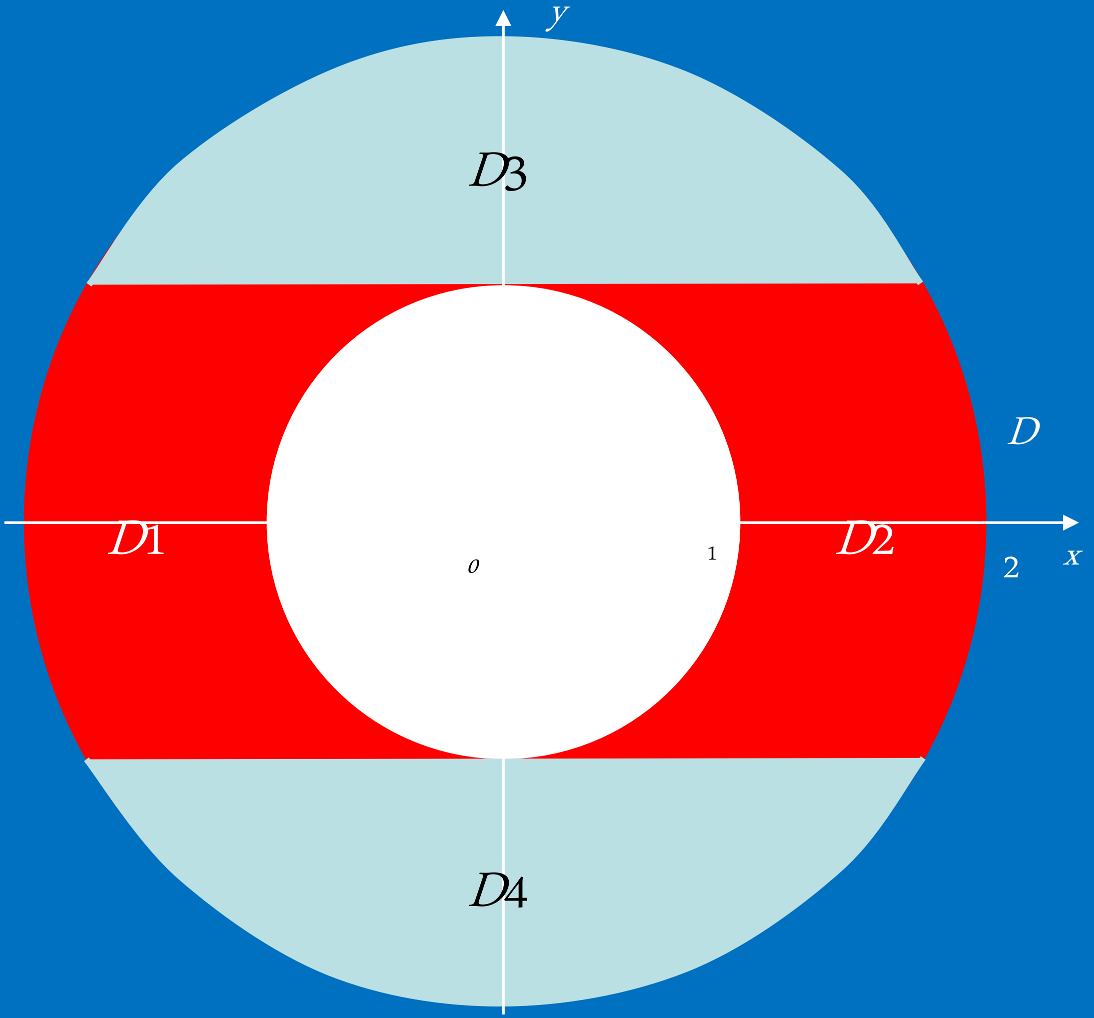
极坐标系下二重积分的计算方法

$$D: 1 \le x^2 + y^2 \le 4$$

计算 $\iint\limits_D f(x, y)\mathrm{d}x\mathrm{d}y$

$$\iint\limits_D = \iint\limits_{D_1} + \iint\limits_{D_2} + \iint\limits_{D_3} + \iint\limits_{D_4}$$



