時間序列分析

第三章：平穩時間序列分析

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第3講：AR模型的統計性質

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- AR(p)模型

$$x_t = \phi_0 + \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + \varepsilon_t$$

- 假設此AR(p)模型滿足平穩性條件，在等式兩邊取期望，有

$$E(x_t) = E(\phi_0 + \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + \varepsilon_t)$$

均值μ

均值μ

均值μ

$E(\varepsilon_t) = 0$

$$\mu = \phi_0 + \phi_1 \mu + \cdots + \phi_p \mu$$

$$\mu = \frac{\phi_0}{1 - \phi_1 \ldots - \phi_p}$$

均值

中心化AR(p)模型，$\phi_0 = 0$，均值為0

AR模型的方差

AR模型的方差

- AR(p)模型

$$x_t = \phi_0 + \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + \varepsilon_t$$

- 方差

$$DX_t = E(X_t - \mu)^2 = E\left(\phi_0 + \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + \varepsilon_t - \mu\right)^2$$

$$E(x_{t-1}x_{t-2}), E(x_{t-1}x_{t-3}), \ldots$$

- 怎麼求？

AR模型的方差（Green函數）

- 需要借助**Green函數**來求平穩AR(p)模型的方差。
- 中心化AR(p)模型

$$\boldsymbol{x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + \varepsilon_t}$$

$$x_t - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p} = \varepsilon_t$$

$$x_t - \phi_1 B x_t - \cdots - \phi_p B^p x_t = \varepsilon_t$$

$$\left(1 - \phi_1 B - \cdots \phi_p B^p\right) x_t = \varepsilon_t$$

$$\Phi(B) x_t = \varepsilon_t$$

$$\boxed{\Phi(B) = 1 - \phi_1 B - \cdots \phi_p B^p}$$

把$\boldsymbol{x_t}$寫成只與$\boldsymbol{\varepsilon_t}$、延遲算子B和常數有關的形式

$$\boldsymbol{x_t = \frac{\varepsilon_t}{\Phi(B)}}$$

AR模型的方差（Green函數）

$$1-6x+8x^2 = (1-4x)(1-2x)$$

根：$\frac{1}{4}$和$\frac{1}{2}$

$$\Phi(B) = 1-\phi_1 B - \cdots \phi_p B^p = (1-\lambda_1 B)(1-\lambda_2 B)\ldots(1-\lambda_p B) = \prod_{i=1}^{p}(1-\lambda_i B)$$

$$\frac{1}{(1-4x)(1-2x)} = \frac{2}{1-4x} + \frac{-1}{1-2x}$$

$$\frac{1}{1-x} = \sum_{j=0}^{\infty} x^j$$

$$x_t = \frac{\varepsilon_t}{\Phi(B)}$$

$$= \frac{\varepsilon_t}{\prod_{i=1}^{p}(1-\lambda_i B)}$$

$$= \sum_{i=1}^{p} \frac{k_i}{1-\lambda_i B}\varepsilon_t$$

$$= \sum_{i=1}^{p}\sum_{j=0}^{\infty} k_i(\lambda_i B)^j \varepsilon_t$$

$$= \sum_{j=0}^{\infty}\sum_{i=1}^{p} k_i \lambda_i^j \varepsilon_{t-j}$$

$$\stackrel{\text{def}}{=} \sum_{j=0}^{\infty} G_j \varepsilon_{t-j}$$

$$G_j = \sum_{j=1}^{p} k_i \lambda_i^j$$

AR模型的傳遞形式，G_j為Green函數。
因為λ_i在單位圓裏，所以Green函數呈負指數下降且j → ∞極限為0

AR模型的方差（傳遞形式）

$$x_t = \frac{\varepsilon_t}{\Phi(B)} = \sum_{j=0}^{\infty} G_j \varepsilon_{t-j} = \mathrm{G(B)}\varepsilon_t$$

$$G(B) = \sum_{j=0}^{\infty} G_j B^j$$

$$\Phi(B)G(B)\varepsilon_t = \varepsilon_t$$

待定系數法求G(B)

$$\left(1 - \sum_{k=1}^{p} \phi_k B^k\right)\left(\sum_{j=0}^{\infty} G_j B^j\right)\varepsilon_t = \varepsilon_t$$

$$\left(G_0 + \sum_{j=1}^{\infty}\left(G_j - \sum_{k=1}^{j} \phi'_k G_{j-k}\right)B^j\right)\varepsilon_t = \varepsilon_t$$

$$\phi'_k = \begin{cases} \phi_k, & k \le p \\ 0, & k > p \end{cases}$$

對ε_t，有$G_0\varepsilon_t = \varepsilon_t$，得$G_0 = 1$

對ε_{t-1}，有$j = 1$，即$G_1 - \phi'_1 G_0 = 0$，得$G_1 = \phi'_1 G_0$

對ε_{t-j}，有$G_j - \sum_{k=1}^{j} \phi'_k G_{j-k} = 0$，得$G_j = \sum_{k=1}^{j} \phi'_k G_{j-k}$

AR模型的方差（傳遞形式）

$$x_t = \sum_{j=0}^{\infty} G_j \varepsilon_{t-j}$$

$$\begin{cases} G_0 = 1 \\ G_j = \sum_{k=1}^{j} \phi'_k G_{j-k} \end{cases}$$

$$\phi'_k = \begin{cases} \phi_k, & k \le p \\ 0, & k > p \end{cases}$$

$$j = 0, 1, 2, \ldots$$

AR模型的方差

$$\boldsymbol{x_t} = \sum_{j=0}^{\infty} G_j \varepsilon_{t-j}$$

- AR(p)的方差為

$$Var(x_t) = \sum_{j=0}^{\infty} G_j^2 Var(\varepsilon_{t-j}) = \sum_{j=0}^{\infty} G_j^2 \sigma_\varepsilon^2$$

$\boldsymbol{Var(\varepsilon_t) = \sigma_\varepsilon^2}$

Green函數G_j呈負指數下降且j → ∞極限為0。
說明$\sum_{j=0}^{\infty} G_j^2 < \infty$，平穩序列方差有界。

練習

- 求平穩AR(1)模型$x_t = 0.5x_{t-1} + \varepsilon_t$的方差。
- $\boldsymbol{x_t} = \sum_{j=0}^{\infty} G_j \varepsilon_{t-j}$

$$\begin{cases} G_0 = 1 \\ G_j = \sum_{k=1}^{j} \phi'_k G_{j-k} \end{cases}$$

得

$$G_0 = 1$$

$$G_1 = \phi'_1 = \phi_1 = 0.5$$

$$G_2 = \phi'_1 G_1 + \phi'_2 G_0 = \phi_1 G_1 = 0.5^2$$

$$G_j = 0.5^j$$

$$Var(x_t) = \sum_{j=0}^{\infty} G_j^2 \sigma_\varepsilon^2 = (1 + 0.5^2 + 0.5^4 + \cdots \ldots)\sigma_\varepsilon^2 = \frac{\sigma_\varepsilon^2}{1-0.5^2} = 4\sigma_\varepsilon^2$$

練習

- 求平穩AR(1)模型$x_t = -0.5x_{t-1} + \varepsilon_t$的方差。
- $\boldsymbol{x_t} = \sum_{j=0}^{\infty} G_j \varepsilon_{t-j}$

$$\begin{cases} G_0 = 1 \\ G_j = \sum_{k=1}^{j} \phi'_k G_{j-k} \end{cases}$$

得

$G_0 = 1$

$G_1 = \phi'_1 = \phi_1 = -0.5$

$G_2 = \phi'_1 G_1 + \phi'_2 G_0 = \phi_1 G_1 = (-0.5)^2$

$G_j = (-0.5)^j$

$$Var(x_t) = \sum_{j=0}^{\infty} (-0.5)^{2j} \sigma_\varepsilon^2 = (1 + 0.5^2 + 0.5^4 + \cdots \ldots)\sigma_\varepsilon^2 = \frac{\sigma_\varepsilon^2}{1-0.5^2} = 4\sigma_\varepsilon^2$$

