Summer 2018 MSAN 502 Homework 2, due 7/19 11:59pm

1. Unrelated questions that will hopefully fit in one page.

   (a) [Related to Stats] We have a series of n measurements of the random variable X and Y as $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$. Let $\bar{x}$ and $\bar{y}$ be their respective sample mean. What does it mean for these two sets of measurements to be statistically uncorrelated? Express your answer using dot product.

   (b) Simplify $(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1}$

   (c) Prove that AA^T is symmetric for any matrix A.

2. Let $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}$. Compute the following or write undefined.

   (a) $2A^T + C$ (b) trace of D (c) BA (d) $(2D^T - D)A$ (e) $(4B)C + 2B$ (f) $(-AC)^T + 5D^T$

3. Find an upper triangular U (not diagonal) such that $U^2 = I$.

4. Find all values of a, b, c, d for which the matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ commute.

5. If $(5A^T)^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$, find A.

6. Related to cancellation in matrix multiplication.

   (a) Construct a counter example to disprove "$AB = AC$ implies $B = C$."

   (b) Show that if A is invertible and $AB = AC$, then $B = C$.

7. Related to block matrix.

   (a) What rows or columns or matrices do you multiply to find

      (i) the third column of AB

      (ii) the first row of AB.

   (b) Let A be partitioned into a 2 by 2 block such as $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$. What is A^T?

   (c) Let B be some $m \times n$ matrix, C be some $n \times p$ matrix, and $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$ which is an $n \times n$ diagonal matrix.

      (i) Use appropriate partitioning of B to explain the effect of $B\Lambda$ (on B).

      (ii) Use appropriate partitioning of C to explain the effect of ΛC.

8. $A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$

   (a) Compute the inverse of A using row operations.

   (b) Find $\det(A)$ using the work in (a) without doing any more computations.

   (c) Find $\det(A)$ by expanding along a row of your choosing.

   (d) Find $\det(A)$ by expanding along a column of your choosing.

   (e) What is $\det(A^{-1})$? Confirm your answer by computing it.

9. Let A be 4×4 with $\det(A) = -2$. Fill in the blank.
   (a) $\det(-A)$
   (b) $\det(A^{-1})$
   (c) $\det(2A^T)$
   (d) $\det(A^3)$

10. (a) Find the value of k for which the matrix $A = \begin{bmatrix} k & 2 \\ 2 & k \end{bmatrix}$ is invertible.
    (b) If we have a 4×4 matrix B such that column 1 + column 2 = column 3, find a nontrivial solution of $Bx = 0$.

11. Prove that any rank-1 matrix is equal to uv^T for some vector u and v.

12. Are they subspaces of $\mathbb{R}^3$? Justify your answer.
    (a) All vectors of the form $(a, 0, 0)$
    (b) All vectors of the form $(a, 1, 1)$
    (c) All vectors of the form (a, b, c), where $b = a + c$

13. All 2×2 invertible matrices is not a subspace of $M_{2,2}$ because it is not closed under addition. Find two invertible matrices whose addition is not invertible.

14. Which of the following are linear combinations of $u = (0, -2, 2), v = (1, 3, -1)$?
    (a) (2,2,2) (b) (0,4,5) (c) (0,0,0)

15. Describe geometrically (in words) all linear combinations of
    (a) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$

16. For problems 4, 8, 13 from HW1, describe the solution sets using span. Which ones are subspace?

17. Matrix as linear transformation.
    (a) Find a matrix A, if exists, such that $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \\ 0 \end{bmatrix}$
    (b) Find a matrix B, if exists, such that $B \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xy \\ x-y \\ 0 \end{bmatrix}$
    (c) Find a 2 by 2 matrix that rotates vectors by $\frac{\pi}{4}$ clockwise.
    (d) Find the three elementary matrices that corresponds to the 3 row operations that you performed in doing HW1-10. Also find their inverses by thinking about the reverse row operation.

18. True or false. Provide justification when false as usual.
    (a) If AB and BA are defined, then A and B are square.
    (b) If AB and BA are defined, then AB and BA are square.
    (c) If A is invertible, then A^{-1} and A^2 are invertible.
    (d) A square matrix is invertible if and only if its determinant is nonzero.

(e) For square matrices A and B of the same size, $(A+B)^2 = A^2 + 2AB + B^2$.

(f) If A is a singular matrix, then $Ax = 0$ has infinitely many solutions.

(g) The transpose of an upper triangular matrix is lower triangular.

(h) All entries of a symmetric matrix are determined by the entries occurring on and above the diagonal.

(i) $\det A + \det B = \det(A + B)$.

Python problems

A. Write a python function `REF_basic(B)` that will return the REF (pivots do not have to be 1) of any $n \times (n+1)$ augmented matrix. Row swapping should be included. Your input should be any $n \times (n+1)$ augmented matrix such that the coefficient matrix is invertible. This is how you should be checking invertibility: during the iteration process, your code should make sure every column of the coefficient matrix has a pivot. If not, then the program stops and returns an error message. Test your function at least on

(a) The augmented matrix in HW1-7 (b) np.array([[1,2,3,4,6], [1,2,5,2,-2], [1,1,5, 5,1], [1, 4, 1, 5, 0]])

A+. Extra credit: Write a python function `REF(A)` that will return the REF (pivots do not have to be 1) of any $m \times n$ matrix A.

B. Write a python function `MySolve(A, b)` that will solve $Ax = b$ if A is invertible. This should be as simple as calling functions that you have already written. Compare your function with linalg.solve on randomly generated systems to make sure that your code is working.

C. Lets computationally test the complexity of your `MySolve(A, b)`. Do so by doing 7 runs of random matrices of size 2^k, for $k = 0, ..., 10$. Let T_k = average time for a matrix of size 2^k. Plot the $\log_2(T_k)$ against k. How can we tell the leading order complexity from this plot, i.e. why does this plot show exactly that?