Chapter 2 Limits（极限）

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2.1-2.2 Concept of Limits

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2.1 Introduction to Limits (极限引论)

Overview:

- The concept of limit is one of the ideas that distinguish calculus from algebra and trigonometry.
  (极限概念是微积分区别于代数和三角的关键思想)

- All the important concepts will be explained by use of limits, such as Derivative, Integral.
  (微积分中所有重要概念都建立在极限概念基础之上)

- No rigorous limits, no rigorous Calculus.
  (没有严格的极限论，就没有严格的微积分.)

Calculus is the study of limits.

1. Problems leading to the limit concept

- Area of a planar region bounded by curves.
  (曲线围成的平面区域的面积)

- Tangent lines of curves
  (曲线的切线)

- Instantaneous Velocity
  (瞬时速度)

Instantaneous Velocity 瞬时速度

An object moves along a coordinate line.

Its position at time t is given by $s = f(t)$.

At time c the object is at $f(c)$;

at the nearby time $c+h$, it is at $f(c+h)$.

Thus the average speed (平均速度) on this interval is

$$v_{\text{avg}} = \frac{f(c+h) - f(c)}{h}$$

We can now define instantaneous velocity (瞬时速度):

$$v_s = \lim_{h \to 0} v_{avg} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

2. An Intuitive Understanding of Limits (极限的直观理解)

Consider the function defined by

$$f(x) = \frac{x^3 - 1}{x - 1}$$

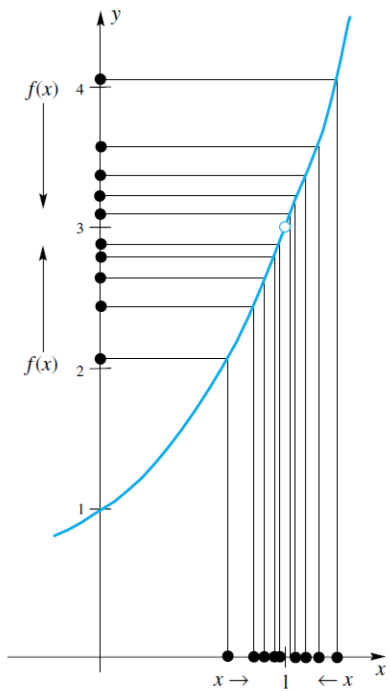
Note that it is not defined at $x = 1$ since at this point $f(x)$ has the form $\frac{0}{0}$, which is meaningless. We can, however, still ask what is happening to $f(x)$ as x approaches 1. More precisely, is $f(x)$ approaching some specific number as x approaches 1?

Example 1: Let $f(x)=\dfrac{x^3-1}{x-1}$. As $x\to 1$, $f(x)\to$?

Analysis: try by some different methods based on numerical computation (数值计算).

x	$y=\dfrac{x^3-1}{x-1}$
1.25	3.813
1.1	3.310
1.01	3.030
1.001	3.003
↓	↓
1.000	?
↑	↑
0.999	2.997
0.99	2.970
0.9	2.710
0.75	2.313

Table of values



Graph of $y=f(x)=\dfrac{x^3-1}{x-1}$

Conclusion:

$$\lim_{x\to 1}\frac{x^3-1}{x-1}=3$$

Definition: Intuitive Meaning of Limit (极限的直观描述)

Definition Intuitive Meaning of Limit

To say that $\lim_{x \to c} f(x) = L$ means that when x is near but different from c then $f(x)$ is near L.

If the values of $f(x)$ get closer and closer to L, when x gets closer and closer to c (but is different from c), then we write $\lim_{x \to c} f(x) = L$.

The limit of *f(x)* as *x* approaches *c* is *L*, or , *f(x)* approaches *L* as *x* approaches *c* .

More Precise Understanding of Limit of a Function at a point

To examine the existence of limit of $f(x)$ as x approaches c, we have to find a real number L which should satisfy the following condition:

$f(x)$ can become as close as we like to L(no matter how close $f(x)$ is to L). This requirement can be realized if we take values of x close enough to c(but is different from c).

Such number L is called the limit of $f(x)$ as x approaches c or the limit of $f(x)$ at c. We write $\lim_{x \to c} f(x) = L$ or $f(x) \to L$ as $x \to c$.

If such a number L does not exist, we say that $f(x)$ has no limit as x approaches c, or the limit of $f(x)$ at c does not exist.

函数极限的直观解释

假设函数 $f(x)$ 在 c 点附近（可不包括 c 点）有定义。“当 x 趋向 c 时 $f(x)$ 的极限是实数 L” 的含义是：无论我们要求 $f(x)$ 与 L 多么接近，只要 x 与 c 足够接近（但 $x \neq c$）就能够实现。L 也称为函数 $f(x)$ 在 c 点的极限。如果这样的实数 L 不存在，我们称 $f(x)$ 当 x 趋向 c 时的极限不存在。

Remarks.

(1) A necessary condition of existence of limit of $f(x)$ at c is that $f(x)$ has definition near c(not at c) on both sides of c. If $f(x)$ has definition on only one side of c, we may consider one sided limit of $f(x)$ at c.

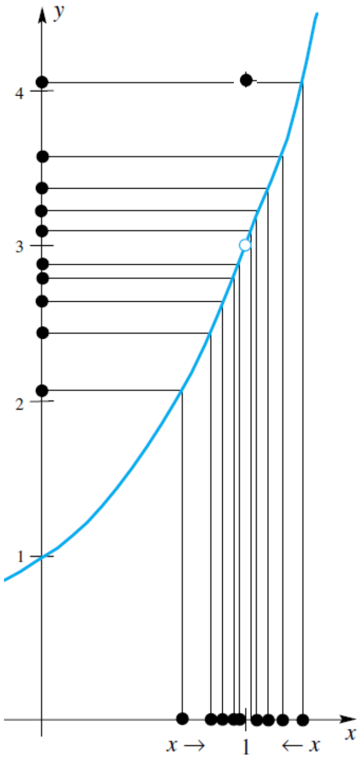
(2) The existence and the value of limit of $f(x)$ at c (if the limit exists) depend only on the behavior of $f(x)$ near c (not at c). Even if $f(x)$ has definition at c, the limit of $f(x)$ at c may not exist; even if the limit exists, it may not be equal to $f(c)$.

(3) From the mathematical point of view, the word “near” or “close” in the Intuitive Definition of Limit is not very clear, We will introduce rigorous (precise) definition of limit in section 2.2, which is very important for understanding Calculus theoretically.

Example 2. Let $g(x) = x^2 + x + 1, (x \neq 1)$ and $g(1) = 4$.

Question: (1) What is $\lim_{x \to 1} g(x)$?

(2) From the result $\lim_{x \to 1} g(x)$ can we get $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$?



Graph of $y = g(x)$

Solution. 1) By both methods we used in Example 1 we can find $\lim_{x \to 1} g(x) = 3$ easily. Note that $\lim_{x \to 1} g(x) \neq g(1)$.

In fact $f(x) = \frac{x^3 - 1}{x - 1}$ is the function in Example 1.

When $x \neq 1$,

$$f(x) = \frac{x^3 - 1}{x - 1} = \frac{(x-1)(x^2+x+1)}{x - 1} = x^2 + x + 1 = g(x).$$

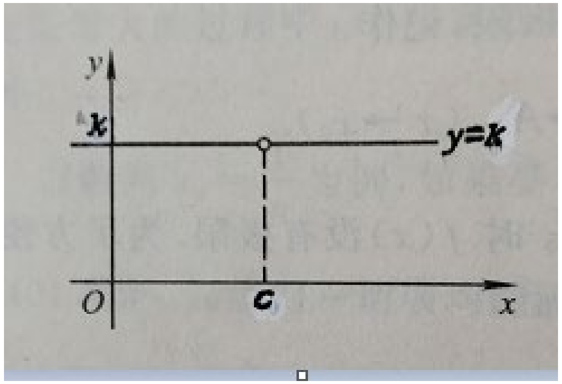
Because "limit of the function as $x \to 1$" depends only on the behaviors of function values when x is near 1 but not at 1, and the two functions $f(x)$ and $g(x)$ are identical when $x \neq 1$. So their function values have the same behaviors when x is near 1(not at 1). Therefore $\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = 3$.

