Fourier Transform (II)

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Image Representation

PRIORI BASIS FOR NATURAL IMAGES

Continuous-time Fourier Series

- Suppose $x(t)$ is a continuous-time **periodic** signal: $x(t) = x(t + kT_0)$
  - The basic signals are $e^{jk\omega_0 t}$ $(k = 0, \pm 1, \pm 2, \ldots)$ $(\omega_0 = 2\pi/T_0)$
- Analysis

$$\boxed{a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt}$$

  – The coefficients $\{a_k\}$ are often called the Fourier series coefficients or the spectral coefficients of $x(t)$

- Synthesis

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}}$$

  – If $x(t)$ is a real signal, then $a_k = a_{-k}$.

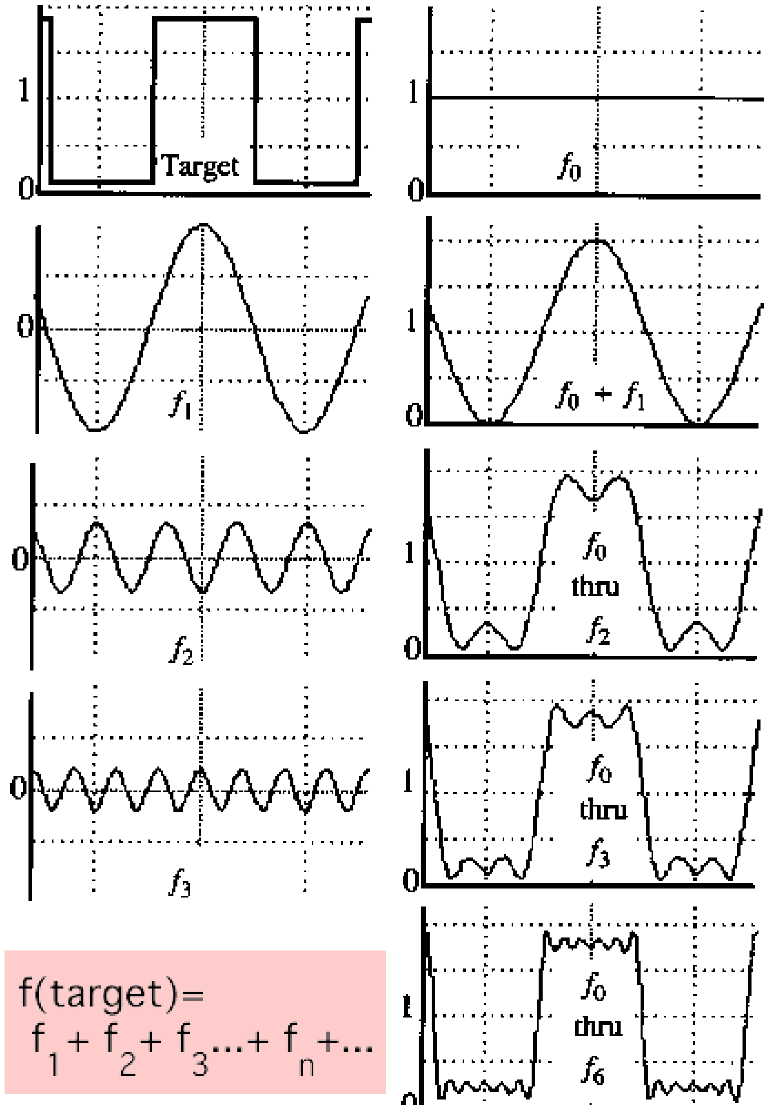
Dirichlet Conditions

- A **periodic** signal $x(t)$, has a Fourier series only if it satisfies the following conditions:
  - 1) $x(t)$ is absolute integrable over any period, namely

$$\boxed{\int_{\alpha}^{\alpha+\mathrm{T}} |x(t)|\, dt < \infty \quad \alpha \in R}$$

  - 2) $x(t)$ has only a finite number of maximum and minima over any period
  - 3) $x(t)$ has only a finite number of discontinuities over any period

An Periodic Square Wave



f(target)=
$f_1 + f_2 + f_3 ... + f_n + ...$

Discrete-time Fourier Series

- How about $x[n]$ is a *discrete*-time **periodic** signal?
  - $x[n] = x[n + N]$ or $x[n] = x[n + kN]$ ($k \in Z$)
  - Now if we divide the circle 2π into N points, we will get N different discrete frequencies $e^{jk\frac{2\pi}{N}n}$, $k \in \{0, 1, \ldots, N-1\}$ or $k \in \{1, 2, \ldots, N\}$, and so on. ($k = <N>$)
  - $e^{jk_1\frac{2\pi}{N}n}$ and $e^{jk_2\frac{2\pi}{N}n}$ are orthogonal to each other whenever $k_1 \neq k_2$ and k_1 , $k_2 \in <N>$ (the set of N consecutive integer numbers)
  - **An important distinction** between the set of harmonically related signals in discrete-time and continuous-time is
    - There are **only** N different signals $e^{jk\frac{2\pi}{N}n}$ in the set $k = 0, \pm 1, \pm 2, \ldots$
    - Whereas all of the $e^{jk\omega_0 t}$ ($k = 0, \pm 1, \pm 2, \ldots$) are **distinct**.

Discrete-time Fourier Series

- Suppose $x[n]$ is a periodic signal in discrete-time domain
  - Remember that we only have N different signals in the set $k = 0, \pm 1, \pm 2, \ldots$

- Analysis

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\frac{2\pi}{N}n}$$

- Synthesis

$$x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n}$$

Discrete-time Fourier Transform

- Now suppose $x[n]$ is an **aperiodic** signal in *discrete*-time domain
  - Analysis

$$\boxed{X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}}$$

  - Synthesis

$$\boxed{x[n] = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega}$$

  - The continuous periodic in the frequency domain

$$\boxed{X(\Omega + 2\pi) = X(\Omega)}$$

Summary of Basis Signals

- **Continuous-time**
  - Fourier Series for periodic signals — $e^{jk\omega_0 t}$
  - Fourier Transform for aperiodic Signals — $e^{j\omega t}$
- **Discrete-time**
  - Discrete-time Fourier series for periodic signals — $e^{jk\frac{2\pi}{N}n}$
  - Discrete-time Fourier Transform for aperiodic Signals — $e^{j\Omega n}$

References

- [1] A. V. Oppenheim, A. S. Willsky and I. T. Young, Signals and Systems, Prentice-Hall, 1983.

Thank You!

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