Lec 05 : 2D linear problem; Theoretical analysis

- Basics of functional analysis
- Finite element error analysis

Basics of functional analysis

Vector space

Def : A vector space over of field $\mathbb{F}$ (which for us will be either $\mathbb{R}$ or $\mathbb{C}$) is a set V together with two laws of composition:

* Vector addition : $V \times V \to V$, written $(\underline{v}, \underline{w}) \mapsto \underline{v} + \underline{w}$
* Scalar multiplication: $\mathbb{F} \times V \to V$, written $(\alpha, \underline{v}) \mapsto \alpha \underline{v}$

that satisfy the following properties:

(i) $\underline{u} + \underline{v} = \underline{v} + \underline{u}$, $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$

(ii) there exists a unique $\underline{0} \in V$ such that for all $\underline{x} \in V$, $\underline{x} + \underline{0} = \underline{0} + \underline{x} = \underline{x}$

(iii) for all $\underline{x} \in V$, there exists a unique $-\underline{x} \in V$ such that $\underline{x} + (-\underline{x}) = (-\underline{x}) + \underline{x} = \underline{0}$

(iv) $1 \underline{x} = \underline{x}$, $\alpha(\beta \underline{x}) = (\alpha\beta) \underline{x}$

(v) $\alpha(\underline{u} + \underline{v}) = \alpha \underline{u} + \alpha \underline{v}$, $(\alpha + \beta) \underline{u} = \alpha \underline{u} + \beta \underline{u}$

Ex. $\mathbb{R}^n$, the usual notion of vectors with n real components is a vector space (over $\mathbb{R}$). For $\underline{x}, \underline{y} \in \mathbb{R}^n$, we write $\underline{x} = (x_1, \cdots, x_n)$, $\underline{y} = (y_1, \cdots y_n)$ for $x_i, y_i \in \mathbb{R}$, and vector addition and scalar multiplication are defined by

$$\underline{x} + \underline{y} = (x_1 + y_1, \cdots, x_n + y_n) \ , \quad \alpha \underline{x} = (\alpha x_1, \cdots, \alpha x_n), \ \alpha \in \mathbb{R}$$

Ex. The set of all infinite sequences $\underline{x} = (x_1, x_2, \cdots)$ of real or complex numbers, denoted $c(\mathbb{N})$, is a vector space, with

$$\underline{x} + \underline{y} = (x_1 + y_1, \cdots, x_j + y_j, \cdots) \ , \quad \alpha \underline{x} = (\alpha x_1, \alpha x_2, \cdots, \alpha x_j, \cdots)$$

In addition, one usually assumes that $\sum_{j=1}^{\infty} |x_j| < \infty$ or $\sum_{j=1}^{\infty} |x_j|^2 < \infty$, in which case these are $\ell^1(\mathbb{N})$ and $\ell^2(\mathbb{N})$.

Ex. Real or complex-valued functions defined over an open subset $\Omega \subset \mathbb{R}^n$ form a vector space:

$C^0(\Omega) = \{$ continuous functions on the set $\Omega \}$

$C^k(\Omega) = \{$ functions on the set Ω whose k-th derivative is continuous $\}$

Normed space

Def. Let V be a real vector space. A norm is a mapping $\|\cdot\|: V \to [0,\infty)$, such that (i) $\|\underline{u}\| = 0 \iff \underline{u} = \underline{0}$ (non-degeneracy)

(ii) $\|\alpha \underline{u}\| = |\alpha| \cdot \|\underline{u}\|$, $\alpha \in \mathbb{R}$, $\underline{u} \in V$ (positive scalability)

(iii) $\|\underline{u} + \underline{v}\| \le \|\underline{u}\| + \|\underline{v}\|$, $\underline{u}, \underline{v} \in V$ (triangle inequality)

The pairing $(V, \|\cdot\|)$ is called a normed space.

Loosely speaking, a norm measures "how big" an element is in a vector space.

Ex. The Euclidean norm for $\mathbb{R}^n$ is given by $\|\underline{x}\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$, $\underline{x} \in \mathbb{R}^n$.

Ex. The p-norm for $\mathbb{R}^n$ is given by $\|\underline{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, $\underline{x} \in \mathbb{R}^n$.