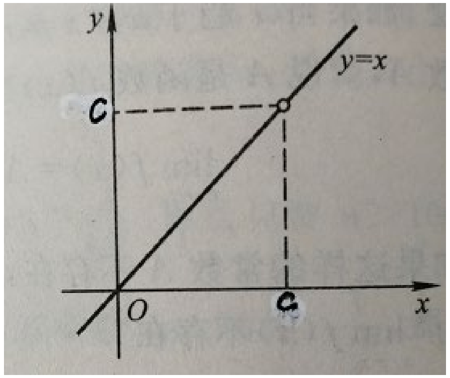
3. Several Important Examples

Example 4. Two basic limits

(1) $\lim_{x \to c} k = k$, where k is a constant, c is any real number;

(2) $\lim_{x \to c} x = c$, where c is any real number





Example 5: Find $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$ (**An Important Limit**)

No algebraic trick will simplify our task.
Numerical computation will help us to get an idea of the limit.
Rigorous proof by Squeeze Theorem will be given in Sec 2.3.

x	$\frac{\sin x}{x}$
1.0	0.84147
0.1	0.99833
0.01	0.99998
↓	↓
0	?
↑	↑
−0.01	0.99998
−0.1	0.99833
−1.0	0.84147

Figure 4

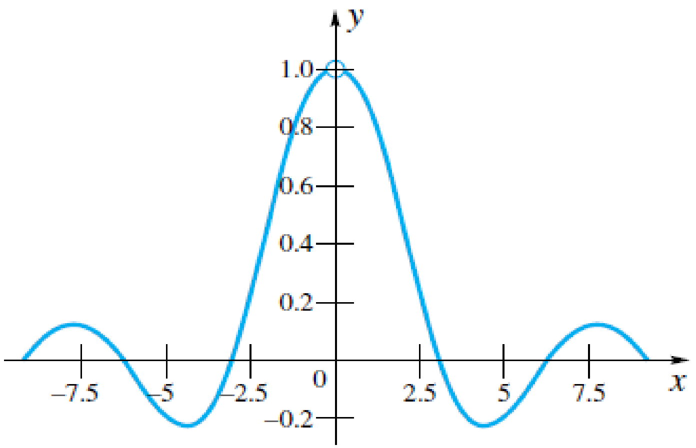
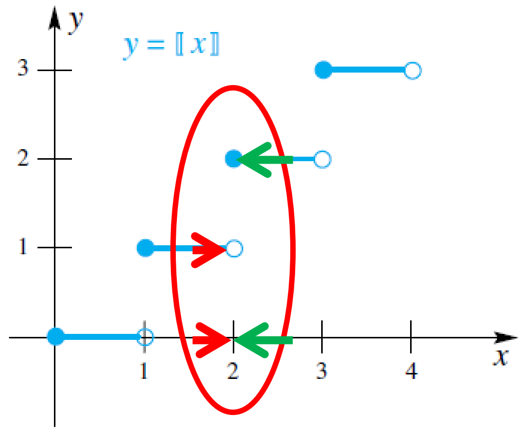


Figure 5

Two Examples that the limit does not exist

Example 6. No limit at a jump Find $\lim_{x\to 2}[x]$



Remark. This function is called **Greatest Integer Function** Can be expressed by also $[\![x]\!]$.

For all numbers x less than 2 but near 2, $[x]=1$.

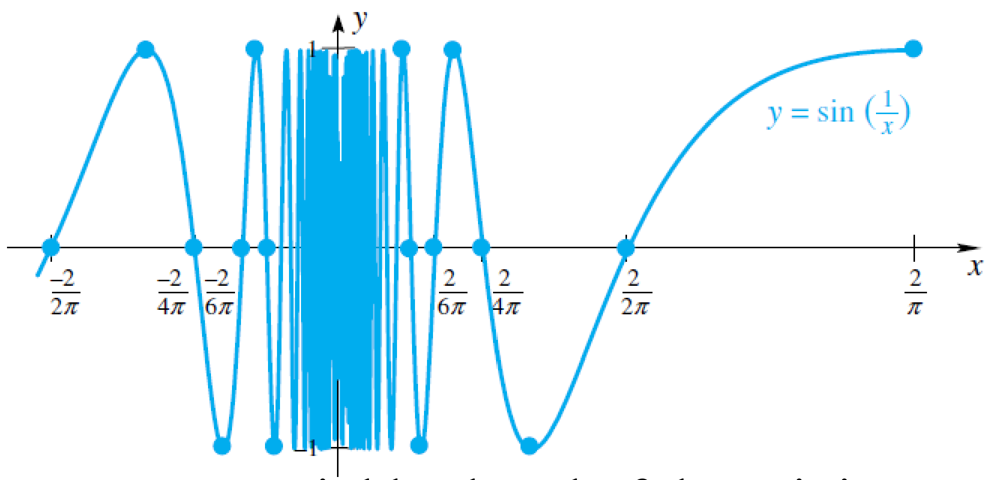
For all numbers x grater than 2 but near 2, $[x]=2$

$f(x)$ is not near a single number,

so $\lim_{x\to 2}[x]$ does not exist.

Example 7. Too many wiggles（摆动） Find $\lim_{x\to 0}\sin(1/x)$

x	$\sin\frac{1}{x}$
$2/\pi$	1
$2/(2\pi)$	0
$2/(3\pi)$	-1
$2/(4\pi)$	0
$2/(5\pi)$	1
$2/(6\pi)$	0
$2/(7\pi)$	-1
$2/(8\pi)$	0
$2/(9\pi)$	1
$2/(10\pi)$	0
$2/(11\pi)$	-1
$2/(12\pi)$	0
$\downarrow$	$\downarrow$
0	?



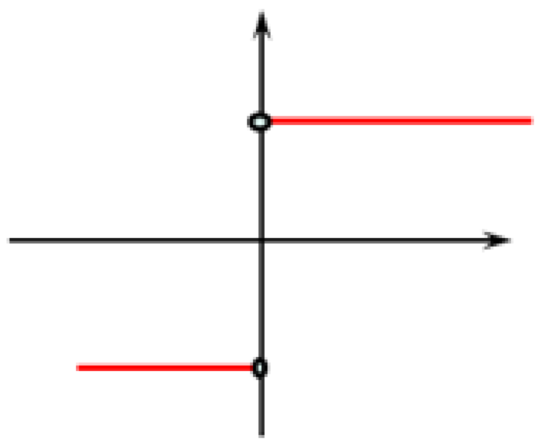
In any neighborhood of the origion, the graph wiggles up and down between -1 and 1 infinitely many times, not close to a single number. So $\lim_{x\to 0}\sin(1/x)$ does not exist.

4. One-Sided Limits(单侧极限)

Definition: Right and Left-Hand Limits

(1) To say that $\lim\limits_{x \to c^+} f(x) = L$ means that when x is near but to the right of c then $f(x)$ is near L.

(2) To say that $\lim\limits_{x \to c^-} f(x) = L$ means that when x is near but to the left of c then $f(x)$ is near L.



$$y = f(x) = \frac{x}{|x|} \quad (x \neq 0)$$

$$\lim_{x \to 0^+} f(x) = 1, \quad \lim_{x \to 0^-} f(x) = -1$$

Theorem A

$$\lim_{x \to c} f(x) = L \text{ if and only if } \lim_{x \to c^-} f(x) = L \text{ and } \lim_{x \to c^+} f(x) = L$$