練習

- 求平穩AR(1)模型$x_t = 0.1x_{t-1} + \varepsilon_t$的方差。
- $\boldsymbol{x_t} = \sum_{j=0}^{\infty} G_j \varepsilon_{t-j}$

$$\begin{cases} G_0 = 1 \\ G_j = \sum_{k=1}^{j} \phi'_k G_{j-k} \end{cases}$$

得

$$G_0 = 1$$

$$G_1 = \phi'_1 = \phi_1 = 0.1$$

$$G_2 = \phi'_1 G_1 + \phi'_2 G_0 = \phi_1 G_1 = 0.1^2$$

$$G_j = 0.1^j$$

$$Var(x_t) = \sum_{j=0}^{\infty} 0.1^j \sigma_\varepsilon^2 = (1 + 0.1^2 + 0.1^4 + \cdots \ldots)\sigma_\varepsilon^2 = \frac{\sigma_\varepsilon^2}{1-0.1^2} \approx 1.01\sigma_\varepsilon^2$$

練習

- 求平穩AR(1)模型$x_t = \phi_1 x_{t-1} + \varepsilon_t$的方差。
- $Var(x_t) = \sum_{j=0}^{\infty} G_j^2 \sigma_\varepsilon^2$ $\quad \begin{cases} G_0 = 1 \\ G_j = \sum_{k=1}^{j} \phi'_k G_{j-k} \end{cases}$
- $G_0 = 1$
- $G_j = \phi_1 G_{j-1} = \phi_1^j$
- $Var(x_t) = \sum_{j=0}^{\infty} \phi_1^{2j} \sigma_\varepsilon^2 = \frac{\sigma_\varepsilon^2}{1-\phi_1^2}$

對於平穩AR(1) 模型：

$$Var(x_t) = \frac{\sigma_\varepsilon^2}{1-\phi_1^2}$$

AR模型的協方差函數

AR模型的協方差函數

- 平穩的中心化AR(p)模型$x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + \varepsilon_t$，兩邊同乘$x_{t-k}(\forall k \geq 1)$，再求期望，有協方差

$$\boldsymbol{E(\varepsilon_t x_{t-k}) = 0}$$

$$\boldsymbol{E(x_t x_{t-k}) = \phi_1 E(x_{t-1} x_{t-k}) + \phi_2 E(x_{t-1} x_{t-k}) + \cdots + E(x_{t-p} x_{t-k}) + E(\varepsilon_t x_{t-k})}$$

- 因此有

$$\boldsymbol{\gamma_k = \phi_1 \gamma_{k-1} + \cdots + \phi_p \gamma_{k-p}}$$

練習

- 求平穩AR(1)模型的自協方差函數遞推公式
- $x_t = \phi_1 x_{t-1} + \varepsilon_t$
- $E(x_t x_{t-k}) = \phi_1 E(x_{t-1} x_{t-k})$
- $\gamma_k = \phi_1 \gamma_{k-1} = \phi_1^2 \gamma_{k-2} = \phi_1^k \gamma_0 = \phi_1^k Var(x_t) = \phi_1^k \frac{\sigma_\varepsilon^2}{1-\phi_1^2}$

$$Var(x_t) = \frac{\sigma_\varepsilon^2}{1-\phi_1^2}$$

對於平穩AR(1) 模型：

$$\boldsymbol{\gamma_k = \phi_1^k \frac{\sigma_\varepsilon^2}{1-\phi_1^2}}$$

練習

- 求平穩AR(2)模型的自協方差函數遞推公式
- $x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \varepsilon_t$有$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}$
- k=1時有$\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1 \ \Rightarrow \gamma_1 = \frac{\phi_1}{1-\phi_2}\gamma_0$
- 求γ_0： $x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \varepsilon_t$兩邊同乘$x_t$求期望

$$E(x_t x_t) = E\big((\phi_1 x_{t-1} + \phi_2 x_{t-2} + \varepsilon_t)(\phi_1 x_{t-1} + \phi_2 x_{t-2} + \varepsilon_t)\big)$$

$$\gamma_0 = \phi_1^2 \gamma_0 + \phi_2^2 \gamma_0 + 2\phi_1 \phi_2 \gamma_1 + \sigma_\varepsilon^2$$

$$\gamma_0 = \frac{1-\phi_2}{(1+\phi_2)(1-\phi_1-\phi_2)(1+\phi_1-\phi_2)}\sigma_\varepsilon^2$$

練習

對於平穩AR(2) 模型：

$$\gamma_0 = \frac{1-\phi_2}{(1+\phi_2)(1-\phi_1-\phi_2)(1+\phi_1-\phi_2)}\sigma_\varepsilon^2$$

$$\gamma_1 = \frac{\phi_1}{1-\phi_2}\gamma_0$$

$$\gamma_k = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2}$$

AR模型的自相關系數

AR模型的自相關系數

- 根據自相關系數定義$\rho_k = \frac{\gamma_k}{\gamma_0}$，在自協方差函數遞推公式兩邊同除以$\gamma_0$，則可以自相關系數的遞推公式。

$$E(x_t x_{t-k}) = \phi_1 E(x_{t-1} x_{t-k}) + \phi_2 E(x_{t-1} x_{t-k}) + \cdots + E\left(x_{t-p} x_{t-k}\right) + E(\varepsilon_t x_{t-k})$$

$$\gamma_k = \phi_1 \gamma_{k-1} + \cdots + \phi_p \gamma_{k-p}$$

- 兩邊同除以γ_0：

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-1} + \cdots + \phi_p \rho_{k-p}$$

AR模型的自相關系數

對於平穩AR(1) 模型：

$$\rho_0 = \frac{\gamma_0}{\gamma_0} = 1$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \phi_1^k$$

對於平穩AR(2) 模型：

$$\rho_0 = \frac{\gamma_0}{\gamma_0} = 1$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\phi_1}{1 - \phi_2}$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}$$

AR模型的自相關係數

AR(p)模型的自相關係數性質

① 拖尾性

② 呈負指數衰減

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-1} + \cdots + \phi_p\rho_{k-p}$$

自相關係數的遞推公式是一個p階齊次差分方程，通解為

$$\rho_k = \sum_{i=1}^{p} c_i\lambda_i^k$$

自相關係數的遞推公式的特徵方程與對應的AR(p)特徵方程相同，因此同平穩，$|\lambda_i| < 1$。$c_1, \ldots, c_p$不能全為0。

c_i不全為0，ρ_k始終是非0的，拖尾性。

$|\lambda_i| < 1$，呈負指數衰減。

AR模型的自相關系數

AR(p)模型的自相關系數性質

① 拖尾性

② 呈負指數衰減

$$x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + \varepsilon_t$$

由表達式可見x_t受隨機誤差ε_t和前p期的序列值$x_{t-1}, \ldots, x_{t-p}$影響。

但x_{t-p}又會受$x_{t-p-1}, \ldots, x_{t-2p}$影響。所以實際上，$x_t$會受過去每一期序列值的影響，因此自相關系數（當前$x_t$與過去$x_{t-k}$的相關性）表現拖尾。