$p=1$: $\|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i|$ ("Manhattan" norm)

$p=2$: Euclidean norm

$p \to \infty$: $\|\underline{x}\|_\infty = \max_{i \in \{1,\dots,n\}} |x_i|$

Ex. The uniform norm for the vector space $C([0,1])$ consisting of continuous functions defined on $[0,1]$ is $\|f\| = \max_{x \in [0,1]} |f(x)|$.

Proof. (i) $\|f\| \geq 0$ for all f and $\|f\| = 0 \Leftrightarrow f(x) = 0$ are obvious.

(ii) $\|\alpha f\| = \max_x |\alpha f(x)| = \max_x |\alpha| \, |f(x)| = |\alpha| \max_x |f(x)| = |\alpha| \, \|f\|$

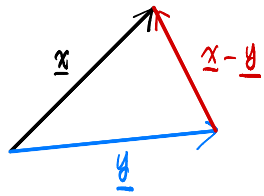
(iii) $\|f+g\| = \max_x |f(x) + g(x)| \leq \max_x \left(|f(x)| + |g(x)|\right) \leq \max_x |f(x)| + \max_x |g(x)| = \|f\| + \|g\|$

Def. A sequence $\{\underline{v}_n\}$ converges to $\underline{v}$ w.r.t. the norm $\|\cdot\|$ if

$$\lim_{n\to\infty} \|\underline{v}_n - \underline{v}\| = 0.$$

Proposition. For $\underline{x}, \underline{y}$ in a space V with norm $\|\cdot\|$, $\left| \|\underline{x}\| - \|\underline{y}\| \right| \leq \|\underline{x} - \underline{y}\|$.

Proof. $\|\underline{x}\| = \|\underline{x} - \underline{y} + \underline{y}\| \leq \|\underline{x} - \underline{y}\| + \|\underline{y}\|$



$\Rightarrow \|\underline{x}\| - \|\underline{y}\| \leq \|\underline{x} - \underline{y}\|$

Exchanging the roles of $\underline{x}$ and $\underline{y}$ gives $\|\underline{y}\| - \|\underline{x}\| \leq \|\underline{y} - \underline{x}\|$, and hence the result.

Proposition. The norm function $\|\cdot\| : V \longrightarrow \mathbb{R}$ is a continuous mapping.

Proof. Consider a sequence $\{\underline{x}_n\}$ converging to $\underline{x}$, i.e., $\underline{x}_n \longrightarrow \underline{x}$. We have $\left| \|\underline{x}_n\| - \|\underline{x}\| \right| \leq \|\underline{x}_n - \underline{x}\| \to 0$, so $\|\underline{x}_n\| \to \|\underline{x}\|$. hence the norm is continuous.

Def. Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be norms on a vector space V. Then $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent if and only if there exists $\alpha, \beta > 0$ s.t.

$$\alpha \|\underline{x}\|_a \leq \|\underline{x}\|_b \leq \beta \|\underline{x}\|_a, \quad \forall \underline{x} \in V.$$

Def. A sequence $\{v_n\} \subset V$ is called Cauchy sequence, if $\sup\{\|v_n - v_m\| : n, m > k\} \to 0$ for $k \to \infty$. A normed space is called complete if every Cauchy sequence converges to an element $v \in V$. A complete normed space is called Banach space.

Inner product space

Def. An <u>inner product</u> on a vector space V over $\mathbb{F}$ is a function

$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{F}$ satisfying

(i) $\langle \underline{v}, \underline{v} \rangle \geq 0$ and $\langle \underline{v}, \underline{v} \rangle = 0 \iff \underline{v} = \underline{0}$ (positivity and non-degeneracy)

(ii) $\langle \alpha \underline{u} + \beta \underline{v}, \underline{w} \rangle = \alpha \langle \underline{u}, \underline{w} \rangle + \beta \langle \underline{v}, \underline{w} \rangle$ (linearity)

(iii) $\langle \underline{u}, \underline{v} \rangle = \overline{\langle \underline{v}, \underline{u} \rangle}$ (conjugate symmetry)

Note: If $\mathbb{F} = \mathbb{R}$, (iii) reduces to symmetry, i.e., $\langle \underline{u}, \underline{v} \rangle = \langle \underline{v}, \underline{u} \rangle$. Also, the combination of linearity and conjugate symmetry imply sesquilinearity:

$$\langle \underline{u}, \alpha \underline{v} \rangle = \overline{\langle \alpha \underline{v}, \underline{u} \rangle} = \bar{\alpha} \overline{\langle \underline{v}, \underline{u} \rangle} = \bar{\alpha} \langle \underline{u}, \underline{v} \rangle .$$

Ex. The function $\langle \underline{x}, \underline{y} \rangle := \sum_{i=1}^{n} x_i \bar{y}_i$ is an inner product on $\mathbb{C}^n$. However, the function $\langle \underline{x}, \underline{y} \rangle := \sum_{i=1}^{n} \bar{x}_i y_i$ is not.