如果

二重积分的积分区域 D 用极坐标表示比较简单；

被积函数的二重积分在直角坐标系下无法计算或很难计算时，

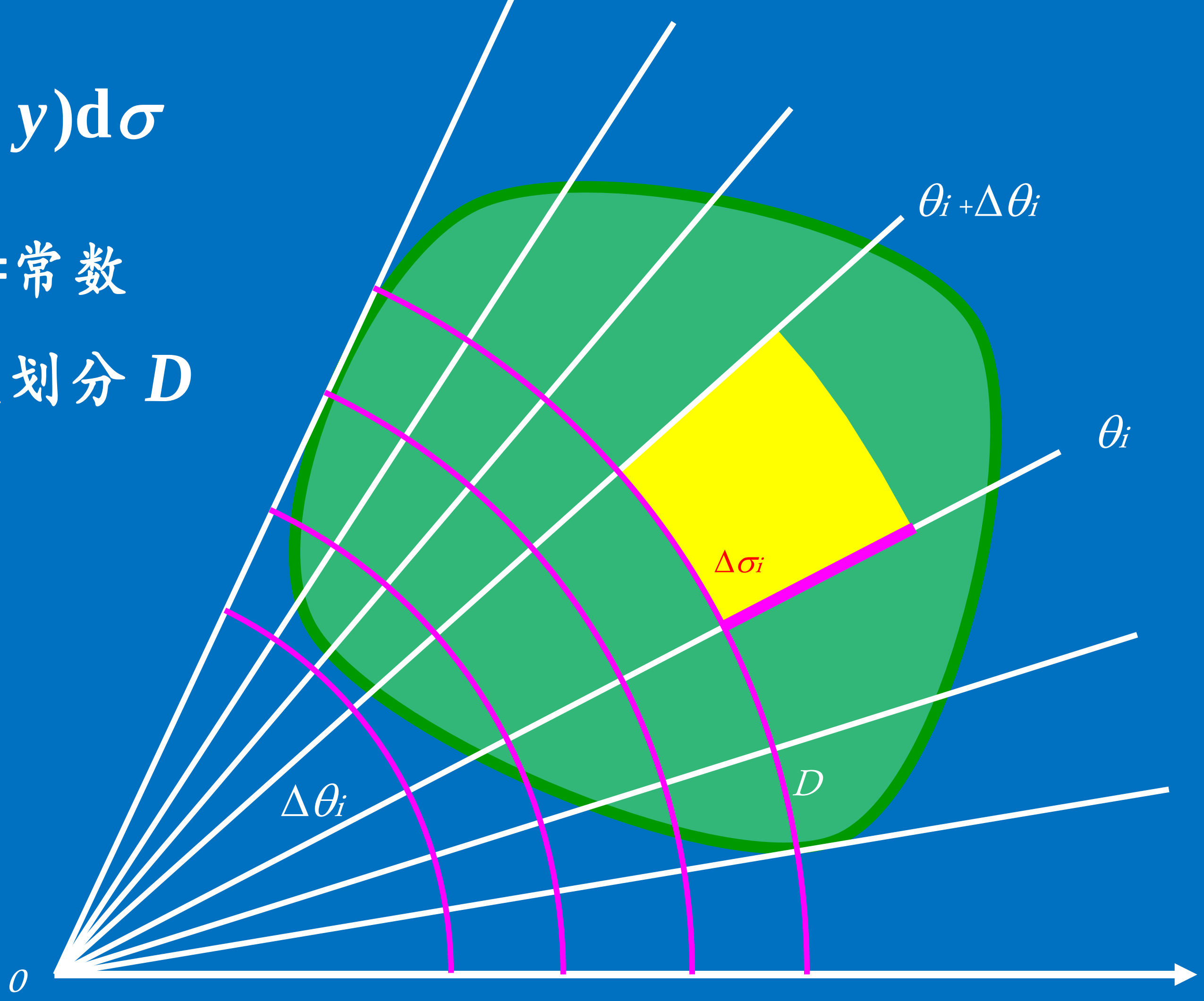
是不是可以考虑在极坐标系下计算二重积分？

计算 $I=\iint\limits_{D} f(x,y)\mathrm{d}\sigma$

用一族同心圆 ρ =常数

一族射线 θ =常数划分 D

$\mathrm{d}\sigma=?$