同時ρ_k會呈負指數衰減。

自相關圖判別平穩序列時，「短期相關」的原因。平穩序列通常只有近期序列值對當前序列值影響較明顯，很久前的序列值影響很小。

課上作業

p3.5.ipynb
solve_root.ipynb

① $x_t = 0.8x_{t-1} + \varepsilon_t$

② $x_t = -0.8x_{t-1} + \varepsilon_t$

③ $x_t = x_{t-1} - 0.5x_{t-2} + \varepsilon_t$

④ $x_t = -x_{t-1} - 0.5x_{t-2} + \varepsilon_t$

對於平穩AR(1) 模型：

$$\rho_0 = \frac{\gamma_0}{\gamma_0} = 1$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \phi_1^k$$

對於平穩AR(2) 模型：

$$\rho_0 = \frac{\gamma_0}{\gamma_0} = 1$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{\phi_1}{1-\phi_2}$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}$$

- 畫出以上四個平穩AR模型的自相關圖，序列長度n分別設成101和1001。（p3.5.ipynb）
- 計算以上四個模型的自相關系數的理論值。
- 比較自相關系數的實驗值和理論值。
- **把自相關圖和理論值寫進報告中。提交代碼和報告（word或pdf文檔）。**

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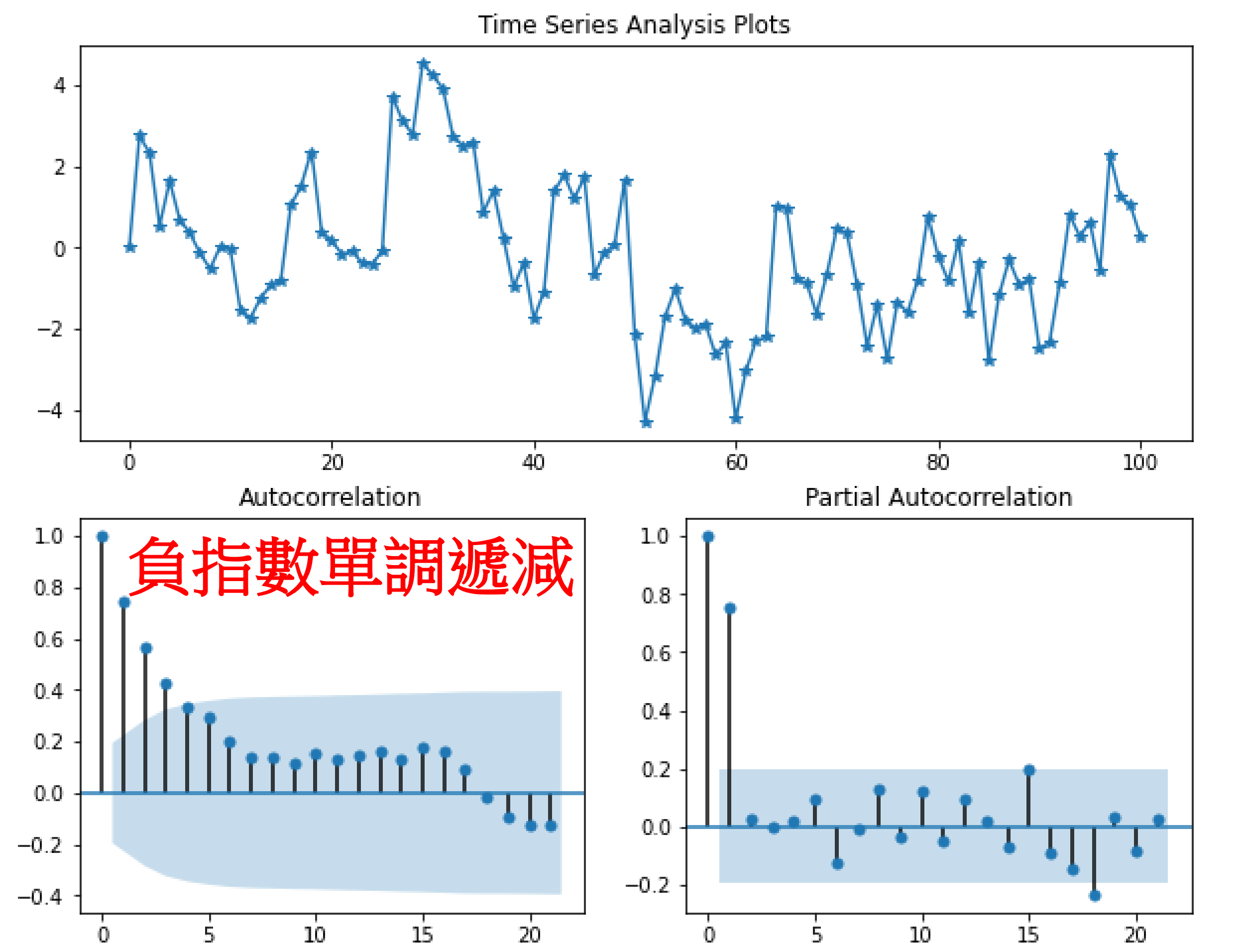
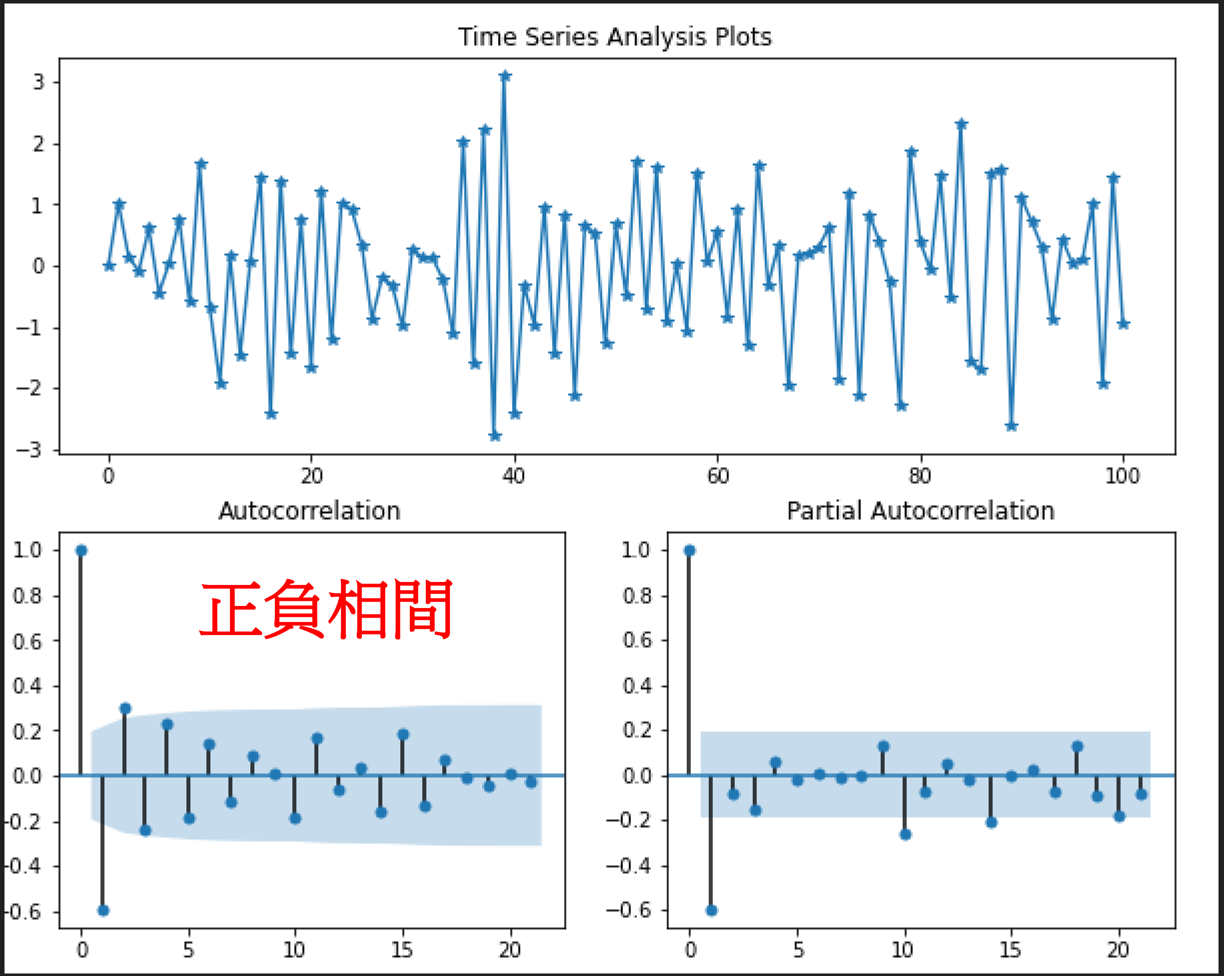
練習



