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Multibody Systems Made Simple and Efficient

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Outline of this presentation

- History of natural coordinates
- Acknowledgements
- Early developments and applications of natural coordinates
  - Back to the basics
  - Multibody simulation plus realistic 3-D graphics
  - Global formulations: Improving the efficiency
  - Biomechanics
  - Flexible multibody systems
- Improving the efficiency throughout topological formulations
  - Rigid multibody systems
  - Flexible multibody systems
- Live demos
- Conclusive remarks

Aknowledgements

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Part I: History of natural coordinates

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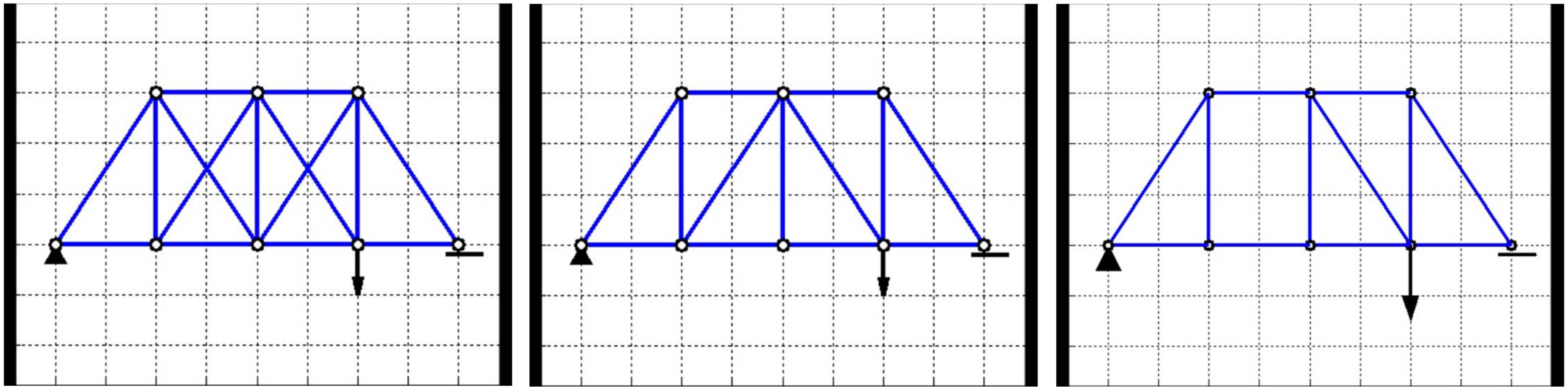
1.1 How the idea arose

- F. Beaufait, Basic Concepts of Structural Analysis, Prentice Hall, 1977
- G. Strang and G. Fix, An Analysis of the Finite Element Method, Wellesley-Cambridge Press (1973)
- G. Strang, Linear Algebra and Its Applications, Academic Press, First Edition (1976)

The beginning of natural coordinates 1/2

- 1971-77
  - Lecturer on Structural Analysis and Numerical Methods
  - Doctoral Thesis on “Finite Element Thermal Stresses in Railways Wheels”
- 1978-79
  - Strong pressure to complete in four weeks an original research work on mechanisms
  - An idea urgently needed: A mechanism as an unstable structure!



Statically indeterminate — Statically determinate — Unstable

- The stiffness matrix has the same null space as the Jacobian matrix of the constant distance constraint equations $(\mathbf{Kx} = \mathbf{0})$

The beginning of natural coordinates 2/2

- Mathematical evolution
  - From minimum elastic potential to minimum constraint violation, and then to nonlinear constraint equations: constant distances, angles, areas,…
- Extension to three-dimensional systems (1981)
  - Straightforward when only points are considered
  - Each rigid body has at least three non-aligned points
  - Revolute joints introduced by sharing two points
  - Cylindrical joints introduced with four points aligned
  - High number of points needed
- Unit vectors introduced (1984)
  - Points and unit vectors contribute to define 3-D orientation
  - A single unit vector may define the direction of several parallel axes
  - Revolute joints introduced by sharing a point and a unit vector
  - Cylindrical joints introduced with two points aligned with a shared unit vector
  - Important size reductions achieved

Example 1: Constraint equations of robot ERA

Body 2
$$\begin{cases} (x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2-d_{12}^2=0 \\ (x_2-x_1)u_{1x}+(y_2-y_1)u_{1y}+(z_2-z_1)u_{1z}=0 \\ u_{1x}^2+u_{1y}^2+u_{1z}^2-1=0 \\ u_{1z}=0 \end{cases}$$

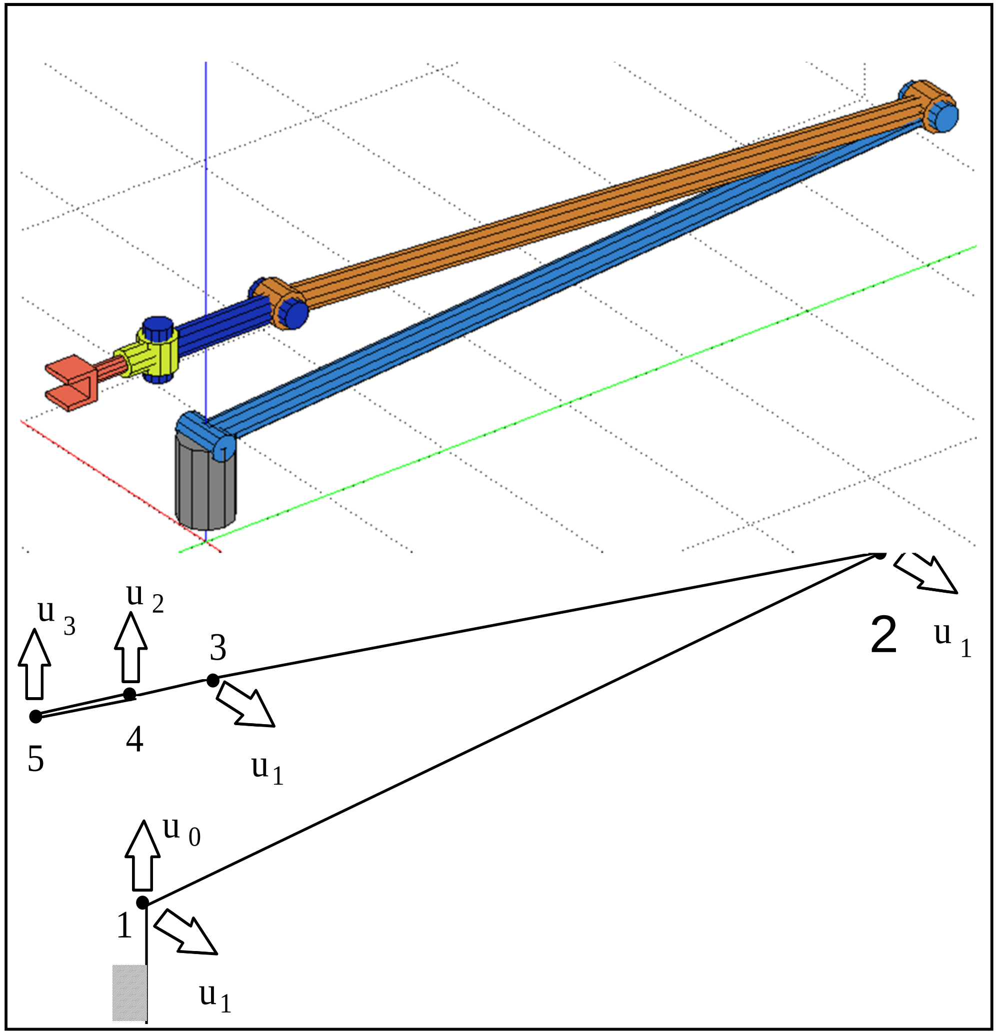
Body 3
$$\begin{cases} (x_3-x_2)^2+(y_3-y_2)^2+(z_3-z_2)^2-d_{23}^2=0 \\ (x_3-x_2)u_{1x}+(y_3-y_2)u_{1y}+(z_3-z_2)u_{1z}=0 \end{cases}$$

Body 4
$$\begin{cases} (x_4-x_3)^2+(y_4-y_3)^2+(z_4-z_3)^2-d_{34}^2=0 \\ (x_3-x_4)u_{1x}+(y_3-y_4)u_{1y}+(z_4-z_3)u_{1z}=0 \\ (x_3-x_4)u_{2x}+(y_3-y_4)u_{2y}+(z_4-z_3)u_{2z}=0 \\ u_{2x}u_{1x}+u_{2y}u_{1y}+u_{2z}u_{1z}=0 \\ u_{2x}^2+u_{2y}^2+u_{2z}^2-1=0 \end{cases}$$

Body 5
$$\begin{cases} (x_5-x_4)^2+(y_5-y_4)^2+(z_5-z_4)^2-d_{45}^2=0 \\ (x_5-x_4)u_{2x}+(y_5-y_4)u_{2y}+(z_5-z_4)u_{2z}=0 \end{cases}$$

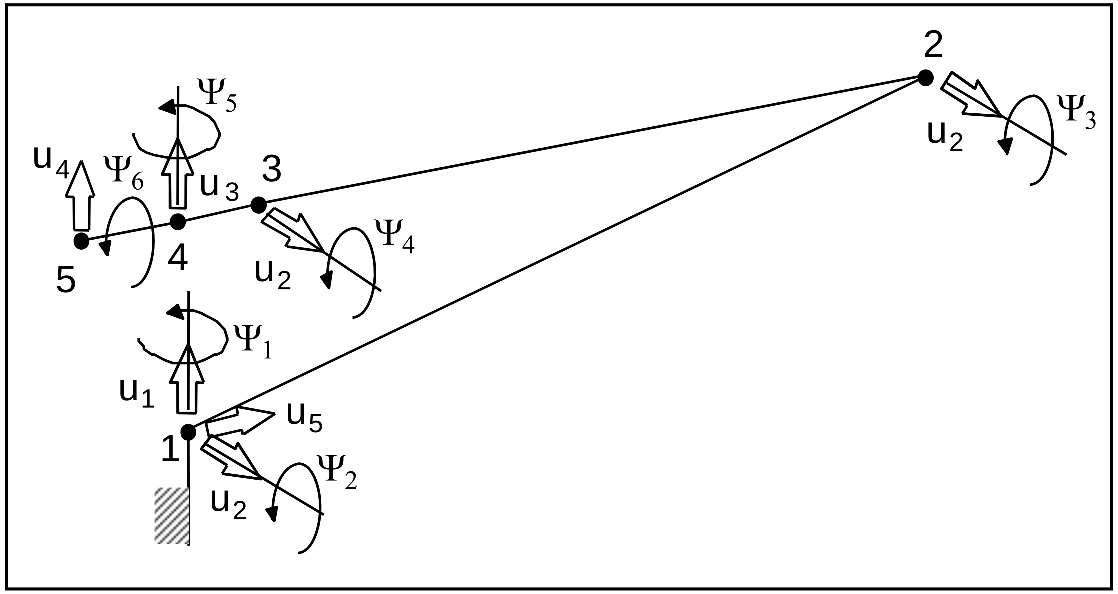
Body 6
$$\begin{cases} (x_5-x_4)u_{3x}+(y_5-y_4)u_{3y}+(z_5-z_4)u_{3z}=0 \\ u_{3x}^2+u_{3y}^2+u_{3z}^2-1=0 \end{cases}$$

21 coordinates and 15 constraints:
$\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_4$, $\mathbf{r}_5$, $\mathbf{u}_1$, $\mathbf{u}_2$ y $\mathbf{u}_3$



Example 2: Robot ERA with mixed coordinates

- Natural coordinates allow an easy addition of relative coordinates
  - Angles and distances in Revolute and Prismatic joints, respectively, allow a direct introduction of actuator forces.



- In the ERA robot, angles have been defined in the following way:
  - Angle ψ_1 goes from $\mathbf{u}_2$ to $\mathbf{u}_5$ and it is measured on $\mathbf{u}_1$.
  - Angle ψ_2 goes from $\mathbf{u}_1$ to $(\mathbf{r}_2 - \mathbf{r}_1)$ and it is measured on $\mathbf{u}_2$.
  - Angle ψ_3 goes from $(\mathbf{r}_1 - \mathbf{r}_2)$ to $(\mathbf{r}_3 - \mathbf{r}_2)$ and it is measured on $\mathbf{u}_2$.
  - Angle ψ_4 goes from $(\mathbf{r}_2 - \mathbf{r}_3)$ to $(\mathbf{r}_4 - \mathbf{r}_3)$ and it is measured on $\mathbf{u}_2$.
  - Angle ψ_5 goes from $(\mathbf{r}_3 - \mathbf{r}_4)$ to $(\mathbf{r}_5 - \mathbf{r}_4)$ and it is measured on $\mathbf{u}_3$.
  - Angle ψ_6 goes from $\mathbf{u}_3$ to $\mathbf{u}_4$ and it is measured on $(\mathbf{r}_5 - \mathbf{r}_4)$.
  - Each angle is defined with the dot product of vectors or with the largest component of the cross product of vectors, according to its value.