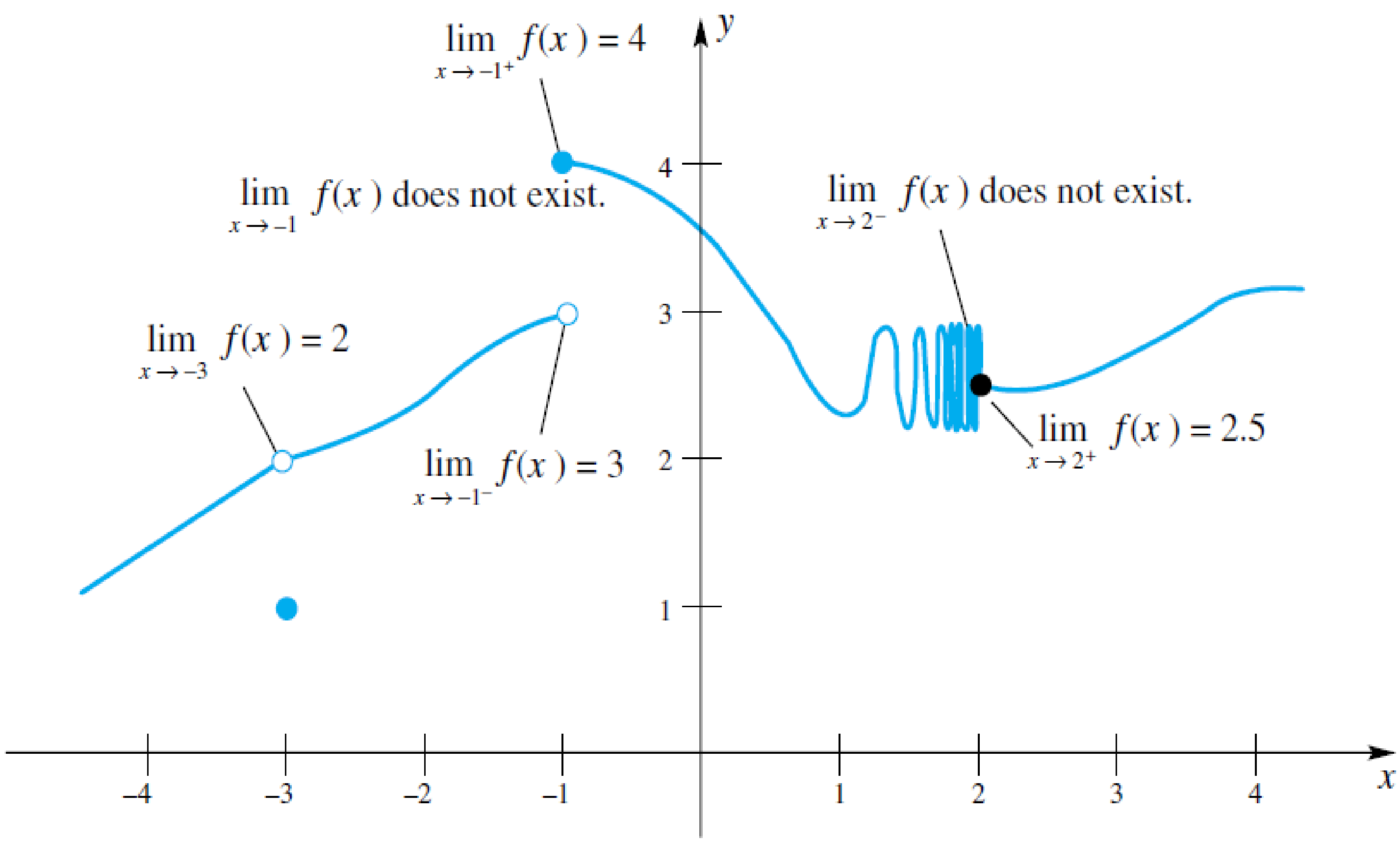


Figure 10

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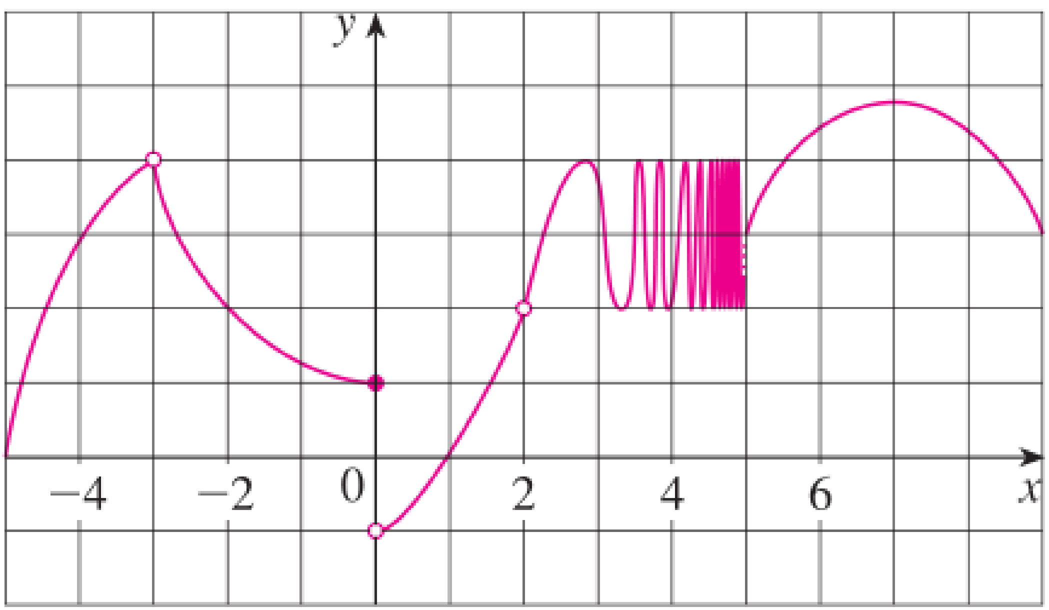
Exercise 1 For the function h whose graph is given, state the value of each quantity, if it exist. If it does not exist, explain why.

(a) $\lim_{x\to -3^-} h(x)$ (b) $\lim_{x\to -3^+} h(x)$ (c) $\lim_{x\to -3} h(x)$

(d) $h(-3)$ (e) $\lim_{x\to 0^-} h(x)$ (f) $\lim_{x\to 0^+} h(x)$

(g) $\lim_{x\to 0} h(x)$ (h) $h(0)$ (i) $\lim_{x\to 2} h(x)$

(j) $h(2)$ (k) $\lim_{x\to 5^+} h(x)$ (l) $\lim_{x\to 5^-} h(x)$



2.2 Rigorous Study of Limits

Definition: Intuitive Meaning of Limit(极限的直观解释)

Definition Intuitive Meaning of Limit

To say that $\lim_{x \to c} f(x) = L$ means that when x is near but different from c then $f(x)$ is near L.

or it is more precise to say that

$f(x)$ can become as close as we like to L(no matter how close $f(x)$ is to L). This requirement can be realized if we take values of x close enough to c (but is different from c).

Mathematical Meaning of "near" and "close" （I）

1) "near" and "close" mean "distance is small";
2) "distance between two numbers a and b" is defined by $|a - b|$;
3) "$f(x)$ is near or close to L" means $|f(x) - L|$ is small, "x is near or close to c but different from c" means $|x - c|$ is small and $x \neq c$(hence $|x - c| > 0$).

Example 1 Use a plot of $y = f(x) = 3x^2$ to determine how close x must be to 2 to guarantee that $f(x)$ is within 0.05 of 12.

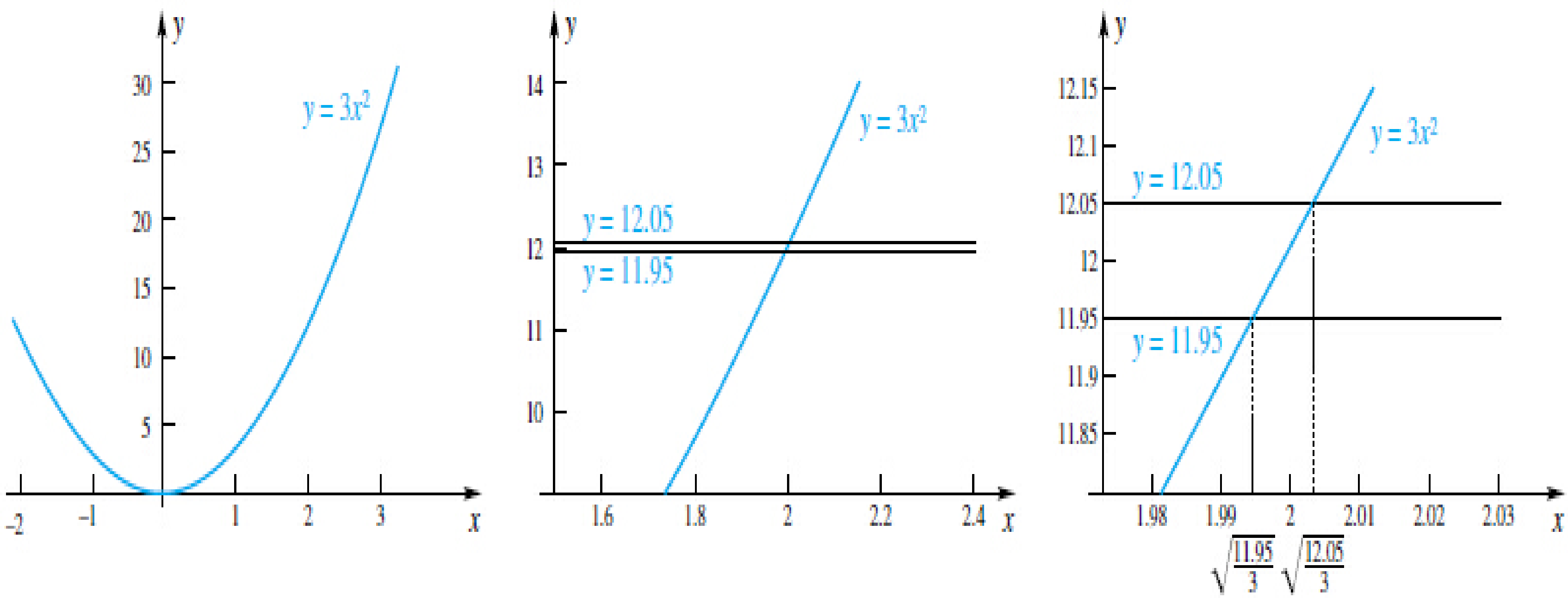


Figure 1

Solution: In order to $f(x) \in (12-0.05, 12+0.05)$, i.e.

