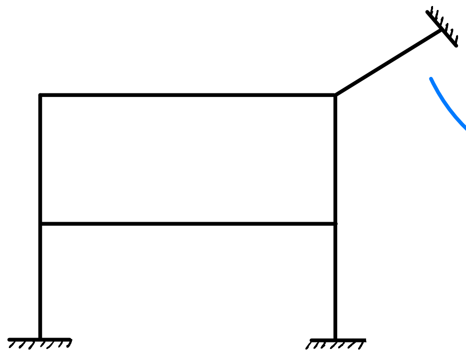
Lec 07: Beams, plates and shells

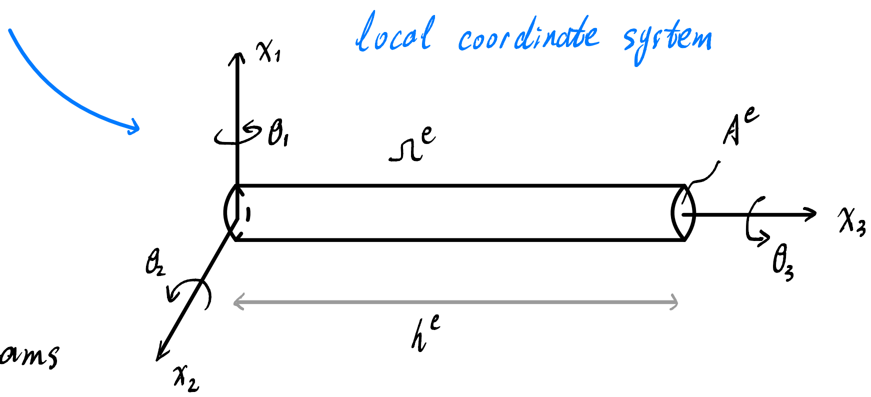
- Beams: Timoshenko beam; Euler-Bernoulli beam
- plates: Reissner-Mindlin plate; Kirchhoff-Love plate
- Shells

Beams

Timoshenko beam



A frame structure made of beams



Total domain:

$$\Omega = \bigcup_{e=1}^{n_{el}} \Omega^e$$

Element domain:

$$\Omega^e = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \in [0, h^e], (x_1, x_2) \in A^e \subset \mathbb{R}^2 \}$$

$$\Big(\text{Assume } \underbrace{\int_{A^e} x_1 \, dA = \int_{A^e} x_2 \, dA}_{\text{centroid}} = \underbrace{\int_{A^e} x_1 x_2 \, dA = 0}_{\text{principal axes}} \Big)$$

Major assumptions on deformation:

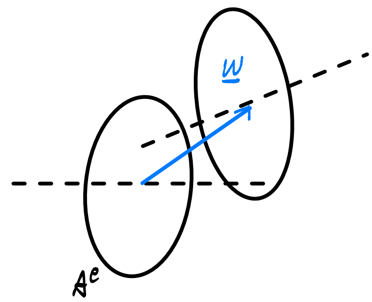
* $\sigma_{\alpha\beta} = 0 \quad (\alpha, \beta \in \{1, 2\})$

(Refer to Hughes book Section 5.4.1)

* $u_1(x_1, x_2, x_3) = w_1(x_3) - x_2 \theta_3(x_3)$

  $u_2(x_1, x_2, x_3) = w_2(x_3) + x_1 \theta_3(x_3)$

  $u_3(x_1, x_2, x_3) = w_3(x_3) - x_1 \theta_2(x_3) + x_2 \theta_1(x_3)$



For a simplified 2D analysis, assume

* $\theta_1 = 0$, $w_2 = 0$ (no out-of-plane bending)
* $\theta_3 = 0$ (no torsional effect)
* $w_3 = 0$ (no axial effect)

$$\Rightarrow \quad \begin{aligned} u_1(x_1, x_2, x_3) &= w_1(x_3) \\ u_2(x_1, x_2, x_3) &= 0 \\ u_3(x_1, x_2, x_3) &= -x_1 \theta_2(x_3) \end{aligned}$$

