MSAN 502 Homework 1.

Tianqi Wang

Question 1

For those three elementary row operations ① Swapping rows is just changing the order of the equations being considered, so it certainly not change the solutions.

② Scalar Multiplication is multiplying both sides of equation, so certainly the equation holds and the solution would not change.

③ For Row Sum, if the solution hold for both row, sum ~~each~~ two equation still get the same result.

Question 2 :

Substitute (x, y) with $(1,1)$, $(2,4)$, $(-1,1)$ respectively, we get:

$$\begin{cases} 1 = a+b+c & (1) \\ 4 = 4a+2b+c. & (2) \\ 1 = a-b+c. & (3). \end{cases} \Rightarrow$$

$(1)-(3) \Rightarrow 2b=0 \Rightarrow b=0$

So we have. $\begin{cases} 1=a+c & (4) \\ 4=4a+c & (5) \end{cases}$

$(5)-(4) \Rightarrow 3=3a. \Rightarrow a=1$ Substitute a for 1 in (4).

We can get $c=0$.

So $\begin{cases} a=1 \\ b=0 \\ c=0 \end{cases}$

Question 3

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow[\rho_3(-4)+\rho_1]{\rho_3(-2)+\rho_2} \left[\begin{array}{ccc|c} 1 & -3 & 0 & -13 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 5 \end{array}\right] \xrightarrow{\rho_2(3)+\rho_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -37 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

So we get
$$\begin{cases} x_1 = -37 \\ x_2 = -8 \\ x_3 = 5. \end{cases}$$

Question 4

$$\left[\begin{array}{cccc|c} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right] \xrightarrow[\rho_3(-8)+\rho_1]{\rho_3(-4)+\rho_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -13 & -10 \\ 0 & 1 & 0 & -13 & -5 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right]$$

So we have: $\begin{cases} x_1 - 13x_4 = -10 \\ x_2 \cancel{\#} -13x_4 = -5 \\ x_3 + x_4 = -2 \end{cases} \Rightarrow \begin{cases} x_1 = 13x_4 - 10 \\ x_2 = 13x_4 - 5 \\ x_3 = -x_4 - 2. \end{cases}$

This system has infinite solutions, x_4 is the free variable.

Question 5

$$\left[\begin{array}{ccc|c} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{array}\right] \xrightarrow{P2(3)+P1} \left[\begin{array}{ccc|c} 1 & 0 & 19 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{array}\right].$$

Since in equation form, this would be:

$$\begin{cases} x_1 + x_3 = 1 \\ x_2 + 4x_3 = 0 \\ \cancel{x_1 + x_2 +} 0 = 5. \end{cases}$$

This Linear system ~~is not solvable~~.
has no solution.

6. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right]$ So we can directly get $\begin{cases} x_1 = -3 \\ x_2 = 0 \\ x_3 = 5 \end{cases}$

Question. 7.

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 0 & 4 & -1 \end{array}\right] \xrightarrow[\rho_1+\rho_2]{\rho_1(-4)+\rho_3} \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & -8 & -4 & -1 \end{array}\right]$$

$$\xrightarrow{\rho_1(4)+\rho_3} \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 4 & -1 \end{array}\right] \xrightarrow[\rho_3(-\frac{4}{4})+\rho_2]{\rho_3(-\frac{1}{2})+\rho_1} \left[\begin{array}{ccc|c} 2 & 2 & 0 & \frac{1}{2} \\ 0 & 7 & 0 & 2 \\ 0 & 0 & 4 & -1 \end{array}\right]$$

$$\xrightarrow[\rho_3\times\frac{1}{4}]{\rho_2(-\frac{2}{7})+\rho_1} \left[\begin{array}{ccc|c} 2 & 0 & 0 & -\frac{1}{14} \\ 0 & 7 & 0 & 2 \\ 0 & 0 & 1 & -\frac{1}{4} \end{array}\right] \xrightarrow[\rho_2\times\frac{3}{7}]{\rho_1\times\frac{1}{2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{28} \\ 0 & 1 & 0 & \frac{2}{7} \\ 0 & 0 & 1 & -\frac{1}{4} \end{array}\right]$$

