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被Hypersingular boundary integral equations have an additional free term[Guiggiani,1995]引用

A new formula for the *C*-matrix in the Somigliana identity

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Abstract. By making use of a convenient decomposition of the fundamental tractions, a new formula for the *C*-matrix in the Somigliana identity for a three- or two-dimensional elastic isotropic body is derived. This kind of formula is more advantageous for analytical and computational *C*-matrix evaluations than the currently well-known formula. A general closed analytical formula of the *C*-matrix for the case of any finite number of tangent planes to the boundary of the body at a non-smooth boundary point, presented in the final section of this paper, demonstrates the usefulness of the new formula.

AMS(MOS) subject classification: 73C02, 73V10, 35E05, 45F15.

1. Introduction

One of the interesting classical approaches to deal with boundary value problems of elasticity is to solve the corresponding boundary integral equation (BIE) that represents a boundary limiting form of the Somigliana identity. The boundary element method (BEM), as an effective computational method for the solution of the BIE, is responsible for a somewhat renewed interest in this classical approach (cf. [1], [3]).

Usually the boundary limiting form of the Somigliana identity, apart from other terms, includes, on the one hand an integral in the sense of Cauchy principal value (CPV) with the fundamental tractions as a strongly singular integral kernel, and on the other hand a free-term with the coefficient *C*-matrix depending on the local characteristic surface (or arc) of the boundary point with respect to the domain. The Somigliana identity for domains with piecewise smooth boundaries (forming edges and vertices) was proved by Hartmann [8]. The closed formula for the *C*-matrix in the case of plane elastostatics was given by Ricardella earlier in [10]. The components of the *C*-matrix in this formula are simple functions of radius vector components of the end-points of the corresponding characteristic arc. The formula for the *C*-matrix in the case of three-dimensional elastostatics derived in [7], [8] includes integrals of simple combinations of trigonometric functions over the

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corresponding characteristic surface. Therefore, it is much more cumbersome to evaluate the C-matrix for three- than for two-dimensional elastostatics. It is, however, a common practice in BEM implementations to evaluate the sum of the corresponding C-matrix and the CPV integrals in an indirect manner using the rigid-body motions (e.g. [3]).

Another approach based on subtraction of Somigliana's integral, with the fundamental tractions multiplied by the rigid-body displacements as integrand, from the Somigliana identity, leads to a regularized boundary limiting form of this identity free of CPV integrals and free of the C-matrix (e.g. [1] §2.3). The resulting BEM implementation, however, is in many respects similar to the more usual approach mentioned above.

There are two obvious reasons why programmers prefer BEM implementations circumventing direct evaluation of CPV integrals and the C-matrix. The first is the lack of general effective and accurate algorithms to compute CPV integrals on non-smooth curves and surfaces. Second, an implementation of the general procedure to evaluate the C-matrix in three-dimensional elastostatics using the formula presented in [7], [8], would be very involved. 复杂的

An accurate and simple algorithm for direct numerical evaluation of CPV integrals on curves in the BEM framework was first proposed in [5]. Recently a non-trivial extension of this algorithm in the case of CPV integrals on surfaces was presented in [6]; and somewhat later, independently by the present author (see [9]).

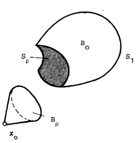
Therefore, it is desirable to find a new formula suitable for direct evaluation of the C-matrix in the case of three-dimensional elastostatics. The main purpose of this paper is to derive that kind of formula. For completeness' sake the same procedure is carried out for plane elastostatics too, but the resultant formula is only a simpler representation of the well-known formula.

In considering the same asymptotic behaviour of singular terms of the fundamental solutions in elastostatics and elastodynamics (e.g. [11], [1] §3.5), it is worth noting that these formulae are directly applicable to the time-harmonic elastodynamic problem, as well.

