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Formulation and numerical treatment of boundary integral equations with hypersingular kernels

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Abstract

Boundary integral equations with strongly singular and hypersingular kernels can be very useful in many fields of applied mechanics. In this paper two basic aspects are addressed. First, a method for the formulation of boundary integral equations with kernels of any order of singularity is outlined. The method never employs arbitrary interpretations in the Cauchy principal value or finite part sense. Essentially it amounts to show that no unbounded terms ultimately arise if the limiting process is properly performed. Second, a semianalytical technique of general applicability for the direct evaluation of singular integrals, as required in any boundary element implementation, is described in detail. The theory is supported by some numerical tests. Further developments and applications are also mentioned.

The method is semi-analytical in the sense that all singular integrations are performed analytically and the limiting process is performed exactly.

1 Introduction

The treatment of singular integrals has always been a chief subject of investigation for the development of the boundary element method (BEM), and many techniques have been proposed so far. We will make no attempt to review these many contributions (see, e.g., Tanaka *et al.* [1] and the references therein), as this is not what this paper is intended for. Instead, the direct approach developed by the present author and some of his students and colleagues will be described in detail. The aim is to give a fairly comprehensive account of a method developed over several years.

The method originated in 1987, in a paper by Guiggiani & Casalini [2], where a complete direct treatment of strongly singular integrals in two-dimensional elasticity was presented. These ideas were soon applied to the Galerkin BEM as well by Parreira & Guiggiani [3]. Additional information can be found in the review paper [4] and in [5, 6]. The non-trivial extension to three-dimensional problems was published in 1990 in a paper by Guiggiani & Gigante [7]. The limiting process was mapped from the surface of the body onto the parameter space, taking into account all the relevant effects.

The evaluation of hypersingular integrals by the direct approach was first presented in 1990 at the IABEM Conference in Rome by Guiggiani *et al.* [8]. Quite surprisingly, the method was first developed for 3D applications. The two-dimensional version was presented by Guiggiani [9] at the GAMM Seminar held in Kiel in 1991. The technique appeared in a journal paper by Guiggiani *et al.* [10] in 1992. Besides providing a direct means for the evaluation of hypersingular integrals, these papers gave some contributions on the theoretical nature of integral equations with apparently non-integrable kernels. In fact, it was shown that there were no unbounded terms around, and that arbitrary interpretations in the finite part sense were not necessary and potentially dangerous (see, e.g. [11]).

【8】Hypersingular boundary integral equations: A new approach to their numerical treatment

Meanwhile, the direct approach for the evaluation of strongly singular integrals was also applied to time-harmonic elastodynamics in Guiggiani [12].

In a paper published in 1994 [13], Guiggiani applied the direct method to the evaluation of the stress tensor at points on the boundary. Quite interestingly, the stress components computed directly by means of hypersingular equations, show much better accuracy than the corresponding components evaluated by the traditional technique that employs displacement derivatives.

少了一个 jump term

In 1995, it was shown by Guiggiani [14] and, independently, by Mantič & París [15], that in hypersingular boundary integral equations (HBIE) a class of free terms had been erroneously omitted. The same conclusion was reached by Young in [16].

[14]Guiggiani, Hypersingular boundary integral equations have an additional free term.
[15] Manti, Existence and evaluation of the two free terms in the hypersingular boundary integral equation of potential theory.
[16] Young, A single-domain boundary element method for 3D elastostatic crack analysis using continuous elements.

可以用作断裂处生成的依据

Quite recently, Guiggiani [17] has found a close link between the residual of HBIE's at boundary points and the sensitivity of approximate BEM solutions with respect to the position of collocation points. This sensitivities seem to provide reliable error indicators as shown in Guiggiani & Lombardi [6].

Finally, Bonnet & Guiggiani [18] have shown that hypersingular boundary integral equations are, indeed, the derivatives of standard BIE's with respect to the position of collocation points. Moreover, some new insights on the continuity requirements of the density function at the collocation points are presented (see also [19]).

2 Some classical theorems

Much of the present analysis is based on Gauss' theorem

$$\int_\Omega \frac{\partial u}{\partial x_i}\,\mathrm{d}\Omega = \int_\Gamma u\, n_i\,\mathrm{d}\Gamma, \tag{1}$$

where Ω is a domain bounded by a Kellogg's regular surface Γ with unit outward normal $\mathbf{n}(\mathbf{x})$.

Remarkable applications of such theorem are the well known Green's second identity for scalar problems

$$\int_\Omega (u\,\nabla^2 w - w\,\nabla^2 u)\,\mathrm{d}\Omega = \int_\Gamma \left(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\right)\,\mathrm{d}\Gamma, \tag{2}$$

and Betti's theorem for vector problems. The two generic functions u and w in eqn (2) belong to $C^2(\Omega)\cap C^1(\Gamma)$.

If the two functions u and w are also *harmonic* in Ω, that is $\nabla^2 u = \nabla^2 w = 0$, the integrand on the l.h.s. in (2) vanishes. As a consequence, the integral on Γ becomes equal to zero

$$\int_\Gamma \left(u\,\frac{\partial w}{\partial n} - w\,\frac{\partial u}{\partial n}\right)\,\mathrm{d}\Gamma = 0. \tag{3}$$

3 General form of boundary integral identities

In this section the limiting process for boundary integral equations with either strongly singular or hypersingular kernels will be described. It will be shown that no unbounded terms ultimately arise even when the singular point $\mathbf{y}$ is on the boundary Γ. Therefore, *a priori* interpretations, either in the Cauchy principal value or in the finite part sense, to make bounded otherwise unbounded integrals, are never necessary. This is a very important, often overlooked, aspect. As a matter of fact, any recourse to such concepts would make the whole procedure totally arbitrary. Without loss of generality, the analysis will be presented for potential and elastic problems since they are a paradigma for any elliptic problem.

To obtain a boundary integral equation we just have to take the following three steps from eqn (3):

$$U_{ij}(\mathbf{s},\mathbf{x}) = \frac{1}{16\pi\mu(1-\nu)r}\left[(3-4\nu)\delta_{ij} + r_{,i}r_{,j}\right],$$

$$\int_\Gamma \left(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\right)\,\mathrm{d}\Gamma = 0. \quad (3)$$

1. replace, e.g. $w(\mathbf{x})$ by the fundamental solution $U(\mathbf{y},\mathbf{x})$, that is by a two-point singular function that satisfies the governing differential equation in any point $\mathbf{x}$, provided $\mathbf{x}\neq\mathbf{y}$, where $\mathbf{y}$ is the singular point;

   奇异点是固定点

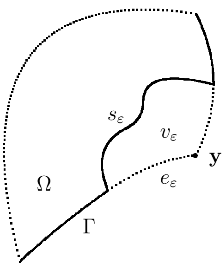
2. owing to the singular behaviour of U as $\mathbf{x}\to\mathbf{y}$, exclude the singular point $\mathbf{y}$ by a (vanishing) neighbourhood v_ε, thus getting the subdomain $\Omega_\varepsilon = \Omega - v_\varepsilon$ on which we can safely write the Green's second identity. Notice that v_ε can have *any shape*. If, as in Figure 1, the pole $\mathbf{y}$ belongs to the boundary Γ, the subdomain Ω_ε has boundary $\partial\Omega_\varepsilon = (\Gamma - e_\varepsilon) + s_\varepsilon = \Gamma_\varepsilon$;

3. take the limit for $\varepsilon\to 0$.

The overall result is the following identity [10, 13]

$$\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon}[T(\mathbf{y},\mathbf{x})u(\mathbf{x}) - U(\mathbf{y},\mathbf{x})q(\mathbf{x})]\mathrm{d}\Gamma\right\} = 0, \tag{4}$$

$$T(\mathbf{y},\mathbf{x}) = \partial U/\partial n(\mathbf{x}). \qquad q = \partial u/\partial n$$



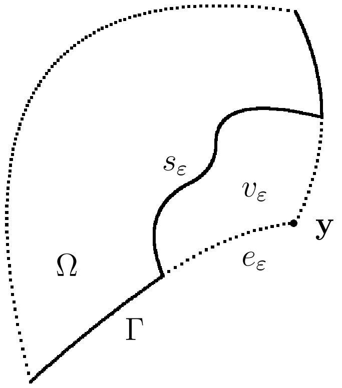


Figure 1. Exclusion of the singular point $\mathbf{y}$ by a vanishing neighbourhood v_ε.

$$\int_\Omega (u\,\nabla^2 w - w\,\nabla^2 u)\,\mathrm{d}\Omega = \int_\Gamma \left(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n} \right) \mathrm{d}\Gamma, \tag{2}$$

where the normal derivatives have been indicated by $q = \partial u/\partial n$ and $T(\mathbf{y}, \mathbf{x}) = \partial U/\partial n(\mathbf{x})$.

$$\lim_{\varepsilon \to 0} \left\{ \int_{\Gamma_\varepsilon} [T(\mathbf{y}, \mathbf{x}) u(\mathbf{x}) - U(\mathbf{y}, \mathbf{x}) q(\mathbf{x})] \mathrm{d}\Gamma \right\} = 0, \tag{4}$$

It is important to realize that Ω_ε is just a subdomain of the original domain Ω. In fact, the Green's second identity (2) holds true in any sufficiently regular subset of Ω. The function u in eqn (4) is still the solution of the problem governed by the Laplace equation and formulated on Ω. This is to say that we are not considering any "deformed" domain, as sometimes erroneously written.

If $r = |\mathbf{x} - \mathbf{y}|$ denotes the distance between the source point $\mathbf{y}$ and integration point $\mathbf{x}$, the fundamental solution U has a weak (integrable) singularity when $r \to 0$. This means that $U = O(\ln r)$ in two-dimensional problems, and $U = O(r^{-1})$ in three-dimensional problems. The other kernel function T has a strong singularity, that is $T = O(r^{-1})$ in 2D and $T = O(r^{-2})$ in 3D. This kind of singularity makes T non-integrable and has created a lot of problems for a speedy development of the boundary element method (BEM).

Despite these singularities, the value of the integral of the l.h.s. in eqn (4) is bounded. In fact, it is identically *zero* for any ε, and so it is zero in the limit. To prove that the limit in eqn (4) is equal to zero notwithstanding the non-integrable singularities, we have just to observe that, by hypothesis, both u and U are harmonic in Ω_ε and, therefore, *must* satisfy Green's second identity (3) for any $\varepsilon > 0$.

The kernel function $U(\mathbf{y}, \mathbf{x})$ can be differentiated with respect to any co-ordinate y_k of the source point $\mathbf{y}$ thus getting the following new set of kernel functions 源点，奇异点，是固定点

$$W_k(\mathbf{y}, \mathbf{x}) = \frac{\partial U}{\partial y_k}.$$

Since, in general, the derivative of an harmonic function is harmonic as well, the functions W_k still satisfy the Laplace equation at any point $\mathbf{x} \neq \mathbf{y}$. Of course,

1. replace, e.g. $w(\mathbf{x})$ by the fundamental solution $U(\mathbf{y},\mathbf{x})$, that is by a two-point singular function that satisfies the governing differential equation in any point $\mathbf{x}$, provided $\mathbf{x} \neq \mathbf{y}$, where $\mathbf{y}$ is the singular point;
奇异点是固定点
2. owing to the singular behaviour of U as $\mathbf{x} \to \mathbf{y}$, exclude the singular point $\mathbf{y}$ by a (vanishing) neighbourhood v_ε, thus getting the subdomain $\Omega_\varepsilon = \Omega - v_\varepsilon$ on which we can safely write the Green's second identity. Notice that v_ε can have *any shape*. If, as in Figure 1, the pole $\mathbf{y}$ belongs to the boundary Γ, the subdomain Ω_ε has boundary $\partial\Omega_\varepsilon = (\Gamma - e_\varepsilon) + s_\varepsilon = \Gamma_\varepsilon$;
3. take the limit for $\varepsilon \to 0$.

differentiation raises the order of singularity and W_k have a strong singularity of $O(r^{-1})$ in 2D and of $O(r^{-2})$ in 3D. 作用在eq:3 $\int_\Gamma \left(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\right) \mathrm{d}\Gamma = 0.$ (3)

Taking the same three steps as above, but with w replaced by W_k , we obtain the following boundary integral equation

$$\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} [V_k(\mathbf{y},\mathbf{x})u(\mathbf{x}) - W_k(\mathbf{y},\mathbf{x})q(\mathbf{x})]\,\mathrm{d}\Gamma\right\} = 0, \tag{5}$$

where the new kernels

$$V_k(\mathbf{y},\mathbf{x}) = \frac{\partial W_k}{\partial n(\mathbf{x})} = \frac{\partial T}{\partial y_k} \qquad W_k(\mathbf{y},\mathbf{x}) = \frac{\partial U}{\partial y_k}. \qquad \partial\Omega_\varepsilon = (\Gamma - e_\varepsilon) + s_\varepsilon = \Gamma_\varepsilon;$$

are *hypersingular* of $O(r^{-3})$ in 3D and of $O(r^{-2})$ in 2D, for $r \to 0$.

It is also worth noting that in eqn (5) the integral on Γ_ε is identically zero for any $\varepsilon > 0$. Therefore all singular terms have to cancel out (and without the recourse to any *a-priori*, arbitrary interpretation in the finite-part sense).

Also note that the validity of identities (4) and (5) is not restricted to smooth boundary points (Figure 1). The singular point $\mathbf{y}$ can be located at corners, vertices, etc.

The explicit expressions of all kernels involved are given herein for two-dimensional

$$U = -\frac{1}{2\pi}\ln r; \quad T = -\frac{1}{2\pi r}\frac{\partial r}{\partial n}; \quad W_k = \frac{1}{2\pi r}r_{,k}\,; \quad V_k = -\frac{1}{2\pi r^2}\left[2r_{,k}\frac{\partial r}{\partial n} - n_k\right], \tag{6}$$

and three-dimensional potential problems

$$U = \frac{1}{4\pi r}; \qquad T = -\frac{1}{4\pi r^2}\frac{\partial r}{\partial n}; \quad W_k = \frac{1}{4\pi r^2}r_{,k}\,; \quad V_k = -\frac{1}{4\pi r^3}\left[3r_{,k}\frac{\partial r}{\partial n} - n_k\right], \tag{7}$$

x是field point 强奇异 超奇异

where $r_{,k} = \partial r/\partial x_k$ and $\partial r/\partial n = r_{,i}\,n_i(\mathbf{x})$.

Interestingly enough, and quite contrary to common belief, for the limit in eqns (4) and (5) to exist, we do not have to pose special requirements on the behaviour (continuity) of the density function u at $\mathbf{y}$. Since the continuity features of u at the point $\mathbf{y}$ never come into play in the limiting process in eqn (4) or in eqn (5), it follows that the density function u does not have to satisfy any continuity requirement at $\mathbf{y}$ for the limits to exist. In other words, the integrals on Γ_ε in eqns (4) and (5) are identically zero provided the functions $u(\mathbf{x})$, $U(\mathbf{y},\mathbf{x})$ and $W_k(\mathbf{y},\mathbf{x})$ are harmonic in the punctured domain $\Omega - v_\varepsilon$, which *never* contains the point $\mathbf{y}$ for any $\varepsilon > 0$. Again, the behaviour of $u(\mathbf{x})$ is relevant in any neighbourhood of $\mathbf{y}$, but not at $\mathbf{y}$.

This aspect was discussed in more detail in Guiggiani [19], where it was explicitly shown that the following, fairly unorthodox boundary integral equation does have an identically zero integral on Γ_ε

$$\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon} [V_k(\mathbf{y},\mathbf{x})U(\mathbf{y},\mathbf{x}) - W_k(\mathbf{y},\mathbf{x})T(\mathbf{y},\mathbf{x})]\,\mathrm{d}\Gamma\right\} = 0. \qquad \partial\Omega_\varepsilon = (\Gamma - e_\varepsilon) + s_\varepsilon = \Gamma_\varepsilon;$$

$$\lim_{\varepsilon \to 0} \left\{ \int_{\Gamma_\varepsilon} [T(\mathbf{y},\mathbf{x})u(\mathbf{x}) - U(\mathbf{y},\mathbf{x})q(\mathbf{x})] \mathrm{d}\Gamma \right\} = 0, \tag{4}$$

$$\lim_{\varepsilon \to 0} \left\{ \int_{\Gamma_\varepsilon} [V_k(\mathbf{y},\mathbf{x})u(\mathbf{x}) - W_k(\mathbf{y},\mathbf{x})q(\mathbf{x})] \, \mathrm{d}\Gamma \right\} = 0, \tag{5}$$

However, this statement is mostly of theoretical value. As will be shown later, to obtain a form of the boundary integral equations useful in actual computations, it seems necessary to assume that at the singular point $u \in C^{0,\alpha}$ in eqn (4), and $u \in C^{1,\alpha}$ in eqn (5).

H¨ older continuous

假设在奇异点或者源点，是holder连续 $u \in C^{0,\alpha}(\mathbf{y})$.

If u_j and t_j denote the displacement and traction components, respectively, the counterpart of eqns (4) and (5) for elastic (i.e. vector) problems are given by

公式8,9和公式4,5的区别在于T U V W的下标

eq：8，9是向量形式

$$\lim_{\varepsilon \to 0} \left\{ \int_{\Gamma_\varepsilon} [T_{ij}(\mathbf{y},\mathbf{x})u_j(\mathbf{x}) - U_{ij}(\mathbf{y},\mathbf{x})t_j(\mathbf{x})] \, \mathrm{d}\Gamma \right\} = 0, \tag{8}$$

$$\partial\Omega_\varepsilon = (\Gamma - e_\varepsilon) + s_\varepsilon = \Gamma_\varepsilon;$$

and

$$\lim_{\varepsilon \to 0} \left\{ \int_{\Gamma_\varepsilon} [V_{ikj}(\mathbf{y},\mathbf{x})u_j(\mathbf{x}) - W_{ikj}(\mathbf{y},\mathbf{x})t_j(\mathbf{x})] \, \mathrm{d}\Gamma \right\} = 0, \tag{9}$$

$$U_{ij}(\mathbf{s},\mathbf{x}) = \frac{1}{16\pi\mu(1-\nu)r}\left[(3-4\nu)\delta_{ij} + r_{,i}r_{,j}\right], \tag{2}$$

$$T_{ij}(\mathbf{s},\mathbf{x}) = -\frac{1}{8\pi(1-\nu)r^2}\left\{\frac{\partial r}{\partial n}[(1-2\nu)\delta_{ij} + 3r_{,i}r_{,j}] - (1-2\nu)(r_{,i}n_j - r_{,j}n_i)\right\}, \tag{3}$$

where $W_{ikj} = \partial U_{ij}/\partial y_k$ and $V_{ikj} = \partial T_{ij}/\partial y_k$.