$$\Delta\sigma_i = \frac{1}{2}(\rho_i + \Delta\rho_i)^2\Delta\theta_i - \frac{1}{2}\rho_i{}^2\Delta\theta_i$$

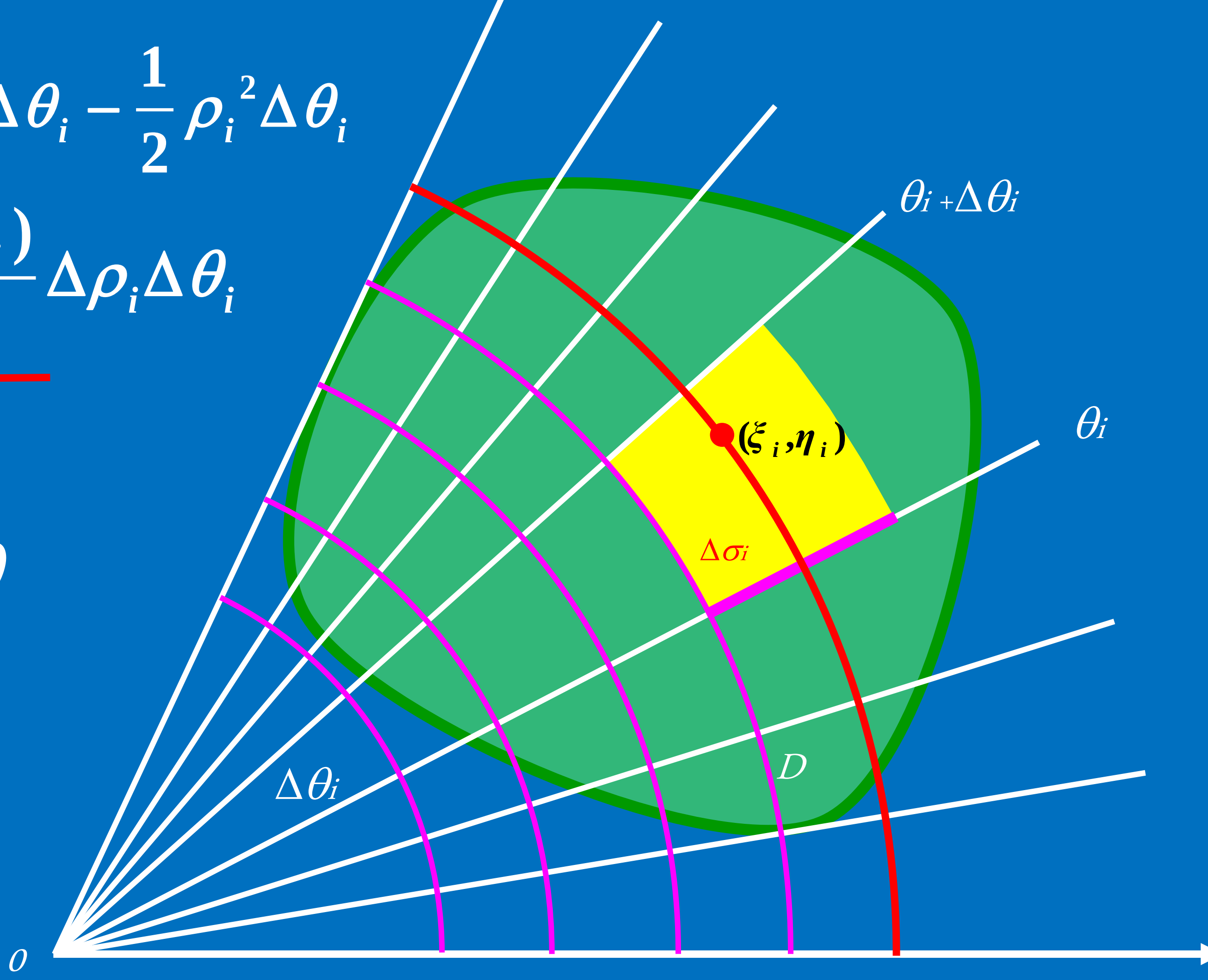
$$= \frac{\rho_i + (\rho_i + \Delta\rho_i)}{2}\Delta\rho_i\Delta\theta_i$$