The next introductory section collects the necessary definitions and known results. For brevity's sake references [7] and [8] are heavily relied upon. The following section presents fundamental tractions of matrix decomposition into the sum of the normal derivative of the fundamental solution of the Laplace equation multiplied by the identity matrix and a term allowing the application of Stokes' theorem to transform corresponding integrals over characteristic surfaces (or arcs) to integrals over its boundary. The corresponding decomposition for the three-dimensional case has been introduced in [4]. The main results of this paper – sought C-matrix formulae – are derived in the fourth section. Finally, the analytical closed formula of the C-matrix for a non-smooth boundary point, with any number of tangent planes, is given. Until now, to the author's knowledge, only the closed analytical C-matrix formula for the case of at most two tangent planes has been published (e.g. [1], [6]).

2. The Somigliana identity and related concepts

Suppose we have a *regular region* $B \subset R^m$, $m = 3$ (or $m = 2$) as defined in [8], which is occupied by a homogeneous linear elastic and isotropic material. Let
x是个固定点 $\mathbf{x}$ be an arbitrary but hereinafter fixed point on the piecewise smooth boundary S of the body B. The surface (contour) S is oriented by the field of the unit outward normal vectors $\mathbf{n}(\mathbf{y})$, $\mathbf{y} \in S$. 单位球



Consider the unit sphere (circle) $\dot{K}_1(\mathbf{x})$ centred at $\mathbf{x} \in S$. The set of all half-tangents to boundary S at point $\mathbf{x}$ cuts out from $\dot{K}_1(\mathbf{x})$ a connected *characteristic surface* $\Omega(\mathbf{x})$ (*characteristic arc* $\omega(\mathbf{x})$) *of the boundary point* $\mathbf{x}$ *relative to* B, see [8]. The orientation of the characteristic surface (arc) is defined by the unit normal vectors $\mathbf{n}(\mathbf{y})$, $\mathbf{y} \in \Omega(\mathbf{x})$ ($\mathbf{y} \in \omega(\mathbf{x})$) pointing to the centre $\mathbf{x}$. As usual, in the two-dimensional case, we can express the unit normal vector $\mathbf{n}(\mathbf{y})$ to $\omega(\mathbf{x})$ by components of the unit tangent vector $\mathbf{t}(\mathbf{y})$ to $\omega(\mathbf{x})$ as

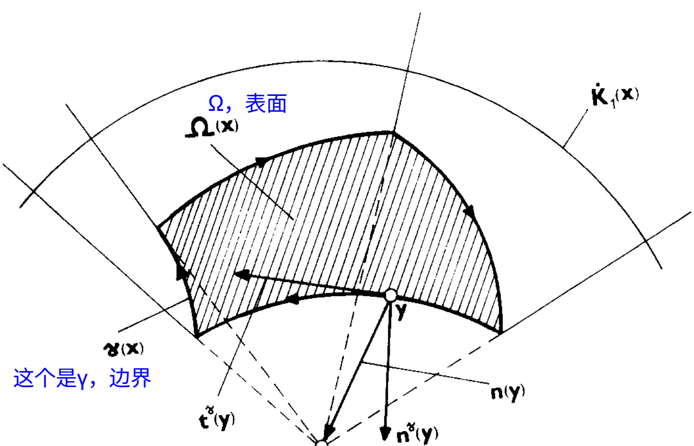
二维空间中，切线和法线互推公式

$$\mathbf{n}(\mathbf{y}) = (t_2(\mathbf{y}), -t_1(\mathbf{y})), \quad \mathbf{y} \in \omega(\mathbf{x}). \tag{2.1}$$

Ω是表面，γ是边界

We denote $\gamma(\mathbf{x})$ as the *boundary of the characteristic surface* (*arc*). 一致地