Def. An inner product space $(V, \langle \cdot , \cdot \rangle)$ can always <u>induce a normed space</u>

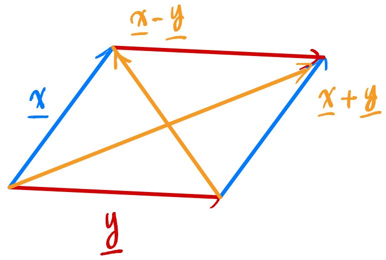
$$\| \underline{x} \| := \sqrt{\langle \underline{x}, \underline{x} \rangle} .$$

Theorem (Cauchy-Schwarz inequality). For any $\underline{x}, \underline{y}$ from an inner product space,

$$| \langle \underline{x}, \underline{y} \rangle | \leq \| \underline{x} \| \| \underline{y} \| . \qquad \text{(Here, } \| \cdot \| \text{ is induced norm)}$$

Proposition (Parallelogram law). For any $\underline{x}, \underline{y}$ from an inner product space,

$$\| \underline{x} + \underline{y} \|^2 + \| \underline{x} - \underline{y} \|^2 = 2 (\| \underline{x} \|^2 + \| \underline{y} \|^2) . \qquad \text{(Here, } \| \cdot \| \text{ is induced norm)}$$



Question: Can normed vector space induce inner product space?

Proposition. Only if a norm satisfying the parallelogram law, can it induce an inner product:

$$\langle \underline{x}, \underline{y} \rangle := \frac{1}{4}\left(\|\underline{x}+\underline{y}\|^2 - \|\underline{x}-\underline{y}\|^2\right) + \frac{i}{4}\left(\|\underline{x}+i\underline{y}\|^2 - \|\underline{x}-i\underline{y}\|^2\right) \quad \text{for } \mathbb{F}=\mathbb{C}$$

or $$\langle \underline{x}, \underline{y} \rangle := \frac{1}{4}\left(\|\underline{x}+\underline{y}\|^2 - \|\underline{x}-\underline{y}\|^2\right) \quad \text{for } \mathbb{F}=\mathbb{R}$$

Def. If an inner product space is complete w.r.t. the norm induced by its inner product, it is said to be a Hilbert space.

Linear transformations

Def. A transformation $T: V \to U$ between vector spaces V and U is linear if for all $\underline{x}, \underline{y} \in V$ and all scalars α, β, we have

$$T(\alpha \underline{x} + \beta \underline{y}) = \alpha T(\underline{x}) + \beta T(\underline{y}).$$

A linear transformation T is also called an operator.

Proposition Every linear transformation between finite dimensional spaces has a matrix representation.

Theorem (Continuity is equivalent to boundedness for linear maps). Let $T: V \to U$ be a linear transformation from one normed space $(V, \|\cdot\|_V)$ to another $(U, \|\cdot\|_U)$. Then the following are equivalent:

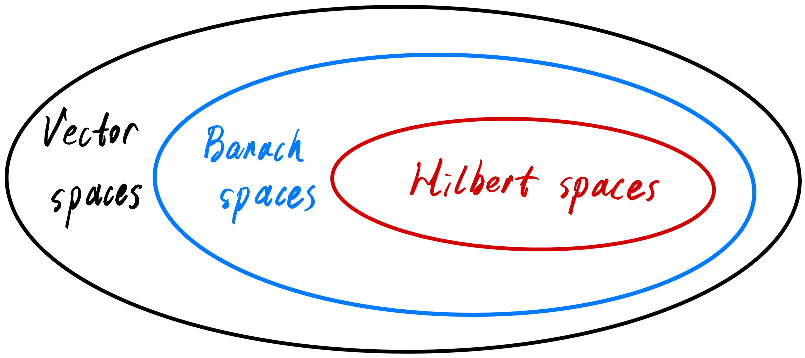
(i) T is continuous.