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Part II: Early developments and applications of natural coordinates

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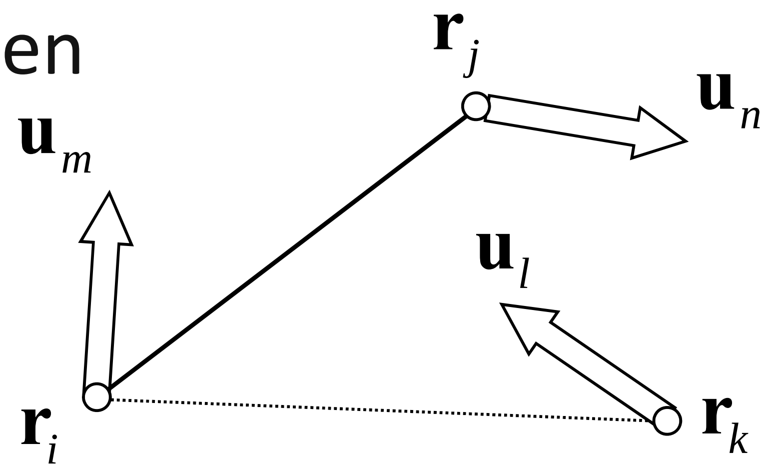
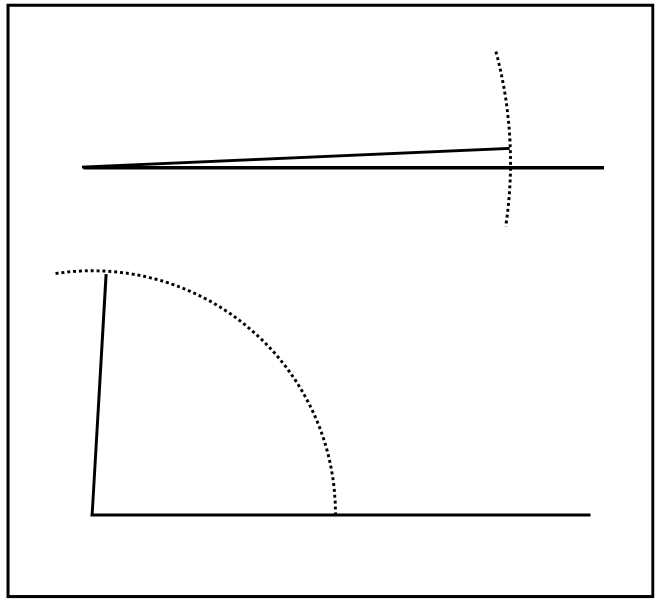
2.1 Back to the basics: singular mass matrices and redundant constraints

- J. García de Jalón and E. Bayo, *Kinematic and Dynamic Simulation of Multi-Body Systems: The Real-Time Challenge*, Springer-Verlag, New-York (1993).
- Udwadia, F. E. and Phohomsiri, P., *Explicit Equations of Motion for Constrained Mechanical Systems with Singular Mass Matrices and Applications to Multi-body Dynamics*, Proceedings of the Royal Society of London, Series A, 462, pp. 2097-2117, (2006).
- J. García de Jalón, *Multibody dynamics with redundant constraints and singular mass matrix: Theory and practice*, paper in preparation, (2011).

Aspects of formulations with natural coordinates



- Singular mass matrices arise very often
  - A constant mass matrix (with no velocity dependent inertia forces) can be defined with two points and two non coplanar unit vectors.
  - If there are more points and vectors their inertia coefficients are null and the mass matrix singular.
- Dependent constraint equations are also frequent
  - for implementation convenience for instance: $\mathbf{u}_j - \mathbf{u}_k = \mathbf{0}, \quad \mathbf{u}_i \times \left(\mathbf{r}_j - \mathbf{r}_k\right) = \mathbf{0}$
  - in over-constrained systems



- These difficulties need always to be dealt with to find an adequate solution, but, When does this solution exists? Is it unique?

Dynamic equations with Lagrange multipliers 1/3

- Motion differential equations
  - Putting together the dynamic equations and all the kinematic constraints:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{T}\boldsymbol{\lambda} = \mathbf{F}$$

$$\mathbf{\Phi}(\mathbf{q},t)=\mathbf{0}, \quad \mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = -\mathbf{\Phi}_{t} \equiv \mathbf{b}, \quad \mathbf{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = -\dot{\mathbf{\Phi}}_{t} - \dot{\mathbf{\Phi}}_{\mathbf{q}}\dot{\mathbf{q}} \equiv \mathbf{c}$$

  that is a system of 3*N*+*M* differential-algebraic equations of index 3.
  - It is also possible to set only the dynamic and the acceleration equations:

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^{T} \\ \mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{c} \end{Bmatrix} \qquad (\text{some stabilization necessary})$$

  - Assume matrix **M** is positive definite and matrix $\mathbf{\Phi}_{\mathbf{q}}$ has full rank

$$\mathbf{M} = \mathbf{M}^{T} \in \mathbb{R}^{n\times n}, \quad \mathbf{\Phi}_{\mathbf{q}} \in \mathbb{R}^{m\times n}, \quad \operatorname{rank}(\mathbf{M}) = n, \quad \operatorname{rank}(\mathbf{\Phi}_{\mathbf{q}}) = m$$

  - Then,

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}\left(\mathbf{F} - \mathbf{\Phi}_{\mathbf{q}}^{T}\boldsymbol{\lambda}\right)$$

$$\mathbf{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} = \mathbf{c} \quad \Rightarrow \quad \mathbf{\Phi}_{\mathbf{q}}\mathbf{M}^{-1}\left(\mathbf{F} - \mathbf{\Phi}_{\mathbf{q}}^{T}\boldsymbol{\lambda}\right) = \mathbf{c} \quad \Rightarrow \quad \boldsymbol{\lambda} = \left(\mathbf{\Phi}_{\mathbf{q}}\mathbf{M}^{-1}\mathbf{\Phi}_{\mathbf{q}}^{T}\right)^{-1}\left(\mathbf{\Phi}_{\mathbf{q}}\mathbf{M}^{-1}\mathbf{F} - \mathbf{c}\right)$$

  - These conditions on **M** and $\mathbf{\Phi}_{\mathbf{q}}$ are too restrictive and shall be relaxed

Dynamic equations with Lagrange multipliers 2/3

- Motion differential equations: the rank *r* of $\mathbf{\Phi_q}$ made visible
  - Index-1 ordinary differential equations may be transformed:

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi_q}^T \\ \mathbf{\Phi_q} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{c} \end{Bmatrix}$$

  - Matrix **M** may be singular or semi-definite
  - Matrix $\mathbf{\Phi_q}$ may have redundant constraints: $\mathbf{P}_{LU}\mathbf{\Phi_q} = \mathbf{L}_{m\times m}\begin{bmatrix} \mathbf{U}_{r\times n} \\ \mathbf{0}_{(m-r)\times n} \end{bmatrix}$

$$\begin{bmatrix} \mathbf{M} & \begin{bmatrix} \mathbf{U}^T_{r\times n} & \mathbf{0}^T_{(m-r)\times n} \end{bmatrix} \mathbf{L}^T_{m\times m} \\ \mathbf{L}_{m\times m}\begin{bmatrix} \mathbf{U}_{r\times n} \\ \mathbf{0}_{(m-r)\times n} \end{bmatrix} & \mathbf{0}_{m\times m} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \mathbf{P}_{LU}\boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{P}_{LU}\mathbf{c} \end{Bmatrix}$$

  - By taking into account that matrix $\mathbf{L}_{m\times m}$ is invertible,

$$\begin{bmatrix} \mathbf{M} & \begin{bmatrix} \mathbf{U}^T_{r\times n} & \mathbf{0}^T_{(m-r)\times n} \end{bmatrix} \\ \begin{bmatrix} \mathbf{U}_{r\times n} \\ \mathbf{0}_{(m-r)\times n} \end{bmatrix} & \mathbf{0}_{m\times m} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \bar{\boldsymbol{\lambda}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \bar{\mathbf{c}} \end{Bmatrix}, \qquad \bar{\boldsymbol{\lambda}} = \mathbf{L}^T_{m\times m}\mathbf{P}_{LU}\boldsymbol{\lambda}, \qquad \bar{\mathbf{c}} = \mathbf{L}^{-1}_{m\times m}\mathbf{P}_{LU}\mathbf{c}$$

  - The existence and uniqueness of solutions are studied next.

Dynamic equations with Lagrange multipliers 3/3

- Dynamic equations: existence and uniqueness of solutions
  - As has seen, these equations are compatible if $\mathbf{\Phi_q}\ddot{\mathbf{q}} = \mathbf{c}$ is compatible,

$$\begin{bmatrix} \mathbf{M} & \begin{bmatrix} \mathbf{U}^T_{r\times n} & \mathbf{0}^T_{(m-r)\times n} \end{bmatrix} \\ \begin{bmatrix} \mathbf{U}_{r\times n} \\ \mathbf{0}_{(m-r)\times n} \end{bmatrix} & \mathbf{0}_{m\times m} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \bar{\boldsymbol{\lambda}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \bar{\mathbf{c}} \end{Bmatrix}, \qquad \bar{\boldsymbol{\lambda}} = \mathbf{L}^T_{m\times m}\mathbf{P}_{LU}\boldsymbol{\lambda}, \qquad \bar{\mathbf{c}} = \mathbf{L}^{-1}_{m\times m}\mathbf{P}_{LU}\mathbf{c}$$

  - Values $\bar{c}_{r+1} = \ldots = \bar{c}_m = 0$ for compatibility. Values $\bar{\lambda}_{r+1}, \bar{\lambda}_{r+2}, \ldots, \bar{\lambda}_m$ are arbitrary.
  - Moreover, if matrix $\begin{bmatrix} \mathbf{M} & \mathbf{U}^T_{r\times n} \end{bmatrix}$ has full rank *n*, these equations are determined in the acceleration vector $\ddot{\mathbf{q}}$ and undetermined in the transformed Lagrange Multipliers $\bar{\boldsymbol{\lambda}}$. In this case, the following matrix is invertible

$$\left[\begin{array}{c|c} \mathbf{M} & \mathbf{U}^T_{r\times n} \\ \hline \mathbf{U}_{r\times n} & \mathbf{0} \end{array}\right]$$

even if matrix **M** is not invertible.

- The QR factorization and the SVD can be applied in a similar way

Three forms to set and solve the dynamic equations 1/4

- Three equivalent formulations
  - Index 1 Lagrange equations

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^{T} \\ \mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{c} \end{Bmatrix}$$

  - Null space method

$$\mathbf{\Phi}_{\mathbf{q}}\mathbf{R} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{R}^{T}\mathbf{M} \\ \mathbf{\Phi}_{\mathbf{q}} \end{bmatrix} \{\ddot{\mathbf{q}}\} = \begin{Bmatrix} \mathbf{R}^{T}\mathbf{F} \\ \mathbf{c} \end{Bmatrix}$$

  - Maggi's method

$$\dot{\mathbf{q}} = \mathbf{R}\dot{\mathbf{z}}, \quad \ddot{\mathbf{q}} = \mathbf{R}\ddot{\mathbf{z}} + \mathbf{S}\mathbf{c}$$

$$\mathbf{R}^{T}\mathbf{M}\mathbf{R}\ddot{\mathbf{z}} = \mathbf{R}^{T}\mathbf{F} - \mathbf{R}^{T}\mathbf{M}\mathbf{S}\mathbf{c}$$

  - The conditions for the existence of a solution $\ddot{\mathbf{q}}$ are:

$$\boxed{\operatorname{rank}\left(\begin{bmatrix} \mathbf{M} \\ \mathbf{\Phi}_{\mathbf{q}} \end{bmatrix}\right) = n \quad \Leftrightarrow \quad \ker \mathbf{M} \cap \ker \mathbf{\Phi}_{\mathbf{q}} = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{R}^{T}\mathbf{M}\mathbf{R} \ \text{ def. pos.}}$$

Three forms to set and solve the dynamic equations 2/4

- The existence of solutions
  - By applying the Rouché-Capelli theorem, in order to be able to express any external force vector as a linear combination of inertia and constraint forces,

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^T \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \mathbf{F} \in \mathbb{R}^{n \times 1} \quad \Rightarrow \quad \boxed{\operatorname{rank}\left(\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^T \end{bmatrix} \right) = n}$$

  - This condition can be transformed into an equivalent one,

$$\operatorname{rank}\left(\begin{bmatrix} \mathbf{M} \\ \mathbf{\Phi}_{\mathbf{q}} \end{bmatrix} \right) = n \quad \Leftrightarrow \quad \begin{bmatrix} \mathbf{M} \\ \mathbf{\Phi}_{\mathbf{q}} \end{bmatrix} \mathbf{x} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \Rightarrow \mathbf{x} = \mathbf{0} \quad \Leftrightarrow \quad \boxed{\ker \mathbf{M} \cap \ker \mathbf{\Phi}_{\mathbf{q}} = \mathbf{0}}$$

which has a more clear physical significance:

"Any physically possible velocity $\left(\dot{\mathbf{q}} \in \ker \mathbf{\Phi}_{\mathbf{q}} \right)$ *is associated with a non-zero (positive) kinetic energy* $\left(\mathbf{M}\dot{\mathbf{q}} \neq \mathbf{0},\ \dot{\mathbf{q}}^T \mathbf{M} \dot{\mathbf{q}} > 0 \right)$."