In the three-dimensional case, the closed contour $\gamma(\mathbf{x})$ is coherently oriented with $\Omega(\mathbf{x})$ by unit tangential vectors $\mathbf{t}^\gamma(\mathbf{y})$, $\mathbf{y} \in \gamma(\mathbf{x})$ (Fig. 1). The *unit outward normal vector* $\mathbf{n}^\gamma(\mathbf{y})$, *to the contour* $\gamma(\mathbf{x})$ *in the surface* $\Omega(\mathbf{x})$, is determined by the



$$\mathbf{n}^\gamma(\mathbf{y}) = \mathbf{t}^\gamma(\mathbf{y}) \times \mathbf{n}(\mathbf{y}) \quad \mathbf{y} \in \gamma(\mathbf{x}).$$

Fig. 1. The characteristic surface configuration at a non-smooth boundary point.

following vector product:

$$\mathbf{n}^{\gamma}(\mathbf{y}) = \mathbf{t}^{\gamma}(\mathbf{y}) \times \mathbf{n}(\mathbf{y}) \quad \mathbf{y} \in \gamma(\mathbf{x}). \tag{2.2}$$

The integral

$$\phi(\mathbf{x}) = \int_{\Omega(\mathbf{x})} 1 \, \mathrm{d}s_y \tag{2.3}$$

fig1中的阴影

立体角；多面角

represents the *internal solid angle* of the boundary point $\mathbf{x}$ relative to B.

The boundary $\gamma(\mathbf{x})$ for the two-dimensional case is formed by two end-points $\mathbf{v}_1$ and $\mathbf{v}_2$ of the open arc $\omega(\mathbf{x})$ (Fig. 2). We define *unit outward normal vectors to the boundary* $\gamma(\mathbf{x})$ *in the curve* $\omega(\mathbf{x})$ obviously as

$$\mathbf{n}_1^{\gamma} = -\mathbf{t}(\mathbf{v}_1), \quad \mathbf{n}_2^{\gamma} = \mathbf{t}(\mathbf{v}_2). \tag{2.4}$$

The absolute value of the angle $\phi(\mathbf{x})$ between vectors $\mathbf{r}_1 = \mathbf{r}(\mathbf{v}_1) = \mathbf{v}_1 - \mathbf{x}$ and $\mathbf{r}_2 = \mathbf{r}(\mathbf{v}_2) = \mathbf{v}_2 - \mathbf{x}$, measured through the region B, is the length of the arc $\omega(\mathbf{x})$.

We now state the boundary limiting form of the *Somigliana identity*, as it was proved in reference [8]. Let $\mathbf{u} \in C^2(B) \cup C^1(\bar{B})$ be a regular solution of Lame's differential equation system, with vector field $\mathbf{f}$ of body forces per unit volume in B. Application of the traction differential operator $\boldsymbol{\tau}$ on the displacement vector field $\mathbf{u}$ gives the corresponding traction vector field $\boldsymbol{\tau}(\mathbf{u})$

a *regular region* $B \subset R^m$, $m = 3$ (or $m = 2$)

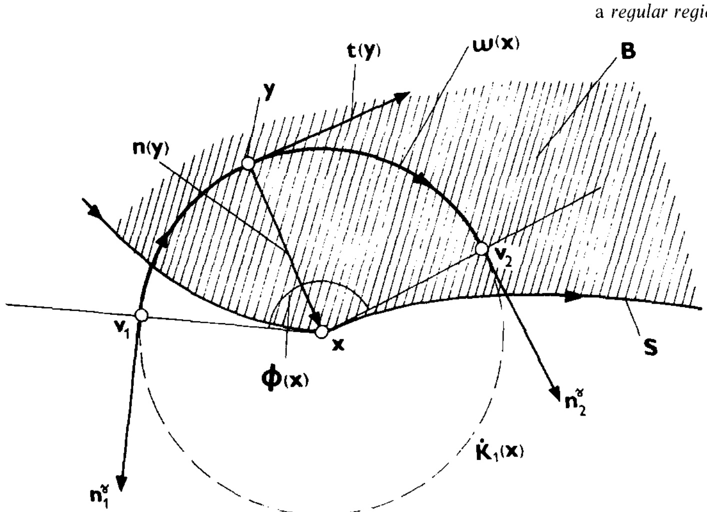


Fig. 2. The characteristic arc configuration at a non-smooth boundary point.

