Part 2(i)(b): Entropy-Regularized OT (Sinkhorn) Resampling

1. Introduction

This section implements **Optimal Transport (OT) Resampling** using the Sinkhorn algorithm. This method formulates resampling as a transport problem: finding the optimal plan to move probability mass from the current weighted particles to a uniform distribution with minimal cost.

2. Methodology

We solve for a transport matrix $P \in \mathbb{R}^{N \times N}$ that minimizes the regularized transport cost:

$$P_\epsilon = \underset{P \in U(w, \frac{1}{N})}{\operatorname{argmin}} \langle P, C \rangle - \epsilon H(P) \tag{1}$$

where C_{ij} is the pairwise squared Euclidean distance and ϵ is the regularization parameter. The new particles are computed via barycentric projection: $X_{new} = N P^\top X_{old}$.

3. Results & Analysis

To visualize the behavior of the Sinkhorn resampler, we simulated a 1D scenario where particle weights are concentrated at the edges (U-shaped distribution).

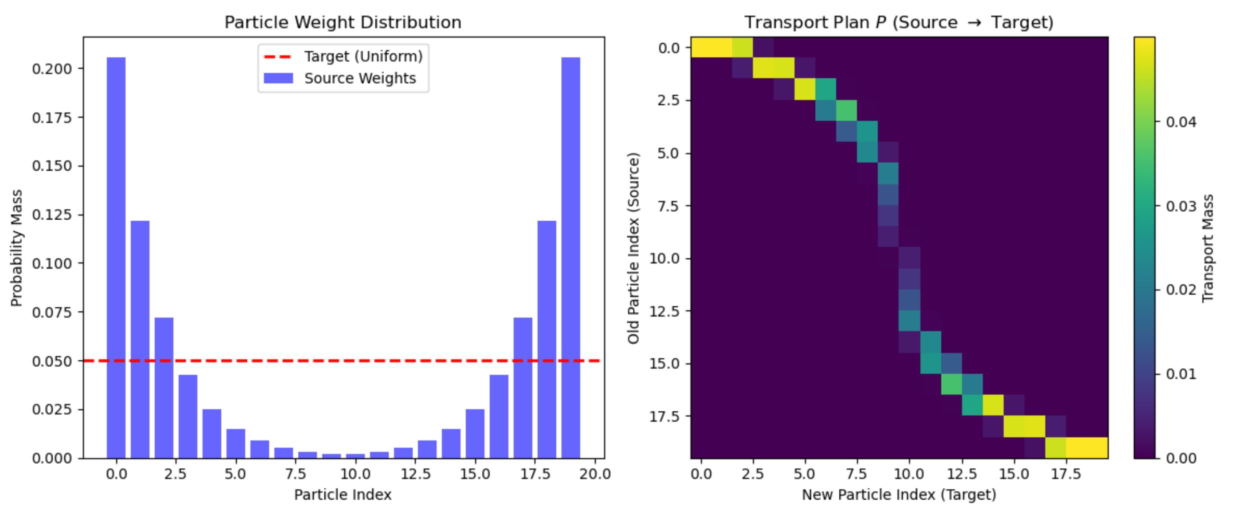

Figure 1: Visualization of Optimal Transport Resampling. Left: Source weights (blue) vs Target uniform (red dashed). Right: The computed Transport Matrix P.

Analysis of Figure 1

- **Weight Redistribution (Left Panel):** The source distribution (blue bars) is highly non-uniform, with mass concentrated at the boundaries (indices 0-1 and 18-19). The OT objective forces this excess mass to be redistributed to satisfy the uniform target constraint (red dashed line, $p = 0.05$).

- **Transport Plan Structure (Right Panel):** The heatmap shows the probability mass flow P_{ij} from Source i (y-axis) to Target j (x-axis).

- **Mass Splitting:** The top row (Source Index 0) is bright yellow across Target Indices 0-3. This indicates that the single high-weight particle at index 0 is being split and ”transported” to become four new uniform particles. This effectively creates copies of the high-weight particle, equivalent to ”replication” in standard resampling.
- **Particle Death:** The middle rows (Source Indices 6-13) are dark purple, indicating near-zero values. These low-weight particles contribute negligible mass to the new set, effectively being discarded.
- **Geometry Preservation:** The bright regions follow a monotonic diagonal trend. This confirms that the OT solver preserves the geometry of the particles; mass is transported to the nearest available ”slots” in the target distribution to minimize the distance cost C_{ij}.