

Principal Component Analysis (PCA)

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Image Representation

POSTERIOR BASIS FOR NATURAL IMAGES

Basis for Natural Images

- **Priori Basis**
 - Fourier transforms
 - DCT
 - ...
- **Posterior Basis (data-driven)**
 - To estimate the linear transform from the data itself, and the transform could be ideally adapted to the kind of data that is being processed.
 - PCA
 - Dictionary based on machine learning

Two Properties of Image Transforms

- Variable decoupling
 - The coefficients of these bases are **less correlated** or become **independent** in ideal cases.
- Dimension reduction
 - The number of bases for approximately reconstructing an image is often much smaller than the number of pixels.

PCA ^[1]

- Either DFT or DCT has “energy compaction” property.
- What is the **optimal** transform in terms of energy compaction?
- Two basic concepts in statistical analysis: *variance* and *covariance* of random vectors

Variance and Covariance

- Consider two random vectors with **zero means** $\mathbf{a} = [a_1, a_2, \dots, a_n]^T$, $\mathbf{b} = [b_1, b_2, \dots, b_n]^T$.
- The **variance** of \mathbf{a} and \mathbf{b} are defined as

$$\sigma_{\mathbf{a}}^2 = \langle a_i \quad a_i \rangle_i = \sum_{i=1}^n a_i^2$$

$$\sigma_{\mathbf{b}}^2 = \langle b_i \quad b_i \rangle_i$$

– where the expectation $\langle \rangle_i$ is the average over n variables.

- The **covariance** between \mathbf{a} and \mathbf{b} is a straight-forward generalization:

$$\sigma_{\mathbf{ab}}^2 = \langle a_i \quad b_i \rangle_i$$

Sample Covariance Matrix

- We generalize two random vectors to an arbitrary number. Suppose we have m random vectors, and each vector has n variables.
- The $n \times m$ observation matrix \mathbf{X} is defined as

$$\mathbf{X} = [\mathbf{x}_1 \quad \cdots \quad \mathbf{x}_m] = \begin{bmatrix} x_{11} & \cdots & x_{m1} \\ \vdots & \ddots & \vdots \\ x_{1n} & \cdots & x_{mn} \end{bmatrix}$$