The expressions of U_{ij} and T_{ij} can be found in any BEM book (e.g. Bonnet [20, page 64–65]). The other kernels have the following expressions

书找不到

$$V_{ikj}(\mathbf{y},\mathbf{x}) = -\frac{A}{r^\beta}\Big\{(1-2\nu)[\beta r_{,k}(n_i r_{,j} - n_j r_{,i}) - n_i\delta_{jk} + n_j\delta_{ik} - n_k\delta_{ij}]$$
$$- \beta n_k r_{,i} r_{,j} + \beta[(1-2\nu)\delta_{ij}r_{,k} + (\alpha+3)r_{,i}r_{,j}r_{,k} - \delta_{ik}r_{,j} - \delta_{jk}r_{,i}]\,\frac{\partial r}{\partial n}\Big\}, \tag{10}$$

and

$$W_{ikj}(\mathbf{y},\mathbf{x}) = \frac{A}{2G\,r^\alpha}[\beta r_{,i} r_{,j} r_{,k} + (3-4\nu)\delta_{ij}r_{,k} - \delta_{ik}r_{,j} - \delta_{jk}r_{,i}], \tag{11}$$

2D 3D 2D 3D

where $\alpha = 1, 2$ and $\beta = 2, 3$ in 2D and 3D problems, respectively. The constant A is equal to $1/[4\,\alpha\,\pi\,(1-\nu)]$. Expressions (10) and (11) have the same singular behaviour, for $r \to 0$, of the corresponding kernels for time-harmonic elastodynamics, as shown, e.g. in Bonnet [20, page 173].

4 Boundary integral equations

前面说明积分方程式有界的，不是发散的

The main point in the former section was to show that ultimately we have to deal only with bounded quantities. Nevertheless, some additional steps need to be taken to transform those integral identities into boundary integral equations (BIE), that is into a form suitable for the subsequent solution by the boundary element method (BEM). Basically, we have to carry out the limiting process for $\varepsilon \to 0$, since we want to deal with Γ, not with Γ_ε. $\partial\Omega_\varepsilon = (\Gamma - e_\varepsilon) + s_\varepsilon = \Gamma_\varepsilon;$

The treatment is essentially the same for standard and hypersingular boundary integral equations (SBIE and HBIE). However, they will be addressed separately for better clarity.

4.1 Standard boundary integral equations (SBIE)

To proceed further we have to make assumptions on the behaviour of u at the singular point $\mathbf{y}$. In particular, we assume u to be Hölder continuous at $\mathbf{y}$, that is to satisfy the following condition when $\mathbf{x} \to \mathbf{y}$

$$|u(\mathbf{x}) - u(\mathbf{y})| \leq k|\mathbf{x} - \mathbf{y}|^{\alpha}, \quad \text{for some constants} \quad k, \alpha > 0. \tag{12}$$

H¨ older continuous

This condition is normally written $u \in C^{0,\alpha}(\mathbf{y})$. Essentially, Hölder continuity means that u is continuous and that $|u(\mathbf{x}) - u(\mathbf{y})|$ goes to zero as some positive power α of r (typically, $\alpha = 1$). In terms of Taylor expansion Holder continuity是泰勒展开的前提

eq：12左侧展开

$$u(\mathbf{x}) = u(\mathbf{y}) + O(r^{\alpha}). \tag{13}$$

By adding and subtracting $u(\mathbf{y}) \int_{s_\varepsilon} T \mathrm{d}\Gamma$ on the l.h.s. of eqn (4) we obtain

$$\partial\Omega_\varepsilon = (\Gamma - e_\varepsilon) + s_\varepsilon = \Gamma_\varepsilon;$$

$$\lim_{\varepsilon \to 0} \left\{ \int_{\Gamma_\varepsilon} [T(\mathbf{y},\mathbf{x}) u(\mathbf{x}) - U(\mathbf{y},\mathbf{x}) q(\mathbf{x})] \mathrm{d}\Gamma \right\} = 0, \tag{4}$$

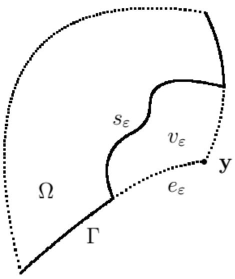
$$\lim_{\varepsilon \to 0} \left\{ \int_{(\Gamma - e_\varepsilon)} [T(\mathbf{y},\mathbf{x})\, u(\mathbf{x}) - U(\mathbf{y},\mathbf{x})\, q(\mathbf{x})] \mathrm{d}\Gamma \right.$$

注意下标

$$\left. + \int_{s_\varepsilon} \Big[T(\mathbf{y},\mathbf{x})[u(\mathbf{x}) - u(\mathbf{y})] - U\, q(\mathbf{x}) \Big] \mathrm{d}\Gamma + u(\mathbf{y}) \int_{s_\varepsilon} T(\mathbf{y},\mathbf{x})\, \mathrm{d}\Gamma \right\} = 0. \tag{14}$$

eq：15表示这一部分为零 这一部分是FREE TERM

At this stage, since the value of the limit taken as a whole is independent on the shape of v_ε (Figure 1), we may select the *most convenient* shape for the vanishing neighbourhood v_ε, which is a ball of radius ε centred at $\mathbf{y}$. It is important to realize that a symmetric neighbourhood is just a matter of convenience, and that in no way it affects the final global result. 球可以左右极限



The selected shape of v_ε also enforces the shape of s_ε and e_ε (Figures 1 and 2). The surface s_ε becomes part of a circle in 2D and part of a sphere in 3D. The excluded surface e_ε from now on becomes a *symmetric* neighbourhood of $\mathbf{y}$. In the numerical treatment we will pay a lot of attention for not destroying such symmetry.

The simple shape of s_ε allows for all integrals on it to be evaluated *analytically*. Because of expansion (13), and since (Figure 2) $\mathrm{d}\Gamma = \varepsilon \mathrm{d}\varphi$ in 2D and $\mathrm{d}\Gamma = \varepsilon^2 \mathrm{d}\omega$ in 3D (where ω is the solid angle at $\mathbf{y}$), we have that 左下角 L=α（弧度）x r(半径) （弧度制） 说的都是在Se上的 s_ε

这是一个专业术语，立体角

$$\int_{s_\varepsilon} \Big[T(\mathbf{y},\mathbf{x})[u(\mathbf{x}) - u(\mathbf{y})] - U(\mathbf{y},\mathbf{x})\, q(\mathbf{x}) \Big] \mathrm{d}\Gamma = O(\varepsilon), \tag{15}$$

so that in the limit it goes to zero.

The second integral on s_ε gives rise to the free-term coefficient

eq:14的左侧最后一个

$$u(\mathbf{y}) \int_{s_\varepsilon} T(\mathbf{y},\mathbf{x})\, \mathrm{d}\Gamma \quad \int_{s_\varepsilon} T(\mathbf{y},\mathbf{x})\, \mathrm{d}\Gamma = c(\mathbf{y}) + O(\varepsilon). \tag{16}$$

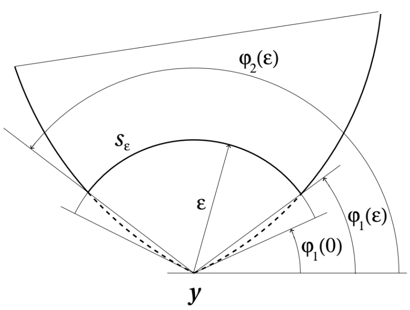


Figure 2. Local geometry at a non-smooth boundary point.

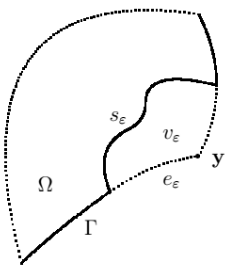


Figure 1. Exclusion of the singular point $\mathbf{y}$ by a vanishing neighbourhood v_ε.

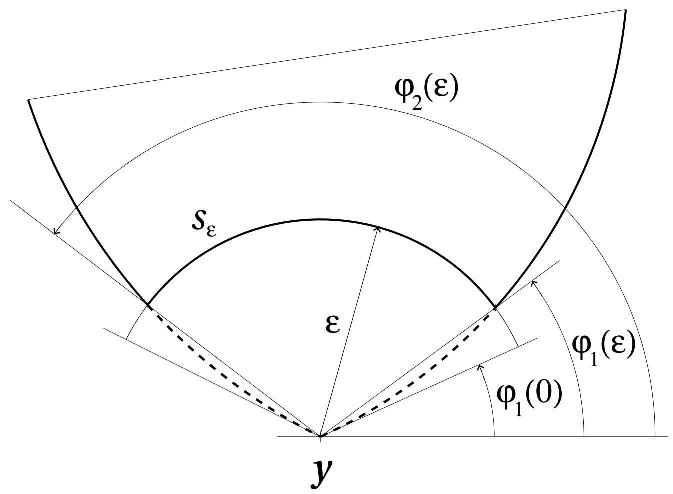
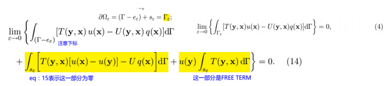


Figure 2. Local geometry at a non-smooth boundary point.

The free-term coefficient only depends upon the local geometry at $\mathbf{y}$. As well known, at smooth boundary points it is $c(\mathbf{y}) = 1/2$.

According to this analysis, the standard boundary integral equation for scalar problems should be written in the following way, with the limiting process still indicated explicitly



$$c(\mathbf{y})\, u(\mathbf{y}) + \lim_{\varepsilon \to 0} \left\{ \int_{(\Gamma - e_\varepsilon)} [T(\mathbf{y},\mathbf{x})\, u(\mathbf{x}) - U(\mathbf{y},\mathbf{x})\, q(\mathbf{x})] \mathrm{d}\Gamma \right\} = 0, \qquad (17)$$

下标

where, consistently with the already computed free term, the neighbourhood e_ε has a *symmetric* shape in the limit.

In case of vector problems, the same steps lead to an equation in the following form

$$c_{ij}(\mathbf{y})\, u_j(\mathbf{y}) + \lim_{\varepsilon \to 0} \left\{ \int_{(\Gamma - e_\varepsilon)} [T_{ij}(\mathbf{y},\mathbf{x})\, u_j(\mathbf{x}) - U_{ij}(\mathbf{y},\mathbf{x})\, t_j(\mathbf{x})] \mathrm{d}\Gamma \right\} = 0, \qquad (18)$$

对角阵，主对角线上为0.5

where $c_{ij} = 0.5\,\delta_{ij}$ at points where the boundary is smooth.

It is common practice to introduce the notation $⨍$ of Cauchy principal value (CPV) integrals in the above equation. This is possible because of the symmetric shape of e_ε, as $\varepsilon \to 0$

对于圆形的必要性

表示柯西主值 CPV

$$c_{ij}(\mathbf{y})\, u_j(\mathbf{y}) + ⨍_{\Gamma} [T_{ij}(\mathbf{y},\mathbf{x})\, u_j(\mathbf{x}) - U_{ij}(\mathbf{y},\mathbf{x})\, t_j(\mathbf{x})] \mathrm{d}\Gamma = 0. \qquad (19)$$

eq：19

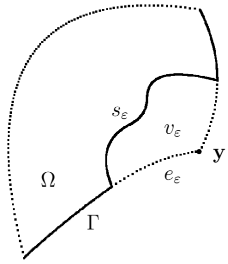
However, this is simply a shorthand to indicate a limiting process with a *symmetric* vanishing neighbourhood. It is misleading to state that CPV "interpretation" is necessary to make the integral meaningful. Actually, it is only because we selected a ball-shaped neighbourhood v_ε that we ended up with a CPV. Should v_ε have a different shape, then the limit of the integral over $\Gamma - e_\varepsilon$ would not be a CPV anymore. Needless to say, the final overall result would have been exactly the same.

In other words, any *a-priori* interpretation would be totally arbitrary, and, in fact, we do not need any *a-priori* CPV definition. It is only at the very end that we may realize that the limiting process in eqn (18) does coincide with the CPV definition (provided a very special shape of v_ε was taken) and use the CPV symbol as a convenient shorthand.

4.2 Hypersingular boundary integral equations (HBIE)

To proceed further with HBIE's we assume that the potential u is differentiable at $\mathbf{y}$, with its derivatives satisfying a Hölder condition 为了得到eq20 (12), that is $u \in C^{1,\alpha}$ at $\mathbf{y}$. As already mentioned, these (quite restrictive) continuity requirements are not strictly necessary for the existence of the limit in eqn (5) and eqn (9). However, they allow for the "extraction" of the free terms which seems a necessary step for the subsequent solution by the boundary element method.

The fulfilment of Hölder conditions makes it possible to expand u and q around $\mathbf{y}$



$|u(\mathbf{x}) - u(\mathbf{y})| \le k|\mathbf{x}-\mathbf{y}|^\alpha$, for some constants $k, \alpha > 0$. (12)

$u(\mathbf{x}) = u(\mathbf{y}) + O(r^\alpha)$. (13)

注意：y是原点，x是积分点或者场点

$$\begin{aligned} u(\mathbf{x}) &= u(\mathbf{y}) + u_{,h}(\mathbf{y})(x_h - y_h) + O(r^{1+\alpha}), \\ q(\mathbf{x}) &= u_{,h}(\mathbf{x})n_h(\mathbf{x}) = u_{,h}(\mathbf{y})n_h(\mathbf{x}) + O(r^\alpha), \end{aligned} \tag{20}$$

where $0 < \alpha \le 1$.

$$\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_\varepsilon}[V_k(\mathbf{y},\mathbf{x})u(\mathbf{x}) - W_k(\mathbf{y},\mathbf{x})q(\mathbf{x})]\,\mathrm{d}\Gamma\right\} = 0, \tag{5}$$

By adding and subtracting in eqn (5) the relevant terms of expansions (20), the following form is obtained (Figure 1)

$$\begin{aligned} \lim_{\varepsilon\to 0}\Bigg\{ &\int_{(\Gamma - e_\varepsilon)} [V_k(\mathbf{y},\mathbf{x})u(\mathbf{x}) - W_k(\mathbf{y},\mathbf{x})q(\mathbf{x})]\mathrm{d}\Gamma \\ &+ \int_{s_\varepsilon}\Big[V_k[u(\mathbf{x}) - u(\mathbf{y}) - u_{,h}(\mathbf{y})(x_h - y_h)] - W_k[q(\mathbf{x}) - u_{,h}(\mathbf{y})n_h(\mathbf{x})]\Big]\mathrm{d}\Gamma \\ &+ u_{,h}(\mathbf{y})\int_{s_\varepsilon}[V_k(x_h - y_h) - W_k n_h(\mathbf{x})]\mathrm{d}\Gamma \\ &+ u(\mathbf{y})\int_{s_\varepsilon} V_k\,\mathrm{d}\Gamma\ \Bigg\} = 0. \end{aligned} \tag{21}$$

第二行在极限中为0

$$\int_{s_\varepsilon}[V_k(x_h - y_h) - W_k n_h(\mathbf{x})]\mathrm{d}\Gamma = c_{kh}(\mathbf{y}) + O(\varepsilon). \tag{23}$$

The free terms are given by the limit for $\varepsilon \to 0$ of the integrals on s_ε in eqn (21).

6 Direct evaluation of singular integrals

The numerical treatment proceeds directly from equations in the form of (17) or (25), with strongly singular and even hypersingular kernels (in eqn (25)). The method is semi-analytical in the sense that all singular integrations are performed *analytically* and the limiting process is performed *exactly*. Therefore, numerical integration has only to deal with regular integrals. 只在非奇异方程中，进行数值积分。

奇异方程，通过解析直接获得。

$$
\lim_{\varepsilon\to 0}\left\{\int_{(\Gamma-e_\varepsilon)}[V_k(\mathbf{y},\mathbf{x})u(\mathbf{x})-W_k(\mathbf{y},\mathbf{x})q(\mathbf{x})]\mathrm{d}\Gamma\right.
$$

$$
+\int_{s_\varepsilon}\Big[V_k[u(\mathbf{x})-u(\mathbf{y})-u_{,h}(\mathbf{y})(x_h-y_h)]-W_k[q(\mathbf{x})-u_{,h}(\mathbf{y})n_h(\mathbf{x})]\Big]\mathrm{d}\Gamma
$$

$$
+u_{,h}(\mathbf{y})\int_{s_\varepsilon}[V_k(x_h-y_h)-W_k n_h(\mathbf{x})]\mathrm{d}\Gamma
$$

$$
\left.+u(\mathbf{y})\int_{s_\varepsilon}V_k\,\mathrm{d}\Gamma\right\}=0. \tag{21}
$$

Once again, the most convenient shape for v_ε seems to be a ball of radius ε centred at $\mathbf{y}$. As a matter of fact, this simple shape of s_ε allows the analytical evaluation of all integrals on s_ε, for $\varepsilon \to 0$.

Because of the expansions (20) it follows that

$$
\int_{s_\varepsilon}\Big[V_k[u(\mathbf{x})-u(\mathbf{y})-u_{,h}(\mathbf{y})(x_h-y_h)]-W_k[q(\mathbf{x})-u_{,h}(\mathbf{y})n_h(\mathbf{x})]\Big]\mathrm{d}\Gamma=O(\varepsilon), \tag{22}
$$

公式21的第二行

so that we only have to consider the other integrals.

The second kind of integrals on s_ε gives rise to the following free-term coefficients

$+u_{,h}(\mathbf{y})\int_{s_\varepsilon}[V_k(x_h-y_h)-W_k n_h(\mathbf{x})]\mathrm{d}\Gamma$

$$
\int_{s_\varepsilon}[V_k(x_h-y_h)-W_k n_h(\mathbf{x})]\mathrm{d}\Gamma=c_{kh}(\mathbf{y})+O(\varepsilon). \tag{23}
$$

公式21的第三行

At smooth boundary points these free-term coefficients simply become $c_{kh} = 0.5\,\delta_{kh}$.

The last integral on s_ε in eqn (21) has the following expansion

b会被抵消掉　对于直角边形状为0

$$
\int_{s_\varepsilon}V_k\,\mathrm{d}\Gamma=\frac{b_k(\mathbf{y})}{\varepsilon}+a_k(\mathbf{y})+O(\varepsilon). \tag{24}
$$

公式21的第四行

In Guiggiani [14] and Mantič & Paris [15] it was pointed out that the free-term coefficients a_k had been erroneously omitted in former papers (e.g. [8, 9, 10, 13]). They depend on the *curvature* of the boudary at $\mathbf{y}$. Details on their evaluation are provided in the next section.

According to this analysis, hypersingular boundary integral equations for scalar problems can be written in the following form

直角情况为0

$$
a_k(\mathbf{y})u(\mathbf{y})+c_{kh}(\mathbf{y})u_{,h}(\mathbf{y})
$$

$$
+\lim_{\varepsilon\to 0}\left\{\int_{(\Gamma-e_\varepsilon)}[V_k(\mathbf{y},\mathbf{x})u(\mathbf{x})-W_k(\mathbf{y},\mathbf{x})q(\mathbf{x})]\,\mathrm{d}\Gamma+\frac{b_k(\mathbf{y})}{\varepsilon}u(\mathbf{y})\right\}=0. \tag{25}
$$

Note that, at non-smooth boundary points between curved elements it is not possible to obtain free-terms only involving potential derivatives.