Isotropic solids:

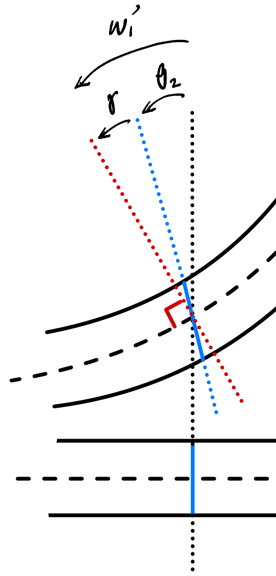
$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu(\sigma_{22} + \sigma_{33}))$$

$$\varepsilon_{22} = \frac{1}{E}(\sigma_{22} - \nu(\sigma_{11} + \sigma_{33}))$$

$$\varepsilon_{33} = \frac{1}{E}(\sigma_{33} - \nu(\sigma_{11} + \sigma_{22}))$$

$$\varepsilon_{12} = \frac{1}{2G}\sigma_{12}, \quad \varepsilon_{13} = \frac{1}{2G}\sigma_{13}, \quad \varepsilon_{23} = \frac{1}{2G}\sigma_{23}$$

Timoshenko beam (2D)



(physical interpretation)

In our case: $\sigma_{13} = 2G\varepsilon_{13}$, $\sigma_{23} = 2G\varepsilon_{23}$, $\sigma_{33} = E\varepsilon_{33}$

and $\varepsilon_{13} = \frac{1}{2}\left(\frac{dw_1}{dx_3} - \theta_2\right) = \frac{1}{2}(w_1' - \theta_2) = \frac{1}{2}\gamma$ (from figure)

$$\varepsilon_{23} = 0, \quad \varepsilon_{33} = -x_1 \frac{d\theta_2}{dx_3} = -x_1 \theta_2'$$

$\gamma = 2\varepsilon_{13}$ is the engineering shear strain, and $\gamma \neq 0$ for Timoshenko beam.

Weak form (Timoshenko beam)

For simplicity, we will derive the weak form for just one beam.

From previous lecture, for general 3D linear elasticity, we have:

$$\int_{\Omega} \bar{u}_{(i,j)} \sigma_{ij} = \int_{\Gamma_N} t_j \bar{u}_j + \int_{\Omega} f_j \bar{u}_j \qquad \forall \bar{\underline{u}} \quad (\bar{\square} \text{ for test function})$$

In our context, $w_1(x_3)$ and $\theta_2(x_3)$ are the major variables instead of $\underline{u}(x_1, x_2, x_3)$, and similarly, $\bar{w}_1(x_3)$ and $\bar{\theta}_2(x_3)$ will be the test functions instead of $\bar{\underline{u}}(x_1, x_2, x_3)$. Then, we have

$$\int_{\Omega} \bar{u}_{(i,j)} \sigma_{ij} = \int_{\Omega^e} 2\bar{u}_{(1,3)} \sigma_{13} + \int_{\Omega^e} \cancel{2\bar{u}_{(2,3)} \sigma_{23}}^{\,0} + \int_{\Omega^e} \bar{u}_{(3,3)} \sigma_{33}$$

$$= \int_{\Omega^e} 2 \cdot \frac{1}{2}(\bar{w}_1' - \bar{\theta}_2)\sigma_{13} + \int_{\Omega^e} - x_1 \bar{\theta}_2' \sigma_{33} = \int_0^{h^e}\int_{A^e} (\bar{w}_1' - \bar{\theta}_2)\sigma_{13} + \int_0^{h^e}\int_{A^e} - x_1\bar{\theta}_2' \sigma_{33}$$

$$= \int_0^{h^e} (\bar{w}_1' - \bar{\theta}_2) \int_{A^e} \sigma_{13} + \int_0^{h^e} \bar{\theta}_2' \int_{A^e} - x_1 \sigma_{33} \quad (*)$$

Caveat: Shear strain (in reality) is not uniform. You need a correction.

