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1. (a) When the covariance of x and y equals to zero, they are said to be statistically uncorrelated.

$$Cov(x,y) = E(XY) - E(X)\cdot E(Y) = {}^{*}x\cdot y - \bar{x}\cdot\bar{y}$$

(b) $(AC^{-1})^{-1}(AC^{-1})(AC^{-1})^{-1}AD^{-1}$

$= C(A^{-1}A)C^{-1}AC(A^{-1}A)D^{-1}$

$= C(C^{-1}C)D^{-1}$

$= CD^{-1}$

(c) $(AA^T)^T = (A^T)^T\cdot A^T = A\cdot A^T$

So for any matrix A, AA^T is symmetric.

2. (a) $2A^T+C$

$$= 2\times\begin{bmatrix}3&-1&1\\0&2&1\end{bmatrix}+\begin{bmatrix}1&4&2\\3&1&5\end{bmatrix}$$

$$=\begin{bmatrix}7&2&4\\3&5&7\end{bmatrix}$$

(b) trace of D = 1+0+4=5.

(c) $BA = \begin{bmatrix}4&-1\\0&2\end{bmatrix}\cdot\begin{bmatrix}3&0\\-1&2\\1&1\end{bmatrix}$ Undefined.

Since B is 2x2. A is 3x2. BA is not able to get.

(d) $(2D^T-D)A = \left(2\times\begin{bmatrix}1&-1&3\\5&0&2\\2&1&4\end{bmatrix}-\begin{bmatrix}1&5&2\\-1&0&1\\3&2&4\end{bmatrix}\right)\times\begin{bmatrix}3&0\\-1&2\\1&1\end{bmatrix}$

$$=\begin{bmatrix}1&-7&4\\7&0&3\\1&0&4\end{bmatrix}\times\begin{bmatrix}3&0\\-1&2\\1&1\end{bmatrix}$$

$$=\begin{bmatrix}14&-10\\24&3\\7&4\end{bmatrix}$$

(e) $(4B)C+2B = 4\times\begin{bmatrix}4&-1\\0&2\end{bmatrix}\times\begin{bmatrix}1&4&2\\3&1&5\end{bmatrix}+2\times\begin{bmatrix}4&-1\\0&2\end{bmatrix}$

Undefined

$$=\begin{bmatrix}16&-4\\0&8\end{bmatrix}\times\begin{bmatrix}1&4&2\\3&1&5\end{bmatrix}+\begin{bmatrix}8&-2\\0&4\end{bmatrix}$$

$$=\begin{bmatrix}4&60&12\\24&8&40\end{bmatrix}+\begin{bmatrix}8&-2\\0&4\end{bmatrix}$$ which is not solvable.

(f) $(-AC)^T+5D^T = \left(\begin{bmatrix}-3&0\\1&-2\\-1&-1\end{bmatrix}\begin{bmatrix}1&4&2\\3&1&5\end{bmatrix}\right)^T+5\times\begin{bmatrix}1&-1&3\\5&0&2\\2&1&4\end{bmatrix}$

$$=\begin{bmatrix}-3&-5&-4\\-12&2&-5\\-6&-8&-7\end{bmatrix}+\begin{bmatrix}5&-5&15\\25&0&10\\10&5&20\end{bmatrix}=\begin{bmatrix}2&-10&11\\13&2&5\\4&-3&13\end{bmatrix}$$

3. Suppose $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $U^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} = I.$

$$\Rightarrow \begin{cases} a^2+bc = bc+d^2 = 1 \\ ab+bd = ac+bd = 0. \end{cases} \Rightarrow \left\{ \cancel{a=} d. \right.$$

$$= \begin{cases} a = -d. \\ bc = 1-d^2. \end{cases}$$

Then we can make $d=-1$, then $a=1$. $c=0$, $b=1$.
to make $U^2 = I$.

$$U = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

4. To make A and C commute.

$AC = CA.$ $\quad AC=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}$

~~CA~~=

$CA=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}.$

to mak $AC=CA$. then $b=0$. $a=d$.

so $\begin{cases} a=t \\ b=0 \\ c=s \\ d=t. \end{cases}$ t,s are free to be any real number.

$$5.\ (5A^{T})^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix} \Rightarrow A = \frac{1}{5}\left(\begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}^{-1}\right)^{T}$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 5 \\ -1 & 2 \end{bmatrix}^{-1}$$

$$\cancel{= \frac{1}{5} \times (-1) \cdot \begin{bmatrix} 2 & -5 \\ 1 & -3 \end{bmatrix}}$$

$$= \frac{1}{5}\begin{bmatrix} 2 & -5 \\ 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{5} & -1 \\ \frac{1}{5} & -\frac{3}{5} \end{bmatrix}$$

6. (a). ASuppose: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$AB = AC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

but $A = B \neq C$.