Similarly, HBIE's for vector problems (e.g. elasticity) can be written as

$$
a_{ikj}(\mathbf{y})u_j(\mathbf{y})+c_{ikjh}(\mathbf{y})u_{j,h}(\mathbf{y}) \tag{26}
$$

$$
+\lim_{\varepsilon\to 0}\left\{\int_{(\Gamma-e_\varepsilon)}[V_{ikj}(\mathbf{y},\mathbf{x})u_j(\mathbf{x})-W_{ikj}(\mathbf{y},\mathbf{x})q_j(\mathbf{x})]\,\mathrm{d}\Gamma+\frac{b_{ikj}(\mathbf{y})}{\varepsilon}u_j(\mathbf{y})\right\}=0.
$$

It is worth noting that the limiting process involves an unbounded term plus an unbounded integral on $\Gamma - e_\varepsilon$. Of course, they cancel each other, and this cancellation is made explicit in eqn (25) and eqn (26). Once again, no *a-priori* interpretation is necessary to discard unbounded quantities. Simply, there are no

将eq26与eq27比较，得知b这一项被抵消了

直角情况为0

$$\boxed{a_k(\mathbf{y})u(\mathbf{y})} + c_{kh}(\mathbf{y})\boxed{u_{,h}}(\mathbf{y})$$
$$+ \lim_{\varepsilon\to 0}\left\{\int_{(\Gamma - e_\varepsilon)}[V_k(\mathbf{y},\mathbf{x})u(\mathbf{x}) - W_k(\mathbf{y},\mathbf{x})q(\mathbf{x})]\,\mathrm{d}\Gamma + \boxed{\frac{b_k(\mathbf{y})}{\varepsilon}}u(\mathbf{y})\right\} = 0. \qquad (25)$$

unbounded terms around if the limit is properly taken. However, to somebody it seems quite appealing to introduce the finite part (FP) formalism. As for the case of the CPV integrals, the FP symbol should be just a shorthand for the limiting process between curly braces in eqn (25) and eqn (26). In this fashion, the integral equation (25) can be rewritten as

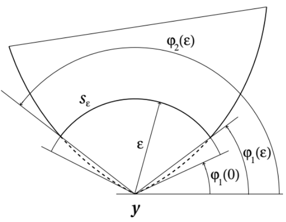
大括号

$$a_k(\mathbf{y})u(\mathbf{y}) + c_{kh}(\mathbf{y})u_{,h}(\mathbf{y}) + \oint\!\!\!\!\!\!=_{\Gamma}[V_k(\mathbf{y},\mathbf{x})u(\mathbf{x}) - W_k(\mathbf{y},\mathbf{x})q(\mathbf{x})]\,\mathrm{d}\Gamma = 0. \qquad (27)$$

这是FP symbol（finite part）

It is manifest that eqn (27), although more compact, conveys less information than eqn (25). For instance, the symmetry of e_ε is completely lost and no rationale is given for the cancellation of unbounded terms.

Our point is that we can use eqn (27) as the final, compact, result of the former analysis, always keeping in mind what is beyind the FP symbol $⨎$. If we start with eqn (27), as it is done quite often, BEM with hypersingular kernels looks like "black magic".



$$\int_{s_\varepsilon}[V_k(x_h - y_h) - W_k n_h(\mathbf{x})]\mathrm{d}\Gamma = c_{kh}(\mathbf{y}) + O(\varepsilon). \qquad (23)$$

公式21的第三行

b会被抵消掉　　对于直角边形状为0

$$\int_{s_\varepsilon} V_k\,\mathrm{d}\Gamma = \frac{b_k(\mathbf{y})}{\varepsilon} + \boxed{a_k(\mathbf{y})} + O(\varepsilon). \qquad (24)$$

公式21的第四行

[14] Hypersingular boundary integral equations have an additional free term.
[15] Existence and evaluation of the two free terms in the hypersingular boundary integral equation of potential theory.
[16] A single-domain boundary element method for 3D elastostatic crack analysis using continuous elements.
[21] On the corner tensor in three-dimensional linear elasticity.

5 Evaluation of the free-term coefficients

这一章没什么实际用处，没有具体公式，参考A new formula for the C-matrix in the Somigliana identity [jump term,非常重要]

In this section, the free-term coefficients c_{kh}, b_k and a_k associated to hypersingular boundary integral equations for two-dimensional scalar problems are obtained. They are defined by expressions (23) and (24). More information on three-dimensional applications can be found in specific papers, such as [14, 15, 16, 21] and in the references reported therein. 三维计算公式

Se是一条圆弧

For convenience, it was assumed s_ε to be an arc of a circle centred at the source point $\mathbf{y}$ and of radius ε (Figure 2). Let (r,φ) be a set of polar coordinates centred at $\mathbf{y}$. Hence the integration point $\mathbf{x}$ on s_ε has coordinates (ε,φ), with $\varphi_1(\varepsilon) \le \varphi \le \varphi_2(\varepsilon)$. Notice that, because of the (possible) curvature of the boundary around $\mathbf{y}$, the interval for φ does depend also on the radius ε, that is on the radial size of s_ε (this is essentially the aspect which was overlooked in Guiggiani *et al.* [10]). 之前错误的原因

Owing to the simple shape of s_ε the following relations hold when $\mathbf{x} \in s_\varepsilon$

$$\begin{array}{ll} r=\epsilon, & \dfrac{\partial r}{\partial n} = -1, \\ r_{,1} = \cos\varphi, & r_{,2} = \sin\varphi, \\ n_1 = -\cos\varphi, & n_2 = -\sin\varphi, \\ x_1 - y_1 = \epsilon\cos\varphi, & x_2 - y_2 = \epsilon\sin\varphi \end{array}$$

$$\text{(A2)}\quad \begin{array}{ll} r = \varepsilon, & \partial r/\partial n = -1, \\ r_{,1} = \cos\varphi, & r_{,2} = \sin\varphi, \\ n_1 = -\cos\varphi, & n_2 = -\sin\varphi, \\ x_1 - y_1 = \varepsilon\cos\varphi, & x_2 - y_2 = \varepsilon\sin\varphi. \end{array} \qquad (28)$$

and $\mathrm{d}\Gamma = \varepsilon\,\mathrm{d}\varphi$.

Accordingly, the kernel functions in the HBIE become

这一步是公式6结合公式28得到的

$$V_1 = \frac{W_1}{\varepsilon} = \frac{\cos\varphi}{2\pi\varepsilon^2}, \qquad V_2 = \frac{W_2}{\varepsilon} = \frac{\sin\varphi}{2\pi\varepsilon^2}. \qquad (29)$$

见公式5，公式9

$$U = -\frac{1}{2\pi}\ln r; \quad T = -\frac{1}{2\pi r}\frac{\partial r}{\partial n}; \quad W_k = \frac{1}{2\pi r}r_{,k}; \quad V_k = -\frac{1}{2\pi r^2}\left[2r_{,k}\frac{\partial r}{\partial n} - n_k\right], \qquad (6)$$

$$W_k(\mathbf{y},\mathbf{x}) = \frac{\partial U}{\partial y_k}. \qquad V_k(\mathbf{y},\mathbf{x}) = \frac{\partial W_k}{\partial n(\mathbf{x})} = \frac{\partial T}{\partial y_k}$$

$$U_{ij}(\mathbf{x}',\mathbf{x}) = \frac{1}{8\pi\mu(1-\nu)}\left\{(3-4\nu)\ln\left(\frac{1}{r}\right)\delta_{ij} + r_{,i}r_{,j}\right\} \qquad \text{(B.2)}$$

$$T_{ij}(\mathbf{x}',\mathbf{x}) = \frac{-1}{4\pi(1-\nu)r}\left\{\frac{\partial r}{\partial n}[(1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}] - (1-2\nu)(r_{,i}n_j - r_{,j}n_i)\right\} \qquad \text{(B.3)}$$

$W_{ikj} = \partial U_{ij}/\partial y_k$ and $V_{ikj} = \partial T_{ij}/\partial y_k$.

$$\int_{s_\varepsilon} V_k \, \mathrm{d}\Gamma = \frac{b_k(\mathbf{y})}{\varepsilon} + a_k(\mathbf{y}) + O(\varepsilon). \qquad (24)$$

$f_k(\varphi) = \mathrm{d}F_k/\mathrm{d}\varphi.$

fk的定义

In both cases we can set

$$V_k = \frac{W_k}{\varepsilon} = \frac{f_k(\varphi)}{\varepsilon^2}. \tag{30}$$

The integral in eqn (24) can now be set in the following form (Figure 2)

$\mathrm{d}\Gamma = \varepsilon \, \mathrm{d}\varphi.$

$$I = \int_{s_\varepsilon} V_k(\mathbf{y}, \mathbf{x}) \mathrm{d}\Gamma = \int_{\varphi_1(\varepsilon)}^{\varphi_2(\varepsilon)} \frac{f_k(\varphi)}{\varepsilon^2} \varepsilon \mathrm{d}\varphi = \frac{1}{\varepsilon}[F_k(\varphi_2(\varepsilon)) - F_k(\varphi_1(\varepsilon))], \tag{31}$$

where $f_k(\varphi) = \mathrm{d}F_k/\mathrm{d}\varphi$.

Since we are interested in the limit of eqn (31) for $\varepsilon \to 0$, we can resort in Taylor series expansions. From differential geometry we obtain (e.g. Farin [22, ch. 11])

eqn (31) for $\varepsilon \to 0$,

$$\varphi_1(\varepsilon) = \varphi_1(0) + \frac{\kappa_1(0)}{2} \varepsilon + O(\varepsilon^2),$$

$$\varphi_2(\varepsilon) = \varphi_2(0) - \frac{\kappa_2(0)}{2} \varepsilon + O(\varepsilon^2). \tag{32}$$

直线的曲率为0

曲率

where $\kappa_i(0)$ are the *signed* curvatures of the boundary Γ around $\mathbf{y}$ (positive curvature means a convex contour). 凸面轮廓

In general, given the parametric equations $(x_1(\xi), x_2(\xi))$ of a curve Γ, the signed curvature κ of a planar curve can be computed with the following expression (e.g. Farin [22, p. 386]), with sign reversed),

对于直线或平面来讲，此公式永远为0

$$\kappa(\xi) = \frac{-x_1'' x_2' + x_2'' x_1'}{[(x_1')^2 + (x_2')^2]^{3/2}}, \tag{33}$$

x1,x2的导数

the prime denoting differentiation with respect to the parameter ξ.

We can also expand $F_k(\varphi(\varepsilon))$ as a funtion of ε

We also set $\varphi_i(0) = \varphi_{i0}$.

$\varphi_1(\varepsilon) \leq \varphi \leq \varphi_2(\varepsilon).$

fai是角度

$$\begin{aligned} F_k(\varphi_1(\varepsilon)) &= F_k(\varphi_1(0)) + \frac{\mathrm{d}F_k}{\mathrm{d}\varphi_1} \frac{\mathrm{d}\varphi_1}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \varepsilon + O(\varepsilon^2) \\ &= F_k(\varphi_{10}) + \frac{\kappa_1(0)}{2} f_k(\varphi_{10}) \, \varepsilon + O(\varepsilon^2), \\ F_k(\varphi_2(\varepsilon)) &= F_k(\varphi_2(0)) + \frac{\mathrm{d}F_k}{\mathrm{d}\varphi_2} \frac{\mathrm{d}\varphi_2}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \varepsilon + O(\varepsilon^2) \\ &= F_k(\varphi_{20}) - \frac{\kappa_2(0)}{2} f_k(\varphi_{20}) \, \varepsilon + O(\varepsilon^2). \end{aligned} \tag{34}$$

where expansions (32) of $\varphi_i(\varepsilon)$ were taken into account. We also set $\varphi_i(0) = \varphi_{i0}$.

$$I = \int_{s_\varepsilon} V_k(\mathbf{y},\mathbf{x})\mathrm{d}\Gamma = \int_{\varphi_1(\varepsilon)}^{\varphi_2(\varepsilon)} \frac{f_k(\varphi)}{\varepsilon^2}\varepsilon\mathrm{d}\varphi = \frac{1}{\varepsilon}[F_k(\varphi_2(\varepsilon)) - F_k(\varphi_1(\varepsilon))], \qquad (31)$$

The integral (31) becomes

$$\begin{aligned} I &= \frac{1}{\varepsilon}\left[F_k(\varphi_{20}) - F_k(\varphi_{10}) - \frac{f_k(\varphi_{20})\kappa_2(0) + f_k(\varphi_{10})\kappa_1(0)}{2}\,\varepsilon + O(\varepsilon^2)\right] \\ &= \frac{F_k(\varphi_{20}) - F_k(\varphi_{10})}{\varepsilon} - \frac{f_k(\varphi_{20})\kappa_2(0) + f_k(\varphi_{10})\kappa_1(0)}{2} + O(\varepsilon) \\ &= \frac{b_k}{\varepsilon} + \underbrace{a_k}_{\text{直线或平面时，恒为零}} + O(\varepsilon). \end{aligned} \qquad (35)$$

This equation shows that the free-term coefficients a_k depend on the local geometry of Γ at $\mathbf{y}$ but in a more subtle way than the other coefficients. They are affected not only by the inner angle at $\mathbf{y}$ but also by the curvature of Γ.

The same computation can be performed in a sligthly different, although equivalent, way. Just observe that

$$I = \int_{\varphi_1(\varepsilon)}^{\varphi_2(\varepsilon)} \frac{f_k(\varphi)}{\varepsilon^2}\varepsilon\mathrm{d}\varphi = \int_{\varphi_1(\varepsilon)}^{\varphi_{10}} + \int_{\varphi_{10}}^{\varphi_{20}} + \int_{\varphi_{20}}^{\varphi_2(\varepsilon)} = \int_{\varphi_{10}}^{\varphi_{20}} - \left[\int_{\varphi_{10}}^{\varphi_1(\varepsilon)} + \int_{\varphi_2(\varepsilon)}^{\varphi_{20}}\right]. \qquad (36)$$

It is now natural to define

这里是b可以相互抵消的原因

$$\frac{b_k(\mathbf{y})}{\varepsilon} = \int_{\varphi_{10}}^{\varphi_{20}} \frac{f_k(\varphi)}{\varepsilon^2}\varepsilon\mathrm{d}\varphi = \frac{F_k(\varphi_{20}) - F_k(\varphi_{10})}{\varepsilon}, \qquad (37)$$

which coincides with eqns (A4) and (A5) in Guiggiani *et al.* [10], and

$$\begin{aligned} a_k(\mathbf{y}) &= -\lim_{\varepsilon\to 0}\left[\int_{\varphi_{10}}^{\varphi_1(\varepsilon)} \frac{f_k(\varphi)}{\varepsilon^2}\varepsilon\,\mathrm{d}\varphi + \int_{\varphi_2(\varepsilon)}^{\varphi_{20}} \frac{f_k(\varphi)}{\varepsilon^2}\varepsilon\,\mathrm{d}\varphi\right] \\ &= -\frac{1}{2}[f_k(\varphi_{20})\kappa_2(0) + f_k(\varphi_{10})\kappa_1(0)]. \end{aligned} \qquad (38)$$

It is worth noting that the coefficients a_k are certainly zero in the following important cases:

当网格是三角网格时候，ak=0；因为每个面都是平面，没有曲率，仔细看图2中y点周围并不是直线

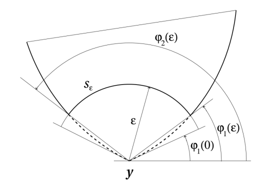


Figure 2. Local geometry at a non-smooth boundary point.

1. $k_1 = k_2 = 0$, that is the contour Γ is piecewise straight around the source point $\mathbf{y}$, regardless of $\mathbf{y}$ being at a corner or not;
2. $k_1 = k_2$ and $\varphi_{20} = \varphi_{10} + \pi$, that is the curvature *and* the tangent vector are continuous at $\mathbf{y}$. This condition can be summarized as Γ having a geometric continuity of order 2 at $\mathbf{y}$, as defined, e.g. in Farin [22, chapter 12] ($\Gamma \in G^2(\mathbf{y})$).

Notice, however, that a_k may be non-zero at a smooth boundary point (unique tangent) with discontinuous curvature.

As for the coefficients c_{kh}, the treatment is similar. From eqn (23)

h,k代表1,2,3=x,y,z

$$c_{hk} = \lim_{\varepsilon\to 0}\int_{s_\varepsilon} [V_h(x_k - y_k) - W_h n_k]\,\mathrm{d}\Gamma = 4\pi\int_{\varphi_{10}}^{\varphi_{20}} f_h(\varphi)\,f_k(\varphi)\,\mathrm{d}\varphi. \qquad (39)$$

At smooth boundary points ($\varphi_{20} = \varphi_{10} + \pi$), we obtain $c_{hk} = 0.5\,\delta_{hk}$.

$$V_k = \frac{W_k}{\varepsilon} = \frac{f_k(\varphi)}{\varepsilon^2}.$$

W1, W2都是三角函数

$$U_{ij}(\mathbf{x}',\mathbf{x}) = \frac{1}{8\pi\mu(1-\nu)}\left\{(3-4\nu)\ln\left(\frac{1}{r}\right)\delta_{ij} + r_{,i}r_{,j}\right\} \qquad \text{(B.2)}$$

$$T_{ij}(\mathbf{x}',\mathbf{x}) = \frac{-1}{4\pi(1-\nu)r}\left\{\frac{\partial r}{\partial n}[(1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}] - (1-2\nu)(r_{,i}n_j - r_{,j}n_i)\right\} \qquad \text{(B.3)}$$

$$V_1 = \frac{W_1}{\varepsilon} = \frac{\cos\varphi}{2\pi\varepsilon^2}, \qquad V_2 = \frac{W_2}{\varepsilon} = \frac{\sin\varphi}{2\pi\varepsilon^2}. \qquad (29)$$

$$\int_{s_\varepsilon} [V_k(x_h - y_h) - W_k n_h(\mathbf{x})]\mathrm{d}\Gamma = c_{kh}(\mathbf{y}) + O(\varepsilon). \qquad (23)$$

$$a_k(\mathbf{y})u(\mathbf{y}) + c_{kh}(\mathbf{y})u_{,h}(\mathbf{y})$$
$$+ \lim_{\varepsilon\to 0}\left\{\int_{(\Gamma-e_\varepsilon)}[V_k(\mathbf{y},\mathbf{x})u(\mathbf{x}) - W_k(\mathbf{y},\mathbf{x})q(\mathbf{x})]\,\mathrm{d}\Gamma + \frac{b_k(\mathbf{y})}{\varepsilon}u(\mathbf{y})\right\} = 0. \quad (25)$$

6 Direct evaluation of singular integrals

The numerical treatment proceeds directly from equations in the form of (17) or (25), with strongly singular and even hypersingular kernels (in eqn (25)). The method is semi-analytical in the sense that all singular integrations are performed *analytically* and the limiting process is performed *exactly*. Therefore, numerical integration has only to deal with regular integrals.只在非奇异方程中，进行数值积分。

奇异方程，通过解析直接获得。

Since all analytical steps are performed in the space of parametric (intrinsic) coordinates for both 2D and 3D problems, the type of boundary elements has only marginal influence on the integration technique. Thus, the boundary elements can be of any order and shape. Of course, the use of intrinsic coordinates demands both the integrand and the limiting process to be "translated" from the Euclidean space into the parameter space. This is a key issue in the development of the proposed method.