$$x_t = 0.8x_{t-1} + \varepsilon_t$$

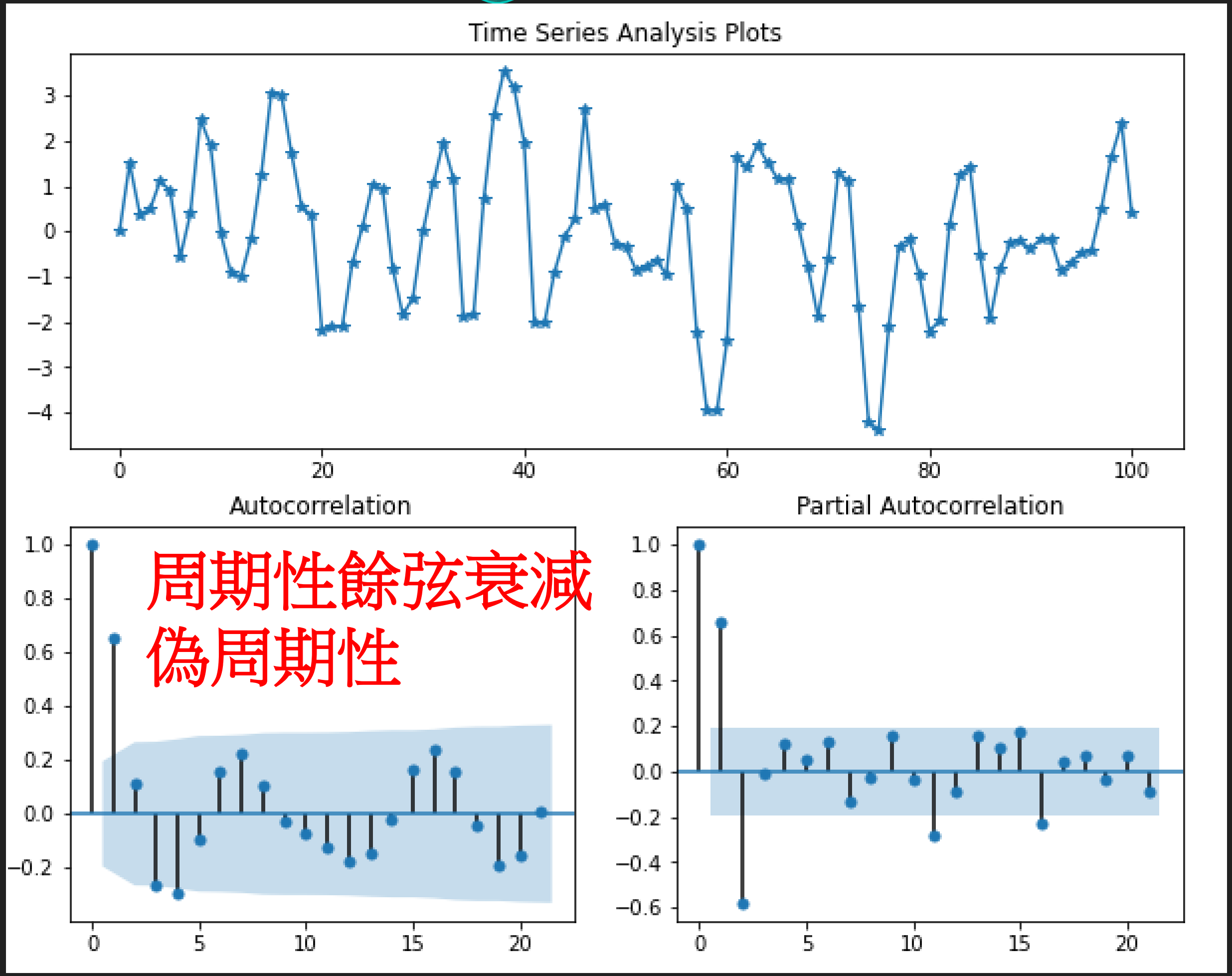
```
lambda= [0.8000000000000000]
abs(lambda)= [0.8000000000000000]
```



$$x_t = -0.8x_{t-1} + \varepsilon_t$$

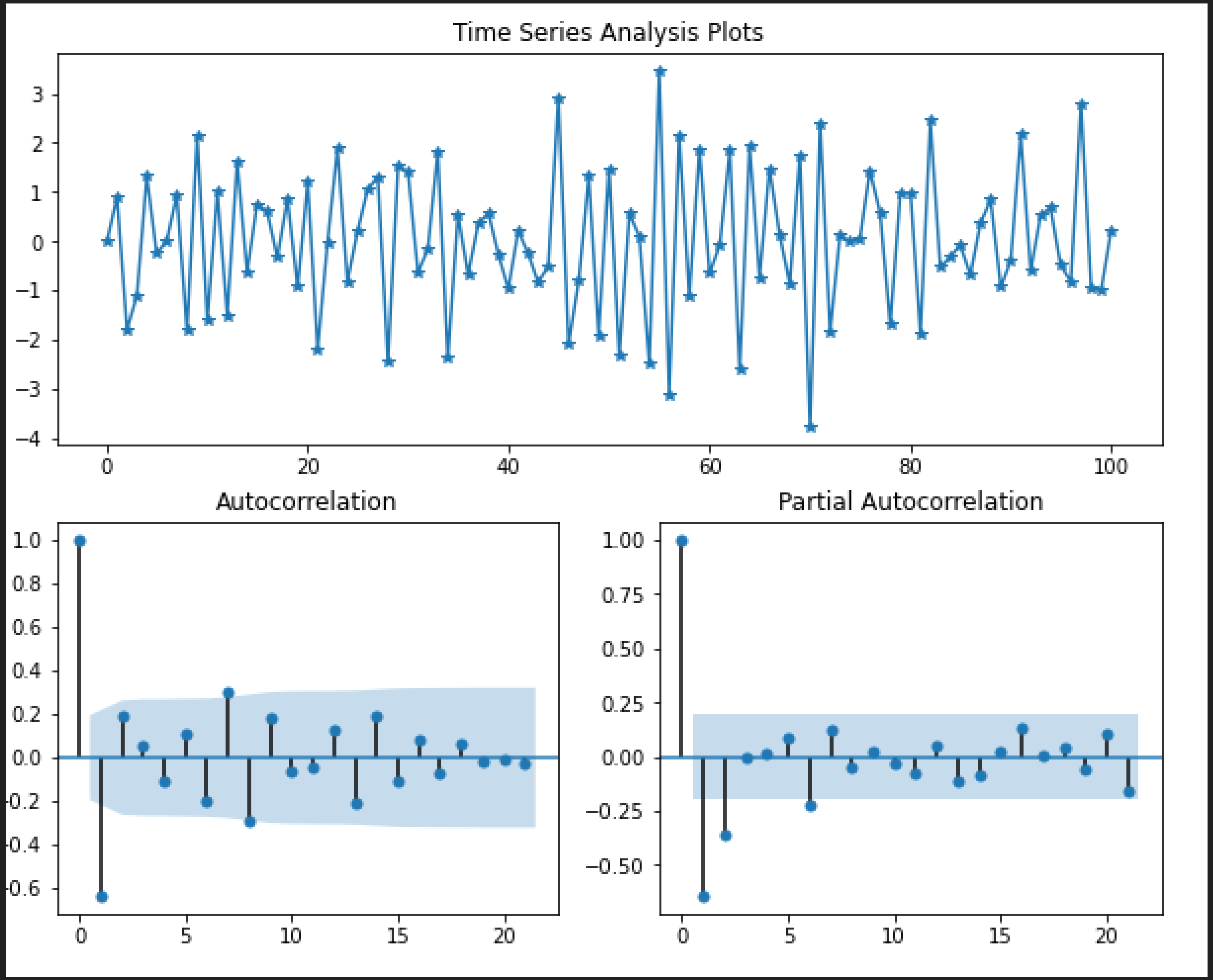
```
lambda= [-0.8000000000000000]
abs(lambda)= [0.8000000000000000]
```