S. Then we may write

$$\mathbf{C}(\mathbf{x})\mathbf{u}(\mathbf{x}) = \int_S [\mathbf{U}(\mathbf{y}, \mathbf{x})\boldsymbol{\tau}(\mathbf{u})(\mathbf{y}) - \mathbf{T}(\mathbf{y}, \mathbf{x})\mathbf{u}(\mathbf{y})]\, \mathrm{d}s_y + \int_B \mathbf{U}(\mathbf{y}, \mathbf{x})\mathbf{f}(\mathbf{y})\, \mathrm{d}v_y, \tag{2.5}$$

where $\mathbf{U}(\mathbf{y}, \mathbf{x})$ is the symmetric matrix of the well-known Kelvin fundamental displacements (e.g. [1], [3]) and the transposed matrix $\mathbf{T}(\mathbf{y}, \mathbf{x})^T$ of the fundamental tractions is given by application of the operator $\boldsymbol{\tau}$ on columns of the matrix $\mathbf{U}(\mathbf{y}, \mathbf{x})$. To be precise, we explicitly describe the elements of the fundamental tractions matrix

$$T_{ij}(\mathbf{y}, \mathbf{x}) = -\frac{1}{4(m-1)\pi(1-\nu)r^{m-1}}\left[\{1-2\nu)\delta_{ij} + mr_{,i}r_{,j}\}\frac{\partial r}{\partial n} - (1-2\nu)(n_j r_{,i} - n_i r_{,j})\right], \tag{2.6}$$

where

$$i, j = 1, \ldots, m, \quad \mathbf{r}(\mathbf{y}) = \mathbf{y} - \mathbf{x},$$

$$r = |\mathbf{r}|, \quad r_{,i} = \frac{r_i}{r} = \frac{\partial r}{\partial y_i}, \quad n_i = n_i(\mathbf{y}), \quad \frac{\partial r}{\partial n} = r_{,k}n_k,$$

δ_{ij} is the Kronecker symbol and ν is Poisson's ratio. As immediately follows from the results presented in [8], the symmetric matrix $\mathbf{C}(\mathbf{x})$ can be defined by the integral

$$\mathbf{C}(\mathbf{x}) = \int_{\Omega(\mathbf{x})} \mathbf{T}(\mathbf{y}, \mathbf{x})\, \mathrm{d}s_y \text{ resp. } \mathbf{C}(\mathbf{x}) = \int_{\omega(\mathbf{x})} \mathbf{T}(\mathbf{y}, \mathbf{x})\, \mathrm{d}s_y. \tag{2.7}$$

$$\tilde{T}_{ij} = r^2 T_{ij}$$

$$c_{ij}(\mathbf{y}) = \int_{\Omega(\mathbf{y})} \tilde{T}_{ij}(\vartheta,\varphi)\sin\vartheta\, d\vartheta\, d\varphi \qquad (22)$$

3. The fundamental tractions decomposition

Let $f(\mathbf{y})$ be an arbitrary smooth function defined in a neighbourhood U of $\Omega(\mathbf{x})$ or $\omega(\mathbf{x})$ ($f \in C^1(U)$). We define an antisymmetric differential operator D_{ij} ($i, j = 1, \ldots, m$) for $\mathbf{y} \in \Omega(\mathbf{x})$ (or $\mathbf{y} \in \omega(\mathbf{x})$) as follows:

$$D_{ij}(f)(\mathbf{y}) = n_i(\mathbf{y})\frac{\partial f}{\partial y_j}(\mathbf{y}) - n_j(\mathbf{y})\frac{\partial f}{\partial y_i}(\mathbf{y}). \tag{3.1}$$

It is apparent from Stokes' theorem in three-dimensions that

We denote $\gamma(\mathbf{x})$ as the *boundary of the characteristic surface (arc)*.