- The **sample** covariance matrix is estimated

$$C_{\mathbf{X}} = \frac{1}{m-1} \mathbf{X}\mathbf{X}^T$$

Goal of PCA

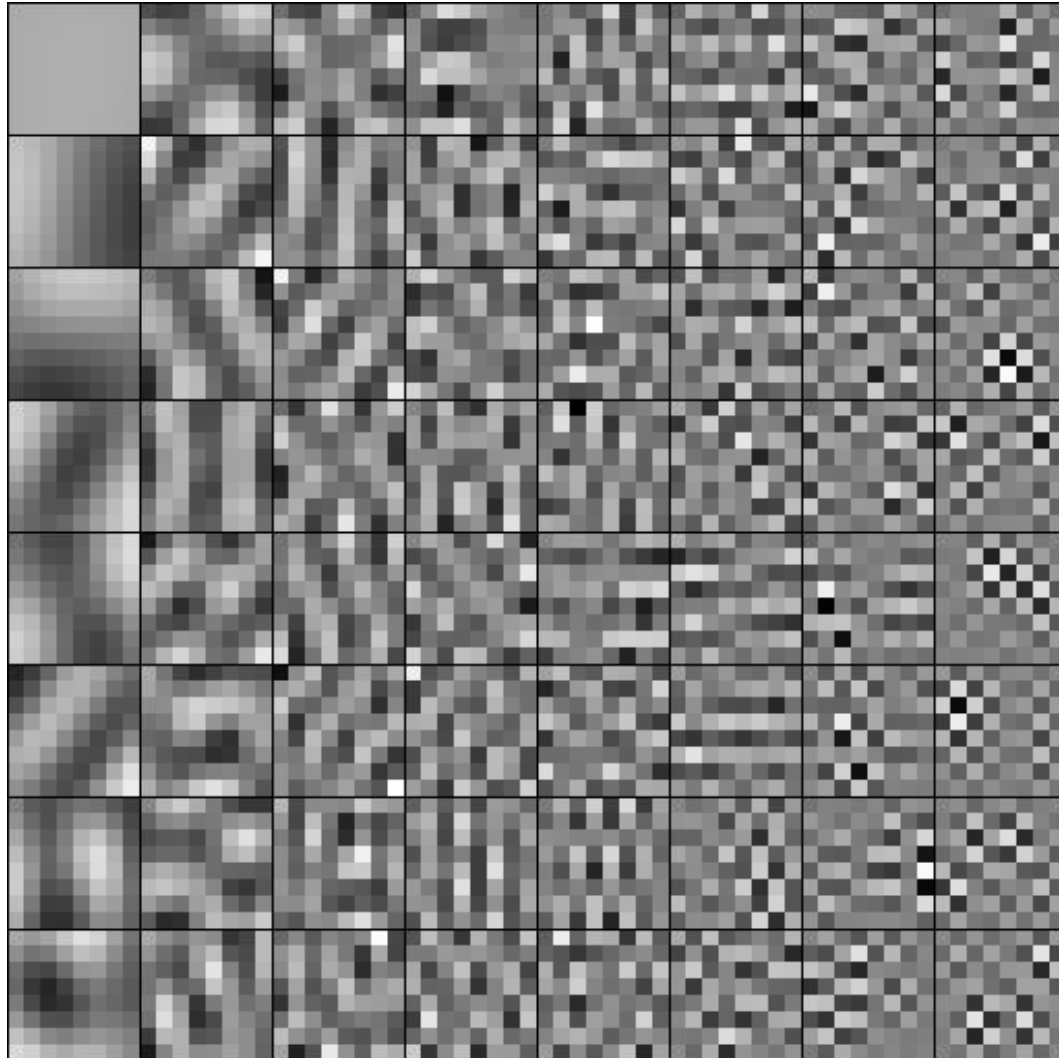
- The goal of PCA is to find *a set of basis* in which the observation matrix \mathbf{X} is transformed to a new observation matrix \mathbf{Y} , and *the covariance matrix* of \mathbf{Y} will be a **diagonal** matrix — to remove correlation among data whereas to preserve the energy as much as possible.
- Assumptions of PCA:
 - Firstly all basis vectors $\{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ are **orthonormal** (i.e. $\mathbf{p}_i \bullet \mathbf{p}_j = \delta_{ij}$), that is all basis vectors construct an orthonormal matrix \mathbf{P} .
 - Secondly, *the directions with the largest variances* are more important than those with low variances, so in the transformed space the energy of signals is kept as compact as possible.

Eigen-Value Decomposition

- In the linear algebra, the covariance matrix $C_{\mathbf{x}}$ can be *diagonalized* by an orthogonal matrix of its eigen-vectors.
- Eigen-value decomposition: $C_{\mathbf{x}} = \mathbf{E}\mathbf{D}\mathbf{E}^T$
- If the orthonormal transform matrix \mathbf{P} is selected as $\mathbf{P} = \mathbf{E}^T$,
 $\mathbf{Y} = \mathbf{P}\mathbf{X} = \mathbf{E}^T\mathbf{X}$

$$\begin{aligned}C_Y &= \frac{1}{m-1} \mathbf{Y}\mathbf{Y}^T \\&= \frac{1}{m-1} (\mathbf{E}^T \mathbf{X})(\mathbf{E}^T \mathbf{X})^T \\&= \frac{1}{m-1} \mathbf{E}^T \mathbf{X}\mathbf{X}^T \mathbf{E} \\&= \mathbf{E}^T C_{\mathbf{x}} \mathbf{E} \\&= \mathbf{E}^T \mathbf{E} \mathbf{D} \mathbf{E}^T \mathbf{E} \\&= \mathbf{D}\end{aligned}$$

64 8×8 principal components (or basic images) learned from the image “Lena”



A toy example of PCA

1) Divide an image with the size of 512×512 into 8×8 blocks

2) Observation Matrix \mathbf{X} : 64×4096

3) The Sampling Covariance Matrix \mathbf{C}_X : 64×64

4) Eigenvalue Decomposition of \mathbf{C}_X : $\mathbf{C}_X = \mathbf{E} \mathbf{D} \mathbf{E}^T$