練習



$$x_t = x_{t-1} - 0.5x_{t-2} + \varepsilon_t$$

```
lambda= [0.5 - 0.5*I 0.5 + 0.5*I]
abs(lambda)= [0.707106781186548 0.707106781186548]
```



$$x_t = -x_{t-1} - 0.5x_{t-2} + \varepsilon_t$$

```
lambda= [-0.5 - 0.5*I -0.5 + 0.5*I]
abs(lambda)= [0.707106781186548 0.707106781186548]
```

AR模型的偏自相關系數

AR模型的偏自相關系數

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + \varepsilon_t$$

- 對於一個平穩AR(p)模型，滯後k自相關系數ρ_k並不是x_t與x_{t-k}之間的單純的相關關系。
- 因為x_t同時受中間k-1個隨機變量$x_{t-1}, \dots, x_{t-k+1}$影響，而$x_{t-1}, \dots, x_{t-k+1}$與$x_t$也有相關關系，所以$\rho_k$裏摻雜了其他變量對$x_t$與$x_{t-k}$的相關影響。
- 因此引入偏自相關系數PACF（partial autocorrelation function）

AR模型的偏自相關系數

- 對於平穩序列$\{x_t\}$，滯後k偏自相關系數是指**給定中間k-1個隨機變量**$x_{t-1},\ldots,x_{t-k+1}$時，(即去掉中間k-1個隨機變量$x_{t-1},\ldots,x_{t-k+1}$的干擾之後)，x_{t-k}對x_t影響的相關關度量。

x_{t-k}	x_{t-k+1}	$\ldots$	x_{t-2}	x_{t-1}	x_t

固定

$$\rho_{x_t,x_{t-k}|x_{t-1},\ldots,x_{t-k+1}} = \frac{E\left[\left(x_t - \hat{E}x_t\right)\left(x_{t-k} - \hat{E}x_{t-k}\right)\right]}{E\left[\left(x_{t-k} - \hat{E}x_{t-k}\right)^2\right]}$$

參考PACF1.jpg, PACF2.jpg, PACF3.jpg, PACF4.jpg

AR模型的偏自相關系數

- 假定$\{x_t\}$為中心化平穩序列，用過去k期序列值$x_{t-1}, x_{t-2}, \dots, x_{t-k}$對$x_t$作k階自回歸擬合

$$x_t = \phi_{k1}x_{t-1} + \phi_{k2}x_{t-2} + \cdots + \phi_{kk}x_{t-k} + \varepsilon_t$$

- 兩邊同乘x_{t-l}並求期望

$$\rho_l = \phi_{k1}\rho_{l-1} + \phi_{k2}\rho_{l-1} + \cdots + \phi_{kk}\rho_{l-k}, \forall l \geq 1$$

取前k個方程構成的方程組

$$\begin{cases} \rho_1 = \phi_{k1}\rho_0 + \phi_{k2}\rho_1 + \cdots + \phi_{kk}\rho_{k-1} \\ \rho_2 = \phi_{k1}\rho_1 + \phi_{k2}\rho_0 + \cdots + \phi_{kk}\rho_{k-2} \\ \cdots \\ \rho_k = \phi_{k1}\rho_{k-1} + \phi_{k2}\rho_{k-2} + \cdots + \phi_{kk}\rho_0 \end{cases}$$

AR模型的偏自相關系數

$$\begin{cases} \rho_1 = \phi_{k1}\rho_0 + \phi_{k2}\rho_1 + \cdots + \phi_{kk}\rho_{k-1} \\ \rho_2 = \phi_{k1}\rho_1 + \phi_{k2}\rho_0 + \cdots + \phi_{kk}\rho_{k-2} \\ \qquad\qquad\qquad \cdots \\ \rho_k = \phi_{k1}\rho_{k-1} + \phi_{k2}\rho_{k-2} + \cdots + \phi_{kk}\rho_0 \end{cases}$$

- Yule-Walker方程，ϕ_{kk}則是滯後k偏自相關系數的值。

$$\begin{bmatrix} 1 & \rho_1 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \cdots & \rho_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \phi_{k1} \\ \phi_{k2} \\ \vdots \\ \phi_{kk} \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_k \end{bmatrix}$$

AR模型的偏自相關系數

$$\boldsymbol{\phi_{kk} = \frac{D_k}{D}}$$

$$D = \begin{vmatrix} 1 & \rho_1 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \cdots & \rho_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & 1 \end{vmatrix} \quad \mathrm{D_{kk}} = \begin{vmatrix} 1 & \rho_1 & \cdots & \rho_1 \\ \rho_1 & 1 & \cdots & \rho_2 \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & \rho_k \end{vmatrix}$$

AR模型的偏自相關系數

參考PACF1.jpg, PACF2.jpg, PACF3.jpg, PACF4.jpg

- 偏自相關系數截尾：平穩AR(p)模型的偏自相關系數具有p階截尾。即$\phi_{kk}=0\ (\forall k>p)$
- AR(p) $x_t=\phi_1 x_{t-1}+\phi_2 x_{t-2}+\cdots+\phi_p x_{t-p}+\varepsilon_t$
- 用過去k期序列值$x_{t-1},x_{t-2},\ldots,x_{t-k}$對$x_t$作k階自回歸擬合

$$x_t=\phi_{k1}x_{t-1}+\phi_{k2}x_{t-2}+\cdots+\phi_{kk}x_{t-k}+\varepsilon_t$$

- 當k>p時， $x_{t-1},x_{t-2},\ldots,x_{t-p}$固定了， x_t也固定了。因此，k>p時， x_{t-k}對x_t沒有影響了，故偏自相關系數為0。

…	x_{t-k}	…	x_{t-p-1}	x_{t-p}	…	x_{t-2}	x_{t-1}	x_t	ε_t

練習

- 考察以下AR模型的偏自相關系數的截尾性。
- $x_t = 0.8x_{t-1} + \varepsilon_t$
- $x_t = -0.8x_{t-1} + \varepsilon_t$
- $x_t = x_{t-1} - 0.5x_{t-2} + \varepsilon_t$
- $x_t = -x_{t-1} - 0.5x_{t-2} + \varepsilon_t$

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練習

- $AR(1)$
- $\rho_1 = \phi_{11}\rho_0 \Rightarrow \phi_{11} = \frac{\rho_1}{\rho_0} = \phi_1$