Let $q_1 = \int_{A^e} \sigma_{13} = \int_{A^e} 2G\varepsilon_{13} = \int_{A^e} G(w_1' - \theta_2) = (w_1' - \theta_2) G A$

q_1: shear force

Let $m_2 = \int_{A^e} x_1 \sigma_{33} = \int_{A^e} x_1 E \varepsilon_{33} = \int_{A^e} x_1 E \cdot - x_1 \theta_2' = - E\theta_2' I_2 \quad \left(I_2 = \int_{A^e} x_1^2 \right)$

m_2: bending moment

$$\Rightarrow (*) = \int_0^{h^e} (\bar{w}_1' - \bar{\theta}_2) GA (w_1' - \theta_2) + \int_0^{h^e} \bar{\theta}_2' E I_2 \theta_2'$$

Next, consider traction B.C. :

$$\int_{\Gamma^N} t_j \bar{u}_j = \int_{A^e, x_3=0} \overset{n_1=n_2=0,\ n_3=-1}{n_i} \sigma_{ij} \bar{u}_j + \int_{A^e, x_3=h^e} \overset{n_1=n_2=0,\ n_3=1}{n_i} \sigma_{ij} \bar{u}_j = \int_{A^e, x_3=0} - \sigma_{3j}\bar{u}_j + \int_{A^e, x_3=h^e} \sigma_{3j} \bar{u}_j$$

$$= \int_{A^e, x_3=0} - \sigma_{31}\bar{u}_1 + \int_{A^e, x_3=0} - \sigma_{33}\bar{u}_3 + \int_{A^e, x_3=h^e} \sigma_{31}\bar{u}_1 + \int_{A^e, x_3=h^e} \sigma_{33}\bar{u}_3 \qquad (\bar{u}_1 = \bar{w}_1,\ \bar{u}_3 = -x_1\bar{\theta}_2)$$

$$= -\bar{w}_1 q_1^N \Big|_{x_3=0} + \bar{\theta}_2 m_2^N \Big|_{x_3=0} + \bar{w}_1 q_1^N \Big|_{x_3=h^e} - \bar{\theta}_2 m_2^N \Big|_{x_3=h^e}$$

$$= \bar{w}_1 q_1^N \Big|_0^{h^e} - \bar{\theta}_2 m_2^N \Big|_0^{h^e} \quad (**)$$

Next, consider body force :

$$\int_{\Omega} f_j \bar{u}_j = \int_{\Omega^e} f_j \bar{u}_j = \int_0^{h^e}\int_{A^e} f_1 \bar{u}_1 + f_3 \bar{u}_3 = \int_0^{h^e}\int_{A^e} f_1 \bar{w}_1 - f_3 x_1 \bar{\theta}_2$$

$$= \int_0^{h^e} \bar{w}_1 \underbrace{\int_{A^e} f_1}_{F_1(x_3)} - \int_0^{h^e} \bar{\theta}_2 \underbrace{\int_{A^e} x_1 f_3}_{C_2(x_3)} = \int_0^{h^e} \bar{w}_1 F_1 - \int_0^{h^e} \bar{\theta}_2 C_2 \qquad (***)$$

applied forces per unit length

applied couples per unit length

Since $(*) = (**) + (***)$, we get the weak form: Find (w_1, θ_2) s.t.

$$\int_0^{h^e} (\bar{w}_1' - \bar{\theta}_2)\, GA\, (w_1' - \theta_2) + \int_0^{h^e} \bar{\theta}_2'\, EI_2\, \theta_2' = \bar{w}_1 q_1^N \Big|_0^{h^e} - \bar{\theta}_2 m_2^N \Big|_0^{h^e} + \int_0^{h^e} \bar{w}_1 F_1 - \int_0^{h^e} \bar{\theta}_2 C_2 \qquad \forall\, \bar{w}_1, \bar{\theta}_2$$