- The existence of solutions (cont.)
  - For the null space equations, taking into account that $\mathbf{\Phi_q R} = \mathbf{0}$,

$$\begin{bmatrix} \mathbf{R}^T\mathbf{M} \\ \mathbf{\Phi_q} \end{bmatrix}\{\ddot{\mathbf{q}}\} = \begin{Bmatrix} \mathbf{R}^T\mathbf{F} \\ \mathbf{c} \end{Bmatrix}, \qquad \operatorname{rank}\left(\begin{bmatrix} \mathbf{R}^T\mathbf{M} \\ \mathbf{\Phi_q} \end{bmatrix}\right) = n \quad \Leftrightarrow \quad \boxed{\ker\left(\mathbf{R}^T\mathbf{M}\right) \cap \ker \mathbf{\Phi_q} = \mathbf{0}}$$

But matrix **M** is positive-semidefinite, so

$$\mathbf{M} = \mathbf{P}^T\mathbf{P}, \qquad \operatorname{rank}(\mathbf{M}) = \operatorname{rank}(\mathbf{P}) \quad \Rightarrow \quad \ker \mathbf{P} = \ker \mathbf{M}$$

Assume vector **x** belongs to the null spaces of $\mathbf{\Phi_q}$ and $\mathbf{R}^T\mathbf{M}$,

$$\left.\begin{array}{l} \mathbf{x} \in \ker \mathbf{\Phi_q} \quad \Rightarrow \quad \mathbf{x} = \mathbf{Rz},\ \mathbf{z} \in \mathbb{R}^{f\times 1} \\ \mathbf{x} \in \ker\left(\mathbf{R}^T\mathbf{M}\right) \quad \Rightarrow \quad \mathbf{R}^T\mathbf{Mx} = \mathbf{0} \end{array}\right\} \Rightarrow \quad \mathbf{R}^T\mathbf{MRz} = \mathbf{0} \quad \Rightarrow \quad \mathbf{z}^T\mathbf{R}^T\mathbf{MRz} = 0$$

$$\mathbf{z}^T\mathbf{R}^T\mathbf{MRz} = \mathbf{z}^T\mathbf{R}^T\mathbf{P}^T\mathbf{PRz} = \left\|\mathbf{PRz}\right\|_2^2 = \mathbf{0} \quad \Rightarrow \quad \mathbf{Rz} \in \ker \mathbf{P} \quad \Rightarrow \quad \boxed{\mathbf{x} \in \ker \mathbf{M}}$$

and the previous condition $\boxed{\ker(\mathbf{M}) \cap \ker \mathbf{\Phi_q} = \mathbf{0}}$ is obtained.

Three forms to set and solve the dynamic equations 4/4

- The existence of solutions(cont.)
  - For the Maggi's equations

$$\mathbf{R}^T\mathbf{M}\mathbf{R}\ddot{\mathbf{z}} = \mathbf{R}^T\mathbf{F} - \mathbf{R}^T\mathbf{M}\mathbf{S}\mathbf{c}, \qquad \ddot{\mathbf{q}} = \mathbf{R}\ddot{\mathbf{z}} + \mathbf{S}\mathbf{c}$$

The accelerations $\ddot{\mathbf{z}}$ and $\ddot{\mathbf{q}}$ are determined if matrix $\mathbf{R}^T\mathbf{M}\mathbf{R}$ is positive-definite.

$$\mathbf{z}^T\mathbf{R}^T\mathbf{M}\mathbf{R}\mathbf{z} > \mathbf{0} \quad \forall\mathbf{z} \in \mathbb{R}^r,\ \mathbf{z} \neq \mathbf{0}$$

This condition can be written as,

$$\mathbf{z}^T\mathbf{R}^T\mathbf{M}\mathbf{R}\mathbf{z} = \mathbf{z}^T\mathbf{R}^T\mathbf{P}^T\mathbf{P}\mathbf{R}\mathbf{z} = \left\|\mathbf{P}\mathbf{R}\mathbf{z}\right\|_2^2 > 0 \quad \Rightarrow \quad \mathbf{x} \equiv \mathbf{R}\mathbf{z} \notin \ker\mathbf{P} = \ker\mathbf{M}$$

Any vector $\mathbf{x} \equiv \mathbf{R}\mathbf{z} \in \ker\mathbf{\Phi}_{\mathbf{q}},\ \mathbf{x} \neq \mathbf{0},$ cannot belong to $\ker\mathbf{M}$. Thus,

$$\ker(\mathbf{M}) \cap \ker\mathbf{\Phi}_{\mathbf{q}} = \mathbf{0}$$

The positive-definiteness of $\mathbf{R}^T\mathbf{M}\mathbf{R}$ has a physical meaning:

"Any physically possible velocity $\left(\dot{\mathbf{q}} \in \ker\mathbf{\Phi}_{\mathbf{q}},\ \dot{\mathbf{q}} = \mathbf{R}\dot{\mathbf{z}}\right)$ *is associated with a positive kinetic energy* $\left(2T = \dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}} = \dot{\mathbf{z}}^T\mathbf{R}^T\mathbf{M}\mathbf{R}\dot{\mathbf{z}} > 0,\ \forall\dot{\mathbf{z}} \neq \mathbf{0}\right)$."

The uniqueness of solutions

- Assume the conditions for the existence are fulfilled,

$$\operatorname{rank}\left(\begin{bmatrix}\mathbf{M}\\ \mathbf{\Phi_q}\end{bmatrix}\right)=\operatorname{rank}\left(\begin{bmatrix}\mathbf{R}^T\mathbf{M}\\ \mathbf{\Phi_q}\end{bmatrix}\right)=n \quad\Leftrightarrow\quad \ker\mathbf{M}\cap\ker\mathbf{\Phi_q}=\mathbf{0} \quad\Leftrightarrow\quad \mathbf{R}^T\mathbf{M}\mathbf{R}\ \text{ def. pos.}$$

  - The uniqueness of the acceleration $\ddot{\mathbf{q}}$ is better studied with the null space formulation

$$\begin{bmatrix}\mathbf{R}^T\mathbf{M}\\ \mathbf{\Phi_q}\end{bmatrix}\ddot{\mathbf{q}}=\begin{Bmatrix}\mathbf{R}^T\mathbf{F}\\ \mathbf{c}\end{Bmatrix}, \qquad \operatorname{rank}\begin{bmatrix}\mathbf{R}^T\mathbf{M}\\ \mathbf{\Phi_q}\end{bmatrix}=\operatorname{rank}\begin{bmatrix}\mathbf{M}\\ \mathbf{\Phi_q}\end{bmatrix}=n$$

  - As the columns are independent, the solution $\ddot{\mathbf{q}}$ is unique, even if there are redundant constraints
  - Also $\ddot{\mathbf{z}}$ is unique if matrix $\mathbf{R}^T\mathbf{M}\mathbf{R}$ is definite-positive
  - In the Lagrange equations of the first kind $\ddot{\mathbf{q}}$ and $\mathbf{\Phi_q}^T\boldsymbol{\lambda}$ are unique, but the multipliers $\boldsymbol{\lambda}$ are not if there are redundant constraints

$$\begin{bmatrix}\mathbf{M} & \mathbf{\Phi_q}^T\\ \mathbf{\Phi_q} & \mathbf{0}\end{bmatrix}\begin{Bmatrix}\ddot{\mathbf{q}}\\ \boldsymbol{\lambda}\end{Bmatrix}=\begin{Bmatrix}\mathbf{F}\\ \mathbf{c}\end{Bmatrix} \qquad \mathbf{\Phi_q}^T\boldsymbol{\lambda}=\mathbf{F}-\mathbf{M}\ddot{\mathbf{q}}, \quad \mathbf{\Phi_q}^T\in\mathbb{R}^{n\times m}, \qquad \operatorname{rank}\mathbf{\Phi_q}^T=r\le m$$

  - The resultant constraint forces are unique, but the particular are not

Redundant constraints and constraint forces

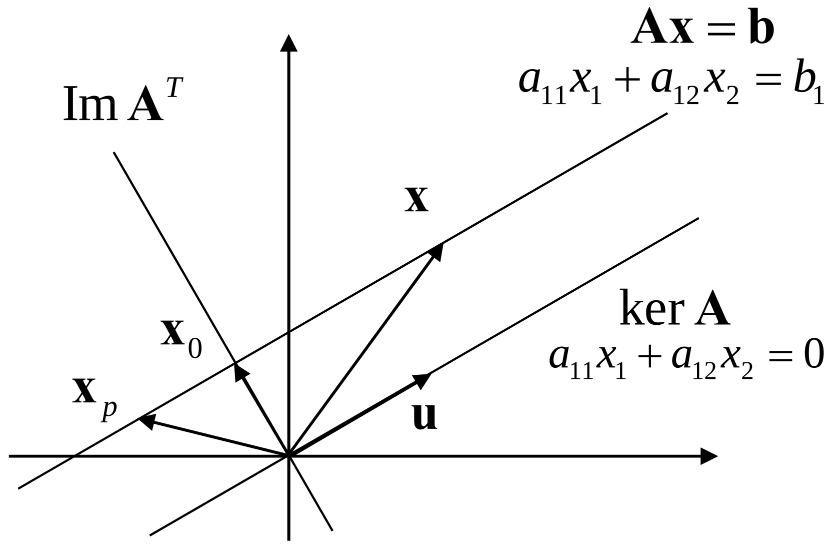
- Assume the MBS is over-constrained and constraint forces are to be determined
  - Remember $\boldsymbol{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda}$ is determined but $\boldsymbol{\lambda}$ is not. Then

$$\boldsymbol{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda} = \mathbf{F}(\mathbf{q},\dot{\mathbf{q}}) - \mathbf{M}\ddot{\mathbf{q}}$$

  - The individual constraint forces are

$$\boldsymbol{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda} = \boldsymbol{\Phi}_{\mathbf{q}}^T(:,1)\lambda_1 + \cdots + \boldsymbol{\Phi}_{\mathbf{q}}^T(:,m)\lambda_m$$

    that are not determined.