$$= \bar{\rho}_i\ \Delta\rho_i\Delta\theta_i$$

$$\mathrm{d}\sigma = \rho\mathrm{d}\rho\mathrm{d}\theta$$

取 $\xi_i = \bar{\rho}_i \cos\bar{\theta}_i$,

$\eta_i = \bar{\rho}_i \sin\bar{\theta}_i$

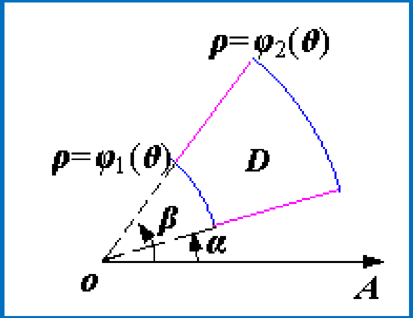


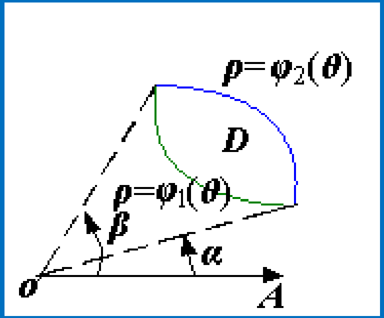
$$I = \iint_D f(x, y)\mathrm{d}\sigma = \lim \sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta\sigma_i$$

$$= \lim \sum_{i=1}^{n} f(\bar{\rho}_i \cos\bar{\theta}_i, \bar{\rho}_i \sin\bar{\theta}_i)\bar{\rho}_i \Delta\rho_i \Delta\theta_i$$

$$= \iint_D f(\rho\cos\theta, \rho\sin\theta)\rho\mathrm{d}\rho\mathrm{d}\theta$$

设 D： $\varphi_1(\theta) \le \rho \le \varphi_2(\theta)$, $\alpha \le \theta \le \beta$.





则 $\iint\limits_D f(\rho\cos\theta, \rho\sin\theta)\rho \mathrm{d}\rho \mathrm{d}\theta$

$$= \int_\alpha^\beta \mathrm{d}\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(\rho\cos\theta, \rho\sin\theta)\rho \mathrm{d}\rho$$

极坐标系下二重积分的例题

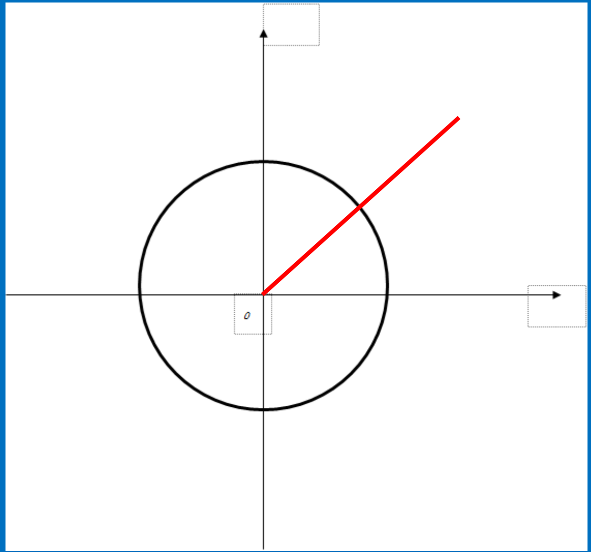
例　计算：$\iint\limits_D e^{-x^2-y^2}\,dxdy$　其中 $D: x^2+y^2\le a^2$.

