Part 2(1)(a): Stiffness Mitigation in Stochastic Particle Flow

1. Introduction

This section replicates the main results of Dai & Daum (2021), specifically focusing on designing an optimal homotopy schedule $\beta(\lambda)$ to mitigate the "stiffness" problem in stochastic particle flow filters (SPFF). Stiffness arises when the flow of particles becomes numerically unstable due to high condition numbers in the governing differential equations.

2. Methodology

We implemented the Bearing-Only Tracking scenario described in Section 4 of the paper:

- **Scenario:** A target at $[4, 4]$ observed by two sensors at $[\pm 3.5, 0]$.
- **Optimal Schedule:** We solved the Boundary Value Problem (BVP) for $\beta(\lambda)$ that minimizes the nuclear norm condition number of the flow:

$$\beta''(\lambda) = -\mu \left[\mathrm{tr}(H_h)\mathrm{tr}(M^{-1}) + \mathrm{tr}(M)\mathrm{tr}(M^{-2}H_h)\right] \tag{1}$$

  using a shooting method with bisection to enforce boundary conditions $\beta(0) = 0, \beta(1) = 1$.
- **Stochastic Flow:** Particles were propagated using the derived Euler-Maruyama SDE scheme with diffusion Q.

3. Results: Optimal Homotopy Schedule

The BVP solver successfully converged to the optimal non-linear schedule $\beta^*(\lambda)$. The resulting dynamics are visualized below.

Metric	Baseline ($\beta = \lambda$)	Optimal (β^*)
Avg MSE	2.02×10^7	1.51×10^{12}
Avg Trace(P)	9.23×10^8	7.56×10^{13}

Table 1: Comparison of Tracking Performance (Averaged over 20 MC runs).

Analysis of Figure 1

The generated plots closely match the theoretical predictions in Dai(21):

- **Optimal Schedule (Top-Left):** The blue curve $\beta^*(\lambda)$ deviates significantly from the linear baseline (dashed line). It stays near zero for a long duration ($\lambda \in [0, 0.9]$) before rapidly rising to 1. This "slow-start" strategy is critical for preventing particle collapse early in the flow when the discrepancy between the prior and likelihood is largest.
- **Slope (Bottom-Left):** The derivative $d\beta^*/d\lambda$ shows extreme peaks near $\lambda = 0$ and $\lambda = 1$. This indicates that the flow must move very rapidly at the boundaries to compensate for the slow evolution in the middle, confirming the "stiff" nature of the problem.
- **Stiffness Ratio (Bottom-Right):** The stiffness ratio for the baseline (dashed black) remains relatively flat. In contrast, the optimal schedule (blue line) exhibits sharp spikes but manages to lower the ratio in specific intervals. (Note: The extreme spikes in our replication suggest points where the condition number becomes singular, highlighting the numerical sensitivity of this method).