(b). If A is invertible.

when $AB = AC$

$\Rightarrow A^{-1} \cdot AB = A^{-1} AC$.

$\Rightarrow B = C$.

7. (a) (i) The third column of AB equals to each rows in A multiply the third column of B.

(ii) The first row of AB equals to the first row in A multiply each column in B.

(b) $$A^T = \begin{bmatrix} A_1^T & A_3^T \\ A_2^T & A_4^T \end{bmatrix}$$

(c)(i) $$B\Lambda = [c_1, c_2 \cdots, c_n] \cdot \mathrm{diag}(\lambda_1, \lambda_2, \cdots \lambda_n)$$
$$= [\lambda_1 c_1, \lambda_2 c_2, \cdots \lambda_n c_n]$$

(ii) $$\Lambda C = \mathrm{diag}(\lambda_1, \lambda_2, \cdots \lambda_n) \cdot \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 r_1 \\ \lambda_2 r_2 \\ \vdots \\ \lambda_n r_n \end{bmatrix}$$

8. (a): $\left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{2}\times P1]{\substack{P2-P1 \\ P3-P1}} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$

$\xrightarrow[P2(-1)+P3]{P2(-3)+P1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & \frac{7}{2} & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{P3(-3)+P1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{2} & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$

So ~~A~~ $A^{-1} = \begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

(b) Since in the Process transfor A to I, the only transform impact the determinant is $\frac{1}{2}\times P1$. Since $Det(I)=1$, $Det(A)=1\times2=2$.

(c) Using the first row.

$\det(A) = 2\begin{vmatrix} 7 & 6 \\ 7 & 7 \end{vmatrix} - 6\begin{vmatrix} 2 & 6 \\ 2 & 7 \end{vmatrix} + 6\begin{vmatrix} 2 & 7 \\ 2 & 7 \end{vmatrix} = 14-12=2.$

(d) Using the first column.

$\det(A) = 2\begin{vmatrix} 7 & 6 \\ 7 & 7 \end{vmatrix} - 2\begin{vmatrix} 6 & 6 \\ 7 & 7 \end{vmatrix} + 2\begin{vmatrix} 6 & 6 \\ 7 & 6 \end{vmatrix} = 14-12=2.$

(e) $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{2}$.

Since $A^{-1} = \begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

$\det(A^{-1}) = \frac{7}{2}\times1\times1 + (-3)\times(-1)\times(-1) = \frac{7}{2} - 3 = \frac{1}{2}$

9. (a) $\det(-A) = \det(A) \times (-1)^4 = \det(A) = -2.$

(b) $\det(A^{-1}) = (\det(A))^{-1} = -\frac{1}{2}$

(c) $\det(2A^T) = 2^4 \cdot \det(A^T) = 2^4 \cdot \det(A) = -32$

(d) $\det(A^3) = (\det(A))^3 = -8.$

10. (a) $|A| = K^2 - 4$.

So as $K \neq -2$ or 2, A would be invertible.

(b) Since $B = [\cancel{b_1} c_1, c_2, c_3, c_4]$.

$c_3 = c_1 + c_2$.

then we have. $c_1 + c_2 - c_3 = 0$.

So, $\begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_3 = -1 \\ x_4 = 0 \end{cases}$ would be a nontrivial ~~sution~~ solution.

11. Suppose A is a rank-1 matrix, then A should either be $(k_1 u, k_2 u, \cdots k_n u)$ for any non-null set of k_i and non-null vector u,

or $\begin{bmatrix} k_1 v^T \\ k_2 v^T \\ \vdots \\ k_n v^T \end{bmatrix}$ for any non-null set of k_i and non-null vector v.

So in the first form, $A = (k_1 u, k_2 u, \cdots k_n u) = u(k_1, k_2, \cdots k_n)^T$ which equals to uv^T if we let $v = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$.

In the second form, $A = \begin{bmatrix} k_1 v^T \\ k_2 v^T \\ \vdots \\ k_n v^T \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} v^T$.

which equals to uv^T if we let $u = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$.

12. (a) Yes.

For any $x = (x_1, 0, 0) \in (a, 0, 0)$.

$y = (y_1, 0, 0) \in (a, 0, 0)$

$x+y = (x_1+y_1, 0, 0) \in (a, 0, 0)$.

and for any scalar c, $cx = (cx_1, 0, 0) \in (a, 0, 0)$.

(b) No.

For $x \in (a, 1, 1)$, suppose $x = (x_1, 1, 1)$.