解　$D: 0\le\theta\le 2\pi,\ 0\le\rho\le a$

$$\iint\limits_D e^{-x^2-y^2}\,dxdy = \iint\limits_D e^{-\rho^2}\rho\,d\rho\,d\theta$$

$$=\int_0^{2\pi}\left[\int_0^a e^{-\rho^2}\rho\,d\rho\right]d\theta$$

$$=\int_0^{2\pi}\left[-\frac{1}{2}e^{-\rho^2}\right]_0^a d\theta = \pi(1-e^{-a^2})$$



例　计算：$\int_0^{+\infty} \mathrm{e}^{-x^2}\mathrm{d}x$

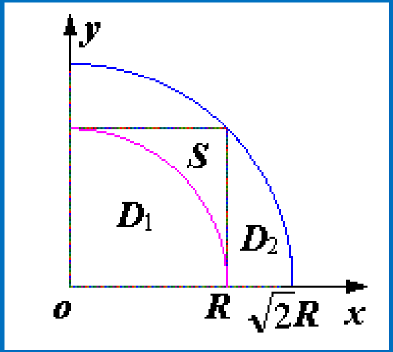
解　记 $I(R)=\int_0^R \mathrm{e}^{-x^2}\mathrm{d}x$，

$$I^2(R)=\left(\int_0^R \mathrm{e}^{-x^2}\mathrm{d}x\right)\left(\int_0^R \mathrm{e}^{-x^2}\mathrm{d}y\right)$$

$$=\int_0^R \mathrm{e}^{-x^2}\mathrm{d}x\int_0^R \mathrm{e}^{-y^2}\mathrm{d}y$$

$$=\iint_S \mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y$$

例　计算：$\int_0^{+\infty} \mathrm{e}^{-x^2}\mathrm{d}x$



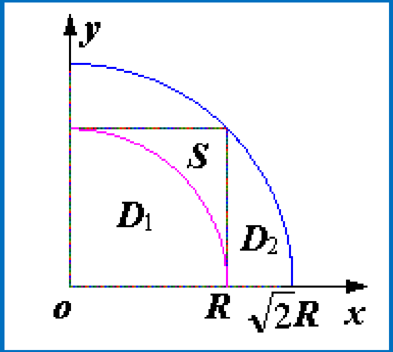
解　设 $S: 0 \le x \le R, 0 \le y \le R$，

$D_1: x^2 + y^2 \le R^2$，　$x > 0, y > 0$，

$D_2: x^2 + y^2 \le 2R^2$，　$x > 0, y > 0$，由于 $\mathrm{e}^{-x^2-y^2} > 0$，所以有

$$\iint_{D_1} \mathrm{e}^{-x^2-y^2}\mathrm{d}x\mathrm{d}y < \iint_{S} \mathrm{e}^{-x^2-y^2}\mathrm{d}x\mathrm{d}y < \iint_{D_2} \mathrm{e}^{-x^2-y^2}\mathrm{d}x\mathrm{d}y$$

$$\iint_{D_1} e^{-x^2-y^2}\,dx\,dy = \frac{\pi}{4}(1-e^{-R^2}),$$



$$\iint_{D_2} e^{-x^2-y^2}\,dx\,dy = \frac{\pi}{4}(1-e^{-2R^2})$$

$$\frac{\pi}{4}(1-e^{-R^2}) < \left(\int_0^R e^{-x^2}\,dx\right)^2 < \frac{\pi}{4}(1-e^{-2R^2})$$

$$\frac{\pi}{4}(1-\mathrm{e}^{-R^2}) < \left(\int_0^R \mathrm{e}^{-x^2}\mathrm{d}x\right)^2 < \frac{\pi}{4}(1-\mathrm{e}^{-2R^2})$$

