Lec09: Time-dependent problems

- General formulation (Rothe's method v.s. Method of lines)
- Time discretization methods
- Modal analysis

General formulation

Recall that for linear elastostatics, we have:

Problem statement: Find displacement $\underline{u} : \Omega \to \mathbb{R}^{n_{sd}}$ such that

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}} + \underline{f} = \underline{0} & \text{in } \Omega \quad (\text{equilibrium}) \\ \underline{u} = \underline{g} & \text{on } \Gamma_D \quad (\text{Dirichlet B.C.}) \\ \underline{n} \cdot \underline{\underline{\sigma}} = \underline{t} & \text{on } \Gamma_N \quad (\text{Neumann B.C.}) \end{cases} \quad \text{or} \quad \begin{cases} \sigma_{ij,i} + f_j = 0 & \text{in } \Omega \\ u_i = g_i & \text{on } \Gamma_D \\ n_i \sigma_{ij} = t_j & \text{on } \Gamma_N \end{cases}$$

For linear elastodynamics, we have:

Problem statement: Find displacement $\underline{u} : \Omega \times [0, T] \to \mathbb{R}^{n_{sd}}$ such that

$$\begin{cases} \boxed{\rho \dfrac{\partial^2 \underline{u}}{\partial t^2}} = \nabla \cdot \underline{\underline{\sigma}} + \underline{f} & (\text{equilibrium}) \\ \underline{u} = \underline{g} & \text{on } \Gamma_D \quad (\text{Dirichlet B.C.}) \\ \underline{n} \cdot \underline{\underline{\sigma}} = \underline{t} & \text{on } \Gamma_N \quad (\text{Neumann B.C.}) \end{cases} \quad \text{or} \quad \begin{cases} \boxed{\rho \, u_{j,tt}} = \sigma_{ij,i} + f_j & \text{in } \Omega \\ u_i = g_i & \text{on } \Gamma_D \\ n_i \sigma_{ij} = t_j & \text{on } \Gamma_N \end{cases}$$

Question: Should we discretize time first and then space (Rothe's method) or discretize space first and then time (method of lines)?

Rothe's method

Let us discretize time first. For example, implicit Euler gives:

$$\begin{cases} \rho \dfrac{\underline{v}^{n+1} - \underline{v}^{n}}{\Delta t} = \underline{\nabla} \cdot \underline{\sigma}(\underline{u}^{n+1}) + \underline{f} \\ \underline{v}^{n+1} = \dfrac{\underline{u}^{n+1} - \underline{u}^{n}}{\Delta t} \end{cases}$$

$\underline{u}^n$: displacement from the previous step

$\underline{v}^n$: velocity from the previous step

Solve for $\underline{u}^{n+1}$ and $\underline{v}^{n+1}$ using standard FEM.

Method of lines

Let us discretize space first. The weak form gives: Find $\underline{u}$ s.t.

$$\int_\Omega \rho \frac{\partial^2 \underline{u}}{\partial t^2} \cdot \underline{w} + \int_\Omega \underline{\sigma} : \underline{\nabla} \underline{w} = \int_{\Gamma^N} \underline{t} \cdot \underline{w} + \int_\Omega \underline{f} \cdot \underline{w} \qquad \forall\, \underline{w} \in V$$

To discretize over space, apply the Galerkin approximation:

$$\underline{u}(\underline{x}, t) \approx \underline{u}^h(\underline{x}, t) = \sum_{A=1}^{n_n} \underbrace{\underline{u}^A(t)}_{\text{time-dependent nodal DoF}} \underbrace{\phi^A(\underline{x})}_{\text{spatial-dependent shape function}}$$

The matrix form is therefore:

$$\underline{\underline{M}}\, \ddot{\underline{U}} + \underbrace{\underline{\underline{K}}\, \underline{U} = \underline{F}}_{\text{same as before}} , \text{ where } \underline{U} = \begin{Bmatrix} [\underline{u}^1] \\ [\underline{u}^2] \\ \vdots \\ [\underline{u}^{n_n}] \end{Bmatrix} , \quad M^{AB}_{ij} = \int_\Omega \rho\, \phi^A \phi^B \delta_{ij}$$

We get a system of ODEs (ordinary differential equations)

Remarks:

* Rothe's method allows us to use standard FEM packages to solve the problem (e.g., FEniCS, Dealii, JAX-FEM). However, it might be cumbersome

for more advanced time stepping methods.

* Method of lines allows us to use standard ODE solvers to solve the problem. However, it is difficult if an adaptive mesh needs to be used.

Time discretization methods

We present some time stepping schemes under the method of lines. These schemes also work for Rothe's method.

Newmark method

Consider the semidiscrete equation of motion:

$$\underline{\underline{M}}\,\ddot{\underline{U}} + \underbrace{\underline{\underline{C}}}_{\text{damping}}\,\dot{\underline{U}} + \underline{\underline{K}}\,\underline{U} = \underline{F}, \quad \text{with} \quad \underbrace{\underline{U}\big|_{t=0} = \underline{U}_0, \ \dot{\underline{U}}\big|_{t=0} = \underline{V}_0, \ \ddot{\underline{U}}\big|_{t=0} = \underline{A}_0}_{\text{initial conditions}}$$