$|f(x)-12| < 0.5 \Leftrightarrow\ 11.95 < f(x) = 3x^2 < 12.05.$

We solve above inequalities to get

$\sqrt{11.95/3} < x < \sqrt{12.05/3}$ (since x is close to 2)

Approximately, $1.99583 < x < 2.00416.$

Then we can choose 0.00416 of 2.

Thus, if $x \in (2-0.00416, 2+0.00416)$, i.e.$(|x-2| < 0.00416)$

then $f(x) \in (12-0.05, 12+0.05)$.

If we asked how close x would have to be to 2 to guarantee that $f(x)$ is with 0.01 of 12?

The solution would proceed along the same lines. We would get a smaller interval than we obtained above.

If we want $f(x)$ to be within 0.001 of 12, we would require an interval that is narrower still.

No matter how close we want $f(x)$ to be in 12, we can accomplish this by taking x sufficiently close 2.

0.05	0.004	$0<\lvert x-2\rvert<0.004 \Rightarrow \lvert f(x)-12\rvert<0.5$
0.01	0.0008	$0<\lvert x-2\rvert<0.0008 \Rightarrow \lvert f(x)-12\rvert<0.01$
0.001	0.00008	$0<\lvert x-2\rvert<0.00008 \Rightarrow \lvert f(x)-12\rvert<0.001$
0.0001	0.000008	
$\vdots$		
ε	δ	$0<\lvert x-2\rvert<\delta \Rightarrow \lvert f(x)-12\rvert<\varepsilon$

Follow the tradition to use the Greek letter ε(epsilon) and δ(delta) to stand for (usually small) arbitrary positive numbers.

Mathematical Meaning of "near" and "close"(II)

4) "$f(x)$ *is as close as we like to* L" means for any positive number(usually is a small number), denoted by ε(epsilon), $|f(x)-L|<\varepsilon$.
5) "x *is sufficiently close to* c *but not equal to* c" means there is a positive number(usually is very small), denoted by $\delta(delta)$, such that $0<|x-c|<\delta$.
6)For the purpose to examine the condition for $\lim_{x\to c} f(x)=L$ (limit exists and equal to L), we need to consider the inequality $|f(x)-L|<\varepsilon$. But it is not necessary to solve this inequality to get exact solution, usually we can find a simpler inequality from which we can find easily a $\delta>0$ such that $0<|x-c|<\delta$ guarantees $|f(x)-L|<\varepsilon$.

Rigorous Definition of Limit 极限的严格定义

To say that $\lim_{x \to c} f(x) = L$ means that for each given $\varepsilon > 0$ (no matter how small) there is a corresponding $\delta > 0$ such that $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$.

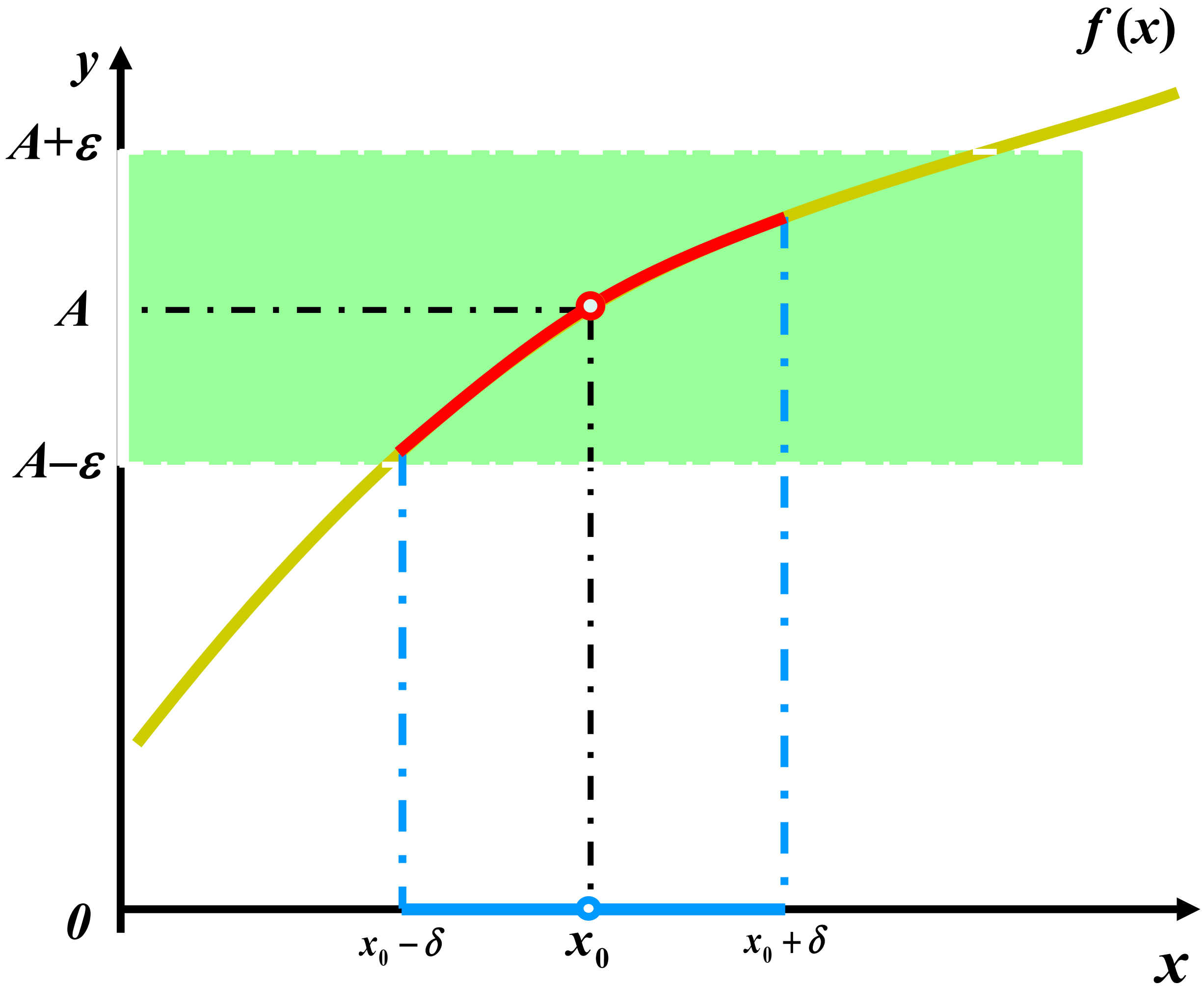
$\lim_{x \to c} f(x) = L$ 的含义是：

对于任意给定的正数 ε (无论多么小)

