

MSAN 502 Homework 1.

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Question 1.

For those three elementary row operations ① swapping rows is just changing the order of the equations being considered, so it certainly not change the solutions.

② Scalar Multiplication is multiplying both sides of equation, so certainly the equation holds and the solution would not change.

③ For Row Sum, if the solution hold for both row sum two equation. still get the same result.

Question 2 :

Substitute (x, y) with $(1, 1), (2, 4), (-1, 1)$ respectively, we get:

$$\begin{cases} l = a + b + c & (1) \\ 4 = 4a + 2b + c. & (2) \\ 1 = a - b + c. & (3) \end{cases}$$

$$(1) - (3) \Rightarrow 2b = 0 \Rightarrow b = 0$$

$$\text{so we have. } \begin{cases} l = a + c & (4) \\ 4 = 4a + c & (5) \end{cases}$$

$$(5) - (4) \Rightarrow 3 = 3a. \Rightarrow a = 1 \quad \text{Substitute } a \text{ for } l \text{ in (4)}$$

We can get $c = 0$.

$$\text{So } \begin{cases} a = 1 \\ b = 0 \\ c = 0 \end{cases}$$

Question 3

$$\left[\begin{array}{ccc|c} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{array} \right] \xrightarrow{\text{P}_3(-2) + P_2} \left[\begin{array}{ccc|c} 1 & -3 & 0 & -15 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 5 \end{array} \right] \xrightarrow{\text{P}_2(3) + P_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -37 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

So we get $\begin{cases} x_1 = -37 \\ x_2 = -8 \\ x_3 = 5 \end{cases}$

Question 4

$$\left[\begin{array}{cccc|c} 1 & 0 & 8 & -5 & 6 \\ 0 & 1 & 4 & -9 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{array} \right] \xrightarrow{\begin{matrix} P_3(-4) + P_2 \\ P_3(-8) + P_1 \end{matrix}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -13 & -10 \\ 0 & 1 & 0 & -13 & -5 \\ 0 & 0 & 1 & 1 & 2 \end{array} \right]$$

So we have :

$$\begin{cases} x_1 - 13x_4 = -10 \\ x_2 - 13x_4 = -5 \\ x_3 + x_4 = -2 \end{cases} \Rightarrow \begin{cases} x_1 = 13x_4 - 10 \\ x_2 = 13x_4 - 5 \\ x_3 = -x_4 - 2 \end{cases}$$

This system has infinite solutions, x_4 is the free variable.

Question 5

$$\left[\begin{array}{ccc|c} 1 & -3 & 7 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{array} \right] \xrightarrow{R_2(3)+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 19 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 5 \end{array} \right].$$

Since in equation form, this would be:

$$\left\{ \begin{array}{l} x_1 + x_3 = 1 \\ x_2 + 4x_3 = 0 \\ x_1 + x_2 + x_3 = 5 \end{array} \right.$$

This linear system is not solvable.

has no solution.

6. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{array} \right]$ So we can directly get $\begin{cases} x_1 = -3 \\ x_2 = 0 \\ x_3 = 5 \end{cases}$

Question. 7.

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ -2 & 5 & 2 & 1 \\ 8 & 0 & 4 & -1 \end{array} \right] \xrightarrow{\frac{P_1(4) + P_3}{P_1 + P_2}} \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 7 & 4 & 1 \\ 6 & -8 & -4 & -1 \end{array} \right]$$

$$\xrightarrow{P_1(4) + P_3} \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 0 & 4 & -1 \end{array} \right] \xrightarrow{\frac{P_3(-\frac{1}{2}) + P_1}{P_3 + \frac{1}{4}P_2}} \left[\begin{array}{ccc|c} 2 & 2 & 0 & \frac{1}{2} \\ 0 & 7 & 0 & 2 \\ 0 & 0 & 4 & -1 \end{array} \right]$$

$$\xrightarrow{\frac{P_2(-\frac{2}{7}) + P_1}{P_3 \times \frac{1}{4}}} \left[\begin{array}{ccc|c} 2 & 0 & 0 & -\frac{1}{14} \\ 0 & 7 & 0 & 2 \\ 0 & 0 & 1 & -\frac{1}{4} \end{array} \right] \xrightarrow{\frac{P_1 \times \frac{1}{2}}{P_2 \times \frac{3}{7}}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{28} \\ 0 & 1 & 0 & \frac{2}{7} \\ 0 & 0 & 1 & -\frac{1}{4} \end{array} \right]$$