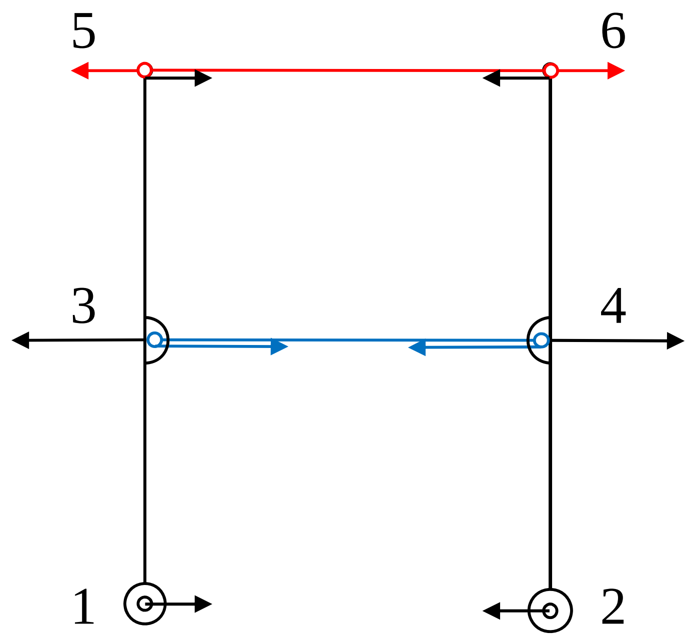
- General and minimum norm solution of $\boldsymbol{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda} = \mathbf{f}$
  - The general solution is $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 + \mathbf{N}\boldsymbol{\alpha},\ \boldsymbol{\alpha} \in \mathbb{R}^{m-r}$, where $\boldsymbol{\lambda}_0$ is the minimum norm solution and $\mathbf{N}\boldsymbol{\alpha}$ is a vector in $\ker\boldsymbol{\Phi}_{\mathbf{q}}^T$
  - The important point is that $\boldsymbol{\Phi}_{\mathbf{q}}^T\mathbf{N}\boldsymbol{\alpha}$ is a set of self-equilibrating constraint forces (see examples)
  - This minimum norm solution can be computed from the system of equations:

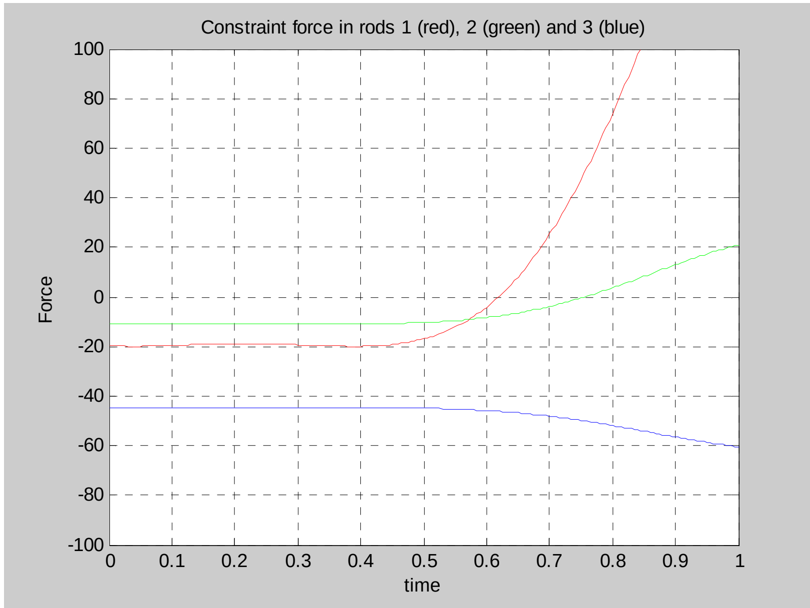
$$\begin{bmatrix} \boldsymbol{\Phi}_{\mathbf{q}}^T \\ \mathbf{N}^T \end{bmatrix} \boldsymbol{\lambda}_0 = \begin{Bmatrix} \mathbf{f} \\ \mathbf{0} \end{Bmatrix}$$

Example 1: 2-D double quad

- Over-constrained system
  - Bar 5-6 can be removed without changing the kinematics of the motion
  - The system cannot be assembled if $L_{34} \neq L_{56}$, unless bars are stretched
  - The minimum norm solution makes the self-equilibrating forces null
- Set of self-equilibrating constraint forces
- Constraint forces for the three constant distances

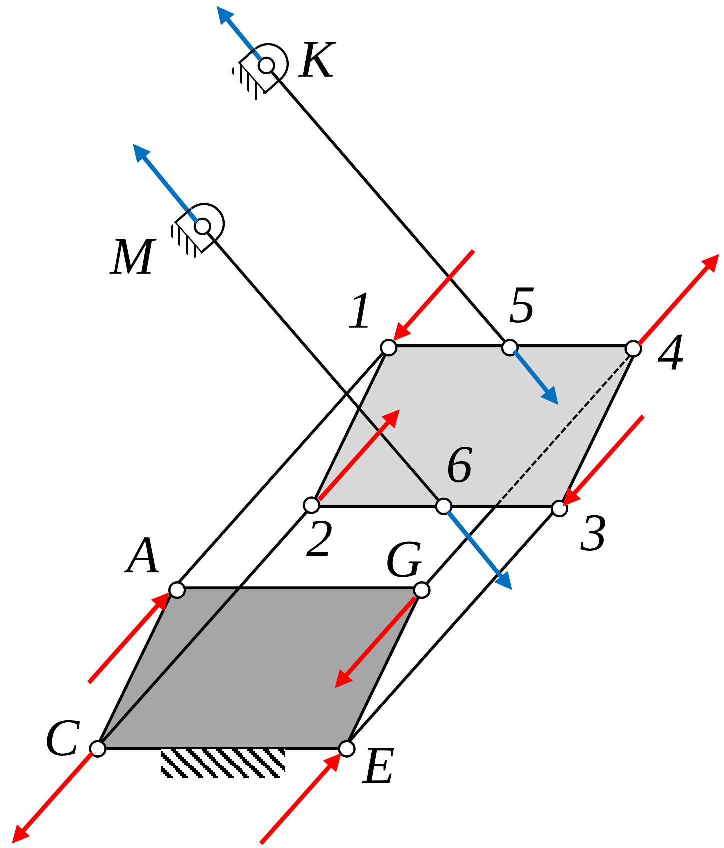
$$L_{12}, L_{34}, L_{56}$$





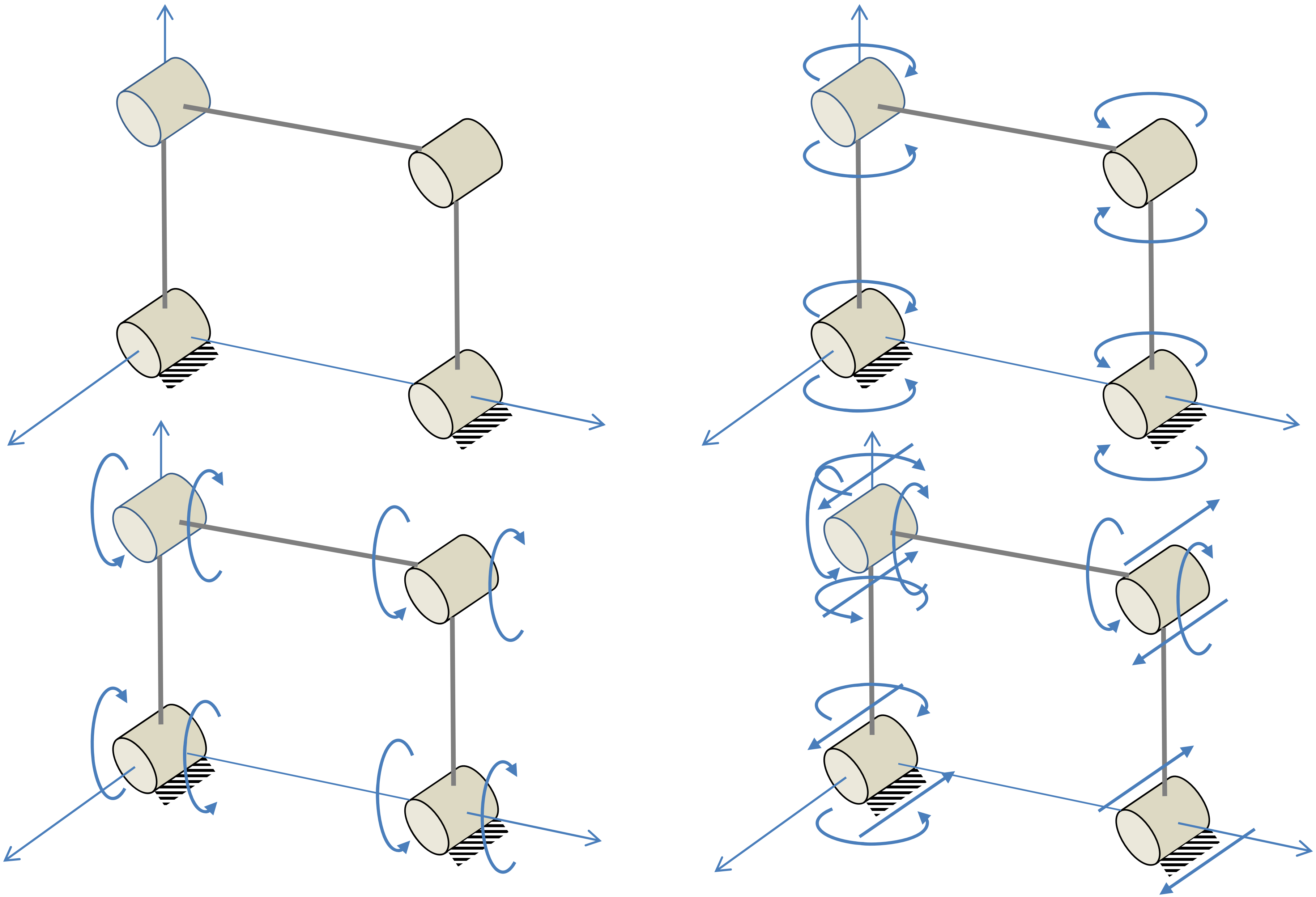
Example 2: 8-link spatial parallelogram

- Frączek & Wojtyra, ECCOMAS Warsaw (2009)
- Platform 1-2-3-4 and bars modeled with natural coordinates
  - 6 rigid body dofs, 6 constant distance conditions and 1 final constrained dof
  - The figure shows the set of self-equilibrating constraint forces
  - The force systems in red and in blue must self-equilibrate separately
  - The forces in blue can not self-equilibrate, and so they are null (proved algebraically by Frączek & Wojtyra)



Example 3: 3-D four bar system

- 3-D four-bar with 3 self-equilibrating sets of constraint forces

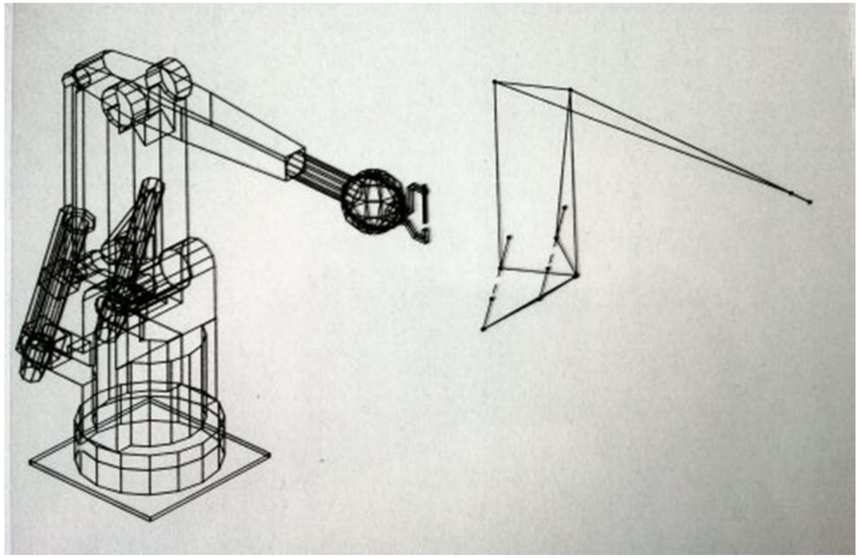


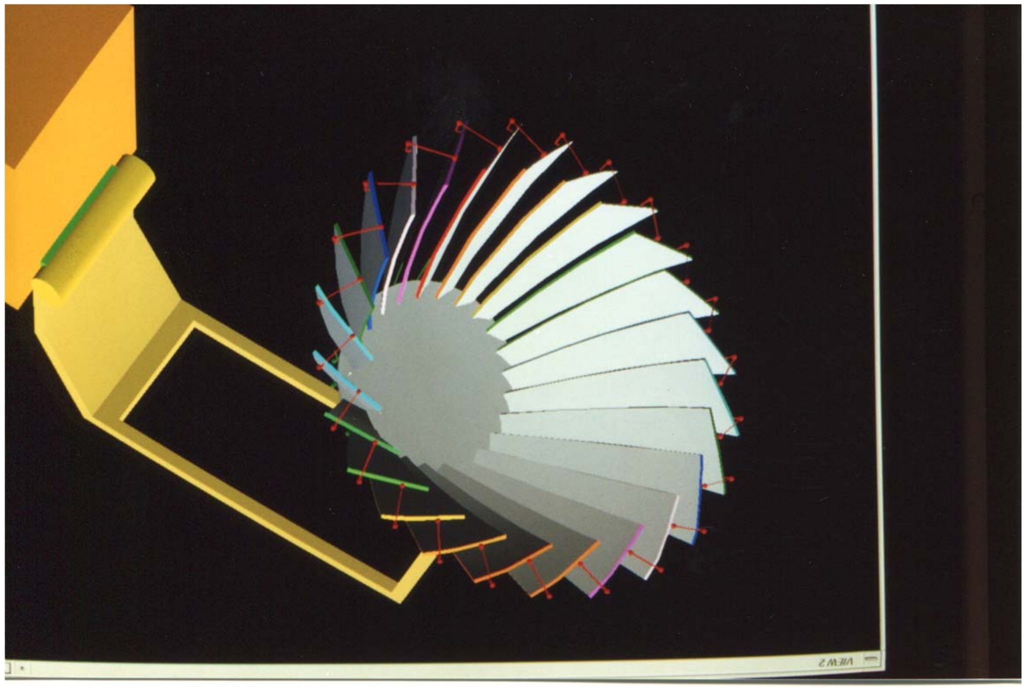
2.2 Multibody simulation with realistic 3-D graphics

- J. García de Jalón, A. García–Alonso and N. Serrano, "Interactive Simulation of Complex Mechanical Systems", Eurographics'93, Barcelona, 1993.
- G. Álvarez, A. García-Alonso, P. Urban, N. Serrano and J. García de Jalón, "Biomechanics: Dynamics & Playback", video presented in SIGGRAPH'93, Annual Conference Series, ISBN 0-89791-602-6, Anaheim CA, USA, 1993.