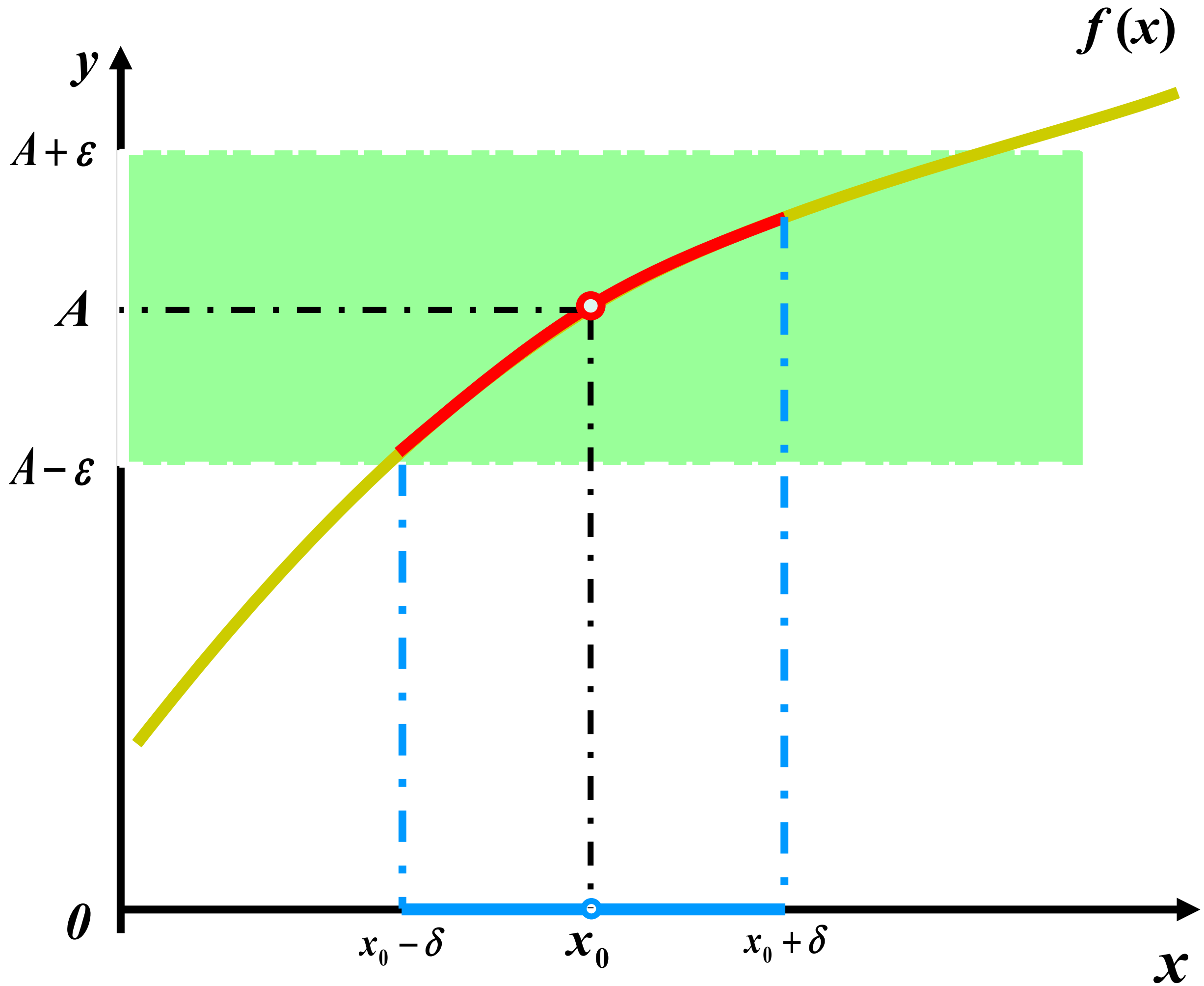
总能够找到一个相应的正数 $\delta > 0$，使得

只要 x 满足 $0 < |x - c| < \delta$, 就有 $|f(x) - L| < \varepsilon$.

Geometric Intuition of $\lim_{x \to x_0} f(x) = A$



Geometric Intuition of $\lim\limits_{x \to x_0} f(x) = A$



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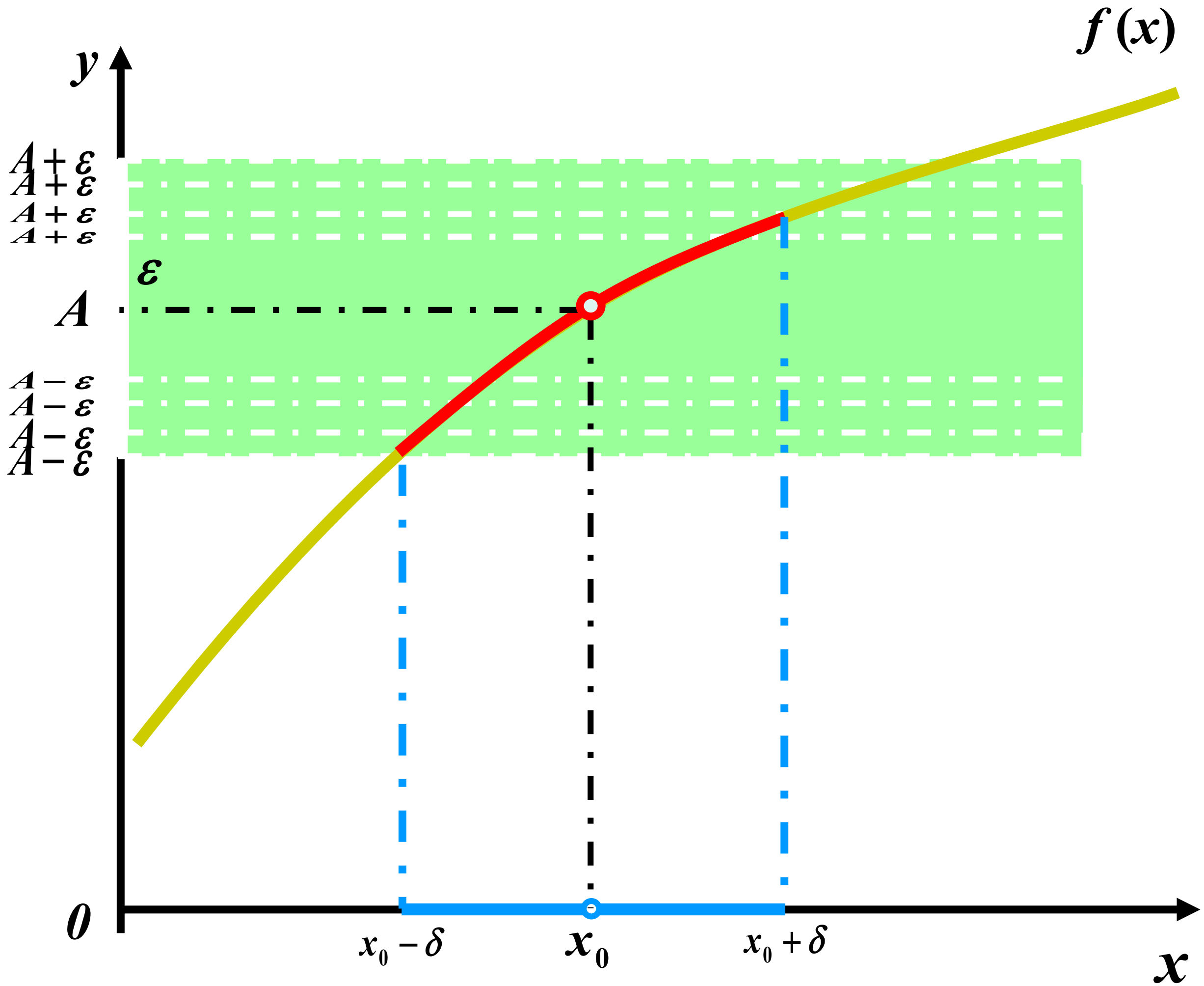

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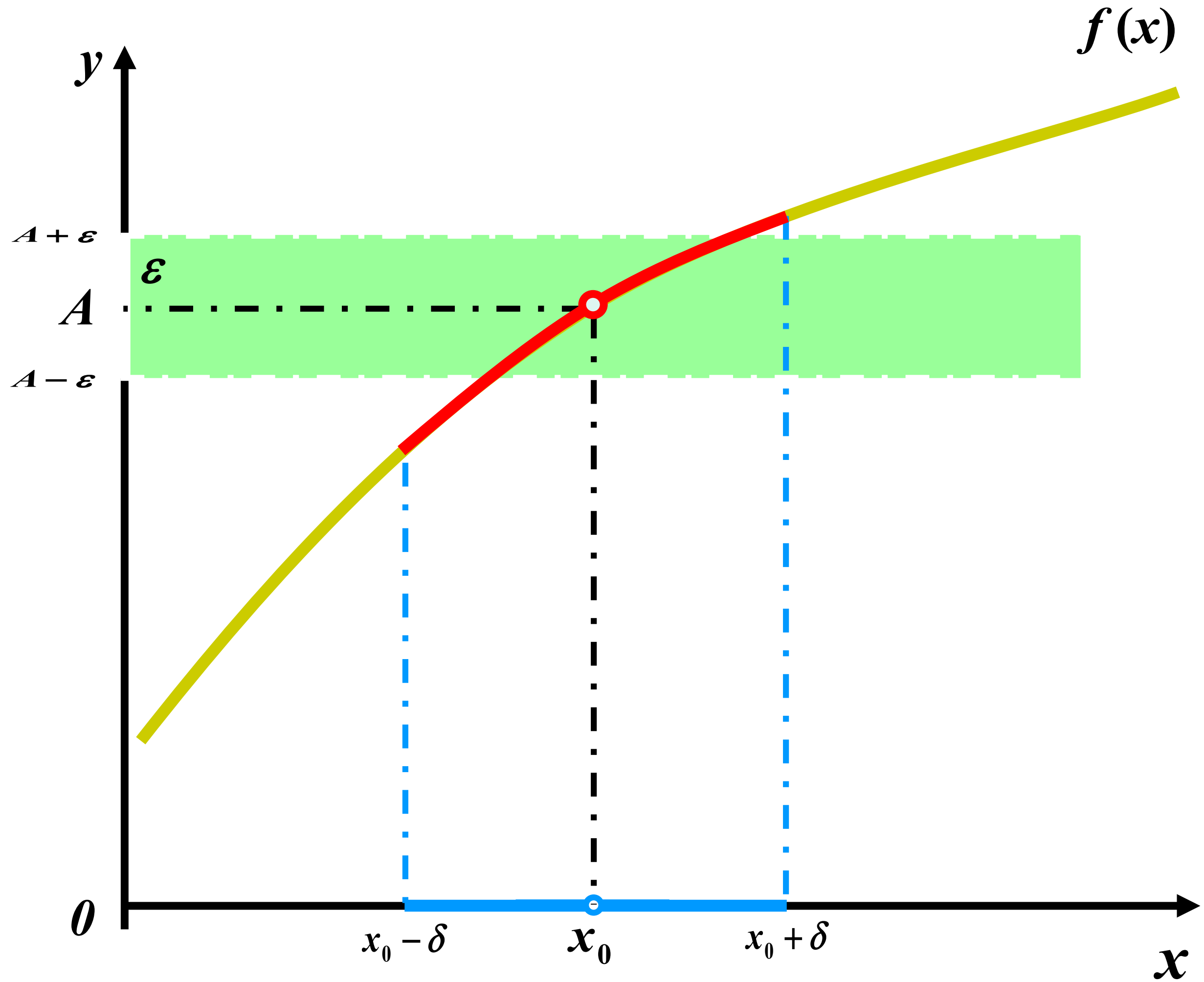