$\boldsymbol{\rho_k = \phi_1^k}$

- $\phi_{kk} = 0, \forall k > 1$

練習

$$\rho_k = \begin{cases} 1, & k = 0 \\ \dfrac{\phi_1}{1-\phi_2}, & k = 1 \\ \phi_1\rho_{k-1} + \phi_2\rho_{k-2}, & k \geq 2 \end{cases}$$

- $AR(2)$
- $\rho_1 = \phi_{11}\rho_0 \Rightarrow \phi_{11} = \dfrac{\rho_1}{\rho_0} = \dfrac{\phi_1}{1-\phi_2}$
- $\begin{cases} \rho_1 = \phi_{21}\rho_0 + \phi_{22}\rho_1 \\ \rho_2 = \phi_{21}\rho_1 + \phi_{22}\rho_0 \end{cases}$
- $\begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \Rightarrow \phi_{22} = \phi_2$

$$\phi_{22} = \frac{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix}} = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2}.$$

$\rho_1 = \dfrac{\phi_1}{1-\phi_2}$, $\rho_2 = \dfrac{\phi_1^2}{1-\phi_2} + \phi_2$. 代入得.

$$\phi_{22} = \frac{(1-\phi_2)\phi_1^2 + (1-\phi_2)^2\phi_2 - \phi_1^2}{(1-\phi_2)^2 - \phi_1^2}.$$

$$\phi_{22} = \frac{-\phi_2\phi_1^2 + \phi_2(1-\phi_2^2)}{(1-\phi_2)^2 - \phi_1^2}$$

$$\phi_{22} = \frac{\phi_2\left[(1-\phi_2)^2 - \phi_1^2\right]}{(1-\phi_2)^2 - \phi_1^2}$$

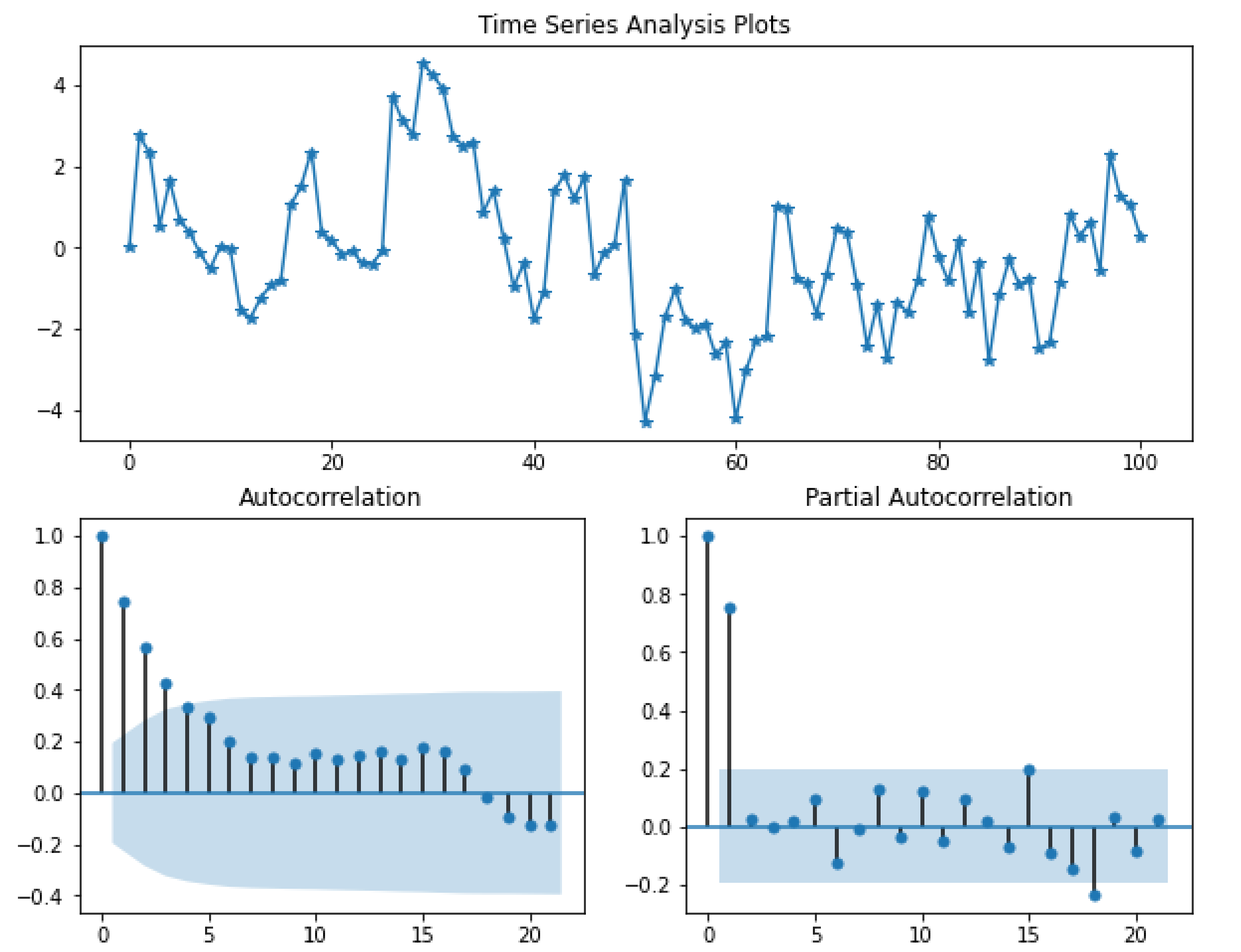
$$\phi_{22} = \phi_2.$$

練習

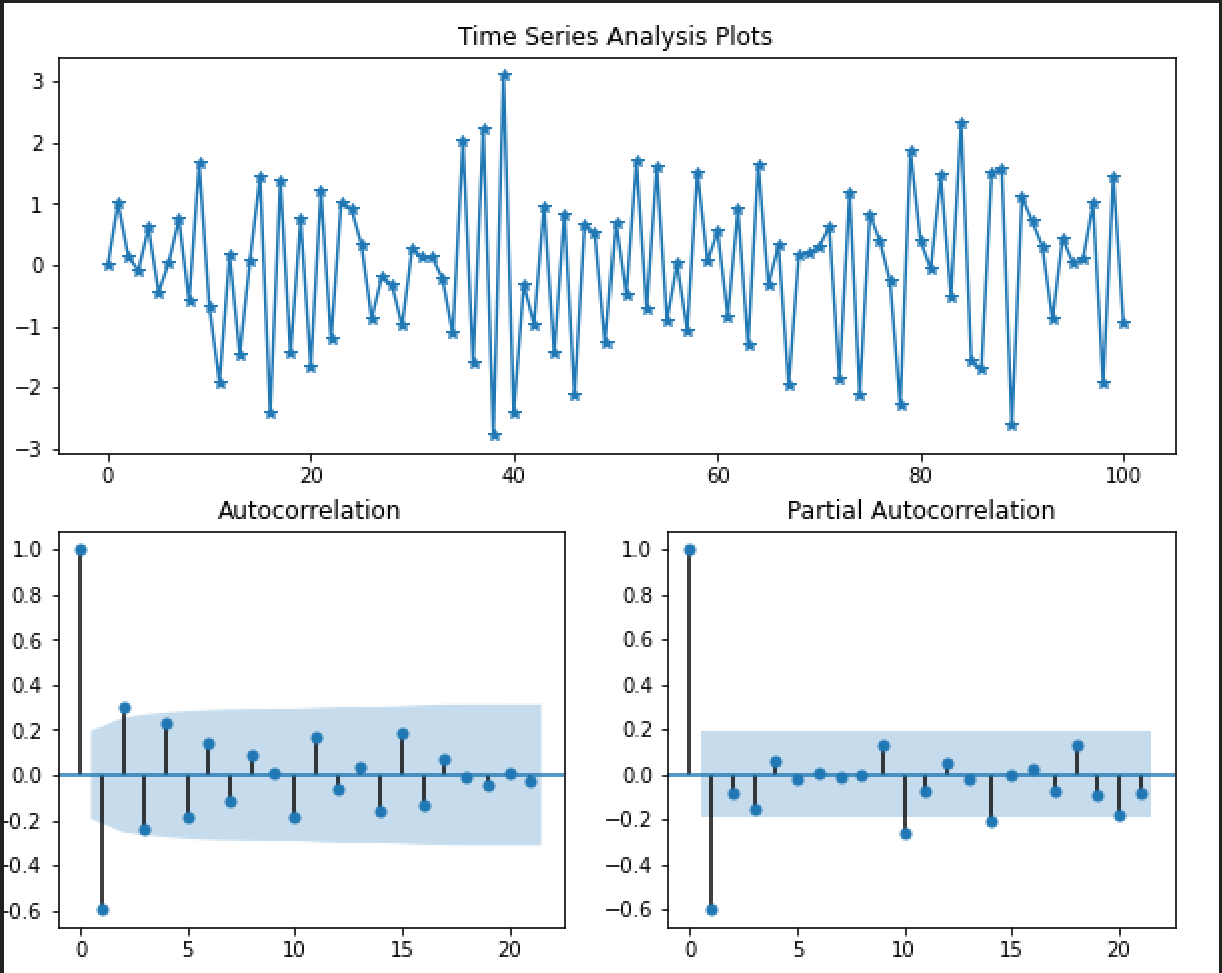
- $AR(1)$：$\phi_{11} = \phi_1$
- $AR(2)$：$\phi_{11} = \frac{\phi_1}{1-\phi_2}$
- $AR(2)$：$\phi_{22} = \phi_2$

