Short Range:

Calculate with CST directly to get wake potential.

Long Range:

Method1: Calculate with CST or ACE3P for a very long time

Method2: Calculate the eigen-frequencies to get R, Q, ω for each mode, then reconstruct the wake potential by inverse Fourier Transformation of the Z(ω).

To get that, we need the wake function (wake potential of delta bunch) of a resonator, which can be found by the Fourier Transformation of the impedance of the resonator:

$$Z_{||} =$$

About Transverse:

We have two ways of calculate the transverse impedance.

- First is directly from the definition:

$$\frac{Z_\perp}{Q} = \frac{{V_\perp}^2}{\omega U}$$

  And $V_\perp$ can be calculated from the integration of the transverse force across the cavity directly.

- The second way is to calculate the gradient of longitudinal potential. The relation between the $V_\perp$ and $V_\parallel$ can be derived be Panofsky-Wenzel Theorem.
  Starting from the change of momentum:

$$\Delta\vec{p} = e\int \left(\vec{E} + \vec{v} \times \vec{B}\right)dt$$

  If we take partial derivative against time and assume $\vec{v} = \frac{d\vec{s}}{dt}$ is constant,

$$\frac{\partial\Delta\vec{p}}{\partial t} = e\int \left(\frac{\partial\vec{\mathrm{E}}}{\partial t} + \frac{d\vec{s}}{dt} \times \frac{\partial\vec{B}}{\partial t}\right)dt$$

  We know that $\frac{\partial\vec{B}}{\partial t} = -\nabla \times \vec{E}$,

$$\frac{\partial\Delta\vec{p}}{\partial t} = e\int \left(\frac{\partial\vec{\mathrm{E}}}{\partial t}dt - d\vec{s} \times \vec{\nabla} \times \vec{E}\right)$$

$$d\vec{s} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}\left(d\vec{s} \cdot \vec{E}\right) - \left(d\vec{s} \cdot \vec{\nabla}\right)\vec{E} = \vec{\nabla}\left(d\vec{s} \cdot \vec{E}\right) - d\vec{s}\frac{\partial\vec{E}}{\partial s}$$

  Substitute in,

$$\begin{aligned}\frac{\partial\Delta\vec{p}}{\partial t} &= e\int \left(\frac{\partial\vec{\mathrm{E}}}{\partial t}dt - \vec{\nabla}\left(d\vec{s} \cdot \vec{E}\right) + \left(d\vec{s} \cdot \vec{\nabla}\right)\vec{E}\right) \\ &= -e\int \vec{\nabla}\left(d\vec{s} \cdot \vec{E}\right) - d\vec{E}_\perp \\ &= -\nabla_\perp \Delta E_k + e\left(\vec{E}_\perp(\infty) - \vec{E}_\perp(-\infty)\right)\end{aligned}$$

Second term is zero, and

$$\Delta E_k = c\Delta p_{\parallel}.$$

Therefore,

$$\frac{\partial \Delta \overrightarrow{p_{\perp}}}{\partial t} = -c\ \nabla_{\perp} \Delta p_{\parallel}$$

Assume sinusoidal field, $\frac{\partial}{\partial t} \rightarrow -j\omega$

$$-j\omega \Delta \overrightarrow{p_{\perp}} = -c\nabla_{\perp} \Delta p_{\parallel}$$

$$\Delta \overrightarrow{p_{\perp}} = \frac{1}{j}\frac{c}{\omega}\nabla_{\perp} \Delta p_{\parallel}$$

Or we can write the deflecting voltage in terms of the gradient of the longitudinal voltage,

$$V_{\perp} = \frac{1}{j}\frac{c}{\omega}\nabla_{\perp} V_{\parallel}$$

Hence, the transverse impedance can be written as,

$$\frac{Z_{\perp}}{Q} = \frac{\left(\frac{c}{\omega}\right)^2 (\nabla_{\perp} V_{\parallel})^2}{U\omega}$$