Fourier Transform (I)

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Image Representation

PRIORI BASIS FOR NATURAL IMAGES

Basic Exponential Signals (I)

- Basic building signals $e^{j\omega t}$ and $e^{j\Omega n}$
  - ω and Ω denote the radian frequencies in the continuous-time and discrete-time domains, respectively.

- The properties of continuous-time exponential signals $e^{j\omega t}$
  - This is a **periodic** signal
    - The fundamental period is $T=2\pi/\omega$

$$\boxed{e^{j\omega t} = e^{j\omega(t+kT)} = e^{j\omega(t+k\frac{2\pi}{\omega})}}$$

  - the larger the magnitude of ω, the higher rate of oscillation of the signal.

  - $e^{j\omega_1 t}$ and $e^{j\omega_2 t}$ are **orthogonal** to each other whenever $|\omega_1| \neq |\omega_2|$

To prove $e^{j\omega_1 t}$ and $e^{j\omega_2 t}$ are **orthogonal** to each other whenever $|\omega_1| \neq |\omega_2|$
Because they are periodic signals, we only need to prove they are orthogonal to each other within a common period.

Now assume T is the least common multiple of $T_1 = 2\pi/\omega_1$ and $T_2 = 2\pi/\omega_2$

$$Assume \quad T = k_1 T_1 = k_1 \frac{2\pi}{\omega_1} \quad and \quad T = k_2 T_2 = k_2 \frac{2\pi}{\omega_2} \quad (k_1, k_2 \in Z^+)$$

$$\int_0^T e^{j\omega_1 t}\overline{e^{j\omega_2 t}}dt = \int_0^T e^{j(\omega_1-\omega_2)t}dt = \frac{1}{j(\omega_1-\omega_2)} e^{j(\omega_1-\omega_2)t}\Big|_0^T$$

$$= \frac{1}{j(\omega_1-\omega_2)}(e^{j(k_1-k_2)2\pi}-1) = 0$$

Basic Exponential Signals (II)

- The properties of discrete-time exponential signals $e^{j\Omega n}$

$$\boxed{e^{j\Omega n} = e^{j\Omega(n+N)} = e^{j(\Omega n + 2\pi m)}}$$

  - is **NOT** periodic for **arbitrary** values of Ω
    - Only when $\Omega/2\pi = m/N$ (m and N are integers, i.e. only when $\Omega/2\pi$ is a rational number)
    - But $e^{j\Omega n}$ is a **periodic** signal *w.r.t.* Ω. $\boxed{e^{j\Omega n} = e^{j(\Omega + 2\pi)n}}$

  - $e^{j\Omega n}$ are **NOT** *distinct*, as the signal with frequency Ω_0 is identical to the signals with frequencies $(\Omega_0 \pm 2\pi)$, $(\Omega_0 \pm 4\pi)$, … and so on $(\Omega_0 \pm 2\pi k)$.

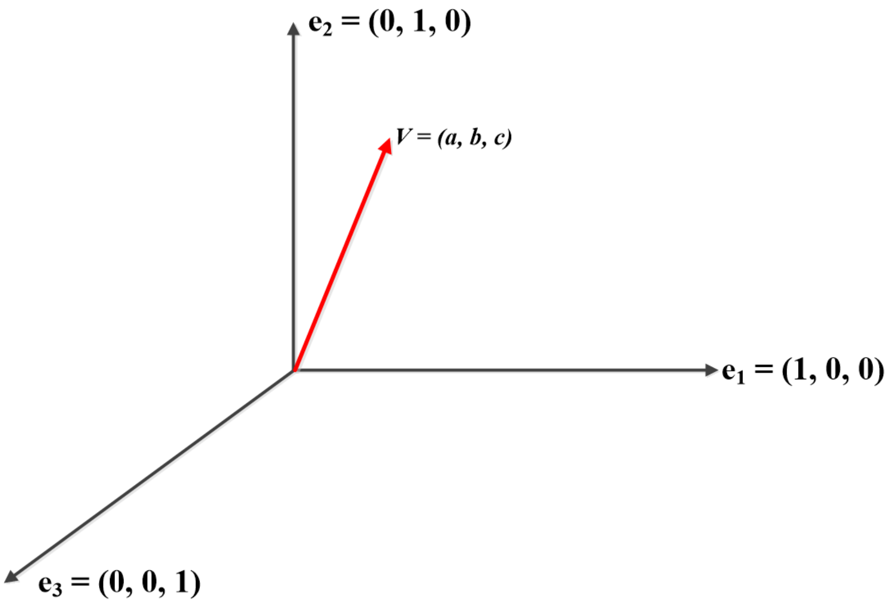
$e^{j\omega t}$ vs. $e^{j\Omega n}$

$e^{j\omega t}$	$e^{j\Omega n}$
Distinct signals for distinct *magnitudes* of ω	Identical signals for exponential at frequencies separated by 2π
Periodic for any choice of ω	Periodic **only if** $\Omega_0 = 2\pi(m/\mathrm{N})$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency if it is periodic: Ω_0 /m
Fundamental period $\omega_0=0$: undefined $\omega_0 \neq 0$: $2\pi/\omega_0$	Fundamental period if it is periodic $\Omega_0=0$: undefined $\Omega_0 \neq 0$: $m(2\pi/\Omega_0)$

$e^{j\omega_1 t}$ and $e^{j\omega_2 t}$ are **orthogonal** to each other whenever $|\omega_1| \neq |\omega_2|$

Euclidean Geometric Space

- 3D Euclidean Geometric Space
  - 3 basic vectors



  - **Analysis**

$$\boxed{a = \mathbf{v} \bullet \mathbf{e}_1 \quad b = \mathbf{v} \bullet \mathbf{e}_2 \quad c = \mathbf{v} \bullet \mathbf{e}_3}$$

  - **Synthesis**

$$\boxed{\mathbf{v} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3}$$

Signal Space

- The number of basic signals is infinite.
- This is a complex space
  - **Analysis**

$$\boxed{X(\omega) = \int_{-\infty}^{+\infty} x(t)\overline{e^{j\omega t}}dt = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt}$$

  - **Synthesis**

$$\boxed{x(t) = \int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega}$$

Continuous-Time Fourier Transform

- A representation of continuous-time **aperiodic** signals

  - Analysis

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt \quad or \quad X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

  - Synthesize

$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{j2\pi ft} df \quad or \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

Dirichlet Conditions

- A continuous-time **aperiodic** signal $x(t)$ has a Fourier transform only if it satisfies the following conditions:
  - 1) $x(t)$ is absolutely integrable, namely

$$\boxed{\int_{-\infty}^{+\infty} |x(t)|\, dt < \infty}$$

  - 2) $x(t)$ has only a finite number of maximum and minima within any finite interval.
  - 3) $x(t)$ has only a finite number of discontinuities within any finite interval.

References

- [1] A. V. Oppenheim, A. S. Willsky and I. T. Young, Signals and Systems, Prentice-Hall, 1983.

Thank You!

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