(ii) T is continuous at $\underline{0}$.

(iii) There exists a constant M such that $\|T(\underline{x})\|_U \leq M \|\underline{x}\|_V$, for all $\underline{x} \in V$.

Def. If T is a continuous linear map between normed spaces, we can define the norm of T to be $\|T\| = \sup_x \frac{\|T(x)\|}{\|x\|}$ ("sup" is smallest upper bound)
This norm is called the operator norm.

Theorem (Riesz representation theorem). Let H be a Hilbert space, and let $\lambda: H \to \mathbb{F}$ be a continuous linear functional on H. Then exists a unique $u \in H$ s.t. $\lambda(v) = \langle v, u \rangle$ for all $v \in H$. Moreover, $\|\lambda\| = \|u\|$.



Finite element error analysis

Sobolev spaces

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$.

Def. $L^2(\Omega)$ space is the vector space containing all functions which are Lebesgue square integrable on Ω, i.e., $L^2(\Omega) = \{ f: \Omega \to \mathbb{R} : \int_\Omega |f(x)|^2 < \infty \}$, endowed with the inner product $(f, g)_0 = \int_\Omega fg$ and the induced norm $\|f\|_0 = \sqrt{(f,f)_0}$.

Note: $L^2(\Omega)$ is a Hilbert space.

Def. For a multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, where $|\alpha| = \sum_{i=1}^d \alpha_i \in \mathbb{N}_0$. We use the notation $D^\alpha \varphi = \frac{\partial^{|\alpha|} \varphi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_d}}$ to denote the derivative of a function $\varphi(x)$.

Subject to some technical conditions, this notation will be used to represent the more generalized "weak derivative".

Def. A Sobolev space of order k on Ω is defined as:

$$H^k(\Omega) := \{ f \in L^2(\Omega) : D^\alpha f \in L^2(\Omega), \ \forall |\alpha| \le k \}$$

Note: $H^k(\Omega)$ is a Hilbert space endowed with inner product

$$(f, g)_k = \sum_{|\alpha| \le k} (D^\alpha f, D^\alpha g)_0$$

and the induced norm

$$\|f\|_k = \sqrt{(f, f)_k}$$

Def. The H^k-seminorm is given by $|f|_k = \sqrt{\sum_{|\alpha|=k} (D^\alpha f, D^\alpha f)_0} = \sqrt{\sum_{|\alpha|=k} \|D^\alpha f\|_0^2}$

Note: Seminorm $|\cdot|$ may not be a norm since $|\underset{\sim}{x}| = 0$ may not imply $\underset{\sim}{x} = \underset{\sim}{0}$.

Ex. $H^0(\Omega) = L^2(\Omega) = \{ f \in L^2(\Omega) \}$

$$H^1(\Omega) = \{ f \in L^2(\Omega) : \frac{\partial f}{\partial x_i} \in L^2(\Omega), \ 1 \le i \le d \}$$

L^2-norm (or H^0-norm): $\|f\|_0^2 = \int_\Omega |f|^2$

H^1-norm: $\|f\|_1^2 = \int_\Omega |f|^2 + \int_\Omega |\nabla f|^2$

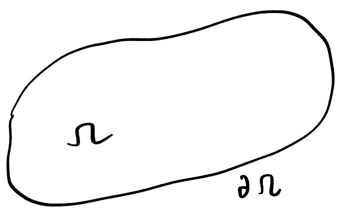
H^k-norm: $\|f\|_k^2 = \|f\|_{k-1}^2 + \sum_{|\alpha|=k} (D^\alpha f, D^\alpha f)_0 = \|f\|_{k-1}^2 + |f|_k$

H^0-seminorm: $|f|_0^2 = \|f\|_0^2$

H^1-seminorm: $|f|_1^2 = \int_\Omega |\nabla f|^2 = \sum_{i=1}^{d} \|\frac{\partial f}{\partial x_i}\|_0^2 = \|\nabla f\|_0^2$

H^2-seminorm: $|f|_2^2 = \sum_{1 \le i \le j \le d} \|\frac{\partial^2 f}{\partial x_i \partial x_j}\|_0^2$

A model problem



Poisson problem with homogeneous Dirichlet B.C.