AR(p)	PACF
$x_t = 0.8x_{t-1} + \varepsilon_t$	$\phi_{kk} = \begin{cases} 0.8, & k = 1 \\ 0, & k \geq 2 \end{cases}$
$x_t = -0.8x_{t-1} + \varepsilon_t$	$\phi_{kk} = \begin{cases} -0.8, & k = 1 \\ 0, & k \geq 2 \end{cases}$
$x_t = x_{t-1} - 0.5x_{t-2} + \varepsilon_t$	$\phi_{kk} = \begin{cases} \frac{2}{3}, & k = 1 \\ -0.5, & k = 2 \\ 0, & k \geq 3 \end{cases}$
$x_t = -x_{t-1} - 0.5x_{t-2} + \varepsilon_t$	$\phi_{kk} = \begin{cases} -\frac{2}{3}, & k = 1 \\ -0.5, & k = 2 \\ 0, & k \geq 3 \end{cases}$

練習

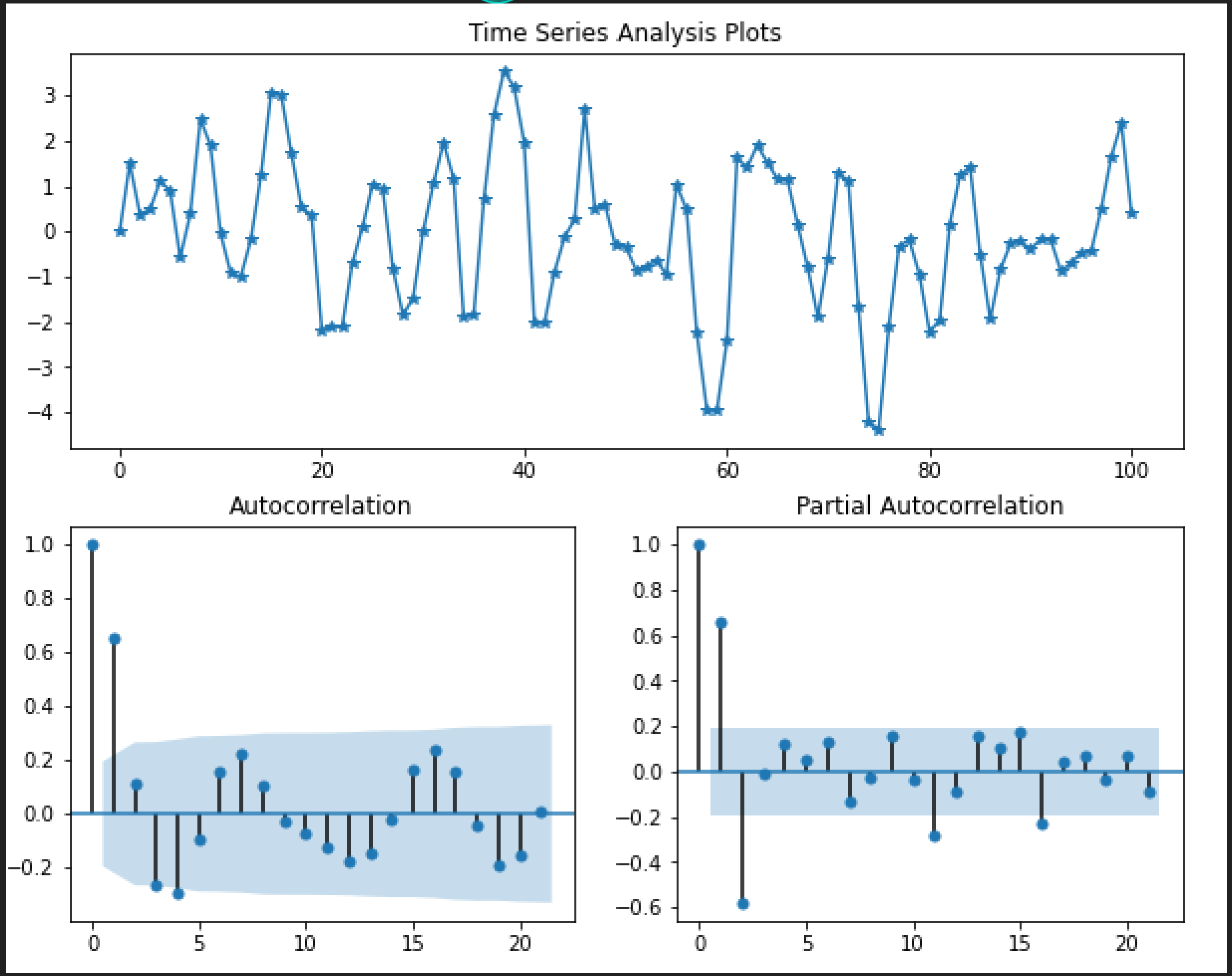


$$x_t = 0.8x_{t-1} + \varepsilon_t$$

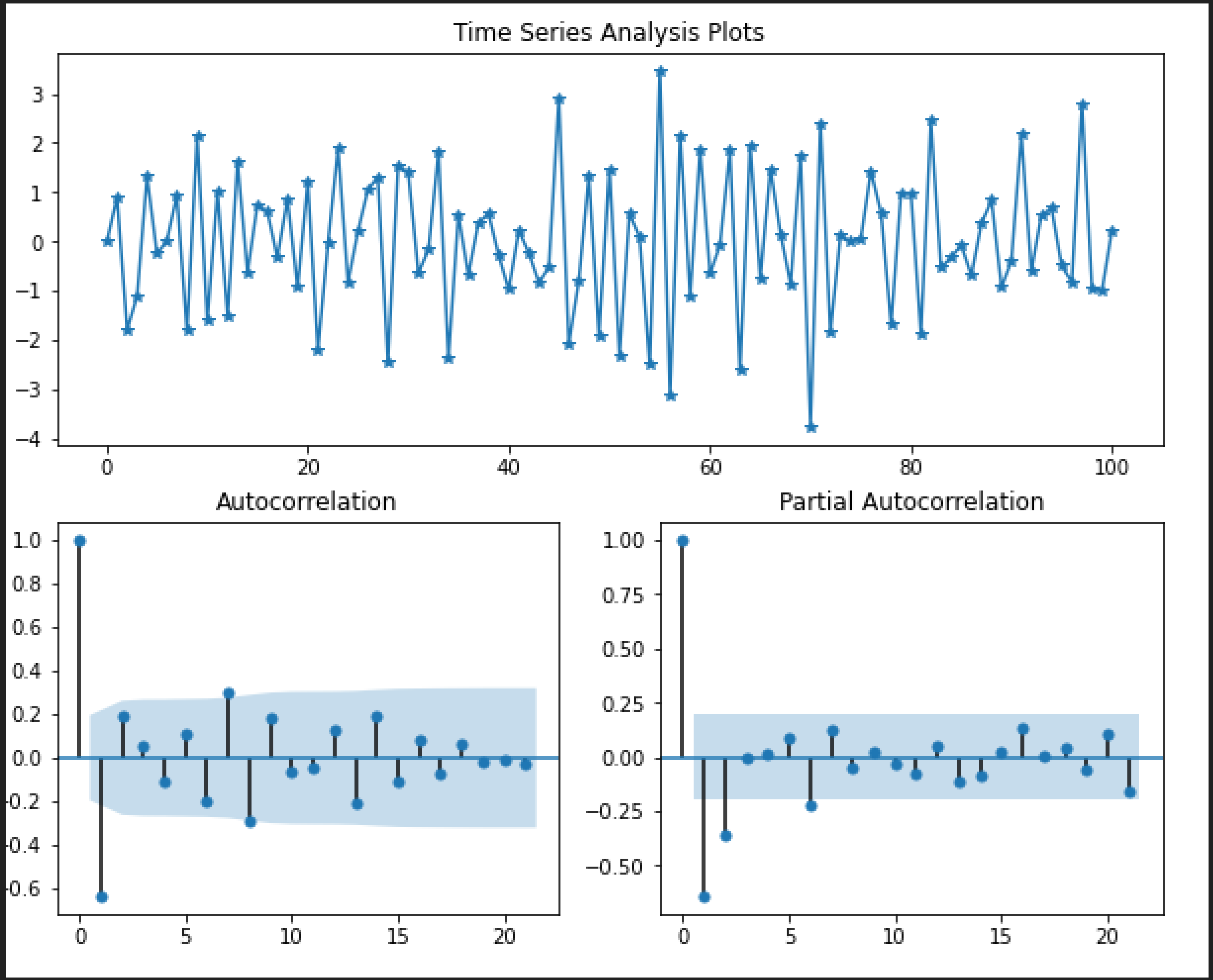


$$x_t = -0.8x_{t-1} + \varepsilon_t$$

練習



$$x_t = x_{t-1} - 0.5x_{t-2} + \varepsilon_t$$



$$x_t = -x_{t-1} - 0.5x_{t-2} + \varepsilon_t$$

練習

- 由於樣本隨機性，樣本的偏自相關系數不會嚴格截尾。
- $PACF$標准差$\sqrt{\frac{1}{n}}$（n為序列長度），在2倍標准差內可看作是截尾。

本節總結

本節總結

- 掌握求AR(p)模型均值、方差、自協方差、自相關系數、偏自相關系數的方法。
- 至少能求AR(1)和AR(2)的各統計值。
- **AR(p)模型，自相關系數拖尾，偏自相關系數p階截尾**

本節總結

- 均值$\mu = \frac{\phi_0}{1-\phi_1}$
- Green函遞推公式$G_j = \phi_1^j$
- 方差$Var(x_t) = \frac{\sigma_\varepsilon^2}{1-\phi_1^2}$
- 自協方差函數$\gamma_k = \phi_1^k \frac{\sigma_\varepsilon^2}{1-\phi_1^2}$
- 自相關系數$\rho_k = \phi_1^k$
- 偏自相關系數$\phi_{11} = \phi_1$

AR(1)

本節總結

AR(2)

- 均值$\mu = \frac{\phi_0}{1-\phi_1-\phi_2}$
- 方差$Var(x_t) = \gamma_0$

自協方差函數γ_k

$$\gamma_0 = \frac{1-\phi_2}{(1+\phi_2)(1-\phi_1-\phi_2)(1+\phi_1-\phi_2)}\sigma_\varepsilon^2$$

$$\gamma_1 = \frac{\phi_1}{1-\phi_2}\gamma_0$$

$$\gamma_k = \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2}$$

自相關系數ρ_k

$$\rho_0 = 1$$

$$\rho_1 = \frac{\phi_1}{1-\phi_2}$$