<u>Strong form (Timoshenko beam)</u>

From the weak form, let's deduce the strong form:

$$(*) = \int_0^{h^e} (\bar{w}_1' - \bar{\theta}_2)\, q_1 - \int_0^{h^e} \bar{\theta}_2'\, m_2$$

$$= -\int_0^{h^e} \bar{w}_1\, q_1' + \bar{w}_1 q_1 \Big|_0^{h^e} - \int_0^{h^e} \bar{\theta}_2\, q_1 + \int_0^{h^e} \bar{\theta}_2\, m_2' - \bar{\theta}_2 m_2 \Big|_0^{h^e}$$

Let $(*) = (**) + (***)$, we get

$$-\int_0^{h^e} \bar{w}_1 (q_1' + F_1) + \int_0^{h^e} \bar{\theta}_2 (-q_1 + m_2' + C_2) + \bar{w}_1 (q_1 - q_1^N) \Big|_0^{h^e} - \bar{\theta}_2 (m_2 - m_2^N) \Big|_0^{h^e} = 0$$

$$\Rightarrow \begin{cases} q_1' + F_1 = 0 \\ m_2' - q_1 + C_2 = 0 \end{cases} \text{(Governing equations)} \qquad \begin{cases} q_1 = q_1^N \text{ at } x_3 = 0 \text{ or } x_3 = h^e \\ m_2 = m_2^N \text{ at } x_3 = 0 \text{ or } x_3 = h^e \end{cases} \text{(Neumann B.C.)}$$

$$\Leftrightarrow \begin{cases} \left((w_1' - \theta_2) G A\right)' + \bar{F}_1 = 0 \\ \left(-E\theta_2' I_2\right)' - (w_1' - \theta_2) G A + C_2 = 0 \end{cases} \Leftrightarrow \begin{cases} GA\left(\dfrac{d^2 w_1}{d x_3^2} - \dfrac{d\theta_2}{d x_3}\right) + \bar{F}_1 = 0 \\ -EI_2 \dfrac{d^2\theta_2}{d x_3^2} - GA\left(\dfrac{dw_1}{dx_3} - \theta_2\right) + C_2 = 0 \end{cases}$$

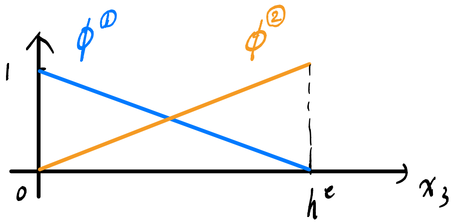
Finite element discretization

We will show the local stiffness matrix and leave the rhs force vector to you.

From $(*) = \displaystyle\int_0^{h^e} (\bar{w}_1' - \bar{\theta}_2)\, GA\, (w_1' - \theta_2) + \int_0^{h^e} \bar{\theta}_2'\, EI_2\, \theta_2'$, let us define

$$w_1^h = w_1^{①} \phi^{①}(x_3) + w_1^{②} \phi^{②}(x_3)$$
$$\theta_2^h = \theta_2^{①} \phi^{①}(x_3) + \theta_2^{②} \phi^{②}(x_3)$$
$$\bar{w}_1^h = \bar{w}_1^{①} \phi^{①}(x_3) + \bar{w}_1^{②} \phi^{②}(x_3)$$
$$\bar{\theta}_2^h = \bar{\theta}_2^{①} \phi^{①}(x_3) + \bar{\theta}_2^{②} \phi^{②}(x_3)$$

where



Use one-point Gaussian quadrature:

$$(*) \approx \left[\bar{w}_1^{①}\ \ \bar{\theta}_2^{①}\ \ \bar{w}_1^{②}\ \ \bar{\theta}_2^{②}\right] \cdot \left(\underbrace{\frac{EI}{h^e}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}}_{\text{bending stiffness}} + \underbrace{\frac{GA}{h^e}\begin{bmatrix} 1 & \frac{h}{2} & -1 & \frac{h}{2} \\ \frac{h}{2} & \frac{h^2}{4} & -\frac{h}{2} & \frac{h^2}{4} \\ -1 & -\frac{h}{2} & 1 & -\frac{h}{2} \\ \frac{h}{2} & \frac{h^2}{4} & -\frac{h}{2} & \frac{h^2}{4} \end{bmatrix}}_{\text{shear stiffness}} \right) \cdot \begin{bmatrix} w_1^{①} \\ \theta_2^{①} \\ w_1^{②} \\ \theta_2^{②} \end{bmatrix}$$

Remarks:

* For a full presentation of 3D frame of beams, refer to Hughes book Section 5.4.