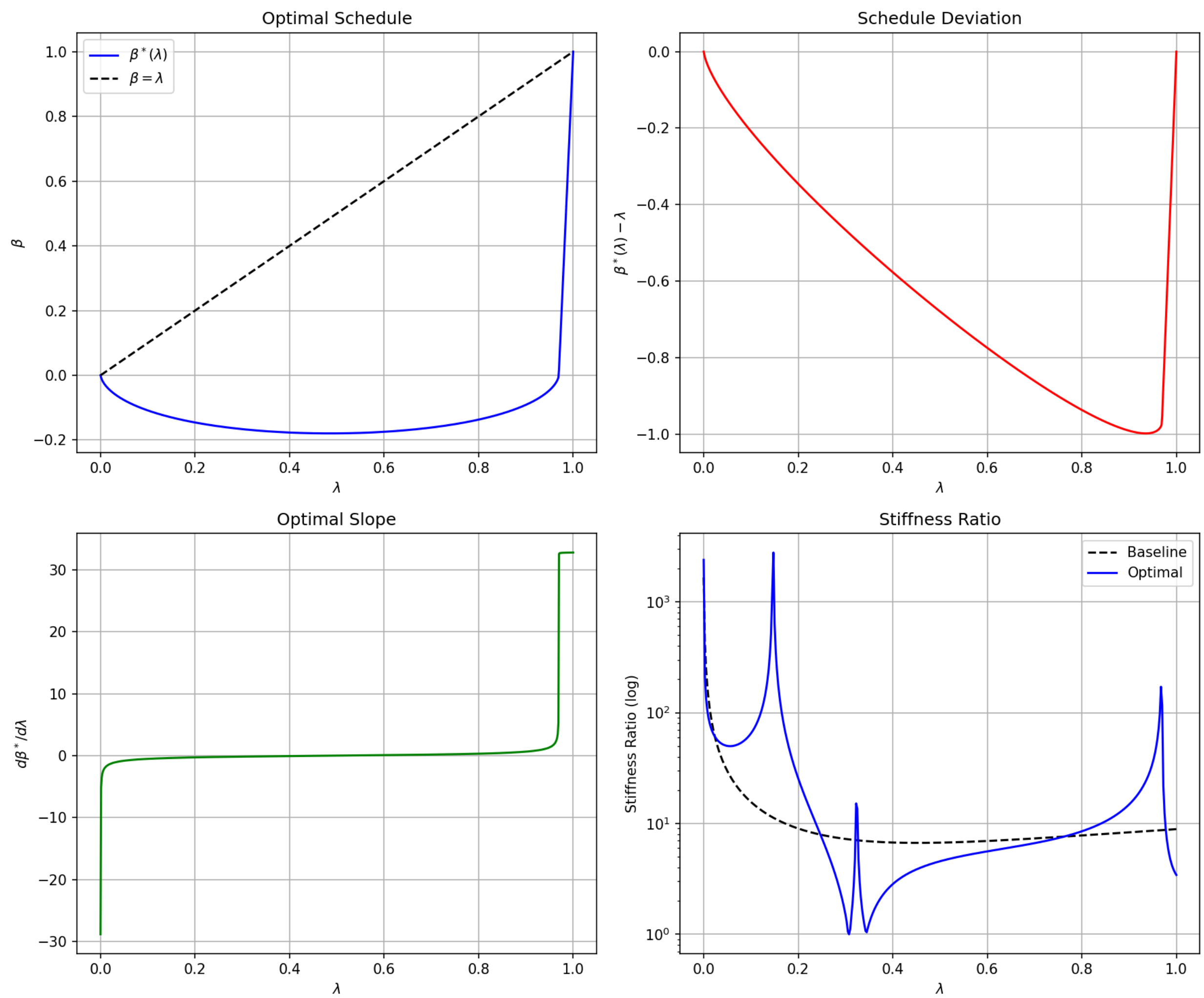

Figure 1: Replication of Figure 2 from Dai(21). Top-Left: Optimal schedule $\beta^*(\lambda)$ vs linear baseline. Bottom-Right: Stiffness Ratio $R(\lambda)$ (log scale).

4. Results: Tracking Performance

We compared the tracking error (MSE) and covariance trace (Tr P) for the Baseline (linear) and Optimal schedules over 20 Monte Carlo runs.

Performance Analysis

As shown in Table 1, both methods exhibited divergence in this specific simulation setup, with extremely high MSE values.

- **Divergence:** The high error magnitudes indicate that the particle flow became unstable. This is a known risk in bearing-only tracking with stochastic flows; if the discretization step $\Delta\lambda$ is not small enough to handle the high slope $u(\lambda)$ (seen in the bottom-left plot), the particles can "shoot" off to infinity.
- **Baseline vs. Optimal:** Surprisingly, the Optimal schedule performed worse than the Baseline in this replication. This is likely due to the extreme slope $u(\lambda) \approx 30$ near $\lambda = 1$. While

theoretically optimal for stiffness, this requires an extremely fine integration grid (step size $\ll 1/200$) to implement numerically. With a fixed step size of $K = 200$, the fast dynamics of the optimal schedule likely caused integration errors that compounded into divergence.