$$\int_{\gamma(\mathbf{x})} \varepsilon_{ijk} f(\mathbf{y})\, \mathrm{d}y_k = \int_{\Omega(\mathbf{x})} n_l(\mathbf{y}) \varepsilon_{lhk} \varepsilon_{ijk} \frac{\partial f}{\partial y_h}(\mathbf{y})\, \mathrm{d}s_y$$

$$= \int_{\Omega(\mathbf{x})} D_{ij}(f)(\mathbf{y})\, \mathrm{d}s_y, \tag{3.2}$$

$$\mathbf{n}(\mathbf{y}) = (t_2(\mathbf{y}), -t_1(\mathbf{y})), \quad \mathbf{y} \in \omega(\mathbf{x}). \tag{2.1}$$

where ε_{ijk} is the Levi-Civita unit antisymmetric tensor. According to (2.1), the operator D_{ij} in two-dimensions can be expressed as

$$D_{ij}(f)(\mathbf{y}) = \varepsilon_{ij3} t_k(\mathbf{y}) \frac{\partial f}{\partial y_k}(\mathbf{y}) = \varepsilon_{ij3} \frac{\partial f}{\partial t}(\mathbf{y}), \quad \mathbf{y} \in \omega(\mathbf{x}). \tag{3.3}$$

Thus,

$$\int_{\omega(\mathbf{x})} D_{ij}(f)(\mathbf{y})\, \mathrm{d}s_y = \varepsilon_{ij3} [f]_{\mathbf{v}_1}^{\mathbf{v}_2} = \varepsilon_{ij3}(f(\mathbf{v}_2) - f(\mathbf{v}_1)). \tag{3.4}$$

From now on we shall denote the function $h_m(r)$ defined for $r > 0$ as follows:

$$h_m(r) = \begin{cases} \dfrac{1}{r}, & m = 3 \\ \ln\left(\dfrac{1}{r}\right), & m = 2. \end{cases} \tag{3.5}$$

By employing notation from (2.6), it is easy to verify that

$$\begin{aligned} D_{ij}(h_m(r)) &= \frac{1}{r^{m-1}} (n_j r_{,i} - n_i r_{,j}), \\ D_{ik}\left(\frac{r_k r_j}{r^m}\right) &= \frac{1}{r^{m-1}} (-\delta_{ij} + m r_{,i} r_{,j}) \frac{\partial r}{\partial n}. \end{aligned} \tag{3.6}$$

After some rearrangements in the right side of (2.6) using relations (3.6), we can arrive at the following *fundamental tractions decomposition*:

$$P_{kj}(\mathbf{y}, \mathbf{x}) = (1 - 2\nu)\delta_{kj} h_m(r) - \frac{r_k r_j}{r^m}.$$

$$T_{ij}(\mathbf{y}, \mathbf{x}) = \frac{1}{2(m-1)\pi} \delta_{ij} \frac{\partial h_m(r)}{\partial n} + \frac{1}{4(m-1)\pi(1-\nu)} D_{ik}(P_{kj})(\mathbf{y}, \mathbf{x}), \tag{3.7}$$

$$T_{ij}(\mathbf{y}, \mathbf{x}) = -\frac{1}{4(m-1)\pi(1-\nu) r^{m-1}} \left[\{1 - 2\nu)\delta_{ij} + m r_{,i} r_{,j}\} \frac{\partial r}{\partial n} - (1-2\nu)(n_j r_{,i} - n_i r_{,j}) \right], \tag{2.6}$$

where

$$P_{kj}(\mathbf{y}, \mathbf{x}) = (1 - 2\nu)\delta_{kj}h_m(r) - \frac{r_k r_j}{r^m}. \tag{3.8}$$