So we get $\begin{cases} x_1 = -\frac{1}{28} \\ x_2 = \frac{2}{7} \\ x_3 = -\frac{1}{4} \end{cases}$

Question 8

$$\left[\begin{array}{cccc|c} 0 & 1 & 3 & -2 & 0 \\ 2 & 1 & -4 & 3 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{array}\right] \xrightarrow{\rho_1 \leftrightarrow \rho_2} \left[\begin{array}{cccc|c} 2 & 1 & -4 & 3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{array}\right]$$

$$\xrightarrow[\rho_1(2)+\rho_4]{\rho_1(-1)+\rho_3} \left[\begin{array}{cccc|c} 2 & 1 & -4 & 3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 2 & 6 & -4 & 0 \\ 0 & -1 & -3 & 2 & 0 \end{array}\right] \xrightarrow[\rho_2+\rho_4]{\rho_2(-2)+\rho_3} \left[\begin{array}{cccc|c} 2 & 1 & -4 & 3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\rho_2(-1)+\rho_1} \left[\begin{array}{cccc|c} 2 & 0 & -7 & 5 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}\rho_1.} \left[\begin{array}{cccc|c} 1 & 0 & -\frac{7}{2} & 5 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So $\begin{cases} x_1 = \frac{7}{2}x_3 - 5x_4 \\ x_2 = -3x_3 + 2x_4. \end{cases}$

x_3 and x_4. are free variables.

There are infinite solutions.

Question 9

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & -2 & 3 & 1 \\ 1 & 2 & a^2-3 & a \end{array}\right] \xrightarrow[P_1(-1)+P_3]{P_1(-2)+P_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -6 & 1 & -3 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right]$$

So when $a^2-4=0$, $a-2=0$ $\Rightarrow a=2$. The rank of augmented matrix equals to the rank of cofficient matrix and less than numbers of variabs. So it would be infinite solutions.

when $a^2-4=0$, $a-2\neq 0$ $\Rightarrow a=-2$, There is no solution.

when $a^2-4\neq 0$, $a-2\neq 0$. The rank of augmented matrix equals to the rank of coefficient matrix and equals to numbers of variables. So there would be exactly one solution.

$$\begin{cases} a=-2 & \text{no solution.} \\ a=2 & \text{infinite solutions} \\ \text{otherwise} & \text{one solution.} \end{cases}$$

Question 10.

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ -1 & -2 & 1 & b \\ 3 & 7 & -1 & c \end{array}\right] \xrightarrow[P1(-3)+P3]{P1(1)+P2} \left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & -2 & -4 & -3a+c \end{array}\right]$$

$$\xrightarrow{P2(2)+P3} \left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & 0 & 0 & -a+2b+c. \end{array}\right]$$

So when $-a+2b+c=0$, the linear system would be consistent, otherwise, this linear system is not consistent.

Question 11

$c_1(-1,0,2) + c_2(2,2,-2) + c_3(1,-2,1) = (-6,12,4)$

holds. when. $-c_1 + 2c_2 + c_3 = -6$

$$2c_2 + -2c_3 = 12$$

$$2c_1 + -2c_2 + c_3 = 4$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & -6 \\ 0 & 2 & -2 & 12 \\ 2 & -2 & 1 & 4 \end{array}\right] \xrightarrow{\rho 1(2)+\rho 3} \left[\begin{array}{ccc|c} -1 & 2 & 1 & -6 \\ 0 & 2 & -2 & 12 \\ 0 & 2 & 3 & -8 \end{array}\right] \xrightarrow[\rho 3-\rho 2]{\rho 1 \times -1}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 6 \\ 0 & 2 & -2 & 12 \\ 0 & 0 & 5 & -20 \end{array}\right] \xrightarrow[\rho 2 \times \frac{1}{2}]{\rho 3 \times \frac{1}{5}} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 6 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & -4 \end{array}\right] \xrightarrow[\rho 3+\rho 2]{\rho 3+\rho 1}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right] \xrightarrow{\rho 2(2)+\rho 1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

So $\begin{cases} c_1 = 6 \\ c_2 = 2 \\ c_3 = -4. \end{cases}$

Question 12.