边界元素的类型对积分只有极轻微影响。

The analysis is first presented for three-dimensional problems, followed by two-dimensional problems. In both cases, the vector case is addressed since it is more general.

$$a_{ikj}(\mathbf{y})u_j(\mathbf{y}) + c_{ikjh}(\mathbf{y})u_{j,h}(\mathbf{y}) \quad (26)$$
$$+ \lim_{\varepsilon\to 0}\left\{\int_{(\Gamma-e_\varepsilon)}[V_{ikj}(\mathbf{y},\mathbf{x})u_j(\mathbf{x}) - W_{ikj}(\mathbf{y},\mathbf{x})q_j(\mathbf{x})]\,\mathrm{d}\Gamma + \frac{b_{ikj}(\mathbf{y})}{\varepsilon}u_j(\mathbf{y})\right\} = 0.$$

6.1 Three-dimensional problems

Let us consider an hypersingular boundary integral equation like, e.g. eqn (26). In principle, we have to compute the following limit

公式26中的非奇异方程部分，安装上述，应该使用数值积分方法获得

$$\lim_{\varepsilon\to 0}\left\{\int_{(\Gamma-e_\varepsilon)}[V_{ikj}(\mathbf{y},\mathbf{x})u_j(\mathbf{x}) - W_{ikj}(\mathbf{y},\mathbf{x})t_j(\mathbf{x})]\,\mathrm{d}\Gamma + u_j(\mathbf{y})\frac{b_{ikj}(\mathbf{y})}{\varepsilon}\right\}, \quad (40)$$

which, we know, is bounded. Notice that, *consistently* with the already obtained coefficients $c_{ikjh}(\mathbf{y})$, $b_{ikj}(\mathbf{y})$ and $a_{ikj}(\mathbf{y})$ the neighbourhood e_ε on Γ is given by $e_\varepsilon = \{\mathbf{x}\in\Gamma$ such that $|\mathbf{x}-\mathbf{y}|\le\varepsilon\}$. 表示三维的

In eqn (40) there are strongly singular kernels W_{ikj} and hypersingular kernels V_{ikj}. However, in the expansion of V_{ikj} there are both hypersingular (of $O(r^{-3})$) and strongly singular (of $O(r^{-2})$) terms. Therefore, the treatment of W_{ikj} is just a particular, simpler case of the method we will discuss for the evaluation of the limit

$$\lim_{\varepsilon\to 0}\left\{\int_{(\Gamma-e_\varepsilon)}V_{ikj}(\mathbf{y},\mathbf{x})u_j(\mathbf{x})\,\mathrm{d}\Gamma + u_j(\mathbf{y})\frac{b_{ikj}(\mathbf{y})}{\varepsilon}\right\}. \quad (41)$$

应该注意的是，将极限分拆的前提是各部分的极限是存在的。

This splitting is possible since in most cases the limits exist separately. As a matter of fact, if the singular point $\mathbf{y}$ is located at a smooth boundary point, the singular kernels are multiplied by different functions, and reciprocal cancelling effects are therefore not possible. On the other hand, if $\mathbf{y}$ is located on an edge or a vertex, both terms must be taken together, as they are in eqn (40).

相互抵消效应

如果y在边上或者顶点上，就要使用公式40的形式

这里是非常重要的，关系到程序的具体实现，也就是使用不连续元和连续元的实现是不同的

这里说的是，公式41的使用范围，考虑到简洁，公式41去掉了W项，而去掉的前提是W的极限是存在的，否则不能去掉。又因为W项存在的前提是通过分子分母相互抵消效应达到的，所以不适用于场点（field point）在边界和顶点处的情况。如果场点在边界或者顶点处，就不能使用公式41,转而使用公式40求解。

6.1.1 Limiting process and discretization of the geometry

We denote a portion of Γ containing the singular point $\mathbf{y}$ by Γ_s. If discontinuous elements with collocation at element interiors are used, then Γ_s consists of just one element; whereas if $C^{1,\alpha}$-continuous element are used to represent u_i, Γ_s consists of all adjacent elements connected to the singular point $\mathbf{y}$.

不连续元，奇异点在元的内部
连续的元奇异点在元的顶点处，所以有相邻元的问题

For simplicity, the case of $\mathbf{y}$ belonging to just one element is considered here. However, the analysis is in no way restricted to this case.

As usual, on each boundary element, the displacement is represented in terms of shape functions and nodal values, $u_i(\mathbf{x}) = N^c(\boldsymbol{\xi}(\mathbf{x}))u_i^c$, where $\boldsymbol{\xi} = (\xi_1, \xi_2)$ are the intrinsic coordinates.

局部坐标下的形函数

From a computational standpoint, we need the evaluation of the quantity I defined on the element Γ_s

这和形函数N的定义有关

$$\lim_{\varepsilon\to 0}\left\{\int_{(\Gamma-e_\varepsilon)} V_{ikj}(\mathbf{y},\mathbf{x})u_j(\mathbf{x})\,\mathrm{d}\Gamma + u_j(\mathbf{y})\frac{b_{ikj}(\mathbf{y})}{\varepsilon}\right\}. \qquad (41)$$

$$N^a = N^a(\boldsymbol{\eta})+\rho\left[\frac{\partial N^a}{\partial\xi_1}\Big|_{\xi=\eta}\cos\theta+\frac{\partial N^a}{\partial\xi_2}\Big|_{\xi=\eta}\sin\theta\right]+O(\rho^2) = N_0^c+\rho N_1^c(\theta)+O(\rho^2). \qquad (74)$$

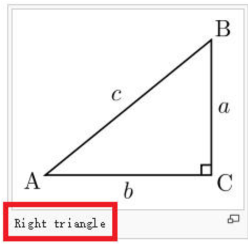
$$I = \lim_{\varepsilon\to 0}\left\{\int_{(\Gamma_s-e_\varepsilon)} V_{ikj}(\mathbf{y},\mathbf{x})N^a(\boldsymbol{\xi}(\mathbf{x}))\,\mathrm{d}\Gamma + N^a(\boldsymbol{\eta})\frac{b_{ikj}(\mathbf{y})}{\varepsilon}\right\}, \qquad (42)$$

intrinsic coordinates

公式74　与公式41的区别在于，边界只是包含奇异点的特定元

where N^a represents those shape functions (usually just one) that are not zero at $\boldsymbol{\eta} = \boldsymbol{\xi}(\mathbf{y})$, the *image* in the local plane of the collocation point $\mathbf{y}$.

usually just one，指的是只包含一个奇异点的元的形函数是“1”。

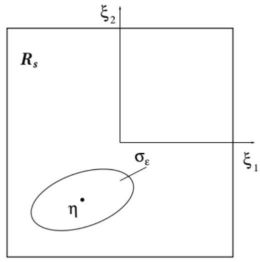


By means of the usual representation for the geometry in terms of shape functions and nodal coordinates

公式42中N和公式43中M的区别：
N是在局部坐标中的形函数，或者说是全局->局部投影的形函数
M是局部->全局的形函数

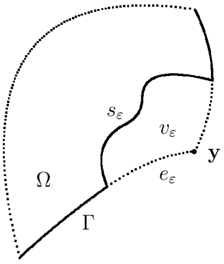
mapped

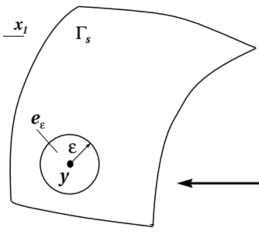
$$x_k(\boldsymbol{\xi}) = M^c(\boldsymbol{\xi})x_k^c, \qquad (43)$$



the boundary element Γ_s is mapped onto a region R_s of standard shape in the $\boldsymbol{\xi}$-space (e.g. a square, or a right triangle). Notice that the shape functions $M^c(\boldsymbol{\xi})$ employed for the geometric description need not be the same used to represent the unknown functions on the boundary.

右边的两个图





Accordingly, the symmetric neighbourhood e_ε of $\mathbf{y}$ in the real space is mapped onto a neighbourhood σ_ε of $\boldsymbol{\eta}$ in the $\boldsymbol{\xi}$-space, as shown in Figure 3. It is important to note that, in general, σ_ε is *not a circle*, although its polar radius still tends to zero with ε.

Thus, in the space of intrinsic coordinates, expression (42) becomes

该方程式建立在局部坐标系上的

$$I = \lim_{\varepsilon\to 0}\left\{\int_{R_s-\sigma_\varepsilon} V_{ikj}(\mathbf{y},\mathbf{x}(\boldsymbol{\xi}))N^a(\boldsymbol{\xi})J(\boldsymbol{\xi})\,\mathrm{d}\xi_1\mathrm{d}\xi_2 + N^a(\boldsymbol{\eta})\frac{b_{ikj}(\mathbf{y})}{\varepsilon}\right\}, \qquad (44)$$

where $\mathrm{d}\Gamma = J(\boldsymbol{\xi})\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2$. It is worth noting that geometric elements can be of any kind and order.

Following a common practice in the BEM, polar coordinates $(\rho,\ \theta)$ centred at $\boldsymbol{\eta}$ (the image of $\mathbf{y}$) are defined in the $\boldsymbol{\xi}$-space (Figure 4)

在局部坐标系的基础上，更进一步，变成局部极坐标系

$$\begin{cases} \xi_1 &= \eta_1 + \rho\cos\theta, \\ \xi_2 &= \eta_2 + \rho\sin\theta, \end{cases}$$

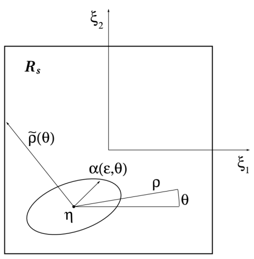


Figure 4. Polar coordinates in the parameter plane.

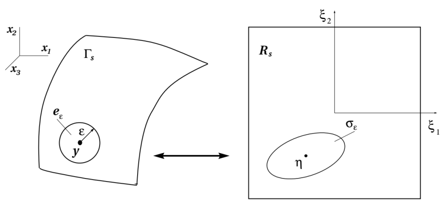


Figure 3. Mapping between the Euclidean space and the space of intrinsic coordinates.

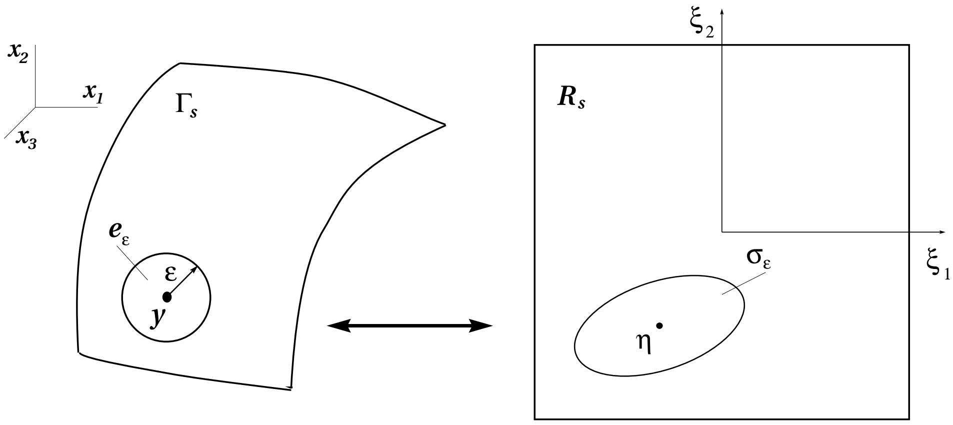


Figure 3. Mapping between the Euclidean space and the space of intrinsic coordinates.

该方程式建立在局部坐标系上的

$$I = \lim_{\varepsilon\to 0}\left\{\int_{R_s-\sigma_\varepsilon} V_{ikj}(\mathbf{y},\mathbf{x}(\boldsymbol{\xi}))N^a(\boldsymbol{\xi})J(\boldsymbol{\xi})\,\mathrm{d}\xi_1\mathrm{d}\xi_2 + N^a(\boldsymbol{\eta})\frac{b_{ikj}(\mathbf{y})}{\varepsilon}\right\}, \qquad (44)$$

$$\left\{\begin{array}{lcl}\xi_1 & = & \eta_1+\rho\cos\theta,\\ \xi_2 & = & \eta_2+\rho\sin\theta,\end{array}\right.$$

so that $\mathrm{d}\xi_1\,\mathrm{d}\xi_2 = \rho\,\mathrm{d}\rho\,\mathrm{d}\theta$. Hence, the quantity I in eqn (44) becomes

$$I = \lim_{\varepsilon\to 0}\left\{\int_0^{2\pi}\int_{\alpha(\varepsilon,\theta)}^{\tilde\rho(\theta)} F_{ikj}(\rho,\theta)\,\mathrm{d}\rho\,\mathrm{d}\theta + N^a(\boldsymbol{\eta})\frac{b_{ikj}(\mathbf{y})}{\varepsilon}\right\}, \qquad (45)$$

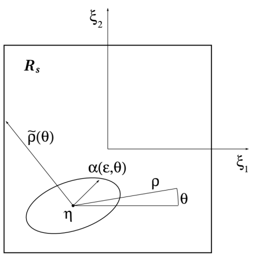


Figure 4. Polar coordinates in the parameter plane.

where $F_{ikj}(\rho,\theta) = V_{ikj}\,N^a\,J\,\rho = O(\rho^{-2})$ is the hypersingular integrand, $\rho = \alpha(\varepsilon,\theta)$ is the equation in polar coordinates of the *distorted* neighbourhood σ_ε (Figure 4), and $\rho = \tilde\rho(\theta)$ is the equation in polar coordinates of the external contour of R_s.

在投影过程中，图3中的圆域变成了椭圆域

这句话针对的是公式44到公式45，在公式44中没有明显的在方程中体现极限操作，而在公式45中通过积分下限来体现极限操作

It is important to realize that the effects of the mapping on the limiting process have been taken into account through the function $\alpha(\varepsilon,\theta)$. Moreover, all manipulations have been done (and will be done) before taking the limit. The distorsion of the exclusion neighbourhood in the parameter plane is quite often erroneously overlooked.

$$F_{ikj}(\rho,\theta) = V_{ikj}\,N^a\,J\,\rho = O(\rho^{-2})$$

Now, let us analyse the singular function $F_{ikj}(\rho,\theta)$. Since it is singular of order ρ^{-2}, we have a (Laurent) series expansion with respect to ρ in the form (subscripts in the expansion are dropped)

$$F_{ikj}(\rho,\theta) = \frac{F_{-2}(\theta)}{\rho^2} + \frac{F_{-1}(\theta)}{\rho} + O(1). \qquad (46)$$

Notice that in the BEM both $F_{-2}(\theta)$ and $F_{-1}(\theta)$ are real functions of θ, even when $F_{ikj}(\rho,\theta)$ is complex valued, as in time-harmonic problems. The dependence on θ is crucial for expansion (46) to actually represent the asymptotic behaviour of $F_{ikj}(\rho,\theta)$, when $\rho\to 0$. Expansion (46) is one of the key ingredients of the present analysis.

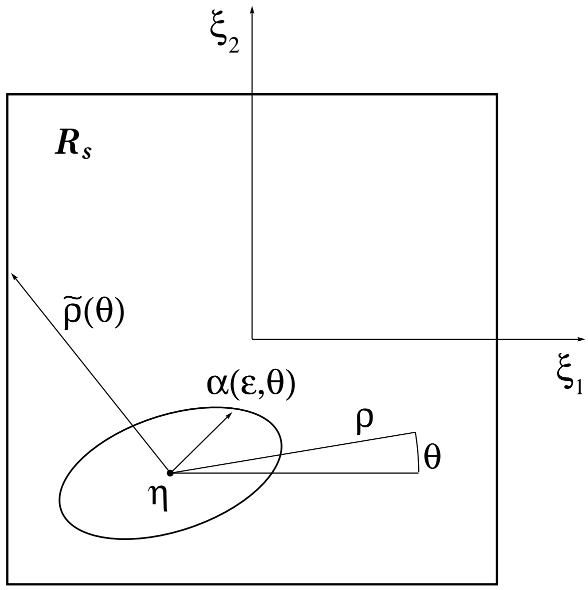


Figure 4. Polar coordinates in the parameter plane.

Also of basic relevance is the Taylor series expansion for $\alpha(\varepsilon, \theta)$, with respect to ε

$$\rho = \alpha(\varepsilon,\theta) = \frac{\varepsilon}{A(\theta)} - \varepsilon^2 \frac{C(\theta)}{A^4(\theta)} + O(\varepsilon^3) = \varepsilon\,\beta(\theta) + \varepsilon^2\gamma(\theta) + O(\varepsilon^3), \tag{77}$$

$$\rho = \alpha(\varepsilon, \theta) = \varepsilon\,\beta(\theta) + \varepsilon^2\gamma(\theta) + O(\varepsilon^3). \tag{47}$$

Note that, in general, $\rho = \varepsilon\,\beta(\theta)$ is the equation of an ellipse. 为什么？

A systematic way of obtaining the explicit expressions of $F_{-2}(\theta)$, $F_{-1}(\theta)$, $\beta(\theta)$ and $\gamma(\theta)$ is presented in Appendix A. It is really an easy task for any kernel function.