5) We select the eigenvectors with the first k largest eigenvalues, and form a PCA transform matrix \mathbf{P} (\mathbf{P} : $64 \times k$, $k \ll 64$)

6) Now we can transform the observation matrix \mathbf{X} into another space by the PCA matrix \mathbf{P} ($\mathbf{Y} = \mathbf{P}^T \mathbf{X}$ and \mathbf{Y} : $k \times 4096$)

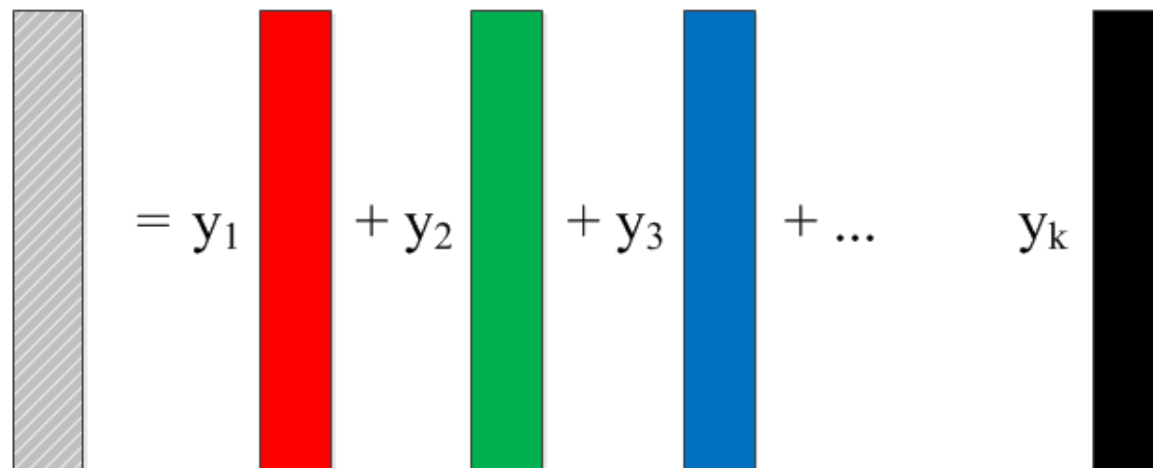
7) Reconstruction: $\mathbf{X}_R = \mathbf{P} \mathbf{Y}$ (Note $\mathbf{X}_R \approx \mathbf{X}$)

PCA Forward Transform: $\mathbf{y} = \mathbf{P}^T \mathbf{x}$



The diagram illustrates the PCA forward transform equation $\mathbf{y} = \mathbf{P}^T \mathbf{x}$. On the left, a vertical gray bar represents the vector \mathbf{y} , with segments labeled y_1 , y_2 , \dots , and y_k . This is followed by an equals sign, an opening parenthesis, a matrix of four horizontal bars (red, green, blue, and black) with a vertical ellipsis between the blue and black bars, and a closing parenthesis. This is followed by a multiplication sign and a tall, thin gray vertical bar representing the matrix \mathbf{P} .

PCA Inverse Transform: $\mathbf{x} = \mathbf{P} \mathbf{y}$



The diagram illustrates the PCA inverse transform equation $\mathbf{x} = \mathbf{P} \mathbf{y}$. On the left, a tall, thin gray vertical bar with diagonal hatching represents the vector \mathbf{x} . This is followed by an equals sign, the label y_1 , a red vertical bar, a plus sign, the label y_2 , a green vertical bar, a plus sign, the label y_3 , a blue vertical bar, a plus sign, an ellipsis, and finally the label y_k next to a black vertical bar. This represents the reconstruction of \mathbf{x} as a weighted sum of the principal components.

Reconstructed images with different number of PCs

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References

- [1] M. J. T. Smith and A. Docef, A Study Guide for Digital Image Processing, Scientific Publishers, Inc. Riverdale, Georgia, 1999.

Thank You!

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