Animated CG essential to understand MBS

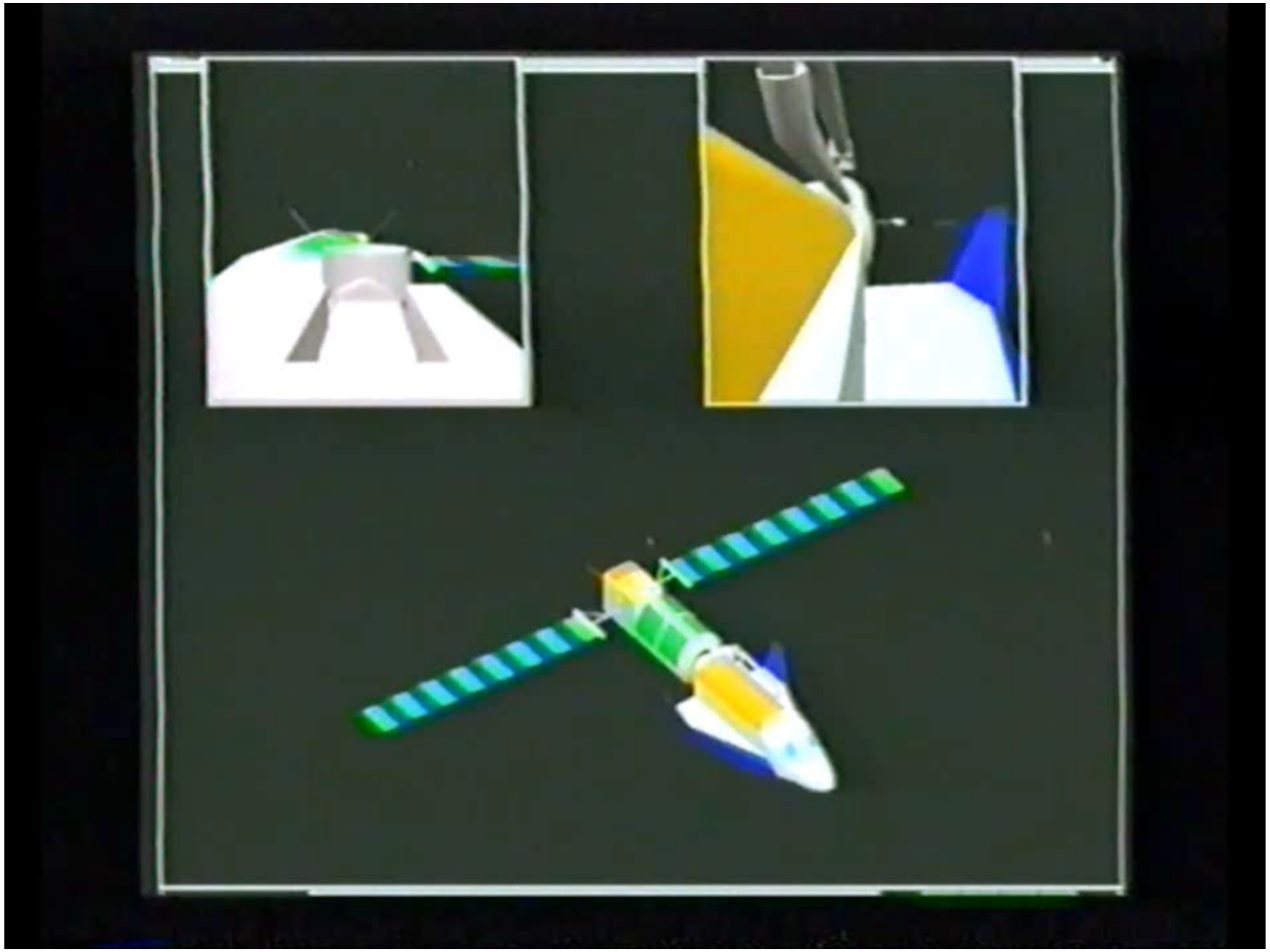
- Computer plots used from 1979
- Tektronix 4114 y 4115 for monochrome vector animations since 1982
- Workstation HP 320SRX with realistic graphics since 1987
  - Motorola 69020 processor plus 3-D graphics accelerator
  - Some examples presented in the VII IFTóMM World Congress on the Theory of Machines and Mechanisms, Seville, September, 17-22, 1987
- Silicon Graphics 4D/240GTX since 1989
  - Four MIPS 3000 processors
- Since 1987 computer graphics produced an interest boom on interactive or even real-time multibody systems





Some MBS animations from the late 80s

- Some of the first examples
  - Kinematics of steering and suspension systems
  - Complete model of a truck
  - Kinematic deployment of an antenna
  - Space dynamic maneuver



2.3 Global formulations: Improving the efficiency

- R. von Schwerin, “Multibody System Simulation: Numerical Methods, Algorithms and Software”, Springer, 338 pages, 1999.
- J. Cuadrado and D. Dopico, “Penalty, semi-recursive and Hybrid Methods for MBS Real-Time Dynamics in the Context of Structural Integrators”, MULTIBODY DYNAMICS 2003, Jorge A.C. Ambrósio (Ed.), IDMEC/IST, Lisbon, Portugal, July 1-4 2003

von Schwerin's Formulation 1/3

- Main characteristics
  - Natural coordinates are used for the system of linear equations $\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^T \\ \mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{c} \end{Bmatrix}$
  - Two methods are considered to solve this system: RSM and NSM
  - The problem is assumed non-stiff and the index 1 DAE system is integrated with a special ABM integrator based on the solution of the *inverse dynamics* that does not need the derivative of the state vector.
  - Instead of using a predictor-corrector formula, von Schwerin enforces the discretized dynamic equations by a modified Newton-Raphson iteration, with an approximate tangent matrix that can be kept for many time steps. At the end, it is possible to obtain a *O(n)* method based on the descriptor form.
  - The index 1 DAE system is stabilized for positions and velocities by using two mass-orthogonal projection with the aforesaid tangent matrix
  - The efficiency can be further improved by considering the system topology through a distinction between the closure of the loop constraints and all the remaining constraints.
- This method seems to have been discontinued in the bibliography

The Range Space method (RSM)

- An LDL^T factorization is assumed and then computed
  - Factorization step (**M** and $\boldsymbol{\Phi}_{\mathbf{q}}$ assumed full rank matrices)

$$\begin{bmatrix} \mathbf{M} & \boldsymbol{\Phi}_{\mathbf{q}}^T \\ \boldsymbol{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{0} \\ \mathbf{L}_{21} & \mathbf{L}_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_f \end{bmatrix} \begin{bmatrix} \mathbf{L}_1^T & \mathbf{L}_{21}^T \\ \mathbf{0} & \mathbf{L}_2^T \end{bmatrix} = \begin{bmatrix} \mathbf{L}_1\mathbf{L}_1^T & \mathbf{L}_1\mathbf{L}_{21}^T \\ \mathbf{L}_{21}\mathbf{L}_1^T & \mathbf{L}_{21}\mathbf{L}_{21}^T - \mathbf{L}_2\mathbf{L}_2^T \end{bmatrix}$$

$$\mathbf{L}_1 = \mathrm{chol}(\mathbf{M}), \quad \mathbf{L}_{21}^T = \mathbf{L}_1^{-1}\boldsymbol{\Phi}_{\mathbf{q}}^T, \quad \mathbf{L}_2 = \mathrm{chol}\left(\mathbf{L}_{21}\mathbf{L}_{21}^T\right)$$

  - Substitution step

$$\begin{bmatrix} \mathbf{L}_1 & \mathbf{0} \\ \mathbf{L}_{21} & \mathbf{L}_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_f \end{bmatrix} \begin{bmatrix} \mathbf{L}_1^T & \mathbf{L}_{21}^T \\ \mathbf{0} & \mathbf{L}_2^T \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{c} \end{Bmatrix}, \qquad \begin{Bmatrix} \mathbf{x} \\ \boldsymbol{\mu} \end{Bmatrix} \equiv \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_f \end{bmatrix} \begin{bmatrix} \mathbf{L}_1^T & \mathbf{L}_{21}^T \\ \mathbf{0} & \mathbf{L}_2^T \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix}$$

$$\begin{bmatrix} \mathbf{L}_1 & \mathbf{0} \\ \mathbf{L}_{21} & \mathbf{L}_2 \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \boldsymbol{\mu} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{c} \end{Bmatrix} \quad \Rightarrow \quad \mathbf{x} = \mathbf{L}_1 \backslash \mathbf{F}, \quad \boldsymbol{\mu} = \mathbf{L}_2 \backslash \left(\mathbf{c} - \mathbf{L}_{21}\mathbf{x}\right)$$

$$\begin{bmatrix} \mathbf{L}_1^T & \mathbf{L}_{21}^T \\ \mathbf{0} & \mathbf{L}_2^T \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{x} \\ -\boldsymbol{\mu} \end{Bmatrix} \quad \Rightarrow \quad \boldsymbol{\lambda} = -\mathbf{L}_2^T \backslash \boldsymbol{\mu}, \quad \ddot{\mathbf{q}} = \mathbf{L}_1^T \backslash \left(\mathbf{x} - \mathbf{L}_{21}^T\boldsymbol{\lambda}\right)$$

  - This method takes into account the matrix symmetry and doesn't need complicated pivoting strategies

The Null Space method (NSM)