* The beam theories can be considered as simplifications of full analysis of 3D elasticity. Sometimes the assumptions may even lead to certain inconsistencies, but they are extremely useful in engineering practices.

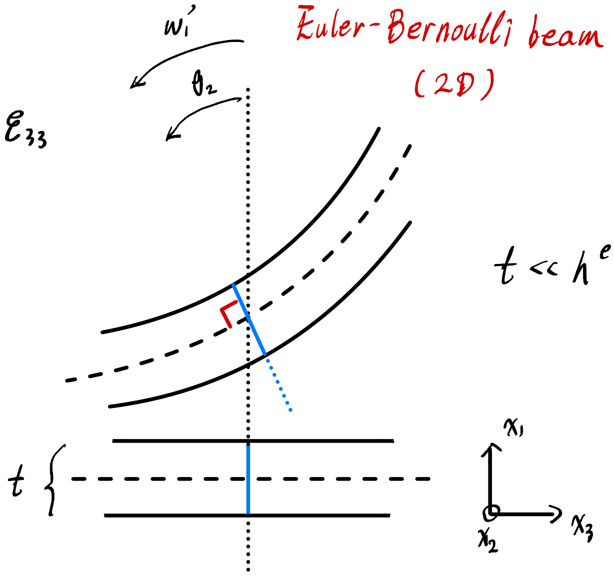
* The "reduced bending flexibility" is often used to improve the performance.

Euler-Bernoulli beam

Reminder: $\sigma_{13} = 2G\,\varepsilon_{13}$, $\sigma_{23} = 2G\,\varepsilon_{23}$, $\sigma_{33} = E\,\varepsilon_{33}$

and $\varepsilon_{13} = \frac{1}{2}\left(\frac{dw_1}{dx_3} - \theta_2\right) = \frac{1}{2}(w_1' - \theta_2) = \frac{1}{2}\gamma$

$\varepsilon_{23} = 0$. $\varepsilon_{33} = -x_1 \frac{d\theta_2}{dx_3} = -x_1\,\theta_2'$



(physical interpretation)

$\gamma = 2\varepsilon_{13}$ is the engineering shear strain, and $\gamma = 0$ for Euler-Bernoulli beam.

Key fact:

$\gamma \approx 0$ when the beam is thin ($t \ll h^e$)

Therefore $\frac{dw_1}{dx_3} = \theta_2$ (We only need w_1 as the major variable!)

Similarly to previous analysis:

Weak form (Euler-Bernoulli beam)

$$\int_0^{h^e} \bar{\theta}_2'\,EI_2\,\theta_2' = \bar{w}_1 q_1^N\Big|_0^{h^e} - \bar{\theta}_2 m_2^N\Big|_0^{h^e} + \int_0^{h^e} \bar{w}_1 F_1 - \int_0^{h^e} \bar{\theta}_2 C_2$$

$$\Rightarrow \int_0^{h^e} \bar{w}_1''\,EI_2\,w_1'' = \bar{w}_1 q_1^N\Big|_0^{h^e} - \bar{w}_1' m_2^N\Big|_0^{h^e} + \int_0^{h^e} \bar{w}_1 F_1 - \int_0^{h^e} \bar{w}_1' C_2$$

Strong form (Euler-Bernoulli beam)

$$\int_0^{h^e} \bar{w}_1''\,EI_2\,w_1'' = -\int_0^{h^e} \bar{w}_1'\,EI_2\,w_1''' + \bar{w}_1'\,EI_2\,w_1''\Big|_0^{h^e}$$

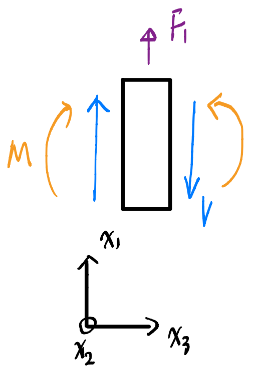
$$= \int_0^{h^e} \bar{w}_1\,EI_2\,w_1'''' - \bar{w}_1\,EI_2\,w_1'''\Big|_0^{h^e} + \bar{w}_1'\,EI_2\,w_1''\Big|_0^{h^e}$$

Also, we may assume there is no applied couples, i.e., $C_2(x_3) = 0$.