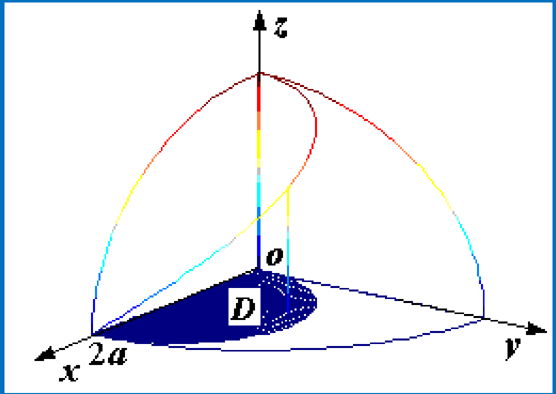
令： $R \to +\infty$，则有

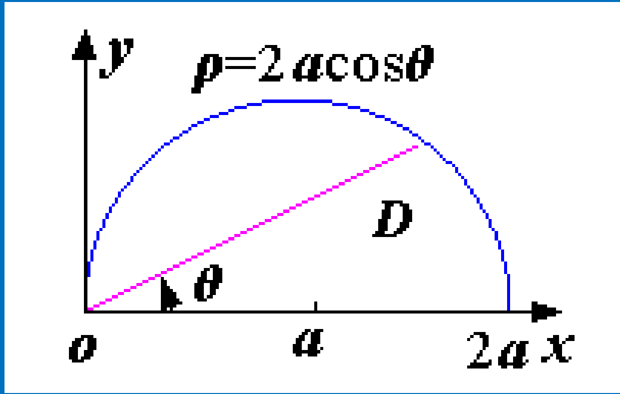
$$\int_0^{+\infty} \mathrm{e}^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

例　求球体 $x^2+y^2+z^2\le 4a^2$ 被圆柱面 $x^2+y^2=2ax$

$(a>0)$ 所截得的(含在圆柱面的内部)立体的体积.

解





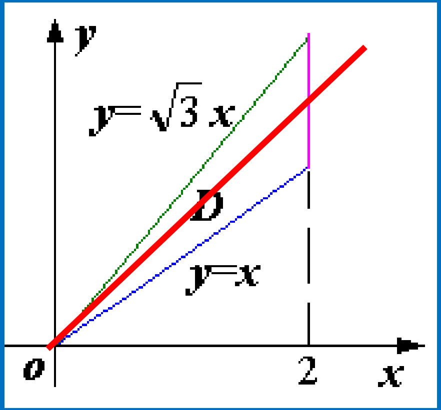
由对称性有 $V=4\iint\limits_D \sqrt{4a^2-x^2-y^2}\,dxdy.$

其中 $D: 0 \le \theta \le \dfrac{\pi}{2},\ 0 \le \rho \le 2a\cos\theta.$

$$V=4\iint\limits_{D}\sqrt{4a^2-x^2-y^2}\,\mathrm{d}x\mathrm{d}y=4\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{2a\cos\theta}\sqrt{4a^2-\rho^2}\,\rho\,\mathrm{d}\rho$$

$$=\frac{32}{3}a^3\int_0^{\frac{\pi}{2}}(1-\sin^3\theta)\mathrm{d}\theta=\frac{32}{3}a^3\left(\frac{\pi}{2}-\frac{2}{3}\right)$$

例　化二次积分 $\int_0^2 \mathrm{d}x \int_x^{\sqrt{3}x} f(\sqrt{x^2+y^2})\mathrm{d}y$ 为极坐标系下的二次积分



解　$D: 0 \le x \le 2,\ x \le y \le \sqrt{3}x$

即 $D: \dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{3},\ 0 \le \rho \le \dfrac{2}{\cos\theta}$

原式$=\iint\limits_D f(\sqrt{x^2+y^2})\mathrm{d}x\mathrm{d}y = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \mathrm{d}\theta \int_0^{\frac{2}{\cos\theta}} f(\rho)\rho \mathrm{d}\rho$