Summer 2018 MSAN 502 Homework 3, due 7/27 9am

1. (a) Determine whether the vectors are linearly independent/dependent in $\mathbb{R}^3$.
   (a) $(-3, 0, 4), (5, -1, 2), (1, 1, 3)$
   (b) $(-2, 0, 1), (3, 2, 5), (6, -1, 1), (7, 0, -2)$

   (b) In the vector space P_2=all polynomials whose degree is less than or equal to 2. Use Theorem 4.3.1 to check whether the 3 polynomials are independent.
   $p_1(x) = -3 + 4x^2, \quad p_2(x) = 5 - x + 2x^2, \quad p_3(x) = 1 + x + 3x^2.$

2. Find a basis for the given subspace, and state its dimension.

   (a) All vectors of the form (a, b, c, d) where $d = a + b$ and $c = a - b$
   (b) The plane $x - 2y + 3z = 0$.
   (c) The intersection of the plane from (b) with the xy plane.
   (d) All vectors perpendicular to the plane from (b).

3. Given $v_1 = (1, 2, -1), v_2 = (4, 1, 3), v_3 = (5, 3, 2), v_4 = (2, 0, 2)$.

   (a) Find the largest possible number of independent vectors among these 4 vectors.
   (b) Find a subset of the given vectors that forms a basis for the space spanned by and then express each vector that is not in the basis as a linear combination of the basis vectors.

4. Find bases for row space and null space of $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$.

5. U comes from A by subtracting row 1 from row 3:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}, U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

   (a) How are the column space, row space, and null space changed from A to U?
   (b) Is your conclusion true in general? Explain your answer.

6. Construct a matrix or state why it does not exist.

   (a) Column space contains $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Row space contains $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

   (b) Column space has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. Row space has basis $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$.

   (c) Column space has basis $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, null space has basis $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

   (d) Row space contains $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Null space contains $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

   (e) Its null space consists of all linear combinations of $v_1 = (1, -1, 3, 2), v_2 = (2, 0, -2, 4)$.

(f) A 3 by 3 matrix whose null space equals its column space.

(g) Row space equals column space, not symmetric, singular.

7. Suppose S is spanned by the vectors $(1,-1,3,2),(2,0,-2,4)$. Find two vectors that span $S^{\perp}$. This is the same as solving $Ax=0$ for which A?

8. These problems have a quick answer.

   (a) If we add an extra column b to the matrix A, then the column space gets larger unless ______. Give an example where the column space gets larger and an example where it doesn't. Why is $Ax=b$ solvable exactly when the column space doesn't get larger?

   (b) As an operator, the range or image of an $m\times n$ matrix A is $\{Ax : x\in\mathbb{R}^n\}$. The image of A is _______ (row space or column space or null space of A). Explain.

   (c) Is $\{u_1=(1,2), u_2=(0,3), u_3=(1,5)\}$ a basis for $\mathbb{R}^2$? Why?

   (d) Is $\{u_1=(-1,3,2), u_2=(6,1,1)\}$ a basis for $\mathbb{R}^3$? Why?

   (e) Let A be a 7×6 matrix such that $Ax=0$ has only the trivial solution. Find the rank and nullity of A.

9. Let A be a 5×7 matrix with rank 4.

   (a) What is the dimension of the solution space of $Ax=0$?

   (b) Is $Ax=b$ consistent for all vectors b in $\mathbb{R}^5$? Explain.

10. Verify that the vectors $v_1=(1,-1,2,-1)$, $v_2=(-2,2,3,2)$, $v_3=(1,2,0,-1)$, $v_4=(1,0,0,1)$ form an orthogonal basis for $\mathbb{R}^4$, and then express $u=(1,1,1,1)$ as a linear combination of v_i's.

11. Projection onto a line.

    (a) Find the orthogonal projection of $b=(1,2,2)$ onto $\text{span}\{a=(1,1,1)\}$. Verify that $e=b-p$ is orthogonal to a. What is the projection matrix that represents projection onto $\text{span}\{(1,1,1)\}$? What is the rank of this projection matrix?

    (b) Project the vector $b=(1,1)$ onto $\text{span}\{a_1=(1,0)\}$ and $\text{span}\{a_2=(1,2)\}$ respectively. Call the projections p_1,p_2 and draw them. Does $p_1+p_2=b$? Change a_1 and play with your drawing to figure out when the two projections will add up to b.

12. Prove that if a vector v is orthogonal to each basis vector for a subspace W, then v is orthogonal to W.

13. True or False

    (a) If $V=\text{span}\{v_1,\cdots,v_n\}$, then $\{v_1,\cdots,v_n\}$ is a basis for V.

    (b) Any set of n independent vectors is a basis for $\mathbb{R}^n$.

    (c) If $A=-A^T$, then the row space of A equals its column space.

    (d) If $P_{a_1}(b)=P_{a_2}(b)$ for every vector b, then $a_1=a_2$.

    (e) If $\dim\{\text{span}\{v_1,\cdots,v_n\}\}=n-1$, then at least one vector v_k is a unique linear combination of the rest $n-1$ vectors.

    (f) If rank(A)=rank(A^T), then A is square.

    (g) Let $\{v_1,\cdots,v_n\}$ be an orthonormal basis of $\mathbb{R}^n$, and let $A=[v_1,\cdots,v_n]$, then $AA^T=I_n$.

    (h) Let $\{v_1,\cdots,v_n\}$ be an orthonormal set of $\mathbb{R}^m$, and let $A=[v_1,\cdots,v_n]$, then $AA^T=I_m$.

We are taking a break from Python this week.