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

Weak form gives : Find $u \in V = H_0^1(\Omega) := \{ v \in L^2(\Omega) : \nabla v \in L^2(\Omega),\ v|_{\partial\Omega} = 0 \}$ s.t.

$$a(v,u) = (v,f), \quad \text{for all } v \in V$$

where $\quad a(v,u) = \int_\Omega \nabla v \cdot \nabla u \quad , \quad (v,f) = (v,f)_0 = \int_\Omega vf \quad (f \in L^2(\Omega))$

Questions : ① Existence and uniqueness of this weak form problem ?

② Why the Galerkin FE method is the "best" approximation ?

③ How fast the Galerkin FE solution converges to the exact solution ?

Question #1 : Existence and uniqueness

* Notice that $a(\cdot,\cdot)$ satisfies all conditions of an inner product over V.

In particular, $a(u,u) = 0 \Rightarrow \nabla u = 0 \Rightarrow u$ is constant . As $u \in V$, $u|_{\partial\Omega} = 0$, we have $u = 0$. Therefore, $a(\cdot,\cdot)$ defines an inner product on V.

* Show $(\cdot, f)$ is a bounded linear functional w.r.t. V and the norm induced by $a(\cdot,\cdot)$. Note that $\sqrt{a(u,u)} = |u|_1$ (H^1-seminorm) is indeed a norm over $V = H_0^1$.

Theorem (Poincaré inequality) : $\|v\|_0 \le C(\Omega)|v|_1, \ \forall v \in H_0^1(\Omega)$

Consider $|(v,f)| = |(v,f)_0| \underset{\substack{\uparrow \\ \text{Cauchy-Schwarz}}}{\le} \|v\|_0 \|f\|_0 \le \underbrace{\|f\|_0 C(\Omega)}_{\text{const.}} |v|_1$

$\Rightarrow (\cdot, f)$ is a bounded linear functional .

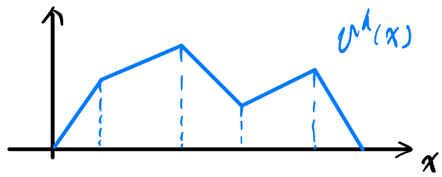
* Apply Riesz representation theorem. For any continuous linear functional, there exists a unique $u \in V$ s.t. $(v, f) = a(v, u)$ for all $v \in V$.

Question # 2: Best approximation

Galerkin method: Find $u^h \in V^h = X^h(\Omega) := \{v^h \in C^0(\bar{\Omega}) : v^h|_K \in \mathbb{P}_k,\ v^h|_{\partial\Omega} = 0\}$ s.t.

polynomial

$$a(v^h, u^h) = (v^h, f), \quad \text{for all } v^h \in V^h$$



Note: $V^h \subset V = H_0^1$.

* Following a similar procedure by replacing V with V^h, we can show that there exists a unique solution $u_h \in V^h$ to the discretized weak form problem.

* Assume $u \in V$ is the exact solution to the weak form and $u^h \in V^h$ is the solution to the Galerkin problem. We have:

(Galerkin orthogonality)

$$\left.\begin{aligned} a(v^h, u) &= (v^h, f) \\ a(v^h, u^h) &= (v^h, f) \end{aligned}\right\} \Rightarrow a(v^h, u - u^h) = 0, \ \forall v^h \in V^h$$

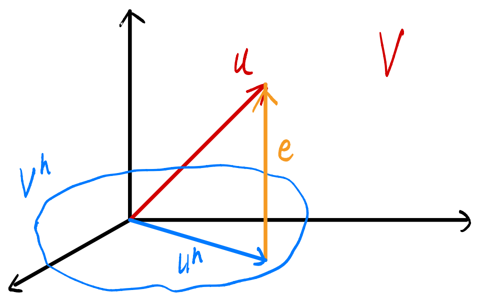
$$\Rightarrow |u - u^h|_1^2 = a(u - u^h, u - u^h) = a(u, u - u^h) = a(u - v^h, u - u^h) \le |u - v^h|_1 |u - u^h|_1$$

$$\Rightarrow |u - u^h|_1 \le |u - v^h|_1 \ \forall v^h \in V^h \Rightarrow |u - u^h|_1 = \min_{v^h \in V^h} |u - v^h|$$ (Best approximation)

* By Poincaré inequality, $\|v\|_0 \le C|v|_1 \Rightarrow \|v\|_0 + |v|_1 = \|v\|_1 \le (C+1)|v|_1$

also, $|v|_1 \le \|v\|_1$ is trivial. Therefore, $|\cdot|_1$ and $\|\cdot\|_1$ are equivalent norms on H_0^1.

