This is a note for myself trying to understand the conservation of phase space area in a geometric way.

For example we have a particle that can be fully described by two variables, q and p, where q is the coordinate and p is the corresponding momentum of that coordinate.

There exist equation of motion that governs the dynamic of the particle:

$$\dot{q} = f(q, p, t)$$
$$\dot{p} = g(q, p, t)$$

Therefore, if we zoom into a quadrilateral in the phase space that is surrounded by four points ABCD:

$$A = (q_A, p_A),$$
$$B = (q_B, p_B),$$
$$C = (q_C, p_C),$$
$$D = (q_D, p_D),$$

then after infinitesimal time interval dt, the new locations of these four points are:

$$A' = (q_A + f_A dt, p_A + g_A dt),$$
$$B' = (q_B + f_B dt, p_B + g_B dt),$$
$$C' = (q_C + f_C dt, p_C + g_C dt),$$
$$D' = (q_D + f_D dt, p_D + g_D dt).$$

The subindexes of f and g indicate that the values of the two functions are taken at corresponding points. We know that the area of the old quadrilateral is:

$$S = \vec{AB} \times \vec{AD}$$

And the are of the new, deformed quadrilateral is:

$$S' = \vec{A'B'} \times \vec{A'D'}$$

Now let's look at this new area and try to find the relation between S and S'.

First we need to prepare some expression for later use:

$$f_B - f_A = \frac{\partial f}{\partial q} q_{AB} + \frac{\partial f}{\partial p} p_{AB}$$

$$g_B - g_A = \frac{\partial g}{\partial q} q_{AB} + \frac{\partial g}{\partial p} p_{AB}$$

$$f_D - f_A = \frac{\partial f}{\partial q} q_{AD} + \frac{\partial f}{\partial p} p_{AD}$$

$$g_D - g_A = \frac{\partial g}{\partial q} q_{AD} + \frac{\partial q}{\partial p} p_{AD}$$

Where

$$q_{AB} = q_B - q_A,$$

$$p_{AB} = p_B - p_A,$$

$$q_{AD} = q_D - q_A.$$

$$p_{AD} = p_D - p_A,$$

are the small difference of coordinates between point B and point A, point D and point A. Then we can start working on the new area:

$$\begin{aligned}
\vec{A'B'} \times \vec{A'D'} &= [q_{AB} + (f_B - f_A)dt, p_{AB} + (g_B - g_A)dt] \times [q_{AD} + (f_D - f_A)dt, p_{AD} + (g_D - g_A)dt] \\
&= (q_{AB} + (f_B - f_A)dt) \cdot (p_{AD} + (g_D - g_A)dt) \\
&- (p_{AB} + (g_B - g_A)dt) \cdot (q_{AD} + (f_D - f_A)dt) \\
&= q_{AB} \cdot p_{AD} - p_{AB} \cdot q_{AD} \\
&+ q_{AB}(g_D - g_A)dt + p_{AD}(f_B - f_A)dt + (f_B - f_A)(g_D - g_A)dt^2 \\
&- (p_{AB}(f_D - f_A)dt + q_{AD}(g_B - g_A)dt + (g_B - g_A)(f_D - f_A)dt^2) \\
&= S \\
&+ q_{AB}\left(\frac{\partial g}{\partial q} q_{AD} + \frac{\partial g}{\partial p} p_{AD}\right) dt + p_{AD}\left(\frac{\partial f}{\partial q} q_{AB} + \frac{\partial f}{\partial p} p_{AB}\right) dt \\
&- p_{AB}\left(\frac{\partial f}{\partial q} q_{AD} + \frac{\partial f}{\partial p} p_{AD}\right) dt - q_{AD}\left(\frac{\partial g}{\partial q} q_{AB} + \frac{\partial g}{\partial p} p_{AB}\right) dt \\
&+ \left[\left(\frac{\partial g}{\partial q} q_{AD} + \frac{\partial g}{\partial p} p_{AD}\right)\left(\frac{\partial f}{\partial q} q_{AB} + \frac{\partial f}{\partial p} p_{AB}\right) - \left(\frac{\partial g}{\partial q} q_{AB} + \frac{\partial g}{\partial p} p_{AB}\right)\left(\frac{\partial f}{\partial q} q_{AD} + \frac{\partial f}{\partial p} p_{AD}\right)\right] \\
&\times dt^2 \\
&= S \\
&+ (q_{AB} \cdot p_{AD} - p_{AB} \cdot q_{AD})\left(\frac{\partial f}{\partial q} + \frac{\partial g}{\partial p}\right) \\
&+ \left[p_{AB} \cdot q_{AD}\left(\frac{\partial g}{\partial q}\frac{\partial f}{\partial p} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial q}\right) + p_{AD} \cdot q_{AB}\left(\frac{\partial g}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}\right)\right] dt^2 \\
&= S\left[1 + \left(\frac{\partial f}{\partial q} + \frac{\partial g}{\partial p}\right) dt + \left(\frac{\partial g}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}\right) dt^2\right]
\end{aligned}$$

We can see, for the area to be conserved, we need two conditions:

$$\frac{\partial f}{\partial q} + \frac{\partial g}{\partial p} = 0$$

and the Poison Bracket

$$[g, f]_{p,q} = 0$$

According to A. Chao's book, for a closed system with Hamiltonian H and

$$f = \frac{\partial H}{\partial p},$$

$$g = -\frac{\partial H}{\partial q},$$

We get the first condition automatically. For the second I'm still not sure...