(a) $2u - v + x = 7x + w. \Rightarrow x = \frac{1}{6}(2u - v - w).$

$$= \frac{1}{6}\cdot\left[(4,-4,6) - (1,-3,4) - (3,6,-4)\right]$$

$$= \frac{1}{6}(0,-7,6).$$

$$= (0, -\frac{7}{6}, 1).$$

(b). distance between $(2,-2,3)$ and $(1,-3,4)$ equals to :

$$\sqrt{(2-1)^2 + (-2+3)^2 + (3-4)^2}. = \sqrt{3}$$

(c) $\|u\| = \sqrt{2^2 + (-2)^2 + 3^2}. = \sqrt{17}$

$\|v\| = \sqrt{1^2 + (-3)^2 + 4^2}. = \sqrt{26}.$

$\|u\| - \|v\| \approx -0.976.$

(d). $u \cdot v = 2\times1 + (-2)\times(-3) + 3\times(-4). = -4$

(e) $(u-v)\cdot(u+2v) = \|u\|^2 - vu + 2uv - 2\|v\|^2. = \|u\|^2 + u\cdot v - 2\|v\|^2.$

$$= 17 - 4 - 2\times26 = -39$$

(f) $\frac{v}{\|v\|} = \left(\frac{1}{\sqrt{26}}, -\frac{3}{\sqrt{26}}, \frac{4}{\sqrt{26}}\right).$

(g). $\cos\theta = \frac{u\cdot v}{\|u\|\,\|v\|} \Rightarrow \cos\theta = \frac{-4}{\sqrt{442}}.$ so $\theta \approx 101°.$

Question 13

(a). Suppose the vector is $X = (x_1, x_2, x_3)$.

Then. to satisfy the condition that orthogonal to u:

$$X \cdot u = 2x_1 - 2x_2 + 3x_3 = 0$$

So as long x_1, x_2, x_3 holds this equation, and. they are not equal to 0 at the same time, the vector X would be orthogonal to u.

(b)

Suppose the vector is $x = (x_1, x_2, x_3)$.

to be orthogonal to u : $2x_1 - 2x_2 + 3x_3 = 0$

to be orthogonal to v : $x_1 - 3x_2 + 4x_3 = 0$.

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & 0 \\ 1 & -3 & 4 & 0 \end{array}\right] \xrightarrow{P_1 \times \frac{1}{2}} \left[\begin{array}{ccc|c} 1 & -1 & \frac{3}{2} & 0 \\ 1 & -3 & 4 & 0 \end{array}\right]$$

$$\xrightarrow{P_1(-1)+P_2} \left[\begin{array}{ccc|c} 1 & -1 & \frac{3}{2} & 0 \\ 0 & -2 & \frac{5}{2} & 0 \end{array}\right]$$

$$\xrightarrow[P_2 \times \frac{1}{2}]{P_2(-\frac{1}{2})+P_1} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{5}{4} & 0 \end{array}\right]$$

So as long as $\begin{cases} x_1 = -\frac{1}{4}x_3 \\ x_2 = \frac{5}{4}x_3 \end{cases} \Rightarrow x_1 : x_2 : x_3 = -\frac{1}{4} : \frac{5}{4} : 1$.

So for ~~any~~ vector $x(-1, 5, 4)$ for any $x \neq 0$. the vector would be orthogonal to u and v.

Question 14.

(a) True.

(b) ~~F~~ True

(c) False. e.g. $w=(1,1,2)$. $u=(1,1,-1)$, $v=(-\sqrt{5},1,1)$.
both u and v are orthogonal to w., but they are not parallel

(d) False. e.g. $u=(1,1,2)$. $v=(1,1,-1)$.
$u\cdot v = 0$ so they are orthogonal.
but $\|u-v\| = \sqrt{3}$.

(e). True.

(f). False. e.g $u=(1,1,2)$ $v=(1,1,-1)$. $w=(-\sqrt{5},1,1)$.
$u\cdot v = u\cdot w$. but $v \neq w$.