6.1.2 Semi-analytical treatment

We can now address the problem of the actual evaluation of I. Adding and subtracting the first two terms of the series expansion (46) in expression (45), we obtain

$$\begin{aligned} I = \lim_{\varepsilon\to 0} \Bigg\{ & \int_0^{2\pi}\int_{\alpha(\varepsilon,\theta)}^{\tilde{\rho}(\theta)} \left[F_{ikj}(\rho,\theta) - \left(\frac{F_{-2}(\theta)}{\rho^2} + \frac{F_{-1}(\theta)}{\rho} \right) \right] \mathrm{d}\rho\,\mathrm{d}\theta \\ & + \int_0^{2\pi}\int_{\alpha(\varepsilon,\theta)}^{\tilde{\rho}(\theta)} \frac{F_{-1}(\theta)}{\rho} \mathrm{d}\rho\,\mathrm{d}\theta + \left[\int_0^{2\pi}\int_{\alpha(\varepsilon,\theta)}^{\tilde{\rho}(\theta)} \frac{F_{-2}(\theta)}{\rho^2} \mathrm{d}\rho\,\mathrm{d}\theta + N^a(\boldsymbol{\eta}) \frac{b_{ikj}(\mathbf{y})}{\varepsilon} \right] \Bigg\} = \\ & = I_0 + I_{-1} + I_{-2}. \end{aligned} \tag{48}$$

Each term I_0, I_{-1}, and I_{-2} in eqn (48) is now analysed separately.

$$I = \lim_{\varepsilon\to 0} \left\{ \int_0^{2\pi}\int_{\alpha(\varepsilon,\theta)}^{\tilde{\rho}(\theta)} F_{ikj}(\rho,\theta)\,\mathrm{d}\rho\,\mathrm{d}\theta + N^a(\boldsymbol{\eta}) \frac{b_{ikj}(\mathbf{y})}{\varepsilon} \right\}, \tag{45}$$

$$F_{ikj}(\rho,\theta) = \frac{F_{-2}(\theta)}{\rho^2} + \frac{F_{-1}(\theta)}{\rho} + O(1). \tag{46}$$

According to eqn (46), in I_0 the integrand is *regular*. Therefore, the limit is straightforward and simply becomes 没有极限了，或者说极限操作已经作用在了积分下标上了

$$I_0 = \int_0^{2\pi} \int_0^{\tilde{\rho}(\theta)} \left[F_{ikj}(\rho,\theta) - \left(\frac{F_{-2}(\theta)}{\rho^2} + \frac{F_{-1}(\theta)}{\rho} \right) \right] \mathrm{d}\rho \, \mathrm{d}\theta \tag{49}$$

重积分

This double integral can be evaluated by standard quadrature rules.

Now, let us consider I_{-1}

$$I = \lim_{\varepsilon \to 0} \left\{ \int_0^{2\pi} \int_{\alpha(\varepsilon,\theta)}^{\tilde{\rho}(\theta)} \left[F_{ikj}(\rho,\theta) - \left(\frac{F_{-2}(\theta)}{\rho^2} + \frac{F_{-1}(\theta)}{\rho} \right) \right] \mathrm{d}\rho \, \mathrm{d}\theta \right.$$
$$\left. + \int_0^{2\pi} \int_{\alpha(\varepsilon,\theta)}^{\tilde{\rho}(\theta)} \frac{F_{-1}(\theta)}{\rho} \mathrm{d}\rho \, \mathrm{d}\theta + \left[\int_0^{2\pi} \int_{\alpha(\varepsilon,\theta)}^{\tilde{\rho}(\theta)} \frac{F_{-2}(\theta)}{\rho^2} \mathrm{d}\rho \, \mathrm{d}\theta + N^a(\boldsymbol{\eta}) \frac{b_{ikj}(\mathbf{y})}{\varepsilon} \right] \right\} =$$
$$= I_0 + I_{-1} + I_{-2}. \tag{48}$$

$$\rho = \alpha(\varepsilon,\theta) = \varepsilon\,\beta(\theta) + \varepsilon^2 \gamma(\theta) + O(\varepsilon^3). \tag{47}$$

$$\begin{aligned} I_{-1} &= \lim_{\varepsilon \to 0} \int_0^{2\pi} \int_{\alpha(\varepsilon,\theta)}^{\tilde{\rho}(\theta)} \frac{F_{-1}(\theta)}{\rho} \mathrm{d}\rho \, \mathrm{d}\theta \\ &= \lim_{\varepsilon \to 0} \int_0^{2\pi} F_{-1}(\theta) [\ln|\tilde{\rho}(\theta)| - \ln|\alpha(\varepsilon,\theta)|] \mathrm{d}\theta \\ &= \int_0^{2\pi} F_{-1}(\theta) \ln|\tilde{\rho}(\theta)| \mathrm{d}\theta - \lim_{\varepsilon \to 0} \int_0^{2\pi} F_{-1}(\theta) \ln|\varepsilon\,\beta(\theta)| \mathrm{d}\theta \\ &= \int_0^{2\pi} F_{-1}(\theta) \ln\left|\frac{\tilde{\rho}(\theta)}{\beta(\theta)}\right| \mathrm{d}\theta - \lim_{\varepsilon \to 0} \left\{ \ln \varepsilon \underbrace{\int_0^{2\pi} F_{-1}(\theta) \mathrm{d}\theta}_{等于0} \right\} \\ &= \int_0^{2\pi} F_{-1}(\theta) \ln\left|\frac{\tilde{\rho}(\theta)}{\beta(\theta)}\right| \mathrm{d}\theta. \end{aligned} \tag{50}$$

$\log_a M + \log_a N = \log_a MN$

ρ ~(θ) is the equation in polar coordinates of the external contour of Rs

分母见公式47，77

Equation (50) shows that I-1 is equivalent to a simple regular one-dimensional integral.

The series expansions (70), (71), (73), and (74) are all we need to obtain $F_{-2}(\theta)$ and $F_{-1}(\theta)$ for *any* hypersingular integrand in the BEM. As a matter

In the derivation of this equation, first we integrated analytically with respect to ρ, then we made use of expansion (47) (limited to the first term), and, finally, we considered the property $\int_0^{2\pi} F_{-1}(\theta)\mathrm{d}\theta = 0$. Equation (50) shows that I_{-1} is equivalent to a simple *regular* one-dimensional integral.

A similar treatment applies to I_{-2} such that

$$\begin{aligned} I_{-2} &= \lim_{\varepsilon \to 0} \left\{ \int_0^{2\pi} \int_{\alpha(\varepsilon,\theta)}^{\tilde{\rho}(\theta)} \frac{F_{-2}(\theta)}{\rho^2} \mathrm{d}\rho \, \mathrm{d}\theta + N^a(\boldsymbol{\eta}) \frac{b_{ikj}(\mathbf{y})}{\varepsilon} \right\} \\ &= \lim_{\varepsilon \to 0} \left\{ \int_0^{2\pi} F_{-2}(\theta) \left[-\frac{1}{\tilde{\rho}(\theta)} + \frac{1}{\alpha(\varepsilon,\theta)} \right] \mathrm{d}\theta + N^a(\boldsymbol{\eta}) \frac{b_{ikj}(\mathbf{y})}{\varepsilon} \right\} \\ &= \lim_{\varepsilon \to 0} \left\{ \int_0^{2\pi} \frac{F_{-2}(\theta)}{\varepsilon\,\beta(\theta)} \left(1 - \varepsilon \frac{\gamma(\theta)}{\beta(\theta)} \right) \mathrm{d}\theta + N^a(\boldsymbol{\eta}) \frac{b_{ikj}(\mathbf{y})}{\varepsilon} \right\} - \int_0^{2\pi} \frac{F_{-2}(\theta)}{\tilde{\rho}(\theta)} \mathrm{d}\theta \\ &= \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon} \right) \left\{ \int_0^{2\pi} \frac{F_{-2}(\theta)}{\beta(\theta)} \mathrm{d}\theta + N^a(\boldsymbol{\eta}) b_{ikj}(\mathbf{y}) \right\} - \int_0^{2\pi} F_{-2}(\theta) \left[\frac{\gamma(\theta)}{\beta^2(\theta)} + \frac{1}{\tilde{\rho}(\theta)} \right] \mathrm{d}\theta \\ &= -\int_0^{2\pi} F_{-2}(\theta) \left[\frac{\gamma(\theta)}{\beta^2(\theta)} + \frac{1}{\tilde{\rho}(\theta)} \right] \mathrm{d}\theta. \end{aligned} \tag{51}$$

Therefore, I_{-2} is also equivalent to just a one-dimensional *regular* integral. Owing to the higher order of singularity of the integrand, in this case both terms $\varepsilon\,\beta(\theta)$ and $\varepsilon^2\gamma(\theta)$ must be retained in the expansion (47) for $\alpha(\varepsilon,\theta)$.

$$\rho = \alpha(\varepsilon,\theta) = \varepsilon\,\beta(\theta) + \varepsilon^2 \gamma(\theta) + O(\varepsilon^3). \tag{47}$$

$$
\begin{aligned}
&a_{ikj}(\mathbf{y})u_j(\mathbf{y}) + c_{ikjh}(\mathbf{y})u_{j,h}(\mathbf{y}) \\
&+ \lim_{\varepsilon \to 0} \left\{ \int_{(\Gamma - e_\varepsilon)} [V_{ikj}(\mathbf{y}, \mathbf{x})u_j(\mathbf{x}) - W_{ikj}(\mathbf{y}, \mathbf{x})q_j(\mathbf{x})] \, \mathrm{d}\Gamma + \frac{b_{ikj}(\mathbf{y})}{\varepsilon} u_j(\mathbf{y}) \right\} = 0.
\end{aligned} \tag{26}
$$

The singularity cancellation we have been speaking about is made explicit in eqn (51), where $\{\int_0^{2\pi} [F_{-2}(\theta)/\beta(\theta)] \, \mathrm{d}\theta + N^a(\boldsymbol{\eta}) b_{ikj}(\mathbf{y})\} = 0$. The unbounded term $N^a b_{ikj}/\varepsilon$ due to an integral on s_ε (see eqn (26)) is cancelled out by a corresponding unbounded term arising from the integral on $(\Gamma_s - e_\varepsilon)$, so that the final result is perfectly bounded and meaningful. This cancellation is strictly related to the nature of the kernels involved.

$$
\begin{aligned}
I = \lim_{\varepsilon \to 0} \Bigg\{ & \int_0^{2\pi} \int_{\alpha(\varepsilon,\theta)}^{\tilde{\rho}(\theta)} \left[F_{ikj}(\rho, \theta) - \left(\frac{F_{-2}(\theta)}{\rho^2} + \frac{F_{-1}(\theta)}{\rho} \right) \right] \mathrm{d}\rho \, \mathrm{d}\theta \\
&+ \int_0^{2\pi} \int_{\alpha(\varepsilon,\theta)}^{\tilde{\rho}(\theta)} \frac{F_{-1}(\theta)}{\rho} \mathrm{d}\rho \, \mathrm{d}\theta + \left[\int_0^{2\pi} \int_{\alpha(\varepsilon,\theta)}^{\tilde{\rho}(\theta)} \frac{F_{-2}(\theta)}{\rho^2} \mathrm{d}\rho \, \mathrm{d}\theta + N^a(\boldsymbol{\eta}) \frac{b_{ikj}(\mathbf{y})}{\varepsilon} \right] \Bigg\} = \\
&= I_0 + I_{-1} + I_{-2}.
\end{aligned} \tag{48}
$$

6.1.3 Final formula

By collecting the previous results, the following *final formula* for the evaluation of hypersingular integrals in three-dimensional BEM analyses can be given

$$
\begin{aligned}
I = & \int_0^{2\pi} \int_0^{\tilde{\rho}(\theta)} \left\{ F_{ikj}(\rho, \theta) - \left[\frac{F_{-2}(\theta)}{\rho^2} + \frac{F_{-1}(\theta)}{\rho} \right] \right\} \mathrm{d}\rho \, \mathrm{d}\theta \\
&+ \int_0^{2\pi} \left\{ F_{-1}(\theta) \ln \left| \frac{\tilde{\rho}(\theta)}{\beta(\theta)} \right| - F_{-2}(\theta) \left[\frac{\gamma(\theta)}{\beta^2(\theta)} + \frac{1}{\tilde{\rho}(\theta)} \right] \right\} \mathrm{d}\theta.
\end{aligned} \tag{52}
$$

This formula proves that the quantity I, originally given (eqn (48)) by a limiting process involving a hypersingular integral plus an unbounded term is simply equal to a regular double integral plus a regular one-dimensional integral. Hence, it is easily *computable*. We recall that in eqn (52) all singular integrations have been carried out analytically and the limiting process has been performed exactly.

$$
\rho = \alpha(\varepsilon, \theta) = \varepsilon \, \beta(\theta) + \varepsilon^2 \gamma(\theta) + O(\varepsilon^3). \tag{47}
$$

The terms containing $\tilde{\rho}(\theta)$ takes into account the external shape of R_s, while the terms with $\beta(\theta)$ and $\gamma(\theta)$ account for the *distorsion* of σ_ε, that is introduced by the mapping in the originally symmetric neighbourhood e_ε (Figures 3 and 4).

Both integrals in eqn (52) are in polar coordinates defined in the local plane, which allows for a standard numerical implementation. Standard Gaussian quadrature rules of low order provide very good accuracy. Therefore, the idea that singular integrals are intractable for numerical computations should be definitely abandoned.

Formula (52) is completely general. It holds for any kind of boundary elements employed, provided that the necessary continuity requirements for u_i are satisfied at each collocation point. Of course, a formula formally identical to eqn (52) can be given for any other hypersingular boundary integral equation.

Owing to the selected symmetric shape of the exclusion neighbourhood, eqn (52) simplifies even further since

$$
\int_0^{2\pi} F_{-1}(\theta) \ln |\beta(\theta)| = 0,
$$

这种情况，不能用在场点在顶点处，由多个三角面片共享的一般情况。

$$
\int_0^{2\pi} F_{-2}(\theta) \frac{\gamma(\theta)}{\beta^2(\theta)} \, \mathrm{d}\theta = 0.
$$

However, the same does not necessarily hold for more general shapes of e_ε.

If the singular point is shared by more than one element, that is if $\Gamma_s = \bigcup_m \Gamma_m$, formula (52) becomes

$$I = \sum_m \left\{ \int_{\theta_1^m}^{\theta_2^m} \int_0^{\tilde{\rho}_m(\theta)} \left[F_{ikj}^m(\rho,\theta) - \left(\frac{F_{-2}^m(\theta)}{\rho^2} + \frac{F_{-1}^m(\theta)}{\rho} \right) \right] \mathrm{d}\rho \, \mathrm{d}\theta \right.$$
$$\left. + \int_{\theta_1^m}^{\theta_2^m} \left[F_{-1}^m(\theta) \ln \left| \frac{\tilde{\rho}_m(\theta)}{\beta_m(\theta)} \right| - F_{-2}^m(\theta) \left(\frac{\gamma_m(\theta)}{(\beta_m(\theta))^2} + \frac{1}{\tilde{\rho}_m(\theta)} \right) \right] \mathrm{d}\theta \right\}, \quad (53)$$

where the index m refers to one element around the collocation point at a time, and $\theta_1^m \leq \theta \leq \theta_2^m$ on the m-th element. In this case, both β_m and γ_m terms must be retained since their contribution is not zero, in general, even for circular e_ε.

6.1.4 Less singular integrals

Integrals with less than hypersingular kernels can be dealt with by exactly the same procedure as above. Of course, some simplifications occur.

For instance, if the integral has only a *strongly singular* kernel (like T_{ij} or W_{ikj}, or even V_{ikj} but with $N^a(\boldsymbol{\eta}) = 0$) the Laurent expansion (46) has $F_{-2}(\theta) \equiv 0$, and the final formulas (52) and (53) simplify accordingly

$$I = \sum_m \left\{ \int_{\theta_1^m}^{\theta_2^m} \int_0^{\tilde{\rho}_m(\theta)} \left[F_{ikj}^m(\rho,\theta) - \frac{F_{-1}^m(\theta)}{\rho} \right] \mathrm{d}\rho \, \mathrm{d}\theta + \int_{\theta_1^m}^{\theta_2^m} F_{-1}^m(\theta) \ln \left| \frac{\tilde{\rho}_m(\theta)}{\beta_m(\theta)} \right| \mathrm{d}\theta \right\}. \quad (54)$$

强奇异积分方程

This is precisely the result first presented by Guiggiani & Gigante in [7] for CPV integrals in elastic problems (along with a similar result for internal cells in elasto-plastic problems) and by Guiggiani in [12] for time-harmonic elastodynamics.

If the integral has a *weakly singular* kernel (like, e.g. U_{ij}), also $F_{-1}(\theta) \equiv 0$ and the formula simply states that the integrand becomes regular if the integral is evaluated in polar coordinates.

Quite remarkably, we have thus shown that the proposed method is a *unified approach* to the evaluation of singular integrals in the BEM, regardless of their order of singularity. This aspect was first mentioned in Guiggiani *et al.* [10].

6.2 Two-dimensional problems

The procedure for plane problems follows essentially the same steps as for the three-dimensional case. It was first presented in Guiggiani [9] and then applied to the stress evaluation at points on the boundary in Guiggiani [13].

Let Γ_s be a part of the contour Γ containing the singular point $\mathbf{y}$. In general, in two-dimensional BEM analyses the collocation point $\mathbf{y}$ may be either shared by two boundary elements (Figure 5(a)), or it may lie (strictly) within one boundary element (Figure 5(b)). The natural choice in the first case is $\Gamma_s = \Gamma_s^1 \bigcup \Gamma_s^2$, whereas in the second case Γ_s is just the element containing $\mathbf{y}$.

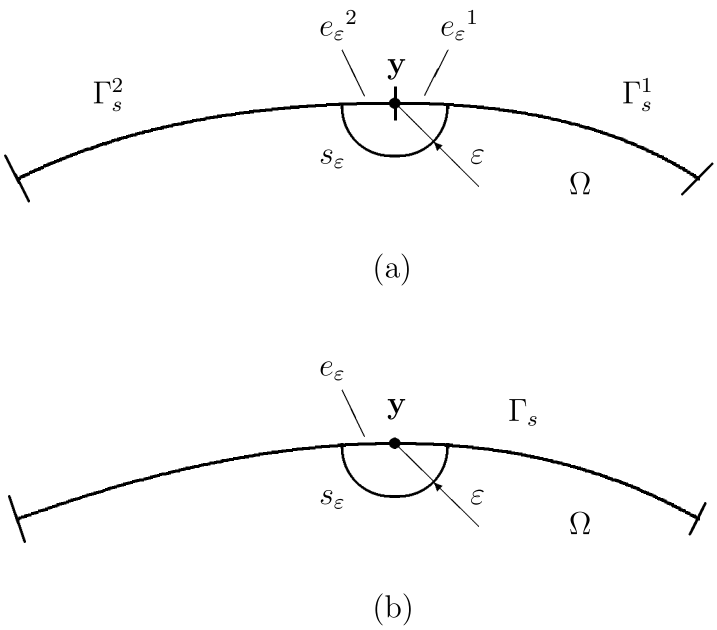


Figure 5. Limiting process on the contour Γ: (a) pole $\mathbf{y}$ between two boundary elements; (b) pole $\mathbf{y}$ within one boundary element.