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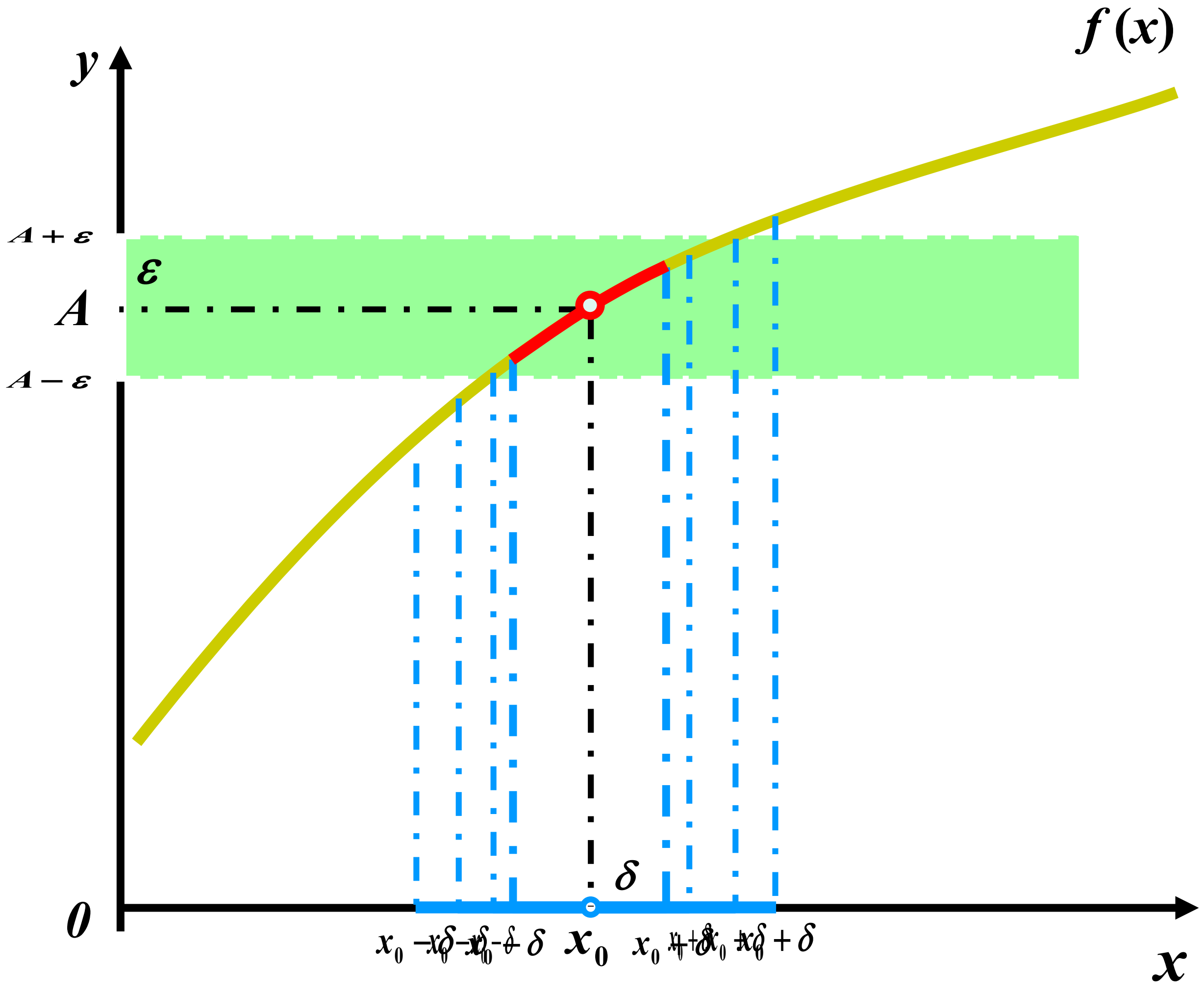
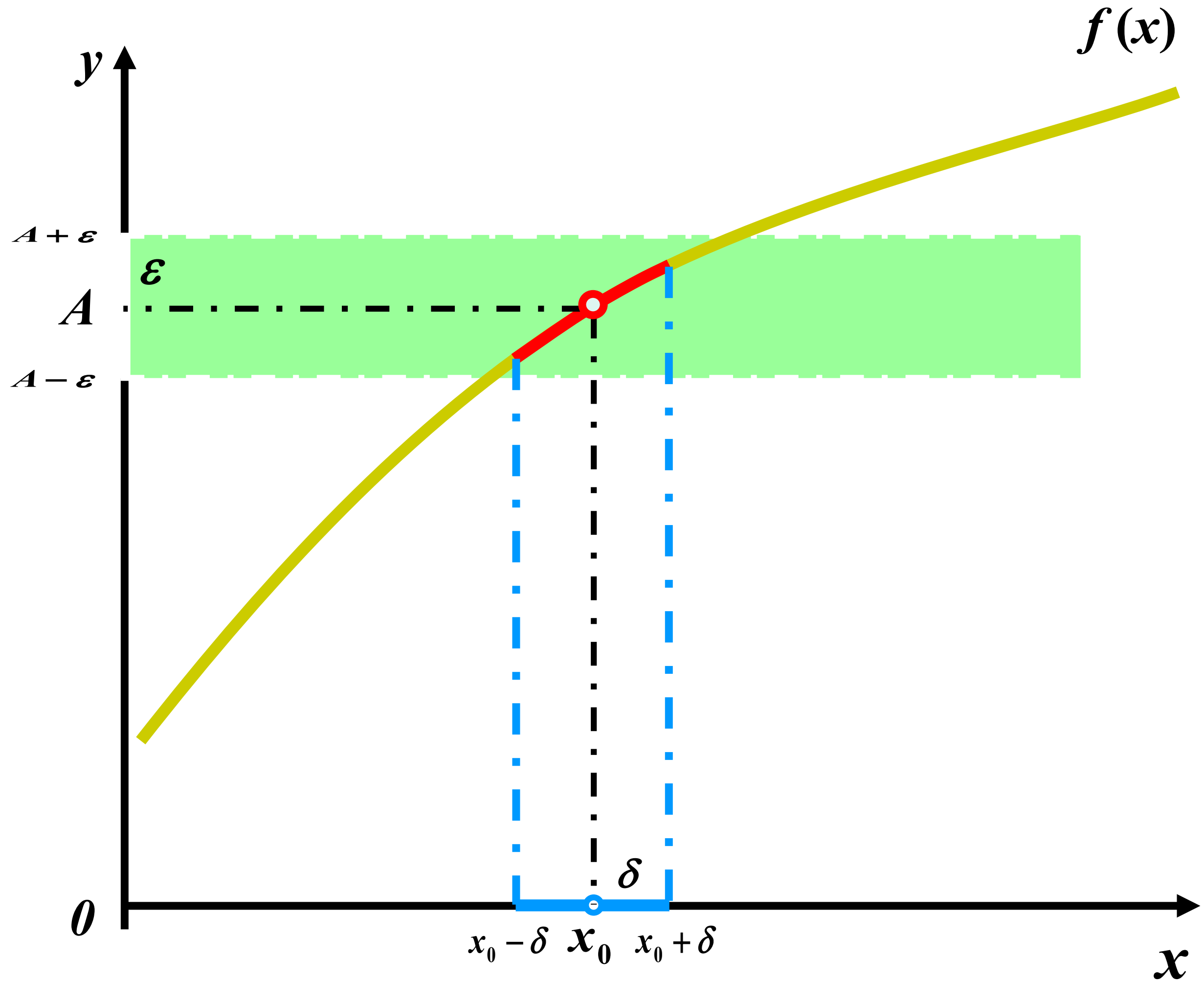
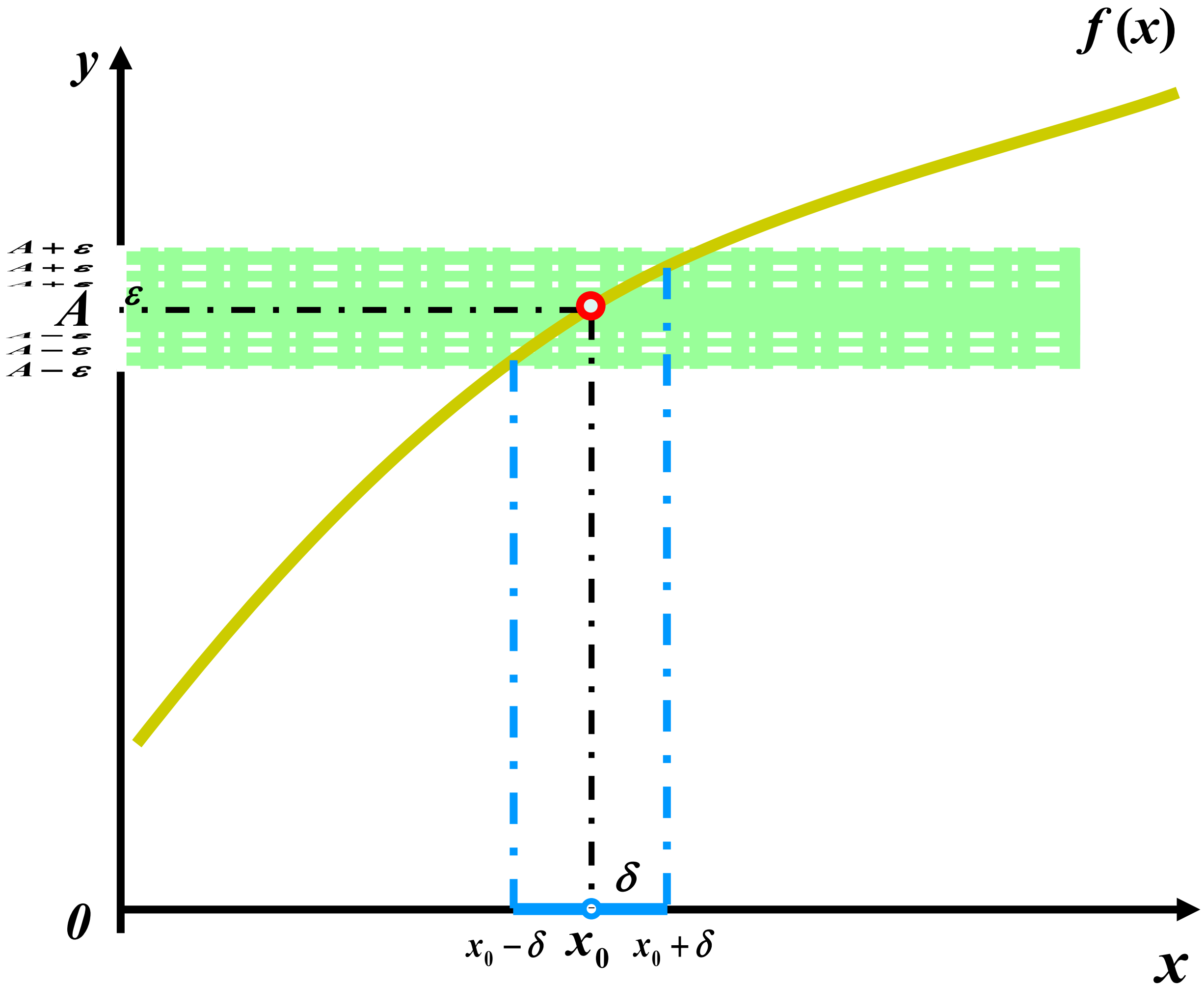