So we get

$$\left\{ \begin{array}{l} x_1 = -\frac{1}{28} \\ x_2 = \frac{2}{7} \\ x_3 = -\frac{1}{4} \end{array} \right.$$

Question 8

$$\left[\begin{array}{cccc|c} 0 & 1 & 3 & -2 & 0 \\ 2 & 1 & -4 & 3 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{array} \right] \xrightarrow{P_1 \leftrightarrow P_2} \left[\begin{array}{cccc|c} 2 & 1 & -4 & 3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{array} \right]$$

$$\xrightarrow{\frac{P_1(-1)+P_3}{P_1(2)+P_4}} \left[\begin{array}{cccc|c} 2 & 1 & -4 & 3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 2 & 6 & -4 & 0 \\ 0 & -1 & 3 & 2 & 0 \end{array} \right] \xrightarrow{\frac{P_2(-2)+P_3}{P_2+P_4}} \left[\begin{array}{cccc|c} 2 & 1 & -4 & 3 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\xrightarrow{P_2(-1)+P_1} \left[\begin{array}{cccc|c} 2 & 0 & -5 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\frac{1}{2}P_1} \left[\begin{array}{cccc|c} 1 & 0 & -\frac{5}{2} & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{So } \begin{cases} x_1 = \frac{7}{2}x_3 - 5x_4 \\ x_2 = -3x_3 + 2x_4 \end{cases}$$

x_3 and x_4 are free variables.
There are infinite solutions.

Question 9

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & \rightarrow 3 & 1 & 1 \\ 1 & 2 & a^2-3 & a \end{array} \right] \xrightarrow{\begin{matrix} P_1(-2)+P_2 \\ P_3(-1)+P_3 \end{matrix}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -6 & 1 & -3 \\ 0 & 0 & a^2+4 & a-2 \end{array} \right] \rightarrow \text{Inconsistent?}.$$

So when $a^2-4=0$, $a-2=0$. The rank of augmented matrix $\Rightarrow a=2$. equals to the rank of coefficient matrix and less than numbers of variables. So it would be infinite solutions.

When $a^2-4=0$, $a-2 \neq 0$, There is no solution.

When $a^2-4 \neq 0$, $a-2 \neq 0$. The rank of augmented matrix equals to the rank of coefficient matrix and equals to numbers of variables. So there would be exactly one solution.

$$\begin{cases} a=-2 & \text{no solution.} \\ a=2 & \text{infinite solutions} \\ \text{otherwise} & \text{one solution.} \end{cases}$$

Question 10.

$$\left[\begin{array}{ccc|c} *1 & 3 & 1 & a \\ -1 & -2 & 1 & b \\ 3 & 7 & -1 & c \end{array} \right] \xrightarrow{\begin{matrix} P_1(1) + P_2 \\ P_1(-3) + P_3 \end{matrix}} \left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & -2 & -4 & 3a+c \end{array} \right]$$

$$\xrightarrow{P_2(2) \times 3} \left[\begin{array}{ccc|c} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & 0 & 0 & -a+2b+c \end{array} \right]$$

So when $-a+2b+c = 0$, the linear system would be consistent, otherwise, this linear system is not consistent.

Question 11

$$C_1(-1,0,2) + C_2(2,2,-2) + C_3(1,-2,1) = (-6, 12, 4)$$

holds. when. $-C_1 + 2C_2 + C_3 = -6$

$$2C_2 + -2C_3 = 12$$

$$2C_1 + -2C_2 + C_3 = 4$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & -6 \\ 0 & 2 & -2 & 12 \\ 2 & -2 & 1 & 4 \end{array} \right] \xrightarrow{P1 \leftrightarrow P2 + P3} \left[\begin{array}{ccc|c} -1 & 2 & 1 & -6 \\ 0 & 2 & -2 & 12 \\ 0 & 2 & 3 & -8 \end{array} \right] \xrightarrow{P1 \times -1} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 6 \\ 0 & 2 & -2 & 12 \\ 0 & 0 & 5 & -20 \end{array} \right] \xrightarrow{P3 - P2}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 6 \\ 0 & 2 & -2 & 12 \\ 0 & 0 & 5 & -20 \end{array} \right] \xrightarrow{P3 \times \frac{1}{5}} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 6 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & -4 \end{array} \right] \xrightarrow{P3 + P1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array} \right] \xrightarrow{P2 \leftrightarrow P1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array} \right]$$

So $\begin{cases} C_1 = 6 \\ C_2 = 2 \\ C_3 = -4 \end{cases}$

Question 12.

