$Q = Q * G(i-1, i, -g(A[i-1][i-1], A[i][i-1]))$

$A = G(i-1, i, g(A[i-1][i-1], A[i][i-1]))^{T} A.$

12. Pf: $\|x\|_2^2 = x^T x$. 展开

$\|Ax-b\|_2 = \min_{y \in \mathbb{R}^n} \|Ay-b\|_2$

$\|A(x+\alpha w)-b\|_2 \geq \|Ax-b\|_2$

$2\alpha w^T A^T(Ax-b) + \alpha^2\|Aw\|^2 \geq 0 \quad \forall \alpha, w.$

~~用结果~~ 设 $\alpha > 0$

$\Rightarrow 2w^T A^T(Ax-b) + \alpha\|Aw\|^2 \geq 0$ 恒成立

若 $A^T(Ax-b) \neq 0$. 设 $(A^T(Ax-b))^{(i)} \neq 0$.

取 $w = [\pm e_i]$. $2w^T A^T(Ax-b) < 0$

令 $\alpha \to 0$ 矛盾.

$\Rightarrow A^T(Ax-b) = 0$. $A^T A x = A^T b$.

第四次书面作业.

1. A_1 在 Jacobi 下的矩阵 $\begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$. 谱半径 $\frac{\sqrt{5}}{2}$ 不收敛

G-S. $\begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$. $\frac{1}{2}$. 收敛

A_2. Jacobi: $\begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix}$. 谱半径. 0. 收敛

G-S. $\begin{pmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{pmatrix}$. 谱半径 2. 不收敛

2. 由谱半径. B 特征值全为0. $B^n = 0$.

$x_n = B^n x_0 + B^{n-1} g + \cdots + Bg + g$. $B^n = 0$

$x_n = (B^{n-1} + \cdots + B + I) g$

$(B x_n + g) = x_n$.

$\Rightarrow x_n$ 为精确解.

3.(1) $x_1^2+x_2^2+x_3^2+2ax_1x_3>0$ 对非零向量恒成立

$a\in(-1,1)$ 时显然满足题意，否则令 $x_1=x_3=1$, $x_2=0$ 矛盾.

$\Rightarrow a\in(-1,1)$

(2). Jacobi $\begin{pmatrix} & & -a \\ & & \\ -a & & \end{pmatrix}$ 特征值 $0, -a, a$.

谱半径 $<1 \Rightarrow a\in(-1,1)$.

(3) J-S. $\begin{pmatrix} & & -a \\ & & \\ & & a^2 \end{pmatrix}$. $0, 0, a^2$.

$<1 \Rightarrow a\in(-1,1)$

4. rmk: $\exists P$ PA左上角元素非零. 否则 $n-1$阶非奇异.

pf: $\det(A)=\sum_i a_{i1}(-1)^{i+1}\det(A_{i1})$.

余子式.

$\det(A)\neq 0 \Rightarrow \exists t.\ a_{t1}(-1)^{t+1}\det(A_{t1})\neq 0$

$a_{t1}\neq 0$ $\det(A_{t1})\neq 0$.

取 $P=E_{1t}$ (交换 1, t 行) 即为所求.

$n=1$ 平凡

归纳 $P=\begin{pmatrix}1 & 0\\ 0 & Q\end{pmatrix}P_0$. 即为所求. #

5. $\|B\|_\infty\le 1$ $\sum_{j=1}^{n}|b_{ij}|=\sum_{j\neq i}\left|\frac{a_{ij}}{a_{i1}}\right|=\frac{\sum_{j\neq i}|a_{ij}|}{|a_{ii}|}<1$

$\Rightarrow \sup_{i=1}\sum_{j=1}^{n}|b_{ij}|\le 1$. 得证

6. 归纳 $n=1$ 平凡

设 n 阶成立. $n+1$:

$m_i \triangleq |a_{ii}|-\sum_{j\neq i}a_{ij}$

A_2. a_{11} 对角元 $|a_{11}a_{kk}-a_{1k}a_{k1}|=\sum_{j=2, j\neq k}^{n+1}|a_{11}a_{kj}-a_{1j}a_{k1}|$

$\geq |a_{11}a_{kk}|-\sum_{j=1, j\neq k}^{n+1}|a_{11}a_{kj}|=m_k|a_{11}|$.