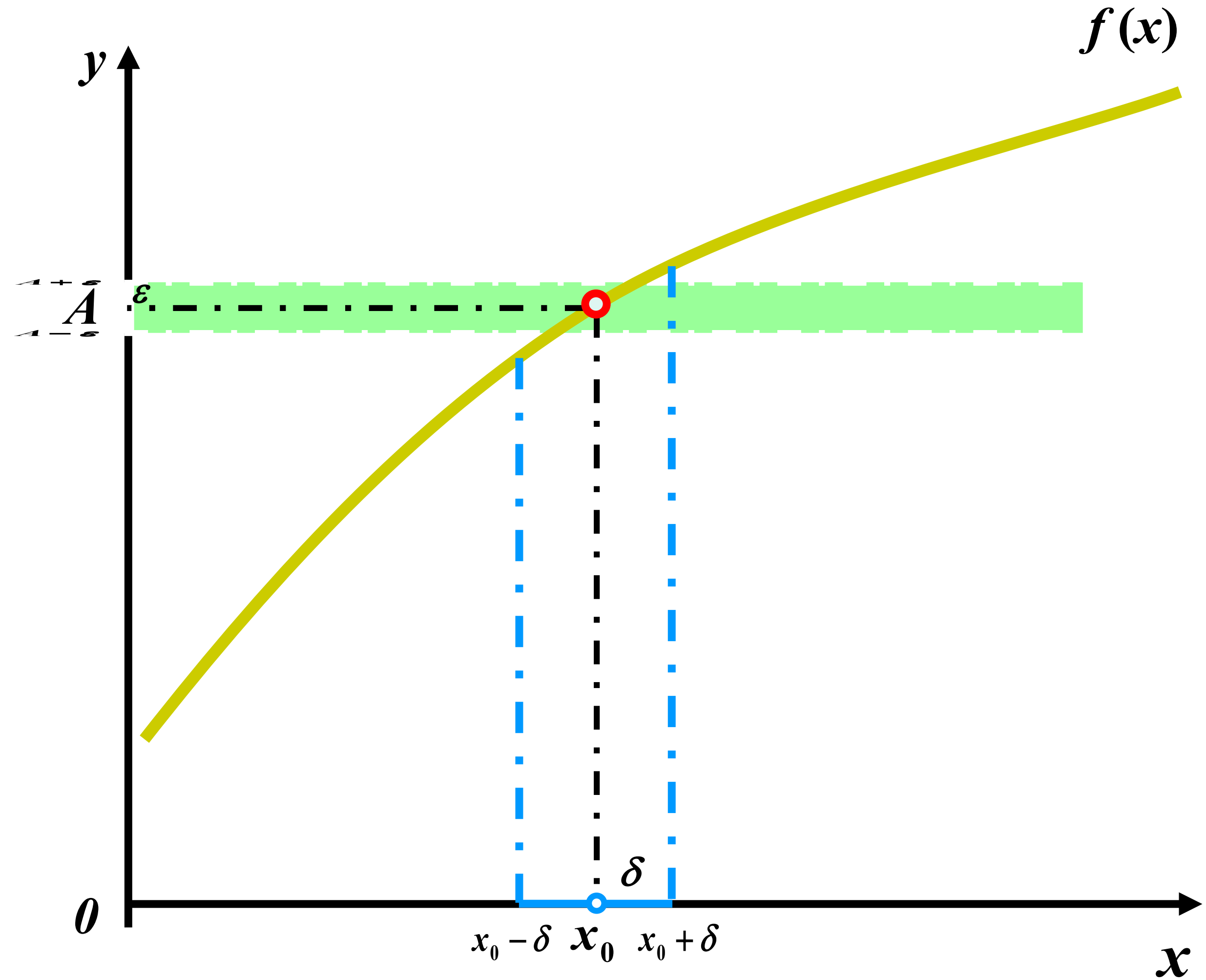
Geometric Intuition of $\lim_{x \to x_0} f(x) = A$