(a) $2u - v + w = 7x + w \Rightarrow x = \frac{1}{6}(2u - v - w).$

$$= \frac{1}{6}[(4, -4, 6) - (1, -3, 4) - (3, 6, -4)]$$

$$= \frac{1}{6}(0, -7, 6).$$

$$= (0, -\frac{7}{6}, 1).$$

(b). distance between $(2, -2, 3)$ and $(1, -3, 4)$ equals to :

$$\sqrt{(2-1)^2 + (-2+3)^2 + (3-4)^2} = \sqrt{3}$$

$$(c) \|u\| = \sqrt{2^2 + (-2)^2 + 3^2} = \sqrt{17}$$

$$\|v\| = \sqrt{1^2 + (-3)^2 + 4^2} = \sqrt{26}.$$

$$\|u\| - \|v\| \approx -0.976.$$

(d). $u \cdot v = 2 \times 1 + (-2) \times (-3) + 3 \times (-4) = -4$

(e) $(u-v) \cdot (u+2v) = \|u\|^2 - vu + 2uv - 2\|v\|^2 = \|u\|^2 + u \cdot v - 2\|v\|^2$

$$= 17 - 4 - 2 \times 26 = -39$$

(f) $\frac{v}{\|v\|} = \left(\frac{1}{\sqrt{26}}, -\frac{3}{\sqrt{26}}, \frac{4}{\sqrt{26}} \right)$

(g). $\cos \theta = \frac{u \cdot v}{\|u\| \|v\|} \Rightarrow \cos \theta = \frac{-4}{\sqrt{4+2}} \text{ so } \theta \approx 101^\circ.$

Question 13

(a). Suppose the vector is $\vec{x} = (x_1, x_2, x_3)$.

Then, \vec{x} to satisfy the condition that orthogonal to u .

$$\vec{x} \cdot u = 2x_1 - 2x_2 + 3x_3 = 0$$

So as long x_1, x_2, x_3 holds this equation, and they are not equal to 0 at the same time, the vector \vec{x} would be orthogonal to u .

(b)

Suppose the vector is $\vec{x} = (x_1, x_2, x_3)$.

$$\text{to be orthogonal to } u : 2x_1 - 2x_2 + 3x_3 = 0$$

$$\text{to be orthogonal to } v : x_1 - 3x_2 + 4x_3 = 0.$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & -2 & 3 & 0 \\ 1 & -3 & 4 & 0 \end{array} \right] \xrightarrow{\text{P1} \times \frac{1}{2}} \left[\begin{array}{ccc|c} 1 & -1 & \frac{3}{2} & 0 \\ 1 & -3 & 4 & 0 \end{array} \right]$$

$$\xrightarrow{\text{P1}(-1) + \text{P2}} \left[\begin{array}{ccc|c} 1 & -1 & \frac{3}{2} & 0 \\ 0 & -2 & \frac{5}{2} & 0 \end{array} \right]$$

$$\xrightarrow{\text{P2} \times \frac{1}{2}} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{5}{4} & 0 \end{array} \right]$$

$$\text{So as long as } \begin{cases} x_1 = -\frac{1}{4}x_3 \\ x_2 = \frac{5}{4}x_3 \end{cases} \Rightarrow x_1 : x_2 : x_3 = -\frac{1}{4} : \frac{5}{4} : 1.$$

So for any vector $\vec{x} = (-\frac{1}{4}, \frac{5}{4}, 1)$ for any $\vec{x} \neq 0$. the vector would be orthogonal to u and v .

Question 14.

(a) True.

(b) False

(c) False. e.g. $w = (1, 1, 2)$. $\nexists u = (1, 1, -1)$, $v = (-\sqrt{5}, 1, 1)$.
both u and v are orthogonal to w , but they are not parallel

(d) False. e.g. $u = (1, 1, 2)$, $v = (1, 1, -1)$.

$u \cdot v = 0$ so they are orthogonal.

$$\text{but } \|u - v\| = \sqrt{3}.$$

(e) True.

(f) False. e.g. $u = (1, 1, 2)$, $v = (1, 1, -1)$, $w = (-\sqrt{3}, 1, 1)$.

$$u \cdot v = u \cdot w \text{ but } v \neq w.$$

Question 15