- A transformation based on a null space basis **R** of $\boldsymbol{\Phi}_{\mathbf{q}}$ is applied
  - The constraint equations for velocities and accelerations allow a transformation based on the null space basis **R**,

$$\boldsymbol{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = \mathbf{b} \quad \Rightarrow \quad \begin{bmatrix} \boldsymbol{\Phi}_{\mathbf{q}}^d & \boldsymbol{\Phi}_{\mathbf{q}}^i \\ \mathbf{0}_{f,m} & \mathbf{I}_{f,f} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}}^d \\ \dot{\mathbf{q}}^i \end{Bmatrix} = \begin{Bmatrix} \mathbf{c} \\ \dot{\mathbf{q}}^i \end{Bmatrix}, \qquad \begin{bmatrix} \boldsymbol{\Phi}_{\mathbf{q}}^d & \boldsymbol{\Phi}_{\mathbf{q}}^i \\ \mathbf{0}_{f,m} & \mathbf{I}_{f,f} \end{bmatrix} \left[\begin{array}{c|c} \mathbf{S}^d & \mathbf{R}^d \\ \mathbf{S}^i & \mathbf{I}_f \end{array}\right] = \begin{bmatrix} \mathbf{I}_{n,n} & \mathbf{0}_{n,m} \\ \mathbf{0}_{m,n} & \mathbf{I}_{m,m} \end{bmatrix}$$

$$\dot{\mathbf{q}} = \begin{Bmatrix} \dot{\mathbf{q}}^d \\ \dot{\mathbf{q}}^i \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_{\mathbf{q}}^d & \boldsymbol{\Phi}_{\mathbf{q}}^i \\ \mathbf{0}_{f,m} & \mathbf{I}_{f,f} \end{bmatrix}^{-1} \begin{Bmatrix} \mathbf{b} \\ \dot{\mathbf{q}}^i \end{Bmatrix} = \left[\begin{array}{c|c} \mathbf{S}^d & \mathbf{R}^d \\ \mathbf{S}^i & \mathbf{R}^i \end{array}\right] \begin{Bmatrix} \mathbf{b} \\ \dot{\mathbf{q}}^i \end{Bmatrix} = \left[\begin{array}{c|c} \left(\boldsymbol{\Phi}_{\mathbf{q}}^d\right)^{-1} & -\left(\boldsymbol{\Phi}_{\mathbf{q}}^d\right)^{-1}\boldsymbol{\Phi}_{\mathbf{q}}^i \\ \mathbf{0} & \mathbf{I}_f \end{array}\right] \begin{Bmatrix} \mathbf{b} \\ \dot{\mathbf{q}}^i \end{Bmatrix} = \mathbf{S}\mathbf{b} + \mathbf{R}\dot{\mathbf{q}}^i$$

$$\ddot{\mathbf{q}} = \begin{Bmatrix} \ddot{\mathbf{q}}^d \\ \ddot{\mathbf{q}}^i \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_{\mathbf{q}}^d & \boldsymbol{\Phi}_{\mathbf{q}}^i \\ \mathbf{0}_{f,m} & \mathbf{I}_{f,f} \end{bmatrix}^{-1} \begin{Bmatrix} \mathbf{c} \\ \ddot{\mathbf{q}}^i \end{Bmatrix} = \begin{bmatrix} \mathbf{R}^d \\ \mathbf{I}_f \end{bmatrix} \ddot{\mathbf{q}}^i + \begin{bmatrix} \mathbf{S}^d \\ \mathbf{0}_{f\times m} \end{bmatrix} \mathbf{c} = \mathbf{R}\ddot{\mathbf{q}}^i + \mathbf{S}\mathbf{c}, \qquad \begin{cases} \mathbf{R}^d = -\left(\boldsymbol{\Phi}_{\mathbf{q}}^d\right)^{-1}\boldsymbol{\Phi}_{\mathbf{q}}^i \\ \mathbf{S}^d = -\left(\boldsymbol{\Phi}_{\mathbf{q}}^d\right)^{-1} \end{cases} \qquad \mathbf{S}\mathbf{c} = \ddot{\mathbf{q}}\big|_{\ddot{\mathbf{q}}^i=\mathbf{0}}$$

  - The accelerations can be computed from the equations

$$\mathbf{R}^T\mathbf{M}\mathbf{R}\ddot{\mathbf{q}}^i = \mathbf{R}^T\left(\mathbf{F} - \mathbf{M}\mathbf{S}\mathbf{c}\right)^i$$

$$\left(\mathbf{R}^{dT}\mathbf{M}^{dd}\mathbf{R}^d + \mathbf{R}^{dT}\mathbf{M}^{di} + \mathbf{M}^{id}\mathbf{R}^d + \mathbf{M}^{ii}\right)\ddot{\mathbf{q}}^i = \mathbf{R}^{dT}\mathbf{F}^d + \mathbf{F}^i - \left(\mathbf{R}^{dT}\mathbf{M}^{dd} + \mathbf{M}^{ii}\right)\left(\left(\boldsymbol{\Phi}_{\mathbf{q}}^d\right)^{-1}\mathbf{c}\right)$$

$$\ddot{\mathbf{q}}^d = -\left(\boldsymbol{\Phi}_{\mathbf{q}}^d\right)^{-1}\left(\boldsymbol{\Phi}_{\mathbf{q}}^i\ddot{\mathbf{q}}_i^i\right) - \left(\boldsymbol{\Phi}_{\mathbf{q}}^d\right)^{-1}\mathbf{c}$$

The RSM and NSM complexities compared

- von Schwerin presented a theoretical study of the complexity of solving the descriptor system of equations using the three methods, depending on the relationship between the number of constraints m and the number of dependent coordinates n.
- His findings are very significant and interesting:
  - For $m/n > 0.63$ the NSM is more efficient than the RSM. It is also more efficient for $m/n > 0.69$ when matrix **M** is constant and the RSM takes advantage of this circumstance.
  - When the ratio m/n is close to 1 (strongly constrained systems with very few degrees of freedom), the NSM requires 7 times less arithmetic operations than the Gauss method, and 3.5 times less operations than the RSM (2.5 if matrix **M** is constant).
  - In the worst circumstances (m/n small and matrix **M** constant), the NSM never requires more than 15% of additional operations with respect to the RSM.
- These results justify the superiority of the methods based on independent coordinates, which are equivalent to the RSM.

Augmented Lagrangian (Cuadrado and Dopico) 1/4

- Description of the multibody formalism
  - Based on natural coordinates (dependent) plus angles and distances.
  - Index-3 augmented Lagrangian equations with generalized-α method.

$$\mathbf{M}\left\{\left(1-\delta_m\right)\ddot{\mathbf{q}}_{n+1}+\delta_m\ddot{\mathbf{q}}_n\right\}+\left(1-\delta_f\right)\left\{\mathbf{\Phi}_{\mathbf{q}}^T\left(\boldsymbol{\alpha}\mathbf{\Phi}+\boldsymbol{\lambda}^*\right)-\mathbf{Q}\right\}_{n+1}$$

$$+\delta_f\left\{\mathbf{\Phi}_{\mathbf{q}}^T\left(\boldsymbol{\alpha}\mathbf{\Phi}+\boldsymbol{\lambda}^*\right)-\mathbf{Q}\right\}_n=\mathbf{0}$$

$$\boldsymbol{\lambda}_{i+1}=\boldsymbol{\lambda}_i+\boldsymbol{\alpha}\mathbf{\Phi}_{i+1}$$

  - Newmark equations to approximate velocities and accelerations.

$$\dot{\mathbf{q}}_{n+1}=\frac{\gamma}{\beta h}\mathbf{q}_{n+1}+\hat{\dot{\mathbf{q}}}_n\ ;\quad \hat{\dot{\mathbf{q}}}_n=-\left\{\frac{\gamma}{\beta h}\mathbf{q}_n+\left(\frac{\gamma}{\beta}-1\right)\dot{\mathbf{q}}_n+\left(\frac{\gamma}{2\beta}-1\right)h\ddot{\mathbf{q}}_n\right\}$$

$$\ddot{\mathbf{q}}_{n+1}=\frac{1}{\beta h^2}\mathbf{q}_{n+1}+\hat{\ddot{\mathbf{q}}}_n\ ;\quad \hat{\ddot{\mathbf{q}}}_n=-\left(\frac{1}{\beta h^2}\mathbf{q}_n+\frac{1}{\beta h}\dot{\mathbf{q}}_n+\left(\frac{1}{2\beta}-1\right)\ddot{\mathbf{q}}_n\right)$$

  - Nonlinear system of equations:

$$\mathbf{f}\left(\mathbf{q}_{t+h}\right)=\mathbf{0}$$

Augmented Lagrangian (Cuadrado and Dopico) 2/4

INSIA-ETSII-UPM

Deo omnis gloria!

- Description of the multibody formalism (cont.)
  - Solution of the nonlinear system by the Newton-Raphson method

$$\mathbf{f}\left(\mathbf{q}\right) = \beta h^2 \left[\mathbf{M}\left\{\left(1-\delta_m\right)\ddot{\mathbf{q}}_{n+1} + \delta_m \ddot{\mathbf{q}}_n\right\} + \left(1-\delta_f\right)\left\{\mathbf{\Phi}_{\mathbf{q}}^T\left(\boldsymbol{\alpha}\mathbf{\Phi} + \boldsymbol{\lambda}^*\right) - \mathbf{Q}\right\}_{n+1} + \delta_f \left\{\mathbf{\Phi}_{\mathbf{q}}^T\left(\boldsymbol{\alpha}\mathbf{\Phi} + \boldsymbol{\lambda}^*\right) - \mathbf{Q}\right\}_n \right] = \mathbf{0}$$

$$\left[\frac{\partial \mathbf{f}\left(\mathbf{q}\right)}{\partial \mathbf{q}}\right]_{t+h}^i \Delta \mathbf{q}_{t+h}^{i+1} = -\left[\mathbf{f}\left(\mathbf{q}\right)\right]_{t+h}^i ; \qquad \mathbf{q}_{t+h}^{i+1} = \mathbf{q}_{t+h}^i + \Delta \mathbf{q}_{t+h}^{i+1}$$

where

$$\left[\frac{\partial \mathbf{f}\left(\mathbf{q}\right)}{\partial \mathbf{q}}\right] \cong \left(1-\delta_m\right)\mathbf{M} + \left(1-\delta_f\right)\gamma h \mathbf{C}_{n+1} + \left(1-\delta_f\right)\beta h^2 \left(\mathbf{\Phi}_{\mathbf{q}}^{\mathrm{T}} \boldsymbol{\alpha} \mathbf{\Phi}_{\mathbf{q}} + \mathbf{K}\right)_{n+1}$$

$$\mathbf{K} = -\frac{\partial \mathbf{Q}}{\partial \mathbf{q}}, \qquad \mathbf{C} = -\frac{\partial \mathbf{Q}}{\partial \dot{\mathbf{q}}}$$

Augmented Lagrangian (Cuadrado and Dopico) 3/4

- Description of the multibody formalism (cont.)
  - Projections of velocities and accelerations onto their constraint manifolds.

$$\left[\frac{\partial \mathbf{f}\left(\mathbf{q}\right)}{\partial \mathbf{q}}\right]\dot{\mathbf{q}} = \mathbf{P}\dot{\mathbf{q}}^{*} - \left(1-\delta_{f}\right)\beta h^{2}\mathbf{\Phi}_{\mathbf{q}}^{T}\boldsymbol{\alpha}\mathbf{\Phi}_{t}$$

$$\left[\frac{\partial \mathbf{f}\left(\mathbf{q}\right)}{\partial \mathbf{q}}\right]\ddot{\mathbf{q}} = \mathbf{P}\ddot{\mathbf{q}}^{*} - \left(1-\delta_{f}\right)\beta h^{2}\mathbf{\Phi}_{\mathbf{q}}^{T}\boldsymbol{\alpha}\left(\dot{\mathbf{\Phi}}_{\mathbf{q}}\dot{\mathbf{q}} + \dot{\mathbf{\Phi}}_{t}\right)$$

$$\mathbf{P} = \left(1-\delta_{m}\right)\mathbf{M} + \left(1-\delta_{f}\right)\gamma h\mathbf{C}_{n+1} + \left(1-\delta_{f}\right)\beta h^{2}\mathbf{K}_{n+1}$$

where

$$\mathbf{K} = -\frac{\partial \mathbf{Q}}{\partial \mathbf{q}}, \qquad \mathbf{C} = -\frac{\partial \mathbf{Q}}{\partial \dot{\mathbf{q}}}$$

Augmented Lagrangian (Cuadrado and Dopico) 4/4

- Implementation details
  - Sparse matrix techniques:
    - CS: Coordinate format (MA27, MA57)
    - CCS: Compressed column format (Cholmod, KLU)
  - Constant mass matrices and gravity forces in rigid-body case
  - Analytical evaluation of Jacobians of forces easy to obtain (spring-damper, contact, tire, brake, etc.)
  - Usually symmetric positive-definite tangent matrices
  - Integration step-sizes attainable: stable with very large step sizes, problems with very small step sizes (bad condition number)
  - Able to modify the system in real-time: adding (removing) bodies (constraints)
- Conclusions on the efficiency obtained
  - Real time for systems with sizes of more than 1000 coordinates
    - Assembly of four–bar linkages with "*n*" loops, CCS, h=1e-3, AMD Athlon 64.
  - Excavator simulator (152 coordinates): about 5 times faster than RT
    - CS, h=5e-3, ASUS U6V laptop computer Intel Core 2 Duo.

2.4 Biomechanics

- ABAT'92 project: Análisis Biomecánico de Alta Tecnología (1988-1992)
- G. Álvarez, A. García-Alonso, P. Urban, N. Serrano and J. García de Jalón, "Biomechanics: Dynamics & Playback", video presented at SIGGRAPH'93, Anaheim CA, USA, 1993.
- G. Alvarez, A. Gutiérrez, N. Serrano, P. Urban and J. García de Jalón, "Computer Data Acquisition, Analysis and Visualization of Elite Athletes Motion", VIth International Symposium on Computer Simulation in Biomechanics, Paris, France, 1993.