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}$$

偏自相關系數ϕ_{11}

$$\phi_{11} = \frac{\phi_1}{1-\phi_2}$$

$$\phi_{22} = \phi_2$$

本節總結

AR(p)

- 均值$\mu = \frac{\phi_0}{1-\phi_1 \ldots -\phi_p}$
- Green函數$x_t = \sum_{j=0}^{\infty} G_j \varepsilon_{t-j}$

$$\begin{cases} G_0 = 1 \\ G_j = \sum_{k=1}^{j} \phi'_k G_{j-k} \end{cases}$$

$$\phi'_k = \begin{cases} \phi_k, & k \le p \\ 0, & k > p \end{cases}$$

$$j = 0, 1, 2, \ldots$$

- 方差$Var(x_t) = \sum_{j=0}^{\infty} G_j^2 \sigma_\varepsilon^2$

$$Var(\varepsilon_t) = \sigma_\varepsilon^2$$

- 自協方差函數$\gamma_k = \phi_1 \gamma_{k-1} + \cdots + \phi_p \gamma_{k-p}$
- 自相關系數$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-1} + \cdots + \phi_p \rho_{k-p}$

本節總結

AR(p)

- 偏自相關系數

$$\phi_{kk} = \frac{D_k}{D}$$

$$D = \begin{vmatrix} 1 & \rho_1 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \cdots & \rho_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & 1 \end{vmatrix} \quad \mathrm{D_{kk}} = \begin{vmatrix} 1 & \rho_1 & \cdots & \rho_1 \\ \rho_1 & 1 & \cdots & \rho_2 \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & \rho_k \end{vmatrix}$$

作業

作業3.3a

作業

- 計算AR(1)模型

$$x_t = 0.8x_{t-1} + \varepsilon_t$$

① 均值

② Green函數遞推公式

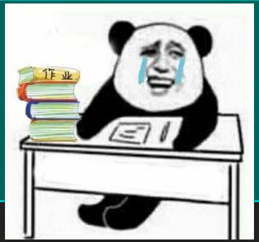
③ 方差

④ 延遲k協方差函數$\gamma_0, \gamma_1, \gamma_2$

⑤ 延遲k自相關系數ρ_0, ρ_1, ρ_2

⑥ 延遲k偏自相關系數$\phi_{11}, \phi_{22}, \phi_{33}$

作業3.3b



作業

- 計算AR(2)模型

$$x_t = 0.8x_{t-1} - 0.64x_{t-2} + \varepsilon_t$$

① 均值

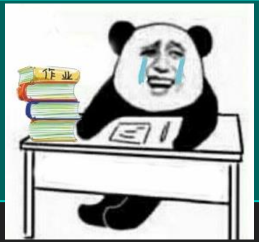
② 延遲k協方差函數$\gamma_0, \gamma_1, \gamma_2$

③ 方差

④ 延遲k自相關係數ρ_0, ρ_1, ρ_2

- 延遲k偏自相關係數$\phi_{11}, \phi_{22}, \phi_{33}$

作業



作業

- 提交HW3-3.docx
- 截止時間：得定