From a computational standpoint we need the evaluation of *bounded* quantities

$$\lim_{\varepsilon \to 0} \left\{ \int_{(\Gamma_s - e_\varepsilon)} V_{ikj}(\mathbf{y}, \mathbf{x}) u_j(\mathbf{x}) ds_x + u_j(\mathbf{y}) \frac{b_{ikj}(\mathbf{y})}{\varepsilon} \right\},$$

with hypersingular kernels $V_{ikj} = O(r^{-2})$. Consistently with the already evaluated coefficients c_{ikjh} and b_{ikj}, the neighbourhood e_ε must be symmetric, that is $e_\varepsilon = \{\mathbf{x} \in \Gamma \mid |\mathbf{x} - \mathbf{y}| \leq \varepsilon\}$. The integrals with the strongly singular kernel W_{ikj} are evaluated in the same way, with obvious simplifications due to the lower order of singularity.

6.2.1 Limiting process and discretization of the geometry

On each boundary element, the displacement components u_j are represented by shape functions $N^c(\xi)$ of local intrinsic coordinate $\xi \in [-1,\ 1]$. Usually, $u_j = u_j[\mathbf{x}(\xi)] = \sum_c N^c(\xi)\, u_j^c$ results in a polynomial representation (u_j^c may or may not be nodal values if, e.g. hierarchical elements are employed [6]). Notice that it is $u_j(\mathbf{x})$ as a function of $\mathbf{x}$ that must be $C^{1,\alpha}$ at $\mathbf{y}$, not $u_j[\mathbf{x}(\xi)]$ as a function of ξ.

Let us consider in detail the case of $\mathbf{y}$ belonging to two boundary elements,

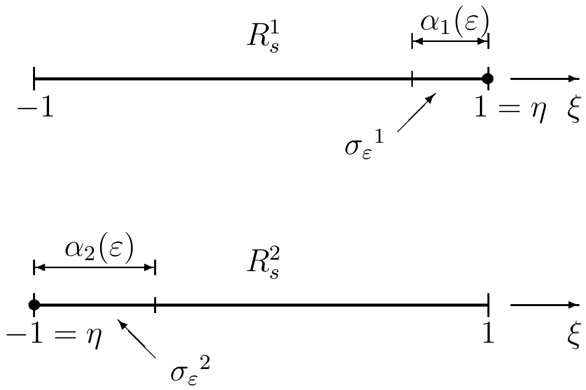


Figure 6. "Translation" of the limiting process.

say Γ_s^1, Γ_s^2 (Figure 5(a)). By means of the usual representation for the geometry in terms of shape functions and nodal coordinates (43) each boundary element is mapped onto the interval $\xi \in [-1,\ 1]$. We denote by R_s^m the interval on the ξ-axis associated to the boundary element Γ_s^m. The elementary arc length is given by $\mathrm{d}\Gamma = J_m(\xi)\,\mathrm{d}\xi$.

Accordingly, the two parts $e_\varepsilon{}^1$ and $e_\varepsilon{}^2$ of the symmetric neighbourhood e_ε are mapped, respectively, onto two portions $\sigma_\varepsilon{}^1$ and $\sigma_\varepsilon{}^2$ of the corresponding intervals R_s^m. It is important to note that, in general, $\sigma_\varepsilon{}^1$ and $\sigma_\varepsilon{}^2$ have different lengths. We denote the length of each $\sigma_\varepsilon{}^m$ by $\alpha_m(\varepsilon)$, $m\ = 1, 2$ (Figure 6).

We are now ready to state the problem of the evaluation of integrals with hypersingular kernel in terms of intrinsic coordinate, which is what we actually need in a BEM framework. We can provide the following equivalent expressions

$$
\begin{aligned}
I &= \lim_{\varepsilon\to 0}\left\{\sum_{m=1}^{2}\int_{\Gamma_s^m - e_\varepsilon{}^m} V_{ikj}(\mathbf{y},\mathbf{x})N^a(\xi(\mathbf{x}))\,\mathrm{d}\Gamma + N^a(\eta)\frac{b_{ikj}(\mathbf{y})}{\varepsilon}\right\} \\
&= \lim_{\varepsilon\to 0}\left\{\sum_{m=1}^{2}\int_{R_s^m - \sigma_\varepsilon{}^m} V_{ikj}(\mathbf{y},\mathbf{x}(\xi))N^a(\xi)J_m(\xi)\,\mathrm{d}\xi + N^a(\eta)\frac{b_{ikj}(\mathbf{y})}{\varepsilon}\right\} \\
&= \lim_{\varepsilon\to 0}\left\{\int_{-1}^{1-\alpha_1(\varepsilon)} F_{ikj}^1(1,\xi)\,\mathrm{d}\xi + \int_{-1+\alpha_2(\varepsilon)}^{1} F_{ikj}^2(-1,\xi)\,\mathrm{d}\xi + N^a(\eta)\frac{b_{ikj}(\mathbf{y})}{\varepsilon}\right\}, \qquad (55)
\end{aligned}
$$

where $F_{ikj}^m(\eta,\xi) = V_{ikj}(\mathbf{x}(\eta),\mathbf{x}(\xi))\,N^a(\xi)\,J_m(\xi)$ are the singular integrands, and η is the image of $\mathbf{y}$ on each interval R_s^m ($\eta = 1$ for $m = 1$, and $\eta = -1$ for $m = 2$) (Figure 6).

Each singular integrand can be expanded in power series in the following form (Laurent series)

$$F_{ikj}^{m}(\eta,\xi) = \frac{F_{-2}^{m}(\eta)}{(\xi-\eta)^2} + \frac{F_{-1}^{m}(\eta)}{\xi-\eta} + O(1). \tag{56}$$

Notice that F_{-1}^{m} and F_{-2}^{m} are local terms that only depend on the first and second derivatives of the shape functions $M^c(\xi)$ and $N^c(\xi)$ evaluated at η [10, 9]. Details on a systematic way for obtaining this kind of expansions are provided in Appendix B.

Since the pole $\mathbf{y}$ is shared by two elements, we also need the Taylor expansion of $\rho = |\xi - \eta| = \alpha_m(\varepsilon)$ in powers of ε, as explained in Appendix B

$$\rho = \alpha_m(\varepsilon) = \varepsilon\,\beta_m(\eta) + \varepsilon^2\gamma_m(\eta)\,\mathrm{sgn}(\delta) + O(\varepsilon^3), \tag{57}$$

where $\delta = \xi - \eta$ and $m = 1, 2$.

6.2.2 Semi-analytical treatment

We are now ready for the actual evaluation of I as given by the last limit in eqn (55). First, let us add and subtract in each integral the first two terms of the series expansion (56) (for brevity, the indices i, k, j have been dropped)

$$\begin{aligned} I = \lim_{\varepsilon\to 0} \Bigg\{ &\sum_{m=1}^{2} \int_{R_s^m - \sigma_\varepsilon{}^m} \left(F^m(\eta,\xi) - \left[\frac{F_{-2}^{m}(\eta)}{(\xi-\eta)^2} + \frac{F_{-1}^{m}(\eta)}{\xi-\eta} \right] \right) \mathrm{d}\xi \\ &+ \int_{R_s^m - \sigma_\varepsilon{}^m} \frac{F_{-1}^{m}(\eta)}{\xi-\eta}\,\mathrm{d}\xi + \left(\int_{R_s^m - \sigma_\varepsilon{}^m} \frac{F_{-2}^{m}(\eta)}{(\xi-\eta)^2}\,\mathrm{d}\xi + N^a(\eta)\frac{b_{ikj}(\mathbf{y})}{\varepsilon} \right) \Bigg\} \\ = {} & I_0 + I_{-1} + I_{-2}. \end{aligned} \tag{58}$$

Let us consider each term separately. From eqn (56) we see that in I_0 the integrand is regular. Therefore the limit is straightforward

$$\begin{aligned} I_0 = \int_{-1}^{1} \Bigg\{ &\left(F^1(1,\xi) - \left[\frac{F_{-2}^{1}}{(\xi-1)^2} + \frac{F_{-1}^{1}}{\xi-1} \right] \right) \\ &+ \left(F^2(-1,\xi) - \left[\frac{F_{-2}^{2}}{(\xi+1)^2} + \frac{F_{-1}^{2}}{\xi+1} \right] \right) \Bigg\}\,\mathrm{d}\xi. \end{aligned} \tag{59}$$

This is regular integral that can be numerically evaluated by standard formulae.

I_{-1} and I_{-2} have very simple integrands. Therefore, they are always integrable in *closed form.* Moreover, thanks to the expansions (57) for $\alpha_m(\varepsilon)$, the limiting process can be performed *exactly.*

For I_{-1} we have

$$
\begin{aligned}
I_{-1} &= \lim_{\varepsilon\to 0}\left\{\int_{-1}^{1-\alpha_1(\varepsilon)} \frac{F_{-1}^1}{\xi-1}\,\mathrm{d}\xi + \int_{-1+\alpha_2(\varepsilon)}^{1} \frac{F_{-1}^2}{\xi+1}\,\mathrm{d}\xi\right\} \\
&= \lim_{\varepsilon\to 0}\left\{F_{-1}^1 \ln\left|\frac{-\alpha_1(\varepsilon)}{-2}\right| + F_{-1}^2 \ln\left|\frac{2}{\alpha_2(\varepsilon)}\right|\right\} \\
&= F_{-1}\lim_{\varepsilon\to 0}\left\{\ln\left|\frac{\varepsilon\beta_1}{2}\right| + \ln\left|\frac{2}{\varepsilon\beta_2}\right|\right\} \\
&= F_{-1}\lim_{\varepsilon\to 0}\{\ln|\varepsilon| + \ln|\beta_1| - \ln|2| + \ln|2| - \ln|\varepsilon| - \ln|\beta_2|\} \\
&= F_{-1}(\eta)\ln\left|\frac{\beta_1(\eta)}{\beta_2(\eta)}\right|,
\end{aligned}
\tag{60}
$$

where $F_{-1} = F_{-1}^1 = F_{-1}^2$. It is worth noting that $\beta_m(\eta) = 1/J_m(\eta)$.

The treatment for I_{-2} is similar. We first integrate analytically, and then perform the limiting process using the expansions for $\alpha_m(\varepsilon)$

$$
\begin{aligned}
I_{-2} &= \lim_{\varepsilon\to 0}\left\{\int_{-1}^{1-\alpha_1(\varepsilon)} \frac{F_{-2}^1}{(\xi-1)^2}\,\mathrm{d}\xi + \int_{-1+\alpha_2(\varepsilon)}^{1} \frac{F_{-2}^2}{(\xi+1)^2}\,\mathrm{d}\xi + N^a(\eta)\frac{b_{ikj}(\mathbf{y})}{\varepsilon}\right\} \\
&= \lim_{\varepsilon\to 0}\left\{F_{-2}^1\left(\frac{1}{\alpha_1(\varepsilon)} - \frac{1}{2}\right) + F_{-2}^2\left(\frac{1}{\alpha_2(\varepsilon)} - \frac{1}{2}\right) + N^a(\eta)\frac{b_{ikj}(\mathbf{y})}{\varepsilon}\right\} \\
&= \lim_{\varepsilon\to 0}\left\{\sum_{m=1}^{2}\left[\frac{F_{-2}^m}{\varepsilon\beta_m}\left(1 - \frac{\varepsilon\,\gamma_m\,\mathrm{sgn}(\delta)}{\beta_m}\right)\right] + N^a(\eta)\frac{b_{ikj}(\mathbf{y})}{\varepsilon}\right\} - \frac{1}{2}\sum_{m=1}^{2}F_{-2}^m \\
&= \lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon}\right)\left\{\sum_{m=1}^{2}\frac{F_{-2}^m}{\beta_m} + N^a(\eta)b_{ikj}(\mathbf{y})\right\} - \sum_{m=1}^{2}\left[F_{-2}^m\left(\frac{\gamma_m\,\mathrm{sgn}(\delta)}{\beta_m^2} + \frac{1}{2}\right)\right] \\
&= -\sum_{m=1}^{2}\left[F_{-2}^m(\eta)\left(\frac{\gamma_m(\eta)\,\mathrm{sgn}(\delta)}{\beta_m^2(\eta)} + \frac{1}{2}\right)\right].
\end{aligned}
\tag{61}
$$

Notice that the unbounded terms arising from the integration are cancelled by the term $N^a(\eta)b_{ikj}/\varepsilon$. This formula is equivalent to expression (51) for the 3D case. In particular $\mathrm{sgn}(\delta)\gamma_m$ corresponds to $\gamma(\theta)$, since $\gamma(\theta) = -\gamma(\theta+\pi)$.

In the above expressions, the "local terms" β_m and γ_m account for the *distortion* of the originally symmetric neighbourhood as introduced by the geometric discretization of the boundary (Figures 5 and 6).

6.2.3 Final formula

By collecting the results in eqns (59),(60) and (61), the following final formula for the actual computation of integrals (55) with hypersingular kernel can be given

$$
I = \sum_{m=1}^{2}\left\{\int_{-1}^{1}\left[F^m(\eta,\xi) - \left(\frac{F_{-2}^m(\eta)}{(\xi-\eta)^2} + \frac{F_{-1}^m(\eta)}{\xi-\eta}\right)\right]\,\mathrm{d}\xi\right.
$$

$$+ F_{-1}^m(\eta) \ln \left| \frac{2}{\beta_m(\eta)} \right| \operatorname{sgn}(\xi - \eta) - F_{-2}^m(\eta) \left[\operatorname{sgn}(\xi - \eta) \frac{\gamma_m(\eta)}{\beta_m^2(\eta)} + \frac{1}{2} \right] \Bigg\}, \quad (62)$$

where $\eta = 1$ for $m = 1$, and $\eta = -1$ for $m = 2$. The numbers 2 are the lengths of the ξ-intervals ($-1 \leq \xi \leq 1$).

Formula (62) holds for the case of $\mathbf{y}$ being shared by two boundary elements (Figure 5(a)), and it is the 2D counterpart of formula (53).

If the collocation point lies within only one element (Figure 5(b)) the corresponding formula is as follows

$$I = \int_{-1}^{1} \left[F(\eta, \xi) - \left(\frac{F_{-2}(\eta)}{(\xi - \eta)^2} + \frac{F_{-1}(\eta)}{\xi - \eta} \right) \right] \mathrm{d}\xi +$$
$$+ F_{-1}(\eta) \ln \left| \frac{1 - \eta}{-1 - \eta} \right| + F_{-2}(\eta) \left(-\frac{1}{1 - \eta} + \frac{1}{-1 - \eta} \right), \quad (63)$$

with $\eta \in (-1,\, 1)$. In this case the two subintervals have length equal to $1 - \eta$ and $\eta - (-1) = \eta + 1$, respectively. There are no local terms in eqn (63) since they are the same on both sides of η.

Formulae (62) and (63) have been obtained through analytical integration of all singular parts and analytical treatment of the limiting process. Therefore, they are exact restatements of the original problem (55). Formulae that are formally identical hold for any other elliptic problem.

Integrands with strongly singular kernels are dealt with in the same way. It just happens that $F_{-2} \equiv 0$. Examples are the integrals with W_{ikj} or T_{ij}, or integrals with V_{ikj} but with $N^a(\eta) = 0$.

It is important to realize that the integral in eqns (62) or (63) is *regular* and can be numerically evaluated by standard quadrature formulae (e.g. Gaussian rules). In Figures 7–8 some examples of the regular integrand in eqn (63) are plotted (solid line: ——). The corresponding singular function F_{ikj} (dashed line: — —), and the function given by the singular terms in the power series (56) (dotted line: - - -) are also plotted in each Figure. The regular function is obtained by subtracting the two-term power series (56) from the singular function in Figure 7, and only the one-term expansion in Figure 8 since the integrand is only strongly singular ($F_{-2} \equiv 0$).

Results in Figures 7 and 8 are for a *curved* quadratic element with the three nodes located at points (1, 0), (0.75, 0.75), and (0, 1), respectively. Poisson ratio was 0.3. In all cases the regular functions have a gentle behaviour. Therefore, the numerical integration requires only Gaussian formulae of low order.

7 Numerical examples

Whenever a method for the evaluation of strongly singular (CPV) or hypersingular integrals is developed, it should be tested on *distorted* meshes, that is on

that is on meshes that do not have any kind of symmetry around the singular point.

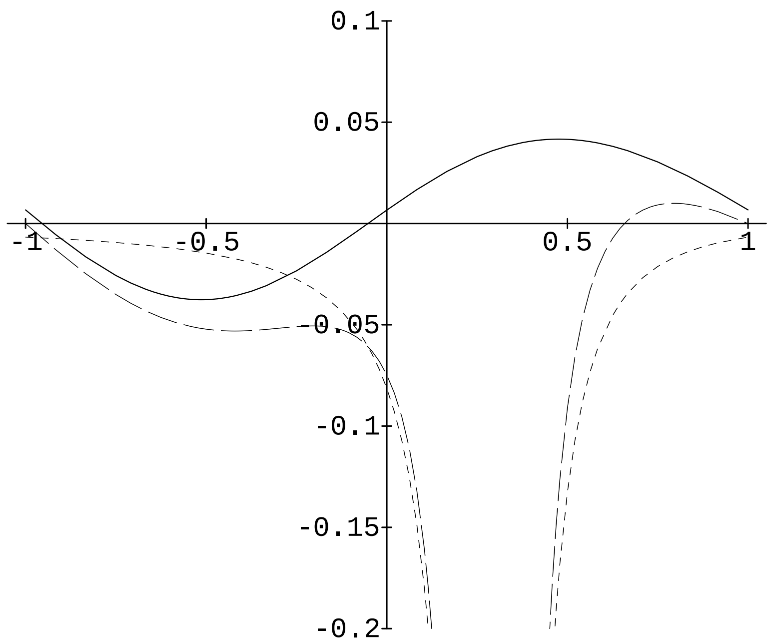


Figure 7. Hypersingular integrand $F_{112} = V_{112} N^2 J$ (— —), two-term power series (- - -) and regular function (——); singular point at $\eta = 0.3$.

meshes that do not have any kind of symmetry around the singular point. Moreover, the insensitivity of the results with respect to the mesh pattern around the pole $\mathbf{y}$ should always be checked.

Owing to the degree of distorsion of the meshes employed, the two tests presented are quite severe and could be considered as possible benchmarks for integration techniques of singular integrals in 3D problems.

7.1 Strongly singular integrals (CPV)

The first test was carried out on a plane region Γ_s in the shape of a $[-1, 1] \times [-1, 1]$ square in the (x_1, x_2) plane of a (x_1, x_2, x_3)-space. The singular point $\mathbf{y}$ was located at the point $\mathbf{y} = (y_1, y_2, y_3) = (0.6, 0, 0)$ (Figure 9).