Then, $$EI_2 \frac{d^4 w_1}{dx_3^4} = F_1 \quad (\text{governing equation})$$

and $$EI_2 \frac{d^3 w_1}{dx_3^3} = -q_1^N \quad \text{at } x_3 = 0 \text{ or } x_3 = h^e$$
$$EI_2 \frac{d^2 w_1}{dx_3^2} = -m_2^N \quad \text{at } x_3 = 0 \text{ or } x_3 = h^e$$
(Neumann B.C.)

Connection to an elementary procedure



$$\frac{dw_1}{dx_3} = \theta_2 \quad (\text{Kinematics})$$
$$EI_2 \frac{d\theta_2}{dx_3} = M \quad (\text{Constitutive law})$$
$$\frac{dM}{dx_3} = V \quad (\text{Equilibrium of moments})$$
$$\frac{dV}{dx_3} = F_1 \quad (\text{Equilibrium of forces})$$

Reminder:
$$q_1 = \int_{A^e} \sigma_{13} = -V$$
$$m_2 = \int_{A^e} x_1 \sigma_{33} = -M$$

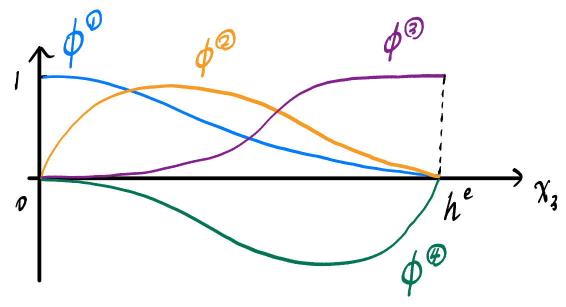
$$\Rightarrow \quad EI_2 \frac{d^4 w_1}{dx_3^4} = F_1 \quad \text{and} \quad EI_2 \frac{d^3 w_1}{dx_3^3} = V \quad \text{at } x_3 = 0 \text{ or } x_3 = h^e$$
$$EI_2 \frac{d^2 w_1}{dx_3^3} = M \quad \text{at } x_3 = 0 \text{ or } x_3 = h^e$$

Finite element discretization

We will construct the local stiffness matrix for $\int_0^{h^e} \bar{w}_1'' EI_2 w_1''$.

Consider Hermite shape functions:

$$w_1^h(x_3) = w^L \phi^{①}(x_3) + \theta^L \phi^{②}(x_3) + w^R \phi^{③}(x_3) + \theta^R \phi^{④}(x_3)$$



$$\phi^{①}(x_3) = -(x_3 - h^e)^2(-h^e - 2x_3)/h^{e3}$$
$$\phi^{②}(x_3) = x_3(x_3 - h^e)^2/h^{e2}$$
$$\phi^{③}(x_3) = x_3^2(3h^e - 2x_3)/h^{e3}$$
$$\phi^{④}(x_3) = x_3^2(x_3 - h^e)/h^{e2}$$