The Newmark family gives:

$$\underbrace{\begin{aligned} &\underline{\underline{M}}\,\underline{A}^{n+1} + \underline{\underline{C}}\,\underline{V}^{n+1} + \underline{\underline{K}}\,\underline{U}^{n+1} = \underline{F} \\ &\underline{U}^{n+1} = \underline{U}^{n} + \Delta t\,\underline{V}^{n} + \frac{\Delta t^2}{2}\left[(1-2\beta)\underline{A}^{n} + 2\beta\,\underline{A}^{n+1}\right] \\ &\underline{V}^{n+1} = \underline{V}^{n} + \Delta t\left[(1-\gamma)\underline{A}^{n} + \gamma\,\underline{A}^{n+1}\right] \end{aligned}}_{\substack{\text{Algebraic equations} \\ \text{No differential terms}}}$$

β, γ : numerical parameters

Note: When $\beta = 0$ and $\gamma = \frac{1}{2}$, we obtain central difference scheme. If mass lumping is applied, we obtain explicit time stepping algorithms. Refer to Hughes book Section 9.1 for stability analysis.

Modal analysis

Time-dependent problems might be solved with eigenvalue approaches.

Consider free vibration: $\underline{\underline{M}}\,\ddot{\underline{U}} + \underline{\underline{K}}\,\underline{U} = \underline{0}$ with initial excitement

$\underline{U}|_{t=0} = \underline{U}_0$ and $\dot{\underline{U}}|_{t=0} = \underline{V}_0$.

Let us guess that the solution $\underline{U}(t) = \underline{\varphi}\, e^{i(wt-\theta)}$, then

$$\underline{\underline{M}}\left((iw)^2 \underline{\varphi}\, e^{i(wt-\theta)}\right) + \underline{\underline{K}}\,\underline{\varphi}\, e^{i(wt-\theta)} = 0$$

$$\Rightarrow \quad w^2 \underline{\underline{M}}\,\underline{\varphi} = \underline{\underline{K}}\,\underline{\varphi} \quad \text{(eigenvalue problem)}$$

Suppose we solve this eigenvalue problem and obtain

$0 < w_1 \le w_2 \cdots w_{n-1} \le w_n$ along with $\underline{\varphi}_1, \underline{\varphi}_2, \cdots, \underline{\varphi}_{n-1}, \underline{\varphi}_n$

eigenvalues $\lambda_i = w_i^2$ eigenvectors

(Suppose $\underline{\underline{M}}$ and $\underline{\underline{K}}$ are symmetric positive definite, and the system has n DoFs)

Notice the linear nature of the problem, we may write the solution as

$$\underline{U}(t) = \sum_{k=1}^{n} \underset{\substack{\uparrow \\ \text{const.}}}{C_k}\, \underline{\varphi}_k\, e^{i(w_k t - \theta_k)}.$$

Since $\underline{U}(t)$ must be real, we

complex conjugate

construct $\underline{U}(t) = \sum_{k=1}^{n} C_k\, \underline{\varphi}_k\, e^{i w_k t} + \bar{C}_k\, \underline{\varphi}_k\, e^{-i w_k t}$ $(\theta_k = 0)$

$$\Leftrightarrow \underline{U}(t) = \sum_{k=1}^{n} \left[A_k \cos(w_k t) + B_k \sin(w_k t)\right] \underline{\varphi}_k \quad \left(A_k = 2\,\mathrm{Re}(C_k),\ B_k = -2\,\mathrm{Im}(C_k)\right)$$

To determine A_k and B_k, let us introduce a fact:

Proposition $\underline{\varphi}_k^T \cdot \underline{\underline{M}} \cdot \underline{\varphi}_\ell = 0$ if $k \neq \ell$.

Proof $w_k^2 \underline{\underline{M}}\, \underline{\varphi}_k = \underline{\underline{K}}\, \underline{\varphi}_k \Rightarrow w_k^2\, \underline{\varphi}_\ell^T \underline{\underline{M}}\, \underline{\varphi}_k = \underline{\varphi}_\ell^T \underline{\underline{K}}\, \underline{\varphi}_k$

$w_\ell^2 \underline{\underline{M}}\, \underline{\varphi}_\ell = \underline{\underline{K}}\, \underline{\varphi}_\ell \Rightarrow w_\ell^2\, \underline{\varphi}_k^T \underline{\underline{M}}\, \underline{\varphi}_\ell = \underline{\varphi}_k^T \underline{\underline{K}}\, \underline{\varphi}_\ell$

since $\underline{\underline{M}}^T = \underline{\underline{M}}$ and $\underline{\underline{K}}^T = \underline{\underline{K}}$, $(w_k^2 - w_\ell^2)\, \underline{\varphi}_k^T \underline{\underline{M}}\, \underline{\varphi}_\ell = 0$

If $w_k \neq w_\ell$, $\underline{\varphi}_k^T \underline{\underline{M}}\, \underline{\varphi}_\ell = 0$;

If $w_k = w_\ell$, use Gram-Schmidt procedure to construct $\underline{\varphi}_k \underline{\underline{M}}\, \underline{\varphi}_\ell = 0$.

Then, $\dfrac{\underline{\varphi}_\ell^T \underline{\underline{M}}\, \underline{U}(0)}{\underline{\varphi}_\ell^T \underline{\underline{M}}\, \underline{\varphi}_\ell} = A_\ell$ and $\dfrac{\underline{\varphi}_\ell^T \underline{\underline{M}}\, \underline{V}(0)}{\underline{\varphi}_\ell^T \underline{\underline{M}}\, \underline{\varphi}_\ell} = B_\ell$

Remarks : You may start with $\underline{U}(t) = \underline{\varphi} \cos(wt - \theta)$ for a real-valued analysis without involving complex numbers.