4. *C*-Matrix formulae

First, let us examine the three-dimensional case. Apparently the following integral over a closed contour $\gamma(\mathbf{x})$ vanishes:

$$\int_{\gamma(\mathbf{x})} \mathrm{d}y_l = 0. \tag{4.1}$$

Note that in the foregoing integral over $\gamma(\mathbf{x})$, we may replace $\mathrm{d}y_l$ by $\mathrm{d}r_l$. Further, for $\mathbf{y} \in \Omega(\mathbf{x})$

$$\frac{\partial(h_3(r))}{\partial n} = -r_i r^{-3} n_i = 1. \tag{4.2}$$

$$\mathbf{C}(\mathbf{x}) = \int_{\Omega(\mathbf{x})} \mathbf{T}(\mathbf{y}, \mathbf{x})\, \mathrm{d}s_y \text{ resp. } \mathbf{C}(\mathbf{x}) = \int_{\omega(\mathbf{x})} \mathbf{T}(\mathbf{y}, \mathbf{x})\, \mathrm{d}s_y. \tag{2.7}$$

Substituting (3.7) into (2.7), and considering relations (4.2), (2.3), (3.2) and (4.1), yields representations of the *C*-matrix components as follows:

$$\begin{aligned} C_{ij}(\mathbf{x}) &= \frac{\phi(\mathbf{x})}{4\pi}\delta_{ij} - \frac{1}{8\pi(1-\nu)}\int_{\gamma(\mathbf{x})} \varepsilon_{ikl} r_k r_j \,\mathrm{d}r_l \\ &= \frac{\phi(\mathbf{x})}{4\pi}\delta_{ij} - \frac{1}{8\pi(1-\nu)}\int_{\gamma(\mathbf{x})} r_i \varepsilon_{jkl} r_k \,\mathrm{d}r_l, \end{aligned} \tag{4.3}$$

where the symmetry of the *C*-matrix has been employed in the second equation. The tensor product of two vectors $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} \otimes \mathbf{b}$, represents a matrix with the components $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$. Then, the sought formula for the *C*-matrix, convenient for analytical and computational evaluations, can be written, in matrix notation, as

$$C(\mathbf{x}) = \frac{\phi(\mathbf{x})}{4\pi}\mathbf{I} - \frac{1}{8\pi(1-\nu)}\int_{\gamma(\mathbf{x})} \mathbf{r} \otimes (\mathbf{r} \times \mathrm{d}\mathbf{r}), \tag{4.4}$$

with $\mathbf{I}$ the unit matrix. Making use $\mathrm{d}\mathbf{r} = \mathbf{t}^{\gamma}(\mathbf{y})\,\mathrm{d}\gamma_y$ on $\gamma(\mathbf{x})$, where $\mathrm{d}\gamma_y$ denotes differential element of length, we get, in view of relations $\mathbf{n}(\mathbf{y}) = -\mathbf{r}(\mathbf{y})$ for

$\mathbf{y} \in \gamma(\mathbf{x})$ and (2.2), a most simple *formula of the C-matrix in three-dimensional case*

$$\phi(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i - (n-2)\pi. \tag{5.4}$$

$$\mathbf{C}(\mathbf{x}) = \frac{\phi(\mathbf{x})}{4\pi}\,\mathbf{I} - \frac{1}{8\pi(1-\nu)} \int_{\gamma(\mathbf{x})} \mathbf{r} \otimes \mathbf{n}^{\gamma}\, \mathrm{d}\gamma_y. \tag{4.5}$$

Notice that the vectors $\mathbf{r}$ and $\mathbf{n}^{\gamma}$ are orthogonal. Consequently, the trace of their tensor product vanishes $\mathrm{tr}(\mathbf{r} \otimes \mathbf{n}^{\gamma}) = 0$. Hence,

$$\mathrm{tr}\,\mathbf{C}(\mathbf{x}) = 3\,\frac{\phi(\mathbf{x})}{4\pi}. \tag{4.6}$$