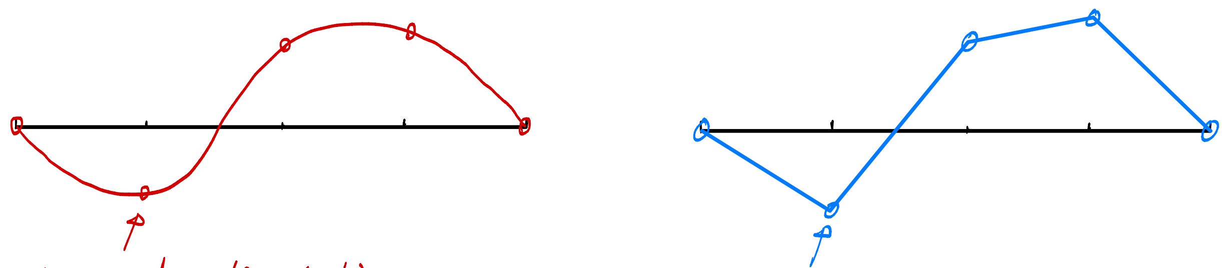
We can verify that $w_1^h(0) = w^L$, $\dfrac{dw_1^h}{dx_3}(0) = \theta^L$

$$w_1^h(h^e) = w^R, \quad \frac{dw_1^h}{dx_3}(h^e) = \theta^R$$

$$\int_0^{h^e} \bar{w}_1'' EI_2 w_1'' \approx \int_0^{h^e} \frac{d^2 \bar{w}_1^h}{dx_3^2} EI_2 \frac{d^2 w_1^h}{dx_3^2} = [\bar{w}^L \ \bar{\theta}^L \ \bar{w}^R \ \bar{\theta}^R] \cdot \frac{EI_2}{h^{e3}} \begin{bmatrix} 12 & 6h^e & -12 & 6h^e \\ 6h^e & 4h^{e2} & -6h^e & 2h^{e2} \\ -12 & -6h^e & 12 & -6h^e \\ 6h^e & 2h^{e2} & -6h^e & 4h^{e2} \end{bmatrix} \cdot \begin{bmatrix} w^L \\ \theta^L \\ w^R \\ \theta^R \end{bmatrix}$$

Remarks:

* The Hermite shape functions enable C^1 continuity, while the Lagrange shape functions only allow C^0 continuity.



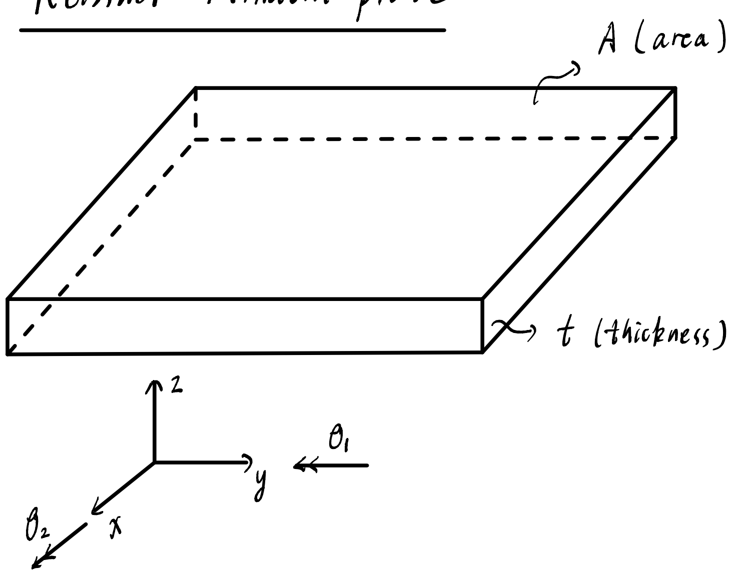
continuous derivative (C^1)

continuous function (C^0)

This is true even for higher order Lagrange shape functions.

Plates

Reissner - Mindlin plate



Major assumptions:

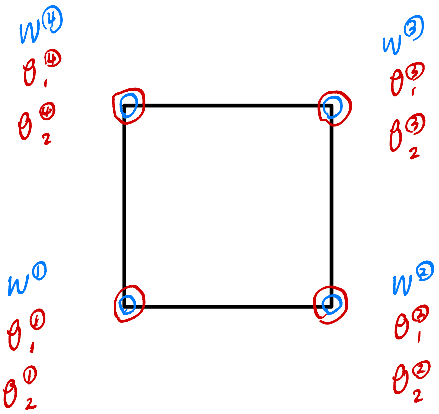
* $\sigma_{33} = 0$
* $u_\alpha(x, y, z) = -z\,\theta_\alpha(x, y)$
* $u_3(x, y, z) = w(x, y)$

