数值代数

书面作业

PB21000030 吕文开

第一次书面作业.

1.1 $LT = I$

设 $T = [T_1 \cdots T_n]$

$LT = [LT_1 \cdots LT_n] = I = [e_1 \cdots e_n]$

$\Rightarrow$ Solve $LT_i = e_i$

```
for j = 1:n
    for k = 1:n-1
        T(k,j) = T(k,j) / L(k,k).
        T(k+1:n,j) = T(k+1:n,j)
                     - T(k,j) L(j+1:n,k).
    end.
    T(n,j) = T(n,j) / L(n,n).
end.
```

1.2 S, T 均为下三角阵 $\Rightarrow \underline{ST} \in L$.

$ST - \lambda I \in L$.

Step 1: $ST = S \cdot T$ $\quad O(n^2)$

Step 2: $L = ST - \lambda I$. $\quad O(n^2)$

Step 3: $L\underset{\sim}{x} = b \Rightarrow x$. $\quad O(n^2)$

1.3 $L_k = I - l_k e_k^T \qquad L_k' \triangleq I + l_k e_k^T$

$$L_k(L_k') = (I - l_k e_k^T)(I + l_k e_k^T)$$

$$= I - l_k e_k^T l_k e_k.$$

$$\underline{e_k^T l_k} = e_k^T (0, 0 \ldots 0, l_{k+1,k}, \cdots l_{n,k})^T,$$

$$= 0$$

$\Rightarrow L_k' = L_k^{-1}$ 显然是 Gauss 变换

1.4

$$L\begin{bmatrix}2\\3\\4\end{bmatrix}=\begin{bmatrix}2\\7\\8\end{bmatrix}$$

$$L=\begin{bmatrix}1&&\\2&1&\\2&0&1\end{bmatrix}.$$

1.5

$$A=L_1U_1=L_2U_2$$

其中令 $\det(L_1)=\det(L_2)=1$

$$L=\underline{L_2^{-1}L_1}=U_2U_1^{-1}$$

① L 为下三角阵. ② $\det L=1 \Rightarrow L=I$

$\downarrow$

$L_2=L_1$. $U_2=U_1$

<u>分解唯一</u>.

1.6

$$A = (a_{ij}) = \begin{cases} 1 & i=j \ / \ j=n \\ -1 & i>j \\ 0 & \end{cases}$$

$$l_{ij} = \begin{cases} 0 & i<j \\ 1 & i=j \\ -1 & i>j \end{cases} \qquad u_{ij} = \begin{cases} 2^{i-1} & j=n. \\ 1 & i=j. \\ 0. & \end{cases}$$

$\underline{A = LU.}$

1.7

设 $L_1 = \begin{bmatrix} 1 & & & \\ -\frac{a_{21}}{a_{11}} & 1 & & \\ & 0 & \ddots & \\ \frac{a_{n1}}{a_{11}} & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & \\ -\frac{a_1}{a_{11}} & I \end{bmatrix}.$

$a_1 \stackrel{d}{=} (a_{21} \cdots a_{n1})^T$

$$A = \begin{bmatrix} a_{11} & a_1^T \\ a_1 & A_{22}^{(0)} \end{bmatrix}.$$

$$L_1 A = [\ \cdot\ \cdot\ ][\ \cdot\ \cdot\ ] = \begin{bmatrix} a_1 & \\ & A_{22}^{(0)} - \frac{a_1 a_1^T}{a_{11}} \end{bmatrix}$$

$$A_2 = A_{22}^{(0)} - \frac{a_1 a_1^T}{a_{11}}$$

1.8

$$A_2 = [\tilde{a}_{ij}]$$

$$\tilde{a}_{ij} = a_{ij} - \frac{a_{i1} a_{1j}}{a_{11}}$$

主对角元: $|\tilde{a}_{ii}| = |a_{ii}| - \left|\frac{a_{i1} a_{1i}}{a_{11}}\right|$

非主对角元 $\sum_{\substack{k=2 \\ k \neq i}}^{n} |\tilde{a}_{ik}| = \sum \left| \_ - \_ \right|$

$$\leq \sum_{k} |a_{ik}| + \frac{|a_{i1}|}{|a_{11}|} \sum_{k} |a_{1k}|$$

由 A 严格对角占优.

$$|a_{kk}| \cancel{> \sum_{j \neq k} |a_{jk}|}$$

$$> \sum_{j \neq k} |a_{kj}|$$

$$\Rightarrow |\tilde{a}_{kk}| > \sum_{j \neq k} |\tilde{a}_{kj}|$$

$\tilde{A}_2$ 严格对角占优

1.9 $A=LU$. $Ax=b$.

$LUx=b$.

$Ux=L^{-1}b$.

① $[A:b] \to [U:L^{-1}b]$

```
for k = 1:n-1
    A(k+1:n, k) = A(k+1:n, k) / A(k, k).
    A(k+1:n, k+1:n) = A(k+1:n, k+1:n)
                      - A(k+1:n, k) A(k, k+1:n)
    b(k+1:n) = b(k+1:n) - A(k+1:n, k) b(k).
end.
```

② $[U:L^{-1}b] \to [I:x]$

共需 $O(\frac{2}{3}n^3)$ 步.