Likewise we shall proceed in two-dimensions. It is clear that for $\mathbf{y} \in \omega(\mathbf{x})$ $h_2(r) = 0$ and

$$\frac{\partial(h_2(r))}{\partial n} = -r_i r^{-2} n_i = 1. \tag{4.7}$$

Taking into account (3.4), and symmetry of the C-matrix, we get by substitution of (3.7) into (2.7) the following representations:

$$\begin{aligned} C_{ij}(\mathbf{x}) &= \frac{\phi(\mathbf{x})}{2\pi}\,\delta_{ij} - \frac{1}{4\pi(1-\nu)}\,[\varepsilon_{ik3} r_k r_j]_{\mathbf{v}_1}^{\mathbf{v}_2} \\ &= \frac{\phi(\mathbf{x})}{2\pi}\,\delta_{ij} - \frac{1}{4\pi(1-\nu)}\,[r_i \varepsilon_{jk3} r_k]_{\mathbf{v}_1}^{\mathbf{v}_2}. \end{aligned} \tag{4.8}$$

From relation (2.1) and from the fact that $\mathbf{n}(\mathbf{y}) = -\mathbf{r}(\mathbf{y})$ for $\mathbf{y} \in \omega(\mathbf{x})$, we immediately obtain

$$t_j = \varepsilon_{jk3} r_k. \tag{4.9}$$

Then the second relation of (4.8) in matrix notation takes the form

$$\mathbf{C}(\mathbf{x}) = \frac{\phi(\mathbf{x})}{2\pi}\,\mathbf{I} - \frac{1}{4\pi(1-\nu)}\,[\mathbf{r} \otimes \mathbf{t}]_{\mathbf{v}_1}^{\mathbf{v}_2}. \tag{4.10}$$

Using (2.4) we arrive at the resultant *formula of the C-matrix in the two-dimensional case* analogous to (4.5)

$$\mathbf{C}(\mathbf{x}) = \frac{\phi(x)}{2\pi}\,\mathbf{I} - \frac{1}{4\pi(1-\nu)} \sum_{i=1}^{2} \mathbf{r}_i \otimes \mathbf{n}_i^{\gamma}. \tag{4.11}$$

5. Example

We shall consider the case of n ($n \geqslant 2$) boundary tangent planes at $\mathbf{x} \in S$. The characteristic surface $\Omega(\mathbf{x})$ takes the shape of a spherical polygon, with an ordered sequence of vertices $\mathbf{v}_i$ ($i = 1, \ldots, n$) (see Fig. 3). In addition, the following vertices are defined $\mathbf{v}_0 = \mathbf{v}_n$ and $\mathbf{v}_{n+1} = \mathbf{v}_1$. An edge of the spherical polygon $\Omega(\mathbf{x})$ between vertices $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$, is formed by an arc $\gamma_{i,i+1}$ on the great circle; which is the intersection of the unit sphere $\dot{K}_1(\mathbf{x})$ and a plane $\pi_{i,i+1}$ tangent to the boundary S at the point $\mathbf{x} \in S$. Thus

$$\gamma(\mathbf{x}) = \bigcup_{i=1}^{n} \gamma_{i,i+1}. \tag{5.1}$$

Making the transition from $\mathbf{v}_i$ to $\mathbf{v}_{i+1}$ along the edge $\gamma_{i,i+1}$, we go in the sense of the orientation of the contour $\gamma(\mathbf{x})$. We indicate $\mathbf{n}_{i,i+1}$ as the outward normal to the boundary S at the point $\mathbf{x}$ defining the plane $\pi_{i,i+1}$, thus $\mathbf{n}_{i,i+1} \perp \pi_{i,i+1}$. It is worthwhile to note that for $\mathbf{y} \in \gamma_{i,i+1}$ is

$$\mathbf{n}^{\gamma}(y) = \mathbf{n}_{i,i+1}. \tag{5.2}$$

The angle α_i of the spherical polygon $\Omega(\mathbf{x})$ at the vertex $\mathbf{v}_i$ is equal to the corresponding angle included between two tangent planes $\pi_{i-1,i}$ and $\pi_{i,i+1}$ (see