Natural coordinates and biomechanics

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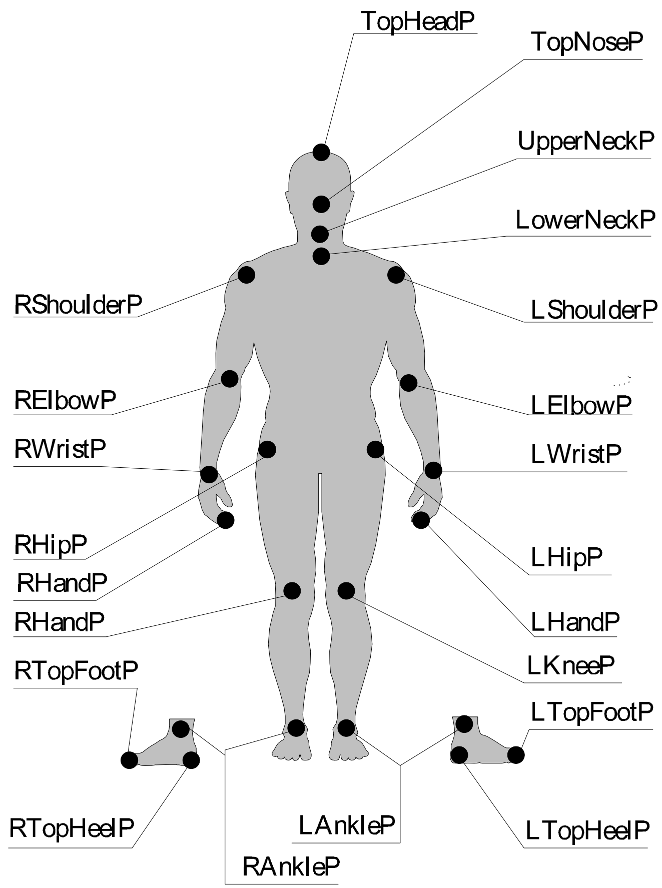
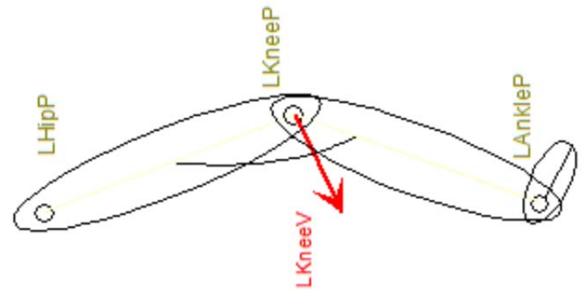
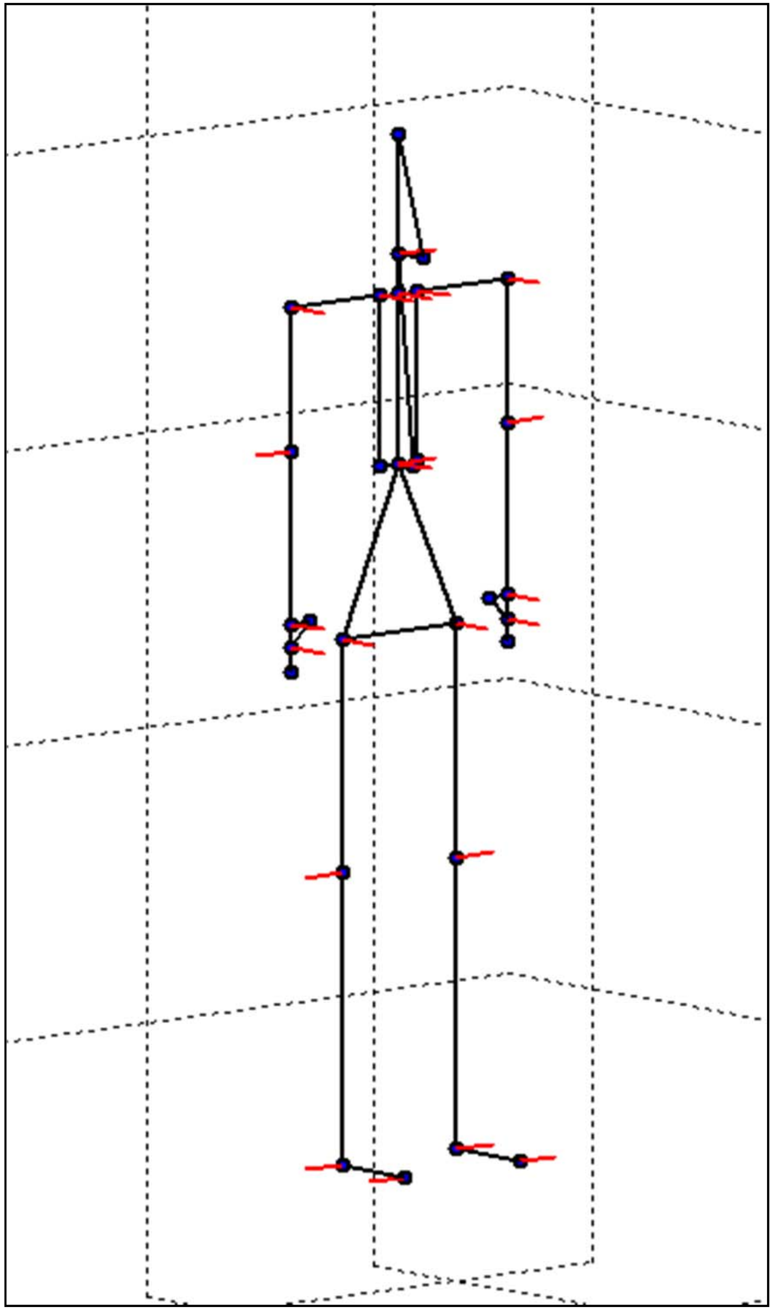
- Step 1: Motion capture: points model
  - Using several synchronized cameras
  - Manual or automatic digitalization
  - 3D coordinates determination with optional filtering
- Step 2: Data conditioning: consistent MBS model
  - Definition of rigid bodies and joints
  - Consistent MBS model: natural coordinates
  - Evaluation of relative angles and filtering
- Step 3: MBS simulation
  - Kinematic simulation
  - Direct dynamic simulation
  - Inverse dynamic simulation





2nd step: MBS model of the human body 1/3

- The HUMAN BODY model
  - Figure shows a typical multi-rigid-body model of the human body, including 37 degrees of freedom.
  - The hip body (consisting of center, right and left points, and three unit vectors) is taken as the base body, in order to show easily the rigid body motions.
- Natural coordinates
  - This model has 30 points (27 movable) and 23 unit vectors (20 movable).
  - The relative coordinates are the angles between the different bodies.
  - So, this human body model has 37 degrees of freedom and 141 dependent coordinates.



2nd step: MBS model of the human body 2/3

- Example: definition of a leg:
  - Points 9, 25 and vectors 15, 16 are a rigid body:

$$(\mathbf{r}_9^T - \mathbf{r}_{25}^T)(\mathbf{r}_9 - \mathbf{r}_{25}) - d_{9,25}^2 = 0$$

$$\mathbf{u}_{15}^T(\mathbf{r}_9 - \mathbf{r}_{25}) = 0$$

$$\mathbf{u}_{16}^T(\mathbf{r}_9 - \mathbf{r}_{25}) = 0$$

  - Points 25, 26 and vectors 16, 17 are a rigid body:

$$(\mathbf{r}_{25}^T - \mathbf{r}_{26}^T)(\mathbf{r}_{25} - \mathbf{r}_{26}) - d_{25,26}^2 = 0$$

$$\mathbf{u}_{16}^T(\mathbf{r}_{25} - \mathbf{r}_{26}) = 0$$

$$\mathbf{u}_{17}^T(\mathbf{r}_{25} - \mathbf{r}_{26}) = 0$$

  - Points 26, 27 and vectors 17, 18 are a rigid body:

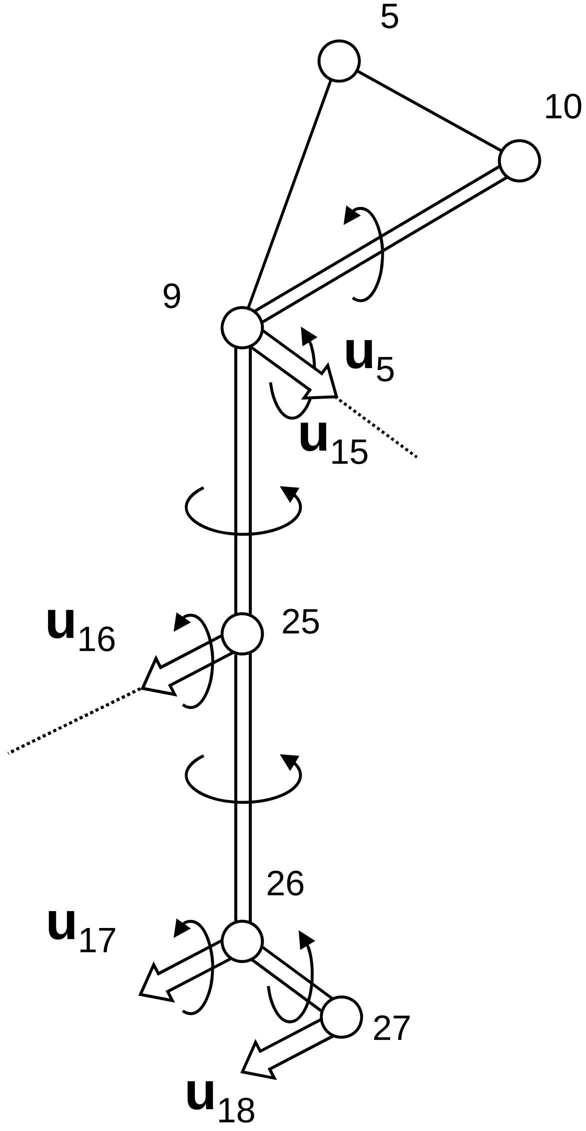
$$(\mathbf{r}_{26}^T - \mathbf{r}_{27}^T)(\mathbf{r}_{26} - \mathbf{r}_{27}) - d_{26,27}^2 = 0$$

$$\mathbf{u}_{17}^T(\mathbf{r}_{26} - \mathbf{r}_{27}) = 0$$

$$\mathbf{u}_{18}^T(\mathbf{r}_{26} - \mathbf{r}_{27}) = 0$$

  - Sometimes foot twist is restricted:

$$\mathbf{u}_{17} \times \mathbf{u}_{18} = 0$$

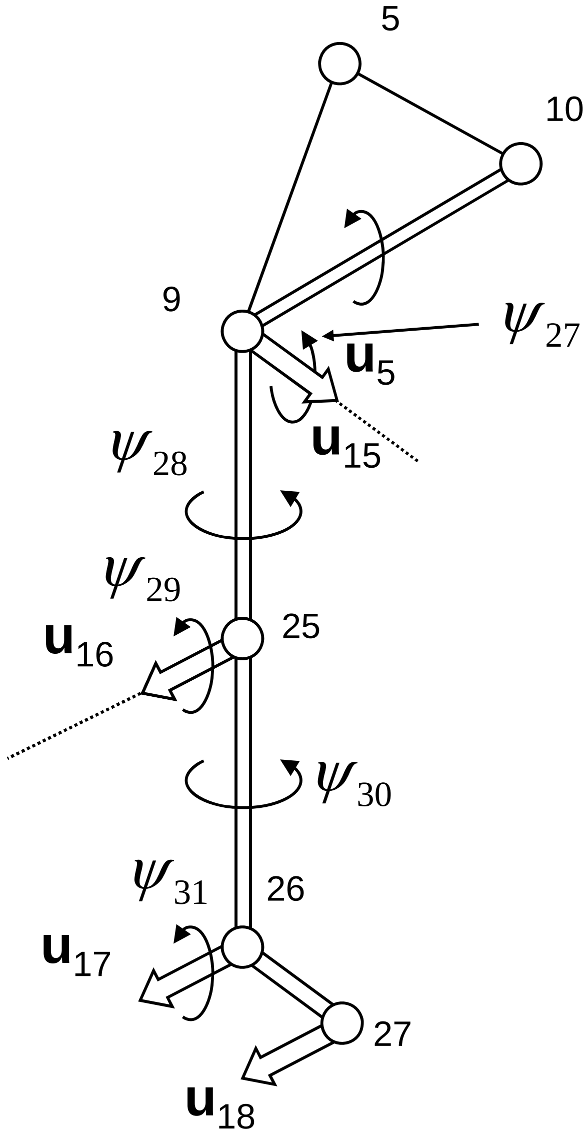


2nd step: MBS model of the human body 3/3

- Relative coordinates used to control the 6 degrees of freedom:
  - ψ_{26} goes from $\mathbf{u}_5$ to $\mathbf{u}_{15}$ measured on $(\mathbf{r}_{19}-\mathbf{r}_{10})$
  - ψ_{27} goes from $(\mathbf{r}_{10}-\mathbf{r}_9)$ to $(\mathbf{r}_{25}-\mathbf{r}_9)$ measured on $\mathbf{u}_{15}$
  - ψ_{28} goes from $\mathbf{u}_{15}$ to $\mathbf{u}_{16}$ measured on $(\mathbf{r}_{25}-\mathbf{r}_9)$
  - ψ_{29} goes from $(\mathbf{r}_{26}-\mathbf{r}_{25})$ to $(\mathbf{r}_9-\mathbf{r}_{25})$ measured on $\mathbf{u}_{16}$
  - ψ_{30} goes from $\mathbf{u}_{16}$ to $\mathbf{u}_{17}$ measured on $(\mathbf{r}_{26}-\mathbf{r}_{25})$
  - ψ_{31} goes from $(\mathbf{r}_{25}-\mathbf{r}_{26})$ to $(\mathbf{r}_{27}-\mathbf{r}_{26})$ measured on $\mathbf{u}_{17}$
- Angle definition constraints:
  - Angle from $(\mathbf{r}_j-\mathbf{r}_i)$ to $(\mathbf{r}_l-\mathbf{r}_k)$ measured on $\mathbf{u}_m$

