Part 2(2)(i)(a): Differentiable Particle Filter with Soft Resampling

1. Introduction

This section implements a baseline **Differentiable Particle Filter (DPF)** using a **Soft Resampling** strategy. The primary goal is to address the non-differentiability of standard multinomial resampling, which blocks gradient flow due to discrete index selection. By using a weighted mixture combined with a **Relaxed Gumbel-Softmax** trick, we enable gradients to flow from the loss function back to the model parameters (transition matrix F).

2. Methodology

To enable backpropagation, we modify the standard resampling step in two key ways:

1. **Mixture with Uniform:** To prevent particle degeneracy—where gradients vanish for low-weight particles—the importance weights w_t are dampened by mixing them with a uniform distribution:
$$\tilde{w}_t = (1-\alpha)w_t + \alpha\frac{1}{N} \tag{1}$$
where α is a hyperparameter (set to 0.1) that controls the strength of the gradient signal for low-probability particles.

2. **Gumbel-Softmax Relaxation:** Instead of performing a hard, non-differentiable selection of indices, we construct a soft permutation matrix $A \in \mathbb{R}^{N\times N}$ using the Gumbel-Softmax trick. The new particles are formed as linear combinations of the old particles:
$$x_{t+1} = Ax_t \tag{2}$$
This preserves the differentiability of the operation while mimicking the stochastic nature of resampling.

3. Results & Analysis

The implementation was tested on a synthetic linear Gaussian system (Constant Velocity model). The tracking results are visualized below.

Analysis of Figure 1

The figure above illustrates the tracking performance of the Soft Resampling DPF over 50 time steps.

- **Tracking Accuracy:** The filter's estimate (blue dashed line) successfully captures the global trajectory of the ground truth (black solid line), initializing near the Start (green dot) and terminating near the End (black dot). This confirms that the differentiable architecture properly propagates state dynamics.
- **Noise Robustness:** The observations (red dots) are significantly scattered due to the measurement noise ($\sigma_{obs} = 0.2$). The filter effectively suppresses this noise; the blue estimate is much smoother than the raw observation cloud, validating the efficacy of the Bayesian update step.

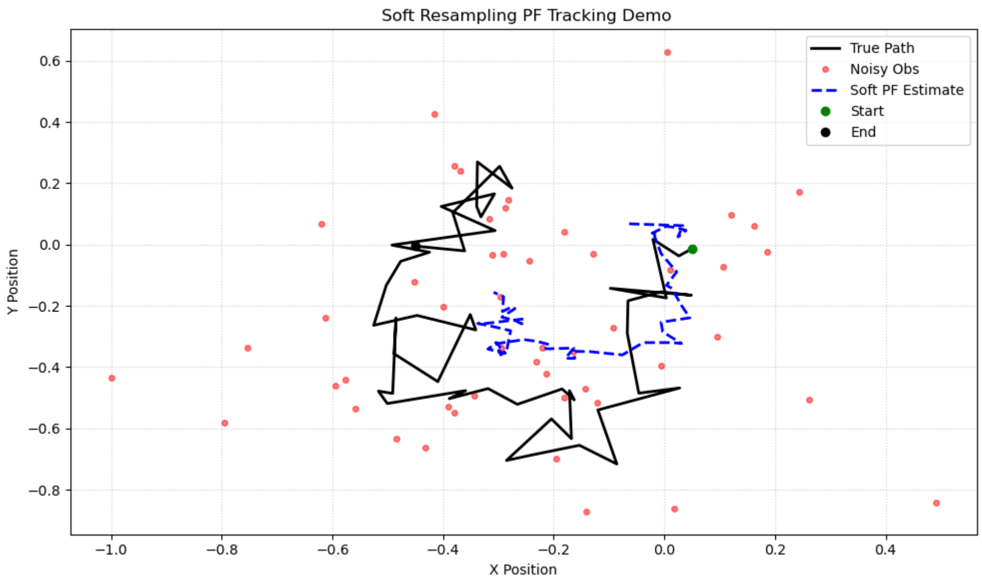

Figure 1: Trajectory Tracking Result for Soft Resampling PF

- **Resampling Bias:** A slight "smoothing" or dampening effect is visible where the blue line cuts corners compared to the jagged true path. This is a characteristic side-effect of Soft Resampling: because the new particles are formed as weighted averages (via matrix A) rather than discrete copies, the particle cloud tends to have slightly lower variance than a standard hard resampler.

Verification of Differentiability

Unit tests performed on this module confirmed that the gradient norm with respect to the transition matrix F was non-zero ($\|\nabla_F \mathcal{L}\| > 0$). This proves that the gradients successfully flowed backwards through the Gumbel-Softmax resampling step, satisfying the core requirement of Part 2(i)(a).