同理 $\det(A_2)\geq m_k$

$|\det A|\geq|a_{11}||\det(A_2)|\geq|a_{11}|\prod_{k=2}^{n+1}m_k\geq\prod_{k=1}^{n+1}m_k$. #

7. 不妨设 $b=0$. $x_{n+1} = (D-L)^{-1}L^T x_n$

$(-D-L)x_{n+1} = L^T x_n$.

同乘 x_n^T x_{n+1}^T

$x_{n+1}^T A x_{n+1} - x_n^T A x_n = -(x_n - x_{n+1})^T D (x_n - x_{n+1}) \le 0$

A不正定 $\exists x \ne 0$. $x^T A x \le 0$.

① 若 $x_n \ne x_{n+1}$ ($\exists n$). D正定. $x_{n+1}^T A x_{n+1} - x_n^T A x_n < 0$

$\Downarrow$

$x_{n+1}^T A x_{n+1} < 0$. 此后不增.

不可能收敛到解.

② 否则 $x_n = x_{n+1}$. 由非零 不是解. 矛盾.

8. $\rho(H) \ge 1$. $\exists \lambda$. $\lambda H = \alpha\lambda$. $|\alpha| \ge 1$.

$\lambda^H B \lambda = (1-|\alpha|^2)\lambda^H P \lambda$. $\lambda = a+bi$.

正定矩阵对于任何复向量λ 都有 $\lambda^H P\lambda > 0$.

$\lambda^H B\lambda > 0$.

矛盾.

9. $\omega=1$时显. $\rho(I-C)<1 \Rightarrow \rho(I-\omega C)<1$ $\omega\in(0,1)$

反证. 设 $(I-\omega C)\lambda = \alpha\lambda$. $|\alpha|\ge 1$

$(I-C)\lambda = (\frac{\alpha-1}{\omega}+1)\lambda$.

$c \triangleq \frac{1}{\omega}$.

$c^2|\alpha|^2 + (c-1)^2 - 2(c-1)c\,\mathrm{Re}(\alpha) \ge c^2|\alpha|^2+(c-1)^2-2(c-1)c|\alpha|$

$= [c(|\alpha|-1)+1]^2 \ge |\alpha|^2 \ge 1$.

矛盾

10. 由于 $I-B = D^{-\frac{1}{2}}(\omega^{-1}D)^{-\frac{1}{2}}A(\omega^{-1}D)^{-\frac{1}{2}}D^{\frac{1}{2}}$

$I+B = D^{-\frac{1}{2}}(2I-(\omega^{-1}D)^{-\frac{1}{2}}A(\omega^{-1}D)^{-\frac{1}{2}})D^{\frac{1}{2}}$

特征值均为正数.

$(\omega^{-1}D)^{-\frac{1}{2}}A(\omega^{-1}D)^{-\frac{1}{2}}$, $2I-(\omega^{-1}D)^{-\frac{1}{2}}A(\omega^{-1}D)^{-\frac{1}{2}}$

正定对称 $\Rightarrow$ 相合. #

11. $\lambda I - J_\omega = (D-\omega L)^{-1}((\lambda+\omega-1)D - \lambda\omega L - \omega U)$

只需证明 $|\lambda|\geq 1$ $(\lambda+\omega-1)D - \lambda\omega L - \omega U$

严格对角占优，1不可约对角占优

同除 ω，$\left(\frac{\lambda-1}{\omega}+1\right)D - \lambda L - U$.

由(题中) $\frac{\lambda-1}{\omega}+1$ 模大于等于 λ.

$\Rightarrow$ 严格对角占优，1不可约对角占优

13. (a) $a_{11}=\sqrt{2}$ $a_{i+1,i} = -\frac{1}{a_{ii}}$.