1.10 $L_1 A = \begin{bmatrix} a_{11} & a_1^T \\ & A_2 \end{bmatrix}$

$$L_1 = \begin{bmatrix} 1 & \\ -\frac{a_1}{a_{11}} & I \end{bmatrix}.$$

$$a_{11} = (a_{21} \cdots a_{n1})^T$$

$$L_1 A L_1^T = \begin{bmatrix} a_{11} & \\ & A_2 \end{bmatrix}$$

L_1 非奇异. $\forall x \in \mathbb{R}^{n-1} \setminus \{0\}$.

构造 $\tilde{x} \triangleq (0, x^T)^T$ $y = L_1^T \tilde{x}$.

正定

$$\underline{0 < y^T A y} = \tilde{x}^T L_1 A L_1^T \tilde{x} = (0, x^T) \begin{bmatrix} a_{11} & \\ & A_2 \end{bmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix}$$

$$= x^T A_2 x.$$

$\forall x$ 成立.

A_2 正定

1.11 A_{11} 有三角分解.

$$L = \begin{array}{cc} \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} & \begin{array}{l} k \\ n-k \end{array} \end{array}$$

其中 $$LA = \begin{array}{cc} \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ & A_{22}^{(k)} \end{bmatrix} & \begin{array}{l} k \\ n-k \end{array} \end{array}$$

$$LA = \begin{bmatrix} L_{11}A_{11} & L_{11}A_{12} \\ L_{21}A_{11} + A_{21} & L_{22}A_{12} + A_{22} \end{bmatrix}$$

$$L_{21} = -A_{21}A_{11}^{-1} \Rightarrow A_{22}^{(k)} = A_{22}A_{11}^{-1}$$

1.12 参考 对角元严格占优证明.

1.13 $PA = LU$

$\underline{(LU)^{-1} e_j} = x_j \Rightarrow LU x_j = e_j$

$(PA)^{-1} = A^{-1}P^{-1} = [x_1 \cdots x_n]$

$A^{-1} = [x_1 \cdots x_n] P$

① $\underline{PA = LU}$

② $LU[x_1 \cdots x_j] = I \Rightarrow LUX = I$

③ $A^{-1} = XP$

1.14 $LUx = e_j$ 计算 $(x_i)_j$ 即为 $A(i,j)$

1.15 $\underline{A^T \in R^{n\times n}}$ 严格对角占优 $\Leftrightarrow$ A 严格 —— .

$A^{(1)} = \begin{bmatrix} a_{11} & a_1^T \\ & A_2 \end{bmatrix} = \underline{L_2 A}$

$\downarrow$

严格对角占优. $|l_{ij}| < 1$

同时主元$\Leftrightarrow$列主元.

$\boxed{1.16}$ ① 设 $N(x,k)=1-xe_k^T$

与 $N(y,k)=1-ye_k^T$

$$N(x,k)\cdot N(y,k)=1-(y+x-xy_k)e_k^T.$$

$$x=\frac{y}{y_k-1}$$

$$\Rightarrow N\left(\frac{y}{y_k-1},k\right).$$

②. $N(y,k)=e_k.$ $x-x_k y=e_k.$

$$y=\frac{(x-e_k)}{x_k}=\left(\frac{x_1}{x_k},\cdots\frac{x_{k-1}}{x_k},1-\frac{1}{x_k},\cdots\right).$$

$\Rightarrow x_k\neq 0.$ 即有 $N(y,k)x_k=e_k.$

③ $A^{-1} = T = [T_1 \cdots T_n]$.

$AT_j = e_j$. $\quad j = 1 \dots n$.

$A^{(0)} = A$, $\quad T^{(0)} = I$.

1. $a_{11}^{(0)} \neq 0$.

$$\begin{cases} y_1 = \left(1 - \frac{1}{a_{11}^{(0)}}, \frac{a_{21}^{(0)}}{a_{11}^{(0)}} \dots \right)^T \\ N_1 = N(y_1, 1). \end{cases}$$

$$\Rightarrow \begin{cases} A^{(1)} = N_1 A^{(0)} \\ T^{(1)} = N_1 (T^{(0)}). \end{cases}$$

2. $a_{22}^{(0)} \neq 0$.

$$\left.\begin{array}{l} y_2 = \left(\frac{a_{12}^{(1)}}{a_{22}^{(1)}}, 1 - \frac{1}{a_{22}^{(1)}}, \dots \right)^T \\ N_2 = N(y_2, 2) \end{array}\right\}$$

$$\begin{cases} A^{(2)} = N_2 A^{(1)}. \\ T^{(2)} = N_2 T^{(1)}. \end{cases}$$

$$\Rightarrow I = A^{(n)} = N_n N_{n-1} \cdots N_1 A.$$

$$T^{(n)} = N_n N_{n-1} \cdots N_1.$$

$$\Rightarrow A^{?} = T^{(n)}.$$ 条件 $a_{kk}^{(k-1)} \neq 0$, $\forall k$.