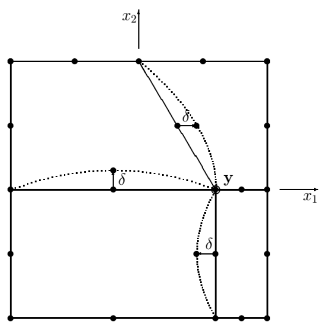
The following Cauchy principal value was considered

$$I = \fint_{\Gamma_s} \frac{r_{,1}}{r^2}\,\mathrm{d}\Gamma = \fint_{\Gamma_s} \frac{1}{r^2} \frac{x_1 - y_1}{r}\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{64}$$

This CPV can be evaluated analytically. The exact value is $I = -0.2114175 \times 10$. The integrand in eqn (64) has all the relevant features of any strongly singular kernel function arising in the BEM.

The region Γ_s was represented by four *distorted* elements as shown in Figure 9. In the initial mesh all elements had straight sides (solid lines in Figure 9).

Figure 9



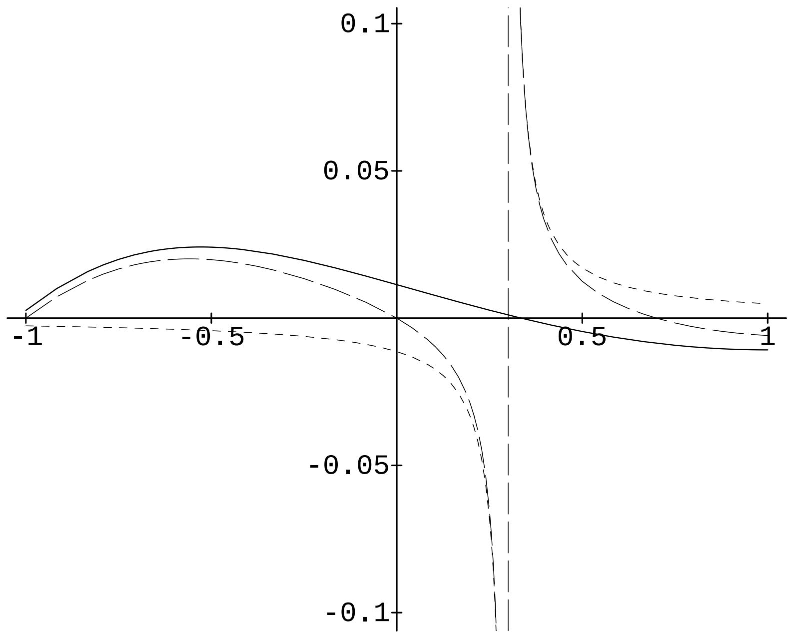


Figure 8. Strongly singular integrand $F_{222} = W_{222}N^3J$ (— —), one-term power series (- - -) and regular function (——); singular point at $\eta = 0.3$.

Table 1. Direct evaluation of CPV integrals on increasingly distorted meshes (Figure 9)

Order n	$\delta = 0.0$	$\delta = 0.05$	$\delta = 0.15$
4	-0.2126244×10	-0.2115070×10	-0.2129408×10
6	-0.2110645×10	-0.2113478×10	-0.2110577×10
8	-0.2114147×10	-0.2114184×10	-0.2114133×10
10	-0.2114189×10	-0.2114176×10	-0.2114190×10
	Exact value: -0.2114175×10		

Although more regular meshes were possible, we chose an irregular mesh for the sake of generality. Two additional calculations were performed by further deforming the mesh around $\mathbf{y}$ (dashed lines in Figure 9). The distance δ gives a measure of the degree of (additional) distorsion. Of course, the exact value of I is not affected by the way we represent Γ_s. Therefore, any robust numerical method should be basically insensitive to the mesh pattern around $\mathbf{y}$.

In this case, on each element it is $F^m(\rho, \theta) = [r_{,1}/r^2]\, J_m(\boldsymbol{\xi})\, \rho$, and therefore $F^m_{-1}(\theta) = [A_1(\theta)/A^3(\theta)] J_m(\boldsymbol{\eta})$.

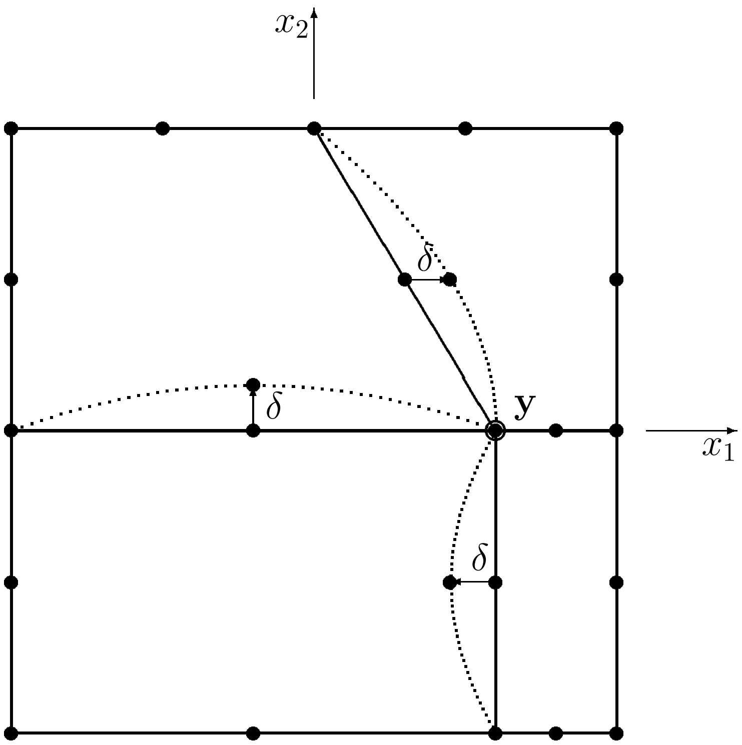


Figure 9. Distorted meshes for testing the integration method of strongly singular integrals. In this case, on each element it is $F^m(\rho,\theta) = [r_{,1}/r^2]\, J_m(\boldsymbol{\xi})\, \rho$, and therefore $F^m_{-1}(\theta) = [A_1(\theta)/A^3(\theta)] J_m(\boldsymbol{\eta})$.

The integral was evaluated by means of formula (54), where $m = 1, 2, 3$ or 4. All integrals were computed using Gaussian quadrature rules of order n (product rules of order $n \times n$ for the double integrals).

Table 1 shows the results for some values of n and δ. The accuracy and stability of the results are remarkable. The value of δ has almost no effect on the accuracy of the numerical values. The error is about 0.17% with $n = 6$ and as low as 0.001% with $n = 8$.

It is also worth noting that the integral $-\sum_m \int_{\theta_1^m}^{\theta_2^m} F^m_{-1}(\theta) \ln[\beta_m(\theta)]\, \mathrm{d}\theta$ in eqn (54) contributes more than all the other integrals to the final result, thus showing that disregarding this local term leads to incorrect results.

7.2 Hypersingular integrals

The second test was still carried out on a plane region Γ_s in the shape of a $[-1,\ 1]\times[-1,\ 1]$ square in the (x_1, x_2) plane of a (x_1, x_2, x_3)-space. The singular point $\mathbf{y}$ was located at the point $\mathbf{y} = (y_1, y_2, y_3) = (2/3, 0, 0)$, as shown in Figure 10.

Without loss of generality, the following hypersingular integral was considered

$$I = \lim_{\varepsilon \to 0} \int_{(\Gamma_s - e_\varepsilon) + s_\varepsilon} -\frac{1}{r^3} \left[3r_{,3} \frac{\partial r}{\partial n} - n_3 \right] \mathrm{d}\Gamma = \lim_{\varepsilon \to 0} \left\{ \int_{(\Gamma_s - e_\varepsilon)} \frac{1}{r^3} \mathrm{d}\Gamma - \frac{2\pi}{\varepsilon} \right\}, \quad (65)$$

where e_ε was chosen (just for convenience) to be a circle centred at $\mathbf{y}$ and of radius ε. The term subtracted in eqn (65) is clearly affected by the selected shape of e_ε and comes from the integral on s_ε (Figure 1).

This integral was chosen for the purpose of comparison since it can be integrated in closed form. Notice, however, that it shows the same relevant features of all hypersingular integrals in the BEM.

The region Γ_s was represented by four *distorted* elements as shown in Figure 10. In the initial mesh all elements had straight sides (solid lines in Figure 10). Although more regular meshes were possible, we chose an irregular mesh for the sake of generality. Two additional calculations were performed by further deforming the mesh around $\mathbf{y}$ (dashed lines in Figure 10). The distance δ gives a measure of the degree of (additional) distorsion. Of course, the exact value of

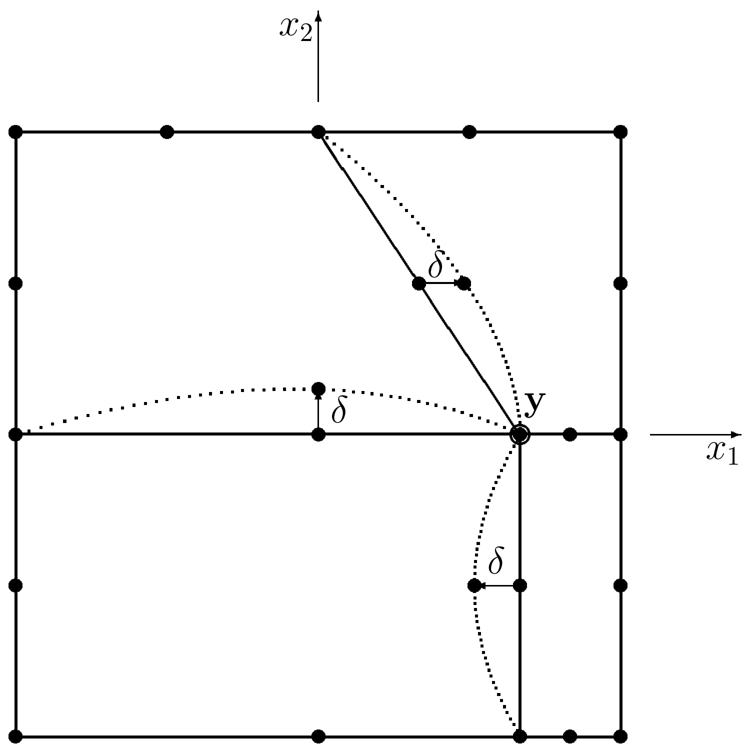


Figure 10. Distorted meshes for testing the integration method of hypersingular integrals.

Table 2. Direct evaluation of hypersingular integrals on increasingly distorted meshes (Figure 10)

Order n	$\delta = 0.0$	$\delta = 0.05$	$\delta = 0.1$
4	0.8770886×10	0.8590618×10	0.8523129×10
6	0.8688872×10	0.8567104×10	0.8564053×10
8	0.8529598×10	0.8546082×10	0.8546989×10
10	0.8547452×10	0.8547932×10	0.8547908×10
	Exact value: 0.85479208×10		

I is not affected by the way we represent Γ_s. Therefore, any robust numerical method should be basically insensitive to the mesh pattern around $\mathbf{y}$.

Formula (53) was employed for the computation, with $F^m = (1/r^3)\, J_m\, \rho$ and $m = 1, 2, 3, 4$. Standard Gauss formulae were used for both the double and the single integrals (an $n \times n$ product rule, and a formula of order n, respectively). For the actual application of Gaussian formulae, each element was subdivided into two triangles with a common vertex at $\boldsymbol{\eta}$, as usual.

Table 2 shows the results for some values of n and δ, along with the exact value. Notice that the value of δ has very little effect on the accuracy of the numerical values. The error is about 0.2% with $n = 8$ and as low as 0.005% with $n = 10$. With less distorted elements, a better accuracy is obtained. It is also worth noting that all terms in eqn (53) contribute significantly to the final results.

8 Related works

Since the proposed method for the evaluation of singular integrals is quite general and simple to use, it has been widely applied, even in commercial codes, like those developed by Numerical Integration Technologies.

The direct evaluation of CPV integrals by means of formula (54) can be found, for instance, in Almeida Pereira & Parreira [24] for infinite and semi-infinite domains, in Qiu *et al.* [25] for steady-state non-linear problems in electrochemistry, or in Schanz & Antes [26] for time-domain BEM.

Papers that deeply rely on the direct evaluation of hypersingular integrals (formulas (52) and (63)) are, e.g. Mi & Aliabadi [27] on fracture mechanics, Rêgo Silva *et al.* [28] on Stokes' flow, Huber *et al.* [29] on the evaluation of the stress tensor in 3D elastostatics, and Lie *et al.* [30] on fracture mechanics for fillet welds.

The proposed method for singular surface integrals has been also revisited by Kieser *et al.* [31] in a more mathematical framework. They found it to work efficiently and stably.

Some researchers have developed further the ideas presented here. Gallego & Domìnguez [33] extended the proposed form of HBIE (26) to time-domain transient elastodynamics, whereas Huber *et al.* [32] extended it to 3D elastoplasticity.

Quite interestingly, the final expressions for singular or hypersingular integrals (eqns (54), (53) and (62)) may provide a firm basis for further theoretical work, as in Bonnet [18], where they have been used to obtain some interesting results on material differentiation of BIE formulations.

$$\rho = \alpha(\varepsilon, \theta) = \varepsilon\,\beta(\theta) + \varepsilon^2\gamma(\theta) + O(\varepsilon^3). \tag{47}$$

The series expansions (70), (71), (73), and (74) are all we need to obtain $F_{-2}(\theta)$ and $F_{-1}(\theta)$ for *any* hypersingular integrand in the BEM. As a matter

$$F_{ikj}(\rho, \theta) = \frac{F_{-2}(\theta)}{\rho^2} + \frac{F_{-1}(\theta)}{\rho} + O(1). \tag{46}$$

APPENDICES

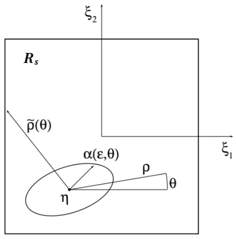


Figure 4. Polar coordinates in the parameter plane.

A Power series expansions for 3D problems

We need the expressions of $F_{-2}(\theta)$ and $F_{-1}(\theta)$ for the expansion of the kernel functions in eqn (46), and of $\beta(\theta)$ and $\gamma(\theta)$ for the expansion (47) of the vanishing neighbourhood. However, it is more convenient to obtain first some more basic expansions. Everything is actually based on the Taylor expansion of $x_i - y_i$ around the image $\boldsymbol{\eta} = (\eta_1, \eta_2)$ of the source point $\mathbf{y}$ (Figure 4)

$$x_i - y_i = \left[\left.\frac{\partial x_i}{\partial \xi_1}\right|_{\xi=\eta} (\xi_1 - \eta_1) + \left.\frac{\partial x_i}{\partial \xi_2}\right|_{\xi=\eta} (\xi_2 - \eta_2) \right] + \tag{66}$$

$$\left[\left.\frac{\partial^2 x_i}{\partial \xi_1^2}\right|_{\xi=\eta} \frac{(\xi_1 - \eta_1)^2}{2} + \left.\frac{\partial^2 x_i}{\partial \xi_1 \partial \xi_2}\right|_{\xi=\eta} (\xi_1 - \eta_1)(\xi_2 - \eta_2) + \left.\frac{\partial^2 x_i}{\partial \xi_2^2}\right|_{\xi=\eta} \frac{(\xi_2 - \eta_2)^2}{2} \right] + \cdots$$

where, as indicated, all derivatives are evaluated at $\boldsymbol{\eta}$.

If polar coordinates (ρ, θ), centred at $\boldsymbol{\eta}$ are introduced in the parameter plane, expansions (66) become

$$x_i - y_i = \rho \left[\left.\frac{\partial x_i}{\partial \xi_1}\right|_{\xi=\eta} \cos\theta + \left.\frac{\partial x_i}{\partial \xi_2}\right|_{\xi=\eta} \sin\theta \right] \quad A_i(\theta)$$

$$+ \rho^2 \left[\left.\frac{\partial^2 x_i}{\partial \xi_1^2}\right|_{\xi=\eta} \frac{\cos^2\theta}{2} + \left.\frac{\partial^2 x_i}{\partial \xi_1 \partial \xi_2}\right|_{\xi=\eta} \cos\theta\,\sin\theta + \left.\frac{\partial^2 x_i}{\partial \xi_2^2}\right|_{\xi=\eta} \frac{\sin^2\theta}{2} \right] + O(\rho^3),$$

$$B_i(\theta)$$

or, more concisely,

i = 1,2,3

$$x_i - y_i = \rho\, A_i(\theta) + \rho^2\, B_i(\theta) + O(\rho^3). \tag{67}$$

Notice that $A_i(\theta)$ and $B_i(\theta)$ are just *simple trigonometric functions* of θ.

Since the series expansion (46) for the hypersingular funtion $F_{ikj}(\rho, \theta)$ involves two terms, the first two terms in the above Taylor expansions have been considered. This is a general rule. The higher the order of singularity, the more terms must be retained. Therefore, if we are integrating a strongly singular (CPV) function, we only need the first term of each expansion, and the overall procedure is much simpler, as shown in [7, 12].

It is now convenient to define

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$$A(\theta) = \left\{ \sum_{k=1}^{3} [A_k(\theta)]^2 \right\}^{1/2} > 0,$$
$$C(\theta) = \sum_{k=1}^{3} A_k(\theta) B_k(\theta). \tag{68}$$

According to the previous results, the Taylor series expansions for the powers of $r = |\mathbf{x} - \mathbf{y}|$ are given by

$$r^n = \rho^n A^n \left(1 + n\,\rho \frac{C}{A^2}\right) + O(\rho^{n+2}), \qquad n = 1, 2, 3, \ldots \tag{69}$$

From eqn (68) and eqn (69) with $n = 1$, the expansions for the derivatives of r can be obtained and are

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$$r_{,i} = \frac{x_i - y_i}{r} = \frac{\rho A_i + \rho^2 B_i + O(\rho^3)}{\rho A\left(1 + \rho \frac{C}{A^2}\right) + O(\rho^3)}$$
$$= \frac{A_i}{A} + \rho\left(\frac{B_i}{A} - A_i \frac{C}{A^3}\right) + O(\rho^2). \tag{70}$$

公式70,71主要用在公式10,11中

Also important is the expansion for r^{-3}

$$\frac{1}{r^3} = \frac{1}{\rho^3 A^3} - \frac{3C}{\rho^2 A^5} + O\left(\frac{1}{\rho}\right). \tag{71}$$

It is worth noting that all the above expansions are valid for any kind of boundary elements. They are simple functions of ρ and θ.

$$\frac{1}{r^3} = \frac{1}{\rho^3 A^3} - \frac{3C}{\rho^2 A^5} + O\left(\frac{1}{\rho}\right). \tag{71}$$

The hypersingular integrand $F_{ikj}(\rho, \theta)$ is given by

$$F_{ikj} = V_{ikj}\, N^a\, J\, \rho = \frac{1}{r^3}\, Q_{ikj}\, N^a\, J\, \rho, \tag{72}$$

$$J_h = n_h J.$$

where the Jacobian is given by $J = \{\sum_{k=1}^{3} J_k^2\}^{1/2}$. An examination of the regular function Q_{ikj} reveals that every term in its expression is multiplied by a component n_h of the unit normal vector. Therefore, the product $Q_{ikj} J$ in eqn (72) can be expressed only in terms of $r_{,h}$ and J_h, since we have that $J_h = n_h J$.