除了 a_{ii} $a_{i+1,i}$ 外的元素均为0.

只考虑 a_{ii} 的递推，T 的对角元为2

$a_{ii}^2 + a_{i+1,i}^2 = 2$.

则 $a_{ii} = \sqrt{\frac{i+1}{i}}$. $a_{i+1,i} = -\frac{\sqrt{i}}{\sqrt{i+1}}$.

(b) 类似(a) $\begin{cases} L: l_{ii}=1 & l_{i+1,i} = -\frac{i}{i+1} \\ U: u_{ii} = \frac{i+1}{i} & u_{i,i+1} = -1 \end{cases}$

(c) T 的特征值互不相同. $\underbrace{\{特征向量\}}_{P\text{ 可逆}}$ = 全空间

$TP = PD$. $T = PDP^{-1}$. D, P 均已知.

$\Rightarrow PDP^{-1}U + UPDP^{-1} = h^2F$.

$U_0 = P^{-1}UP$. $DU_0 + U_0D = h^2P^{-1}FP$

① 计算 P^{-1}. 复杂度 n^3.

② $P^{-1}FP$ $\leq n^3$.

③ D 是对角阵. DU_0+U_0D、可逐元素求解

$DU_0+U_0D = h^2P^{-1}FP$ 复杂度 n^2.

④ $U = PU_0P^{-1}$ n^3

$\Rightarrow O(n^3)$.

4: $S=2$时

$$\begin{pmatrix} D_1 & C_2 \\ B_2 & D_2 \end{pmatrix} = \begin{pmatrix} I & \\ B_2 D_1^{-1} & I \end{pmatrix} \cdot \begin{pmatrix} D_1 & \\ & D_2 - B_2 D_1^{-1} C_2 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} I & D_1^{-1} C_2 \\ & I \end{pmatrix}}$$

$$\det \begin{pmatrix} D_1 & C_2 \\ B_2 & D_2 \end{pmatrix} = \det \begin{pmatrix} D_1 & \\ & D_2 - B_2 D_1^{-1} C_2 \end{pmatrix}$$

$s=k$时成立. $S=k+1$时

左乘 $\begin{pmatrix} I & & \\ & I & \\ & -\mu B_s D_{s-1}^{-1} & I \end{pmatrix}$. 右乘 $\begin{pmatrix} I & & \\ & I & -\mu D_{s-1}^{-1} C_s \\ & & I \end{pmatrix}$

可使右下角元素变为 $D_s - B_s D_{s-1}^{-1} C_s$.

C_s. D_s 部分变为0,

左上部分不变.

$$\det(A) = \det(A_{s-1}) \det(D_s - B_s D_{s-1}^{-1} C_s).$$

(与μ无关).

15. $\det(\lambda I - I\omega) = 0$.

$\Leftrightarrow \det(D - \omega C_L)^{-1} \det((\lambda + \omega - 1) D - \lambda \omega C_L - \omega C_U) = 0$

$\Leftrightarrow \det((\lambda + \omega - 1) D - \lambda \omega C_L - \omega C_U) = 0$

$\Leftrightarrow \det((\lambda + \omega - 1) D \cdot - \lambda^{\frac{1}{2}} \omega C_L - \lambda^{\frac{1}{2}} \omega C_U) = 0$.

$\Leftrightarrow \det\left(\frac{\lambda + \omega - 1}{\lambda^{\frac{1}{2}} \omega} - C_L - C_U\right) = 0$

$\Leftrightarrow \det\left(D^{-1}\left(\frac{\lambda + \omega - 1}{\lambda^{\frac{1}{2}} \omega} \cdot D - C_L - C_U\right)\right) = 0$

$\Leftrightarrow \det\left(\frac{\lambda + \omega - 1}{\lambda^{\frac{1}{2}} \omega} I - B\right) = 0$.

$\mu = \frac{\lambda + \omega - 1}{\lambda^{\frac{1}{2}} \omega}$. . 即为所求.