1.17 $$A = L_1 L_1^T = L_2 L_2^T$$

$$\underbrace{L_1^{-1} L_2}_{\text{下三角阵}} = \underbrace{L_1^T (L_2^T)^{-1}}_{\text{上三角阵}} = D.$$

$\boxed{\text{下三角阵} \quad \text{上三角阵.}} \Rightarrow$ 对角阵

$$L_2 = L_1 D \quad (L_2^T)^{-1} = (D L_1^T)^{-1}$$

$$\Rightarrow D = L_1^T (D L_1^T)^{-1} = D^{-1} = I.$$

$$\Rightarrow L_1 = L_2. \quad \#.$$

1.18 $a_{ij} = 0 \quad |i-j| > m$.

```
for k = 1:n
    A(k,k) = √...
    n_min = min(n, k+m).
    A(k+1:n_min, k) = A(k+1:n_min, k) / A(k,k).
    for j = k+1 : n_min.
        A(j:n_min, j) -= ... A(j,k)
    end
end.
```

$\Rightarrow$ L 也是带状矩阵. 带宽为 $m+1$.

1.19 $A = LL^T$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad L = \begin{bmatrix} L_{11} & \\ L_{21} & L_{22} \end{bmatrix}.$$

$A_{21}^T = A_{12}$

$$A = \begin{bmatrix} A_{11} & A_{21}^T \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} L_{11} & \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} L_{11}^T & L_{21}^T \\ & L_{22}^T \end{bmatrix}$$

$$= \begin{bmatrix} L_{11}L_{11}^T & L_{11}L_{21}^T \\ \cdots & \cdots \end{bmatrix}$$

$\Rightarrow A_{11} = L_{11}L_{11}^T$. Q.E.D.

1.20

存在性：$\exists LU$. $A = LU$.

且除 u_{nn} 外，$u_{ii} \neq 0$.

$D = diag(u_{11} \cdots)$.

$\exists \tilde{U} \Rightarrow U = D\tilde{U}$.

$$A = LD\tilde{U}$$

$$A^T = A$$

$$LD\tilde{U} = \tilde{U}^T D L^T.$$

$$\underline{D\tilde{U}}\,\underline{(L^T)^{-1}} = L^{-1}\tilde{U}^T D. = D.$$

上三角阵　下三角阵

$\Rightarrow \tilde{D}(L^T)^{-1} = L^{-1}\tilde{D}^T = I$. L^T 与 $\tilde{D}$.

$\Rightarrow A = LDL^T$.

D唯一性 $\quad A = L_1 D_1 L_1^T = L_2 D_2 L_2^T$

$$\underbrace{L_2^{-1} L_1}_{\text{下三角}} D_1 = D_2 \underbrace{L_2^T (L_1^T)^{-1}}_{\text{上三角}} = D'$$

$\rightsquigarrow \underline{L_2^{-1} L_1 = I}$. $L_2^T (L_1^T)^{-1} = I$.

$L_1 = L_2$. $\quad D_1 = D_2 = D'$

(1.22)

① $A = LDL^T$

② $LX = 0$

③ $DY = X$.

$\underline{Y = (LD)^{-1}}$

④ $L^T Z = Y \quad \underline{Z = (LDL^T)^{-1} = A^{-1}}$

$\boxed{1.23}$

$$A = LL^T$$

$$L = \begin{bmatrix} 4 & & & \\ 1 & 3 & & \\ 2 & 2 & 2 & \\ 1 & 1 & 3 & 1 \end{bmatrix}$$

$\forall x \in \mathbb{R}^n \setminus \{0\}$. $Lx \neq 0$.

正定

$$x^T A x = x^T L L^T x = \underline{(L^T x)^T L^T x > 0}.$$

$Ly = b$. $y = \begin{pmatrix} 8 \\ 6 \\ 5 \\ 1 \end{pmatrix}$. $L^T x = y$.

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

1-24 $\quad H = A + iB$

$\forall z \in \mathbb{C}.\ z = x + iy.\quad z^* = x^T - iy^T.$

$$z^* H z = (x^T - iy^T)(A + iB)(x + iy).$$

$$= (x^T A x + y^T A y) + i(x^T B x + y^T B y).$$

$\Rightarrow$ H正定 ~ A, B正定.

$\forall z = (x^T, y^T)^T \in \mathbb{R}^{2n}.$

$$z^T C z = (x^T\ y^T)\begin{bmatrix} A & -B \\ B & A \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= x^T A x + y^T A y.$$

H正定 $\Leftrightarrow$ A正定 $\Leftrightarrow$. C正定.

$$(A + iB)(x + iy) = (Ax - By) + i(Ay + Bx)$$

$$\begin{cases} Ax - By = b. \\ Ax + Bx = c. \end{cases} \qquad \underbrace{\begin{bmatrix} A & -B \\ B & A \end{bmatrix}}_{C.}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix}.$$