Weak form :

$$\int_A \underbrace{\bar{\theta}_{(\alpha,\beta)} C_{\alpha\beta\gamma\delta} \theta_{(\gamma,\delta)}}_{\text{bending stiffness tensor}} + \underbrace{\bar{\gamma}_\alpha C_{\alpha\beta} \gamma_\beta}_{\text{shear stiffness tensor}} = \int_A \underbrace{-\bar{\theta}_\alpha C_\alpha}_{\text{applied couple}} + \underbrace{\bar{w} F}_{\text{applied force}} + \int_{\partial A} \underbrace{-\bar{\theta}_\alpha M_\alpha}_{\text{boundary moments}} + \underbrace{\bar{w} Q}_{\text{boundary shear force}}$$

$\gamma_\alpha = \dfrac{dw}{dx_\alpha} - \theta_\alpha \neq 0$ (shear strain , $\alpha = 1,2$)

For finite element discretization , the DoFs are as such for a typical element:



Remarks:

* Reduced integration is needed to deal with shear locking .

* Both bending enery and shear energy contribute , while in the limit when $t \to 0$ (thin plate) , bending will dominate (Kirchhoff - Love plate).

* The Lagrange finite element used is considered as a C^0 approach .

* Refer to Hughes book Section 5 for detailed presentation of the Reissner - Mindlin plate theory .

Kirchhoff - Love plate

Major assumption : $\gamma_\alpha = \dfrac{dw}{dx_\alpha} - \theta_\alpha = 0$ (shear strain , $\alpha = 1,2$) for thin plate ($t \to 0$). Shear energy does NOT contribute.

The weak form is

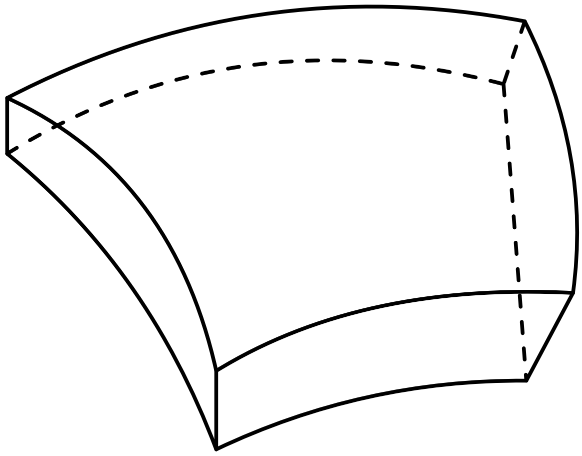
$$\int_A \bar{w}_{,\alpha\beta} \underbrace{C_{\alpha\beta\gamma\delta}}_{\text{bending stiffness tensor}} w_{,\gamma\delta} + \cancelto{0}{\bar{\gamma}_\alpha \underbrace{C_{\alpha\beta}}_{\text{shear stiffness tensor}} \gamma_\beta} = \int_A -\bar{w}_{,\alpha} \underbrace{C_\alpha}_{\text{applied couple}} + \bar{w} \underbrace{F}_{\text{applied force}} + \int_{\partial A} -\bar{w}_{,\alpha} \underbrace{M_\alpha}_{\text{boundary moments}} + \bar{w} \underbrace{Q}_{\text{boundary shear force}}$$

The strong form is $C_{\alpha\beta\gamma\delta}\, w_{,\alpha\beta\gamma\delta} = F + C_{\alpha,\alpha}$ (fourth order PDE)

Remarks:

* We may wish to construct a C^1 continuous finite element approach, but this is extremely difficult in 2D. In practice, people use Reissner-Mindlin plate with small thickness to approximate the behavior of a Kirchhoff-Love plate (of course, shear locking must be properly addressed!).

Shells



physical domain (undeformed)

Remarks:

* Many of the approaches for shell element development are similar to beams and plates.

* Refer to Hughes book Section 6 for C^0-approach for shells.