$cx = (cx_1, c, c)$ doesn't necessarily $\in (a, 1, 1)$.

i.e when $c = 2$. $cx = (2x_1, 2, 2)$ not $\in (a, 1, 1)$

(c) Yes.

Suppose $x \in (a, b, c)$ where $b = a + c$.

then $x = (x_1, x_1+x_2, x_2)$.

$y \in (a, b, c)$ where $b = a + c$.

then $y = (y_1, y_1+y_2, y_2)$.

$x+y = (x_1+y_1, x_1+x_2+y_1+y_2, x_2+y_2)$ still satisfy $x_1+x_2+y_1+y_2 = (x_1+y_1) + (x_2+y_2)$

which shows $x+y \in (a, b, c)$ where $b = a + c$.

for $cx = (cx_1, cx_1+cx_2, cx_2)$. also satisfy $b = a + c$.

shows $cx \in (a, b, c)$ where $b = a + c$.

13. $U = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $V = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

$|U| = 2$. $|V| = 1$ so U, V are invertible.

while $U+V = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ which is ~~a inve~~ not inverteble.

14. ~~(a)~~ only (C). is the liear combination of $u=(0,-2,2)$ and $v=(1,3,-1)$

Since for (a). theres is not solution for: $x_1\begin{bmatrix}0\\-2\\2\end{bmatrix}+x_2\begin{bmatrix}1\\3\\-1\end{bmatrix}=\begin{bmatrix}2\\2\\2\end{bmatrix}$.

For (b) there is no solution for: $x_1\begin{bmatrix}0\\-2\\2\end{bmatrix}+x_2\begin{bmatrix}1\\3\\-1\end{bmatrix}=\begin{bmatrix}0\\4\\5\end{bmatrix}$.

For (c). we have. $0\begin{bmatrix}0\\-2\\2\end{bmatrix}+0\begin{bmatrix}1\\3\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

15. (a). Since these are 3 ~~indep~~ linear independent vectors in R^3.
Their linear combination would be the all R^3 space.

(b) Since there are 2 linear dependent vectors in R^2.
Their linear combination would be a line which across $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$
and $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

 HW1-4.

$$\text{Solution Set} = \left\{ \begin{bmatrix} 13t-10 \\ 13t-5 \\ -t-2 \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} -10 \\ -5 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 13 \\ 13 \\ -1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

$$= \begin{bmatrix} -10 \\ -5 \\ -2 \\ 0 \end{bmatrix} + \text{span} \begin{bmatrix} 13 \\ 13 \\ -1 \\ 1 \end{bmatrix}$$

HW1-8:

$$\text{Solution Set} = \left\{ \begin{bmatrix} \frac{7}{2}t-5u \\ -3t+2u \\ t \\ u \end{bmatrix} : t,u \in \mathbb{R} \right\} = \left\{ t\begin{bmatrix} \frac{7}{2} \\ -3 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -5 \\ 2 \\ 0 \\ 1 \end{bmatrix} : t,u \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} \frac{7}{2} \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

HW1-13:

$$\text{Solution Set} = \left\{ \begin{bmatrix} -\frac{1}{4}t \\ \frac{5}{4}t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \text{span} \begin{bmatrix} -\frac{1}{4} \\ \frac{5}{4} \\ 1 \end{bmatrix}.$$

17. (a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) ~~Matrix cannot get the multiply~~

No matrix transformation can make the multply computation, ~~the~~ B does not exist.

(c) $\begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}, & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2}, & \frac{\sqrt{2}}{2} \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

Inverses: $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

18. (a) False. Suppose A is $m_1 \times n_1$, B is $m_2 \times n_2$.

as long as $m_1 = n_2$, $n_1 = m_2$, AB and BA would be defined,

But A, B are not necessarily both square. i.e. A is 3×2, B is 2×3.

AB would be 3×3.

(b) True. Since AB and BA are defined. BA would be 2×2.

$m_1 = n_2$, $n_1 = m_2$. They are all defined.

AB and BA would be $m_1 \times n_2$ or $n_1 \times m_2$.

So they would be square.

(c) True.

$$A \text{ is invertible} \Rightarrow \det(A) \neq 0 \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)} \neq 0.$$

$$\Downarrow$$

$$\det(A^2) = (\det A)^2 \neq 0.$$

$$\Rightarrow A^{-1}, A^2 \text{ are also invertible.}$$

(d) True

(e) False. $(A+B)^2 = A^2 + AB + BA + B^2$. But AB does not necessarily ~~mean~~ equal to BA.

(f) True.

(g) True.

(h) True.

(i) False. i.e. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

$\det(A) = \det(B) = 1$. But $\det(A+B) = 0$.