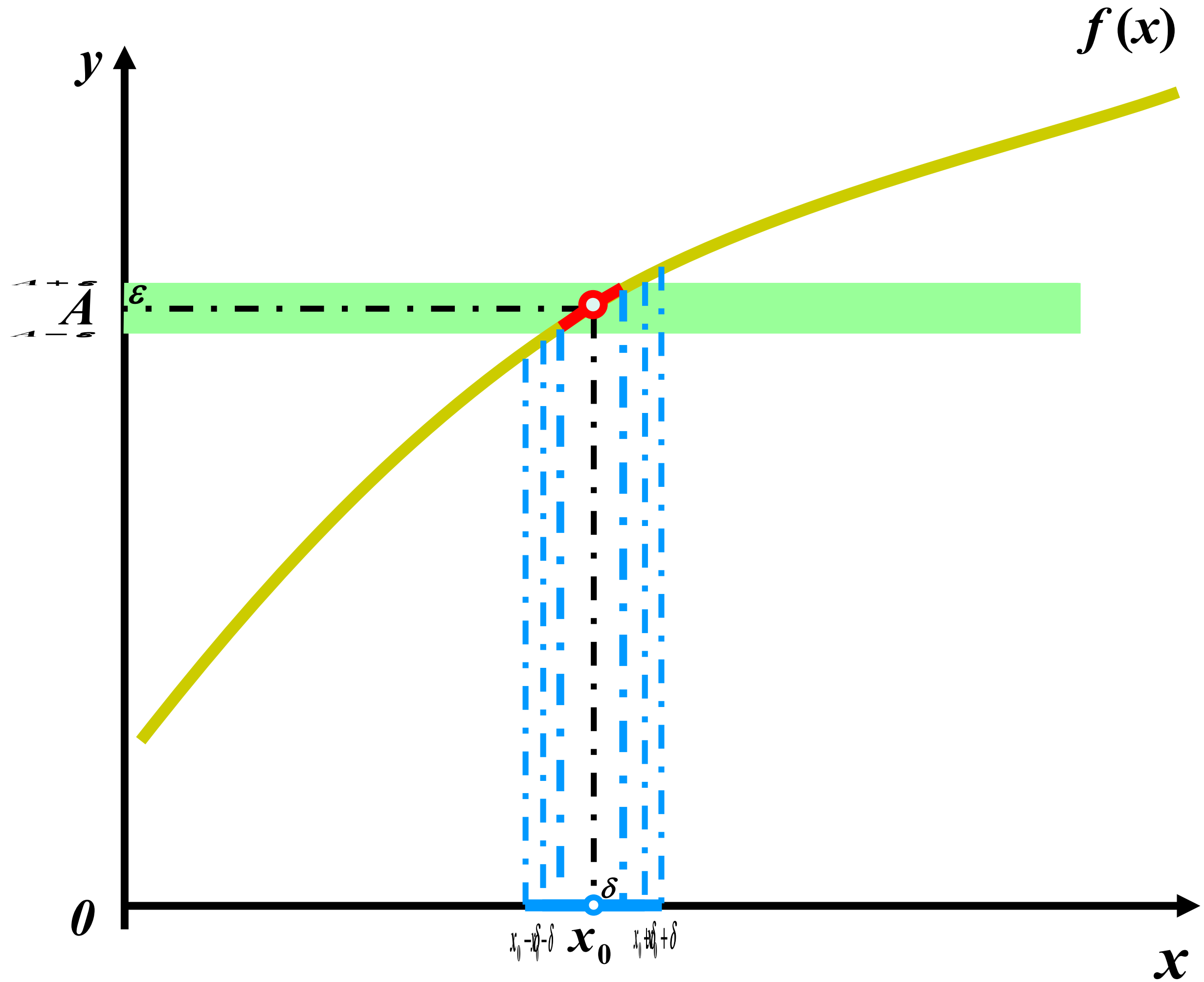
Geometric Intuition of $\lim_{x\to x_0} f(x) = A$



Geometric Intuition of $\lim_{x \to x_0} f(x) = A$



Geometric Intuition of $\lim_{x \to x_0} f(x) = A$



*Limit Proofs

Example 2. Show that $\lim_{x \to 2}(3x + 4) = 10$.

Preliminary Analysis. For any $\varepsilon > 0$,

$$|(3x + 4) - 10| < \varepsilon \Leftrightarrow (6 - \varepsilon)/3 < x < (6 + \varepsilon)/3$$
$$\Leftrightarrow 2 - \frac{\varepsilon}{3} < x < 2 + \frac{\varepsilon}{3}$$
$$\Leftrightarrow |x - 2| < \frac{\varepsilon}{3}$$

So if we take $\delta = \frac{\varepsilon}{3}$, then $0 < |x - 2| < \delta$ guarantee that $|(3x + 4) - 10| < \varepsilon$.

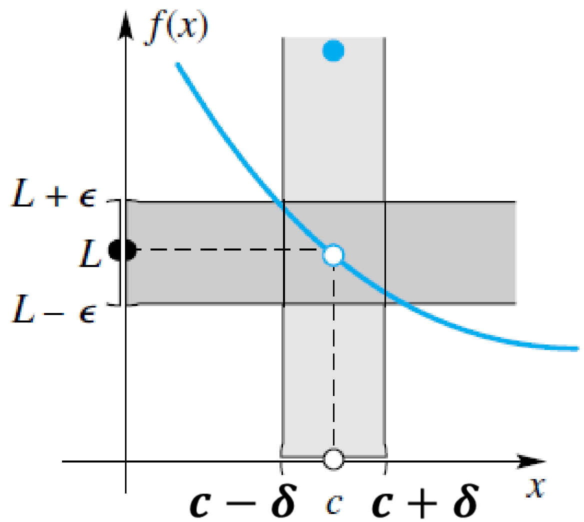
Proof. Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{3}$. Then $0 < |x - 2| < \delta$ implies $|(3x + 4) - 10| = 3|x - 2| < \varepsilon$. By the definition of limit, we have shown that $\lim_{x \to 2}(3x + 4) = 10$.

Example 3. Prove that $\lim_{x \to 2} 3x^2 = 12$.

Proof. For $\forall \varepsilon > 0$, $|3x^2 - 12| < \varepsilon \Leftrightarrow 3|x+2||x-2| < \varepsilon$. Assume that $|x-2| < 1$. Then $|x+2| < 5$ and then $|3x^2 - 12| < 15|x-2|$. Since $15|x-2| < \varepsilon$ provided $|x-2| < \frac{\varepsilon}{15}$
So if $|x-2| < \frac{\varepsilon}{15}$ and $|x-2| < 1$, then $|3x^2 - 12| < \varepsilon$. Without loss of generality, we assume $0 < \varepsilon < 15$. Take $\delta = \frac{\varepsilon}{15}$. Then when $0 < |x-2| < \delta$, $|3x^2 - 12| < \varepsilon$. Therefore $\lim_{x \to 2} 3x^2 = 12$. QED

A Property of Limit

If $\lim_{x \to c} f(x) = L > 0$, then when x is close enough to c, $f(x)$ is very close to L and must be positive.



This is so called Sign Preserving Property of limit.
（极限的保号性）

Theorem A(Sign Preserving Property of Limits).
Suppose $\lim_{x \to c} f(x) = L > 0 (< 0)$. Then there is a $\delta > 0$ such that if $0 < |x - c| < \delta$ then $f(x) > 0 (< 0)$.

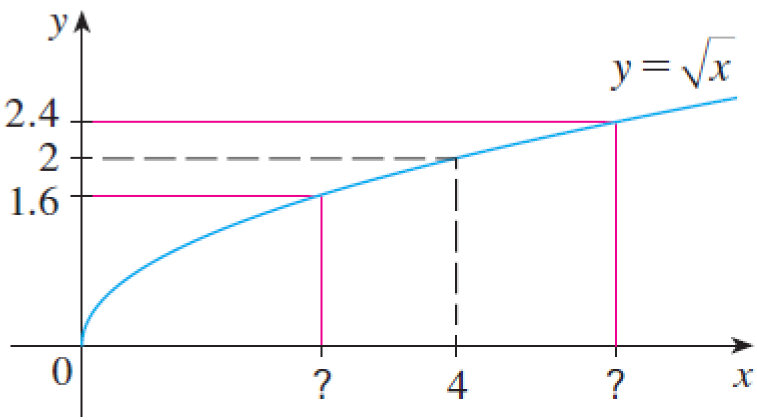
*Proof. Suppose $L > 0$. Take $\varepsilon = \frac{L}{2}$. By definition of limit, We can find $\delta > 0$ such that if $0 < |x - c| < \delta$ then $| f(x) - L | < \frac{L}{2}$. Therefore $f(x) > L - \frac{L}{2} = \frac{L}{2} > 0$.
The case $L < 0$ can be proved similarly.

Corollary. Suppose $\lim_{x \to c} f(x) = L$ and $\exists\, \delta > 0$ such that $f(x) > 0 (< 0)$ whenever $0 < |x - c| < \delta$. Then $L \geq 0$ （≤ 0）.

* Proof. It is proved by contradiction. We consider the case $f(x) > 0$ only. Suppose the conclusion is not true, i.e. $L < 0$. Then by Theorem A, there is a $\delta > 0$ such that $f(x) < 0$ if $0 < |x - c| < \delta$. This is a contradiction to the assumption of the corollary.
The case $f(x) < 0$ can be proved similarly.

Example . Let $f(x) = x^2$. Then $f(x) > 0$ for all $x \neq 0$. It is obvious that $\lim_{x \to 0} f(x) = 0$.

Exercise 2 Given $f(x)=\sqrt{x}$. Find a positive number δ such that if $|x-4|<\delta$ then $|\sqrt{x}-2|<0.4$.



Exercise 3 (P76,Q42) The function $f(x)=x^2$ had been carefully graphed, but during the night a mysterious vistor changed the values of f at a million different places. Does this affect the value of $\lim_{x\to a} f(x)$ at any a? Explain.