I}} = \delta_{ij}\, \underline{e}_i \otimes \underline{e}_j$, so $\underline{\underline{I}}(\underline{a}) = \delta_{ij}\, \underline{e}_i \otimes \underline{e}_j \cdot a_k \underline{e}_k$

$$= a_k\, \delta_{ij}\, \delta_{jk}\, \underline{e}_i$$

$$= a_i\, \underline{e}_i = \underline{a}$$

Transpose of a tensor:

$\underline{\underline{T}} = T_{ij}\, \underline{e}_i \otimes \underline{e}_j$, then $\underline{\underline{T}}^T = T_{ji}\, \underline{e}_i \otimes \underline{e}_j$

or $\underline{\underline{T}}^T = T_{ij}\, \underline{e}_j \otimes \underline{e}_i$

Derivatives

Gradients:

$$\underline{\nabla}(\cdot) = \underline{e}_i \frac{\partial}{\partial x_i}(\cdot)$$ left gradient

$$(\cdot)\underline{\nabla} = \frac{\partial}{\partial x_i}(\cdot)\, \underline{e}_i$$ right gradient

! Other people may use a different convention.

Gradient of a scalar f: $\underline{\nabla} f = f \underline{\nabla} = \frac{\partial f}{\partial x_i} \underline{e}_i$

Gradients of a vector: $\underline{\nabla}\, \underline{v} = \frac{\partial v_i}{\partial x_j} \underline{e}_j \otimes \underline{e}_i$

$$\underline{v}\, \underline{\nabla} = \frac{\partial v_i}{\partial x_j} \underline{e}_i \otimes \underline{e}_j$$

Gradients of a tensor: $\underline{\nabla}\, \underline{\underline{A}} = \frac{\partial A_{ij}}{\partial x_k} \underline{e}_k \otimes \underline{e}_i \otimes \underline{e}_j$

$$\underline{\underline{A}}\, \underline{\nabla} = \frac{\partial A_{ij}}{\partial x_k} \underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k$$

Divergence: $\underline{\nabla} \cdot (\ \cdot\ ) = \underline{e}_i \frac{\partial}{\partial x_i} \cdot (\ \cdot\ )$ left divergence

$$(\ \cdot\ ) \cdot \underline{\nabla} = \frac{\partial}{\partial x_i} (\ \cdot\ ) \cdot \underline{e}_i \quad \text{right divergence}$$

Divergence of a vector: $\underline{\nabla} \cdot \underline{v} = \underline{v} \cdot \underline{\nabla} = \frac{\partial v_i}{\partial x_i}$

Divergence of a tensor: $\underline{\nabla} \cdot \underline{\underline{T}} = \underline{e}_i \frac{\partial T_{jk}}{\partial x_i} \cdot \underline{e}_j \otimes \underline{e}_k$

$$= \frac{\partial T_{jk}}{\partial x_j} \underline{e}_k$$

$$\underline{\underline{T}} \cdot \underline{\nabla} = \frac{\partial T_{jk}}{\partial x_i} \underline{e}_j \otimes \underline{e}_k \cdot \underline{e}_i$$

$$= \frac{\partial T_{jk}}{\partial x_k} \underline{e}_j$$

Laplace : $\Delta f := \nabla \cdot \nabla f = \nabla \cdot \left(\frac{\partial f}{\partial x_i} e_i \right) = \frac{\partial^2 f}{\partial x_i^2}$

Integration theorems

Divergence theorem (Gauss's theorem): $\int_\Omega \frac{\partial u}{\partial x_i} = \int_{\partial\Omega} u n_i$

Note: here, "u" and "v" can be any order tensor.

Ex. Show $\int_\Omega \nabla v = \int_{\partial\Omega} n \otimes v$

Sol. Need to show $\int_\Omega \frac{\partial v_j}{\partial x_i} = \int_{\partial\Omega} n_i v_j$, we just need to take "$u$" to be v_j.

Ex. Show $\int_\Omega \nabla \cdot v = \int_{\partial\Omega} n \cdot v$

Sol. Need to show $\int_\Omega \frac{\partial v_i}{\partial x_i} = \int_{\partial\Omega} n_i v_i$, we just need to take "$u$" to be v_i.

Ex. Show $\int_\Omega \nabla \cdot u\, q = \int_{\partial\Omega} n \cdot u\, q - \int_\Omega u \cdot \nabla q$ (Integration by parts)

Classification of PDEs

Consider an example of a partial differential equation:

$$\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu = f$$

Basic classification:

* Order
* Linearity
* Number of independent variables

In our example, the PDE is second order, linear, and with d independent variables.

If the PDE is second order with two independent variables:

$$\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) = A\frac{\partial^2 u}{\partial x_1^2} + B\frac{\partial^2 u}{\partial x_1 \partial x_2} + C\frac{\partial^2 u}{\partial x_2^2}$$

then the PDE can be classified as:

* Elliptic for $B^2 - 4AC < 0$
* Hyperbolic for $B^2 - 4AC > 0$
* Parabolic for $B^2 - 4AC = 0$

Examples

Transport equation: $\frac{\partial u}{\partial x} + c\frac{\partial u}{\partial y} = 0$ (c is constant) linear first-order

Burger's equation: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ quasilinear first-order

Eikonal equation: $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1$ nonlinear first-order

Laplace's equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 1$ elliptic linear second-order

Wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (c is constant) hyperbolic linear second-order

Heat equation: $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ (k is constant) parabolic linear second-order

Question: What are the PDEs that FEM is "good at" solving?