    $$(\mathbf{r}_j-\mathbf{r}_i)\times(\mathbf{r}_l-\mathbf{r}_k)-\mathbf{u}_m L_{ij} L_{kl}\sin(\psi)=\mathbf{0}$$
  - Angle from $\mathbf{u}_i$ to $\mathbf{u}_j$ measured on $(\mathbf{r}_l-\mathbf{r}_k)$

    $$\mathbf{u}_i\times\mathbf{u}_j-\frac{(\mathbf{r}_l-\mathbf{r}_k)}{L_{kl}}\sin(\psi)=\mathbf{0}$$



3rd step: Kinematic and dynamic simulation

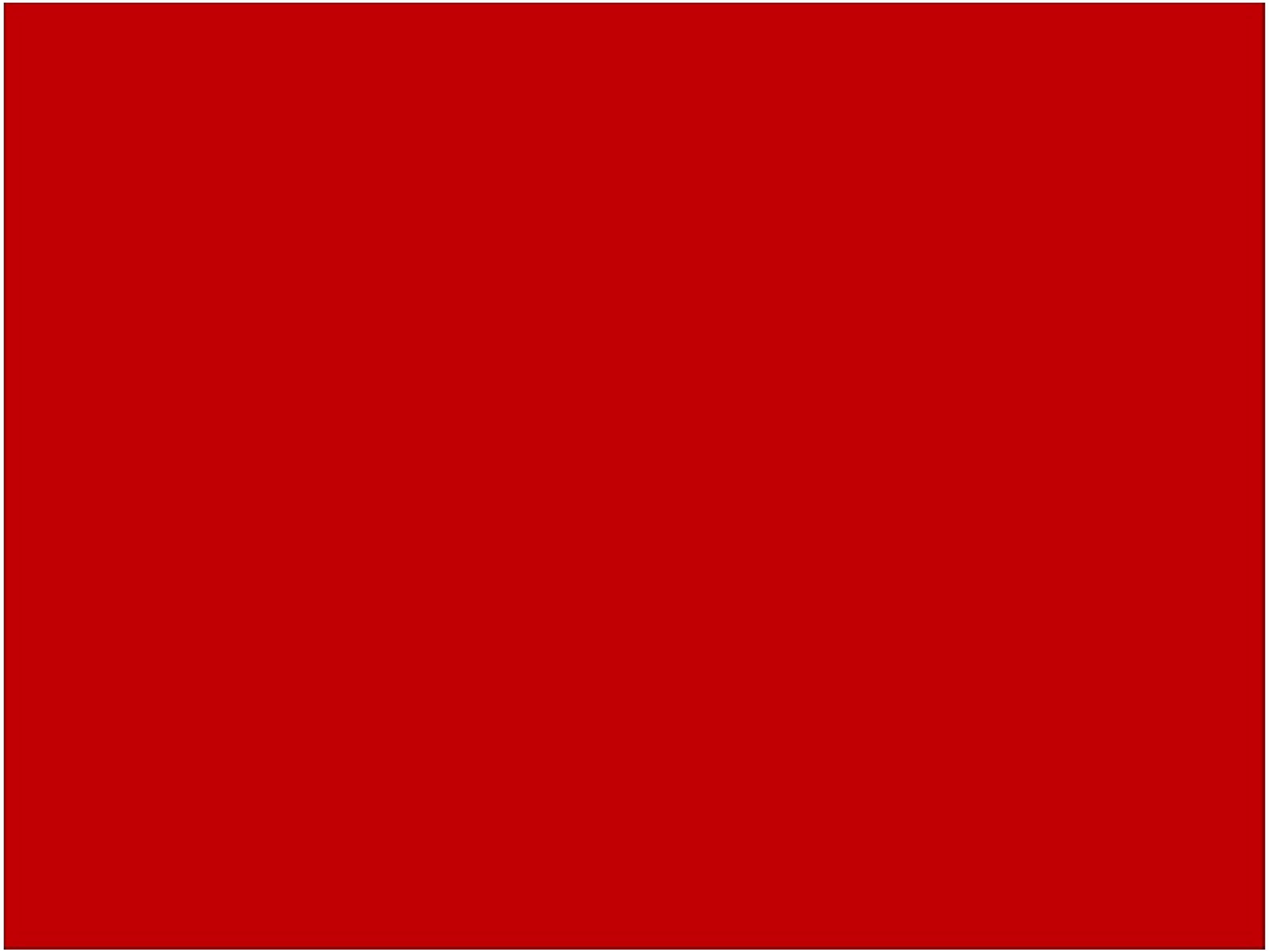
- Kinematic simulation
  - After conditioning, the consistent model with the recorded motion is visualized with realistic graphics from several fixed or moving cameras
  - If necessary, new frames are obtained by interpolation
- Inverse and direct dynamic simulation
  - The inertia properties of the athlete's members are obtained from anthropometric data tables
  - Knowing the motion as a function of time, the external reaction and internal motor-constraint forces can be computed and visualized

Barcelona’ 1992 Olympic Games 1/4

- ABAT’92 Project: Análisis Biomecánico de Alta Tecnología (1988-1992)
  - Developed with CAR (High Performance Sport Training Center)
  - Research project aimed at the multibody analysis of athletes, using the latest technologies in data capturing, analysis of motion and computer graphics
- Two sports were studied: athletics and gymnastics
- Working procedure
  - The trials were recorded in the afternoon
  - The images were manually digitized during the night
  - The digitized data was conditioned, produced and recorded in the morning
  - The resulting images were broadcast at midday news services (too late…)
- Some broadcast images will be shown next
  - Gymnastics, high jump and long jump

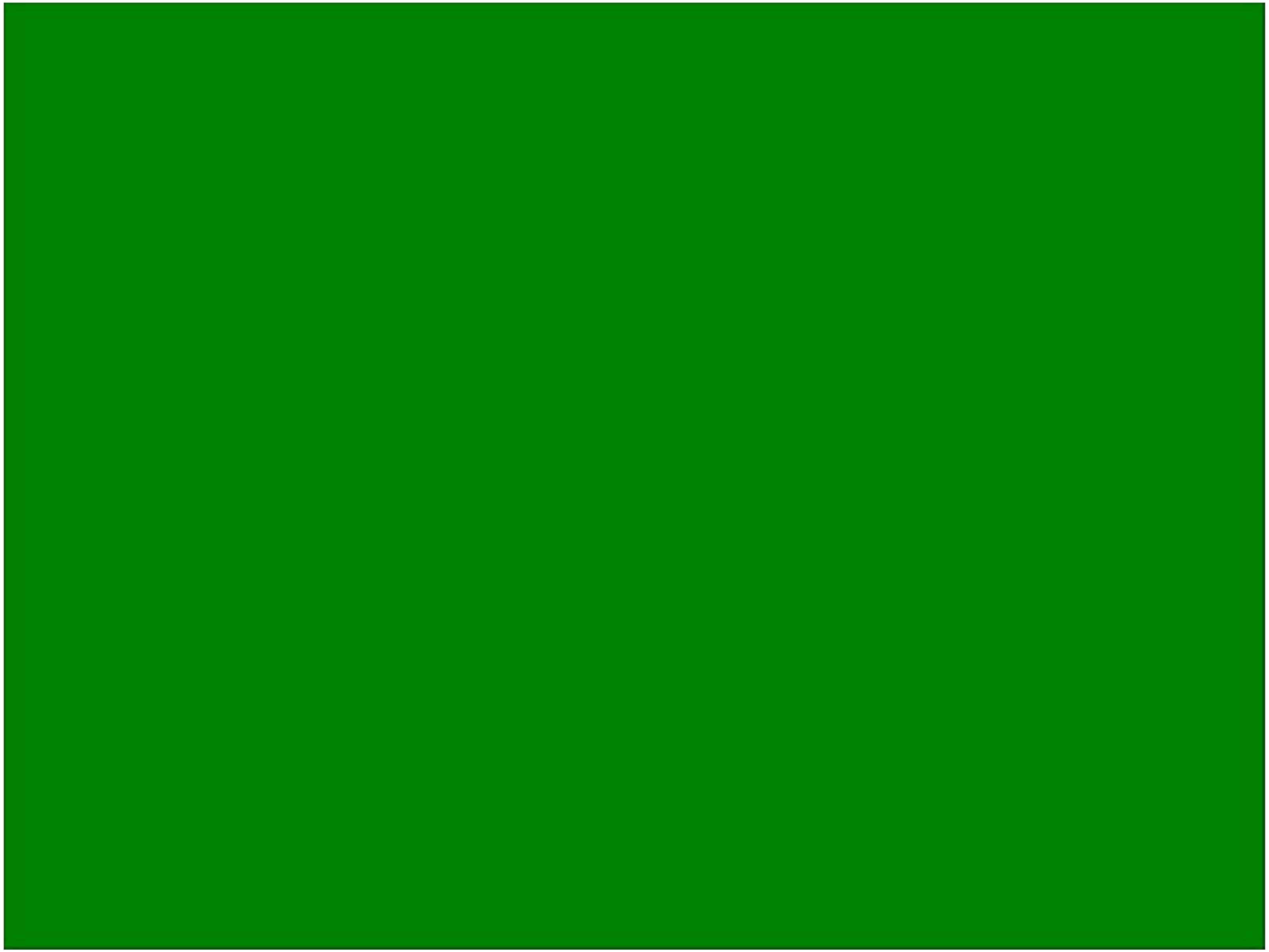
Barcelona’ 1992 Olympic Games 2/4

- Gymnastics



Barcelona' 1992 Olympic Games 3/4

- High jump: Javier Sotomayor



Barcelona’ 1992 Olympic Games 4/4

- Long jump: Lewis vs. Powell

Biomechanics and natural coordinates today 1/3

Deo omnis gloria!

- STT Ingeniería y Sistemas
- Automatic data capturing
  - Gait analysis
  - Bull fighter
  - Golf
- Manual data cap
  - Gymnastic 1
  - Gymnastic 2

Biomechanics and natural coordinates today 2/3

- Kinematic Motion Reconstruction (KMR) at CEIT
- Optimization process
  - The quadratic error between the experimental and theoretical position of the markers, subjected to the kinematic constraint equations, is minimized
- In the next video we will see:
  - Red spheres (●) in the center of joints,
  - Blue spheres (●) in the model points associated to markers
  - Green spheres (●) to represent marker's position experimentally measured
- The error between blue and green spheres is minimized subjected to the constant distance condition between blue and red spheres
- Over-determined and under-determined problems
  - There is over-determined information when more points or more cameras than necessary are used for a body
  - On the other hand, there is for instance an under-determined problem when a body has fewer than necessary points or they are hidden for some cameras

Biomechanics and natural coordinates today 3/3

- A CEIT's REALMAN example: A driver enters and exits a vehicle

2.5 Flexible multibody systems