见公式70 公式73

$$V_{ikj}(\mathbf{y}, \mathbf{x}) = -\frac{A}{r^\beta}\Big\{(1-2\nu)[\beta r_{,k}(n_i r_{,j} - n_j r_{,i}) - n_i \delta_{jk} + n_j \delta_{ik} - n_k \delta_{ij}]$$
$$- \beta n_k r_{,i} r_{,j} + \beta[(1-2\nu)\delta_{ij} r_{,k} + (\alpha+3) r_{,i} r_{,j} r_{,k} - \delta_{ik} r_{,j} - \delta_{jk} r_{,i}]\,\frac{\partial r}{\partial n}\Big\}, \tag{10}$$

and

$$W_{ikj}(\mathbf{y}, \mathbf{x}) = \frac{A}{2G\,r^\alpha}[\beta r_{,i} r_{,j} r_{,k} + (3-4\nu)\delta_{ij} r_{,k} - \delta_{ik} r_{,j} - \delta_{jk} r_{,i}], \tag{11}$$

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where $\alpha = 1, 2$ and $\beta = 2, 3$ in 2D and 3D problems, respectively. The con-

This means that the we need only the expansions of J_h (besides those already obtained) and not of both n_h and J.

$$J_h = J_h(\boldsymbol{\eta}) + \rho \left[\frac{\partial J_h}{\partial \xi_1}\bigg|_{\xi=\eta} \cos\theta + \frac{\partial J_h}{\partial \xi_2}\bigg|_{\xi=\eta} \sin\theta \right] + O(\rho^2) = J_{h0} + \rho\, J_{h1}(\theta) + O(\rho^2), \tag{73}$$

where all derivatives are evaluated at $\boldsymbol{\eta}$. The derivatives $\partial J_h / \partial \xi_j$ can be expressed in terms of the first and second derivatives of $x_i(\xi_1, \xi_2)$.

The last expansion we require is that of the shape function N^a

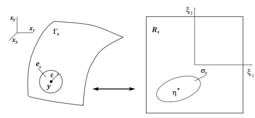
$$N^a = N^a(\boldsymbol{\eta}) + \rho \left[\frac{\partial N^a}{\partial \xi_1}\bigg|_{\xi=\eta} \cos\theta + \frac{\partial N^a}{\partial \xi_2}\bigg|_{\xi=\eta} \sin\theta \right] + O(\rho^2) = N_0^c + \rho\, N_1^c(\theta) + O(\rho^2). \tag{74}$$

Notice that J_{k0} and N_0^c are just constants, and not functions of θ.

The series expansions (70), (71), (73), and (74) are all we need to obtain $F_{-2}(\theta)$ and $F_{-1}(\theta)$ for *any* hypersingular integrand in the BEM. As a matter of fact, even in cases like time-harmonic elastodynamics or acoustics, where the kernels may seem very complicated, it just suffices to observe that their asymptotic behaviour is exactly represented by their static counterpart (see, e.g. Bonnet [20]). Therefore, the expansions are exactly the same. In Guiggiani *et al.* [10, App. C] the case of potential or acoustic problems is completely worked out.

We can now turn our attention to the expansion of $\alpha(\varepsilon, \theta)$ in eqn (47). From the above results, the derivation of $\beta(\theta)$ and $\gamma(\theta)$ is quite easy.

The contour of the neighbourhood e_ε of radius ε is given by (see Figure 3)



$$\varepsilon = r. \tag{75}$$

In polar coordinates in the parameter plane it becomes (see eqn (69) with $n = 1$)

$$r^n = \rho^n A^n \left(1 + n\,\rho \frac{C}{A^2}\right) + O(\rho^{n+2}), \qquad n = 1, 2, 3, \ldots \tag{69}$$

$$\varepsilon = \rho\, A(\theta) + \rho^2 \frac{C(\theta)}{A(\theta)} + O(\rho^3). \tag{76}$$

By using the reversion of the above series, we obtain the expansion in powers of ε of the equation in polar coordinates of the contour of σ_ε (the image of e_ε) (Figures 3 and 4)

$$\rho = \alpha(\varepsilon, \theta) = \frac{\varepsilon}{A(\theta)} - \varepsilon^2 \frac{C(\theta)}{A^4(\theta)} + O(\varepsilon^3) = \varepsilon\, \beta(\theta) + \varepsilon^2 \gamma(\theta) + O(\varepsilon^3), \tag{77}$$

that defines $\beta(\theta)$ and $\gamma(\theta)$. Notice that $\beta(\theta) = \beta(\theta + \pi)$, and $\gamma(\theta) = -\gamma(\theta + \pi)$.

B Power series expansions for 2D problems

We want to obtain the following expansions

$$F_{ikj}(\eta,\xi)=\frac{F_{-2}^{ikj}(\eta)}{(\xi-\eta)^2}+\frac{F_{-1}^{ikj}(\eta)}{\xi-\eta}+O(1), \tag{78}$$

where F_{-2}^{ikj} and F_{-1}^{ikj} are *constants* that only depend on the local properties of $F_{ikj}(\eta,\xi)\ = V_{ikj}\, N^a\, J$, at $\xi=\eta$.

First, some more basic expansions are needed. For brevity, we define $\delta=\xi-\eta$, and $\operatorname{sgn}(\delta)=|\delta|/\delta$.

Everything is actually based on the Taylor expansion of (x_i-y_i), at η

$$\begin{aligned}
x_i-y_i &= \left.\frac{\mathrm{d}x_i}{\mathrm{d}\xi}\right|_{\xi=\eta}(\xi-\eta)+\left.\frac{\mathrm{d}^2x_i}{\mathrm{d}\xi^2}\right|_{\xi=\eta}\frac{(\xi-\eta)^2}{2}+\cdots\\
&= A_i\,(\xi-\eta)+B_i\,(\xi-\eta)^2+\cdots\\
&= A_i\,\delta+B_i\,\delta^2+O(\delta^3),
\end{aligned} \tag{79}$$

where, as indicated, all derivatives are evaluated at η. Therefore, $A_i(\eta)$ and $B_i(\eta)$ are just constants. They can be easily obtained from the parametric equations. Note that for polynomial shape functions, these Taylor expansions always terminate (e.g. there are only two non-zero terms for quadratic shape functions).

It is now convenient to define

$$A=\left(\sum_{k=1}^{2}{A_k}^2\right)^{1/2}=J(\eta), \tag{80}$$

$$C=\sum_{k=1}^{2}A_kB_k. \tag{81}$$

Notice that $s_i=A_i/A$ are the components of the unit tangent vector to Γ at $\mathbf{y}$, with $A>0$.

From the definition of $r=|\mathbf{x}-\mathbf{y}|$ and accordingly to the previous results, the powers of r are given by

$$r^n=A^n|\delta|^n\left(1+n\frac{C}{A^2}\delta\right)+O(\delta^{(n+2)}). \tag{82}$$

Also useful are the expasions for the derivatives $r_{,i}=\partial r/\partial x_i$

$$\begin{aligned}
r_{,i}=\frac{x_i-y_i}{r} &= \frac{A_i\,\delta+B_i\,\delta^2+O(\delta^3)}{A|\delta|\left(1+\frac{C}{A^2}\delta\right)+O(\delta^3)}\\
&= \operatorname{sgn}(\delta)\frac{A_i}{A}+\left(\frac{B_i}{A}-\frac{A_iC}{A^3}\right)|\delta|+O(\delta^2),
\end{aligned} \tag{83}$$

and for the inverse powers of r

$$\frac{1}{r^2} = \frac{1}{A^2\delta^2} - \frac{2C}{A^4\delta} + O(1). \tag{84}$$

In general, the expansions for all inverse powers of r can be obtained directly from eqn (82) with negative n.

The Jacobian is given by $J(\xi) = \sqrt{J_1^2(\xi) + J_2^2(\xi)}$. It is useful to have the expansions of $J_i = n_i\, J = \mathrm{d}x_i/\mathrm{d}\xi_i$

$$\begin{aligned} J_1 &= A_2 + 2B_2\delta + O(\delta^2), \\ J_2 &= -A_1 - 2B_1\delta + O(\delta^2). \end{aligned} \tag{85}$$

Finally, we need the Taylor expansion for the generic shape function

$$N^a(\xi) = N^a(\eta) + \left.\frac{\mathrm{d}N^a}{\mathrm{d}\xi}\right|_{\xi=\eta} (\xi - \eta) + \cdots = N_0^c + N_1^c\delta + O(\delta^2). \tag{86}$$

The explicit expressions of F_{-2}^{ikj} and F_{-1}^{ikj} can be obtained by just by taking the expression of $F_{ikj}(\eta, \xi) \; = V_{ikj}\, N^a\, J$ and replacing each term by its expansion (remember that $n_i\, J = J_i$). For instance, for two-dimensional potential problems, whose hypersingular kernel functions V_i are given in eqn (6), the expansion (56) of $F_i(\eta, \xi) = V_i N^a J$ contains the following terms

$$U = -\frac{1}{2\pi}\ln r; \quad T = -\frac{1}{2\pi r}\frac{\partial r}{\partial n}; \quad W_k = \frac{1}{2\pi r} r_{,k}\,; \quad V_k = -\frac{1}{2\pi r^2}\left[2r_{,k}\frac{\partial r}{\partial n} - n_k\right], \tag{6}$$

$$F_{ikj}^m(\eta, \xi) = \frac{F_{-2}^m(\eta)}{(\xi - \eta)^2} + \frac{F_{-1}^m(\eta)}{\xi - \eta} + O(1). \tag{56}$$

$$\begin{aligned} F_{-2}(\eta) &= \frac{1}{2\pi}\frac{N_0^c J_{i0}}{A^2} = \frac{1}{2\pi}\frac{n_i(\mathbf{y}) N^a(\eta)}{J(\eta)}, \\ F_{-1}(\eta) &= \frac{1}{2\pi}\frac{N_1^c J_{i0}}{A^2} = \frac{1}{2\pi} n_i(\mathbf{y}) \left.\frac{\mathrm{d}N^a}{\mathrm{d}s}\right|_{x=y} = \frac{1}{2\pi}\frac{n_i(\mathbf{y})}{J(\eta)} \left.\frac{\mathrm{d}N^a}{\mathrm{d}\xi}\right|_{\xi=\eta}. \end{aligned}$$

In cases like time-harmonic elastodynamics, kernel functions may have very complicated expressions. However, it turns out that the singular behaviour of the dynamic kernels is exactly reproduced by the corresponding static kernels. Therefore, the expressions of F_{-2}^{ikj} and F_{-1}^{ikj} for the elastostatic case also work for time-harmonic elastodynamics.

It may be useful to observe that in the final expressions of F_{-2}^{ikj} and F_{-1}^{ikj} the absolute value of δ never appears. As a matter of fact, in the final expressions $|\delta|$ always comes in even powers ($|\delta|^{2n} = \delta^{2n}$).

When the pole $\mathbf{y}$ is shared by two elements (eqn (62)), we also need the expansion of $\rho = |\delta| = \alpha_m(\varepsilon)$ in powers of ε

$$\begin{aligned} \rho = \alpha_m(\varepsilon) &= \frac{\varepsilon}{A} - \varepsilon^2\, \frac{C}{A^4}\,\mathrm{sgn}(\delta) + O(\varepsilon^3) \\ &= \varepsilon\,\beta_m + \varepsilon^2\gamma_m\,\mathrm{sgn}(\delta) + O(\varepsilon^3), \end{aligned} \tag{87}$$

where $m = 1, 2$. The above equation shows that $\beta_m = 1/A$, and $\gamma_m = -C/A^4$. It is worth noting that they have the same formal expressions as for three-dimensional problems.

B.1 Alternative technique

An alternative technique for obtaining the coefficients F_{-2}^{ikj} and F_{-1}^{ikj} was suggested by M. Bonnet [23]. The main advantage is that it only employs Taylor expansions of regular functions.

Equation (79) can be written in vector notation in the following way

$$\mathbf{r} = \mathbf{x} - \mathbf{y} = \delta\,[\mathbf{A}(\eta) + \delta\,\mathbf{B}(\eta) + O(\delta^2)] \equiv \delta\,\hat{\mathbf{r}}(\delta;\eta), \tag{88}$$

which also define the function $\hat{\mathbf{r}}(\delta;\eta)$, of $\delta = (\xi - \eta)$ and η, with properties

$$\hat{\mathbf{r}}(0;\eta) = \mathbf{A}(\eta) \neq 0, \tag{89}$$

$$\left.\frac{\partial \hat{\mathbf{r}}}{\partial \delta}\right|_{\delta=0} = \mathbf{B}(\eta). \tag{90}$$

For instance, for the usual quadratic elements

$$\mathbf{A}(\eta) = \sum_{c=1}^{3} N_c'(\eta)\mathbf{x}^c = (\eta - \frac{1}{2})\mathbf{x}^1 - 2\eta\mathbf{x}^2 + (\eta + \frac{1}{2})\mathbf{x}^3,$$

$$\mathbf{B}(\eta) = \frac{1}{2}\sum_{c=1}^{3} N_c''(\eta)\mathbf{x}^c = \frac{1}{2}(\mathbf{x}^1 - 2\mathbf{x}^2 + \mathbf{x}^3).$$

The integrand in the hypersingular integral (2D elasticity) has the form

$$F_{ikj}(\eta,\xi) = \Sigma_{ij\ell,k}(\mathbf{r})J_\ell(\xi)u_j(\xi), \tag{91}$$

where

$$J_\ell(\xi) = n_\ell(\xi)J(\xi), \tag{92}$$

and $\Sigma_{ij\ell}$ is the $j\ell$-component of the Kelvin stress tensor associated to a unit point force applied at $\mathbf{y}$ along the i-direction. From the well-known properties of $\Sigma_{ij\ell}$ (symmetry and homogeneity) the following ones hold

$$\Sigma_{ij\ell,k}(-\mathbf{r}) = \Sigma_{ij\ell,k}(\mathbf{r}), \qquad \Sigma_{ij\ell,k}(a\mathbf{r}) = \frac{1}{a^2}\Sigma_{ij\ell,k}(\mathbf{r}). \tag{93}$$

Thus, using eqns (91), (88) and (93), one has

$$F_{ikj}(\eta,\xi) = \frac{1}{\delta^2}\Sigma_{ij\ell,k}(\hat{\mathbf{r}})J_\ell(\xi)u_j(\xi), \tag{94}$$

so that the two-term Laurent expansion (56) is obtained by means of a *Taylor* expansion of $\Sigma_{ij\ell,k}(\hat{\mathbf{r}})n_\ell(\xi)u_j(\xi)J(\xi)$. Note that the latter quantity is nonsingular at $\delta = 0$ since from eqn (89) one has $\hat{\mathbf{r}}(0;\eta) \neq 0$. The sought-for expansion then comes from

$$\begin{aligned}
&\Sigma_{ij\ell,k}(\hat{\mathbf{r}})n_\ell(\xi)u_j(\xi)J(\xi) \\
&\quad = \Sigma_{ij\ell,k}(\mathbf{A})n_\ell(\eta)u_j(\eta)J(\eta) + \delta\,\frac{\partial}{\partial\delta}\left(\Sigma_{ij\ell,k}(\hat{\mathbf{r}})n_\ell(\xi)u_j(\xi)J(\xi)\right)\Big|_{\delta=0} + O(\delta^2) \\
&\quad = F^{ikj}_{-2}(\eta) + \delta\,F^{ikj}_{-1}(\eta) + O(\delta^2),
\end{aligned}$$

or, in other words

$$F^{ikj}_{-2}(\eta) = \Sigma_{ij\ell,k}(\mathbf{A})J_\ell(\eta)u_j(\eta), \tag{95}$$

$$F^{ikj}_{-1}(\eta) = \frac{\partial}{\partial\delta}\left(\Sigma_{ij\ell,k}(\hat{\mathbf{r}})J_\ell(\xi)u_j(\xi)\right)\Big|_{\delta=0}. \tag{96}$$

We have at this point an explicit expression, eqn (95), for F^{ikj}_{-2} and need to develop eqn (96) for F^{ikj}_{-1}. This is done in the most straightforward way, as follows

$$\begin{aligned}
&\frac{\partial}{\partial\delta}\left(\Sigma_{ij\ell,k}(\hat{\mathbf{r}})J_\ell(\xi)u_j(\xi)\right)\Big|_{\delta=0} \\
&\quad = \Sigma_{ij\ell,km}(\mathbf{A})\frac{\partial \hat{r}_m}{\partial\delta}\Big|_{\delta=0} J_\ell(\eta)u_j(\eta) + \Sigma_{ij\ell,k}(\mathbf{A})\left[\frac{\partial J_\ell}{\partial\delta}u_j + J_\ell\frac{\partial u_j}{\partial\delta}\right]\Big|_{\delta=0}.
\end{aligned}$$

Now, from eqn (90), we have

$$\frac{\partial \hat{r}_m}{\partial\delta}\Big|_{\delta=0} = B_m(\eta),$$

while, using for $\mathbf{u}$ the same quadratic interpolation as for $\mathbf{x}$, one has

$$\frac{\partial u_j}{\partial\delta}\Big|_{\delta=0} = U_j(\eta),$$

where

$$\mathbf{U}(\eta) = (\eta - \frac{1}{2})\mathbf{u}^1 - 2\eta\mathbf{u}^2 + (\eta + \frac{1}{2})\mathbf{u}^3,$$

and, from eqns (85)

$$\frac{\partial J_\ell}{\partial\delta}\Big|_{\delta=0} = 2e_{\ell m}B_m(\eta),$$

(using the 2D permutation symbol $e_{12} = -e_{21} = 1$, $e_{11} = e_{22} = 0$). Combining everything, we get

$$F^{ikj}_{-1}(\eta) = \Sigma_{ij\ell,km}(\mathbf{A})B_m(\eta)J_\ell(\eta)u_j(\eta) + \Sigma_{ij\ell,k}(\mathbf{A})[2e_{\ell m}B_m(\eta)u_j(\eta) + J_\ell(\eta)U_j(\eta)]. \tag{97}$$

The calculation of the Laurent expansion is now complete. The formal expressions of the first and second derivatives of $\Sigma_{ij\ell}(\mathbf{r})$ with respect to Cartesian components of the position vector $\mathbf{r}$ need to be calculated from

$$\Sigma_{ij\ell}(\mathbf{r}) = -\frac{1}{4\pi(1-\nu)r}\left[2r_{,j}r_{,\ell}r_{,i} + (1-2\nu)(\delta_{ij}r_{,\ell} + \delta_{i\ell}r_{,j} - \delta_{j\ell}r_{,i})\right].$$

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