Therefore, $\|u - u^h\|_1 \le C \cdot \|u - v^h\|_1 \ \forall v^h \in V^h$. (Céa's lemma in the symmetric case)

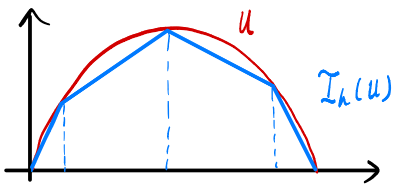


error $e = u - u^h$

Question #3: Convergence rates

Error in H^1-norm

Let $\mathcal{I}_h : V \to V^h$ be the <u>interpolation operator</u>:



<u>Theorem</u> For $u \in H^2(\Omega)$, V^h is the linear finite element space, we have

$$|u - \mathcal{I}_h(u)|_1 \le h \cdot |u|_2$$

Also, assume H^2-regularity condition, i.e., $\|u\|_2 \le C\|f\|_0$.

Then, we have:

$$\underset{\substack{H^1\text{-norm}\\ \text{error}}}{\|u-u^h\|_1} \underset{\substack{\uparrow\\ \text{norm}\\ \text{equivalence}}}{\le} C|u-u^h|_1 \underset{\substack{\uparrow\\ \text{"best"}\\ \text{approx.}}}{\le} C|u-\mathcal{I}_h(u)|_1 \underset{\substack{\uparrow\\ \text{interp.}\\ \text{theorem}}}{\le} Ch|u|_2 \underset{\substack{\uparrow\\ \text{trivial}}}{\le} Ch\|u\|_2 \underset{\substack{\uparrow\\ H^2\text{-reg.}}}{\le} Ch\|f\|_0$$

$\Rightarrow \|u-u^h\|_1 \le C \cdot h \cdot \|f\|_0$ (H^1-norm convergence rate)

Error in L^2-norm (by Nitsche's trick)

$$\|u-u^h\|_0 = \sup_{g \in L^2(\Omega)} \frac{(g, u-u^h)_0}{\|g\|_0} \quad \text{(recall the norm of a linear functional } (\cdot, u-u^h)_0 \text{)}$$

Let $\varphi_g \in H_0^1$ be the solution to the problem " $a(w, \varphi_g) = (w, g)\ \forall w \in H_0^1$ " $(*)$

Then, $(g, u-u^h) = (u-u^h, g) = a(u-u^h, \varphi_g) = a(u-u^h, \varphi_g - v^h) \le |u-u^h|_1 |\varphi_g - v^h|_1$

$\le \|u-u^h\|_1 \cdot \|\varphi_g - v^h\|_1$, $\forall v^h \in V^h$ $\Rightarrow$ $\underline{(g, u-u^h) \le \|u-u^h\|_1 \inf_{v^h \in V^h} \|\varphi_g - v^h\|_1}$

Let $\varphi_g^h \in V^h$ be the Galerkin solution to problem $(*)$. According to H^1-norm convergence, we have $\|\varphi_g - \varphi_g^h\|_1 \le C h \|g\|_0$.

Therefore,
$$\|u-u^h\|_0 = \sup_{g \in L^2(\Omega)} \frac{(g, u-u^h)_0}{\|g\|_0} \le \|u-u^h\|_1 \sup_{g \in L^2(\Omega)} \left(\frac{1}{\|g\|_0} \inf_{v^h \in V^h} \|\varphi_g - v^h\|_1 \right)$$

$$\le \|u-u^h\|_1 \sup_{g\in L^2(\Omega)} \left(\frac{1}{\|g\|_0} \|\varphi_g - \varphi_g^h\|_1 \right) \le \|u-u^h\|_1 \cdot C h \le C h^2 \|f\|_0$$

replace v^h with φ_g^h

H^1-norm convergence

$\Rightarrow \|u-u^h\|_0 \le C h^2 \|f\|_0$ (L^2-norm convergence rate)