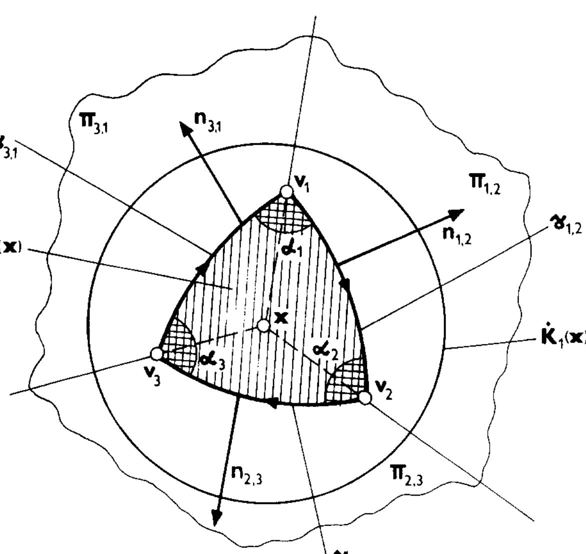


Fig. 3. Geometry of a three boundary tangent planes case.

e.g. [2]). From simple geometrical considerations, we can deduce its following representation

$$\alpha_i = \pi + \operatorname{sgn}((\mathbf{n}_{i-1,i} \times \mathbf{n}_{i,i+1}) \,.\, \mathbf{r}_i) \arccos(\mathbf{n}_{i-1,i} \cdot \mathbf{n}_{i,i+1}), \tag{5.3}$$

where $\mathbf{r}_i = \mathbf{r}(\mathbf{v}_i)$ and the signum function is defined as follows: $\operatorname{sgn}(x) = -1$ for $x < 0$, $\operatorname{sgn}(x) = 0$ for $x = 0$ and $\operatorname{sgn}(x) = 1$ for $x > 0$. Let us be reminded of one of the basic relations in spherical geometry (e.g. [2]):

$$\phi(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i - (n-2)\pi. \tag{5.4}$$

In consequence of (5.1) and (5.2) we have

$$\int_{\gamma(\mathbf{x})} \mathbf{r} \otimes \mathbf{n}^{\gamma} \, d\gamma_y = \sum_{i=1}^{n} \left(\int_{\gamma_{i,i+1}} \mathbf{r} \, d\gamma_y \right) \otimes \mathbf{n}_{i,i+1}. \tag{5.5}$$

To simplify the calculation of an integral in the right side of (5.5) we can consider the following coordinate system with the origin $\mathbf{x}$. It is oriented in such a way that one axis is orthogonal to the plane $\pi_{i,i+1}$ and another being the axis of symmetry of the edge $\gamma_{i,i+1}$. Then we can derive

$$\int_{\gamma_{i,i+1}} \mathbf{r} \, d\gamma_y = (\mathbf{r}_{i+1} - \mathbf{r}_i) \times \mathbf{n}_{i,i+1}. \tag{5.6}$$

Taking into account (4.5), (5.3), (5.4), (5.5) and (5.6), we directly get *a general closed analytical formula for the C-matrix in the case of n boundary tangent planes*

$$\mathbf{C}(\mathbf{x}) = \frac{1}{4\pi} \left(2\pi + \sum_{i=1}^{n} \operatorname{sgn}((\mathbf{n}_{i-1,i} \times \mathbf{n}_{i,i+1}) \,.\, \mathbf{r}_i) \arccos(\mathbf{n}_{i-1,i} \,.\, \mathbf{n}_{i,i+1}) \right) \mathbf{I} \\ - \frac{1}{8\pi(1-\nu)} \sum_{i=1}^{n} ((\mathbf{r}_{i+1} - \mathbf{r}_i) \times \mathbf{n}_{i,i+1}) \otimes \mathbf{n}_{i,i+1}. \tag{5.7}$$

In special cases of two or three boundary tangent planes, this formula can be rewritten and simplified by the reader (cf. [1], [6]).

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