Fourier Transform (III) —DFT

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Image Representation

PRIORI BASIS FOR NATURAL IMAGES

Discrete Fourier Transform (DFT)

- **A finite duration aperiodic signal** $x[n]$, $x[n] = 0$ outside of the interval $0 \le n \le N_1$.
- Construct a periodic signal $\tilde{x}[n]$ with period of N ($N \ge N_1$), over one period $\tilde{x}[n] = x[n]$ ($0 \le n < N$).
  - According to the discrete-time Fourier series, the DFT of $x[n]$ is usually written as

$$\tilde{X}(k) = a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}, \quad k = 0,1,\cdots,N-1$$

  - The synthesis equation

$$x[n] = \sum_{k=0}^{N-1} \tilde{X}(k)e^{jk\frac{2\pi}{N}n}, \quad n = 0,1,\cdots,N-1$$

The DFT Analysis in Matrix-Vector Form

$$\boxed{\tilde{X}(k) = a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n}, \quad k = 0, 1, \cdots, N-1}$$

$$\begin{pmatrix} \tilde{X}(0) \\ \vdots \\ \tilde{X}(N-1) \end{pmatrix} = \begin{pmatrix} W_{0,0} & \cdots & W_{0,N-1} \\ \vdots & \ddots & \vdots \\ W_{N-1,0} & \cdots & W_{N-1,N-1} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[N-1] \end{pmatrix}$$

where $$W_{k,n} = \frac{1}{N} e^{-jk\frac{2\pi}{N}n}$$

DFT Basis

- DFT Matrix

$$W = \frac{1}{N}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j2\pi/N} & \cdots & e^{-j2\pi(N-1)/N} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j2\pi(N-1)/N} & \cdots & e^{-j2\pi(N-1)(N-1)/N} \end{pmatrix}$$

The DFT Analysis in Matrix-Vector Form

$$\begin{pmatrix} \tilde{X}(0) \\ \vdots \\ \tilde{X}(N-1) \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} W_{0,0} & \cdots & W_{0,\text{N-1}} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[N-1] \end{pmatrix} \\ \vdots \\ \begin{pmatrix} W_{\text{N}-1,0} & \cdots & W_{N-1,N-1} \end{pmatrix} \begin{pmatrix} x[0] \\ \vdots \\ x[N-1] \end{pmatrix} \end{pmatrix}$$

Inverse DFT Basis

$$W^H = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{j2\pi/N} & \cdots & e^{j2\pi(N-1)/N} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j2\pi(N-1)/N} & \cdots & e^{j2\pi(N-1)(N-1)/N} \end{pmatrix}$$

The DFT Synthesize in Matrix-Vector Form

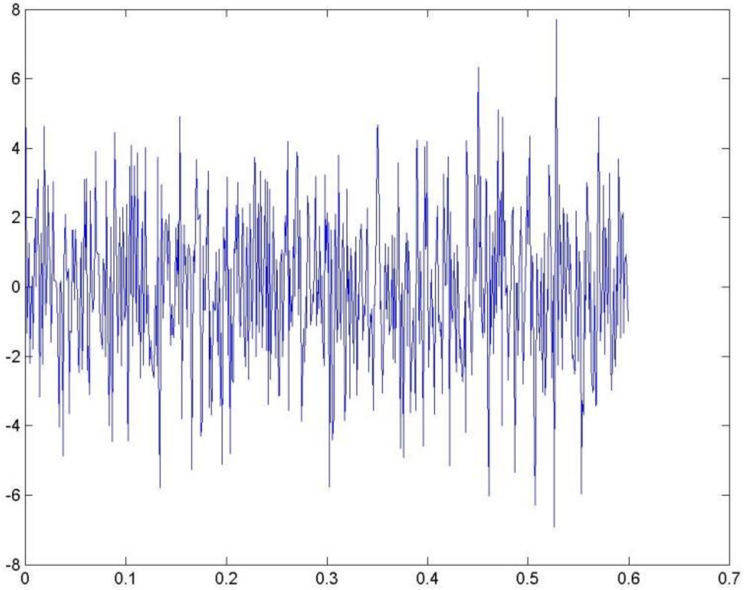
$$x[n] = \sum_{k=0}^{N-1} \tilde{X}(k) e^{jk\frac{2\pi}{N}n}, \quad n = 0, 1, \cdots, N-1$$

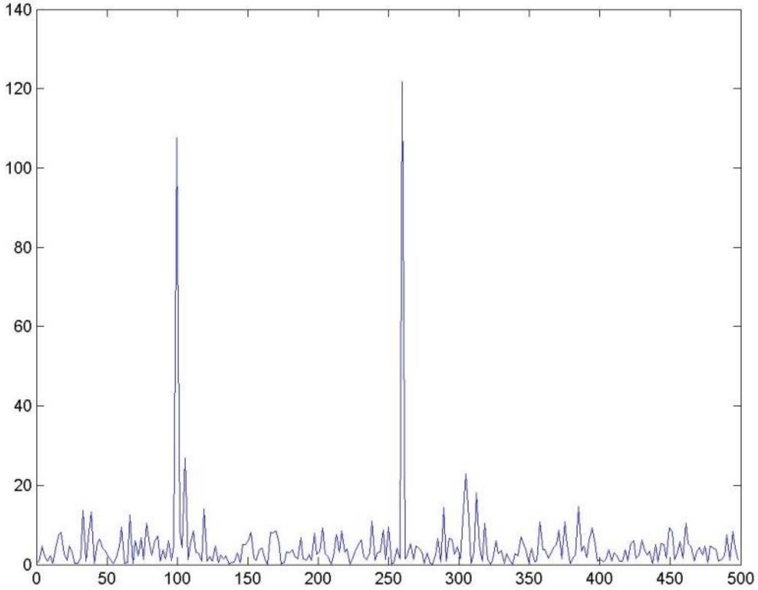
$$\boxed{\begin{pmatrix} x[0] \\ \vdots \\ x[N-1] \end{pmatrix} = \tilde{X}(0) \begin{pmatrix} W^*_{0,0} \\ \vdots \\ W^*_{0,N-1} \end{pmatrix} + \cdots \tilde{X}(N-1) \begin{pmatrix} W^*_{N-1,0} \\ \vdots \\ W^*_{N-1,N-1} \end{pmatrix}}$$

where

$$W^*_{k,n} = e^{jk\frac{2\pi}{N}n}$$

An Fourier Analysis Example





t = 0:0.001:0.6;   f = 1000*(0:255)/512;

x = **cos** (2*pi*100*t)+**sin**(2*pi*260*t);

y = x + 2*randn(size(t)); Y = fft(y, 512);

P = Y.*conj(Y)/512;

figure(1); plot(t, y); figure(2); plot(f, P(1:256));

References

- [1] A. V. Oppenheim, A. S. Willsky and I. T. Young, Signals and Systems, Prentice-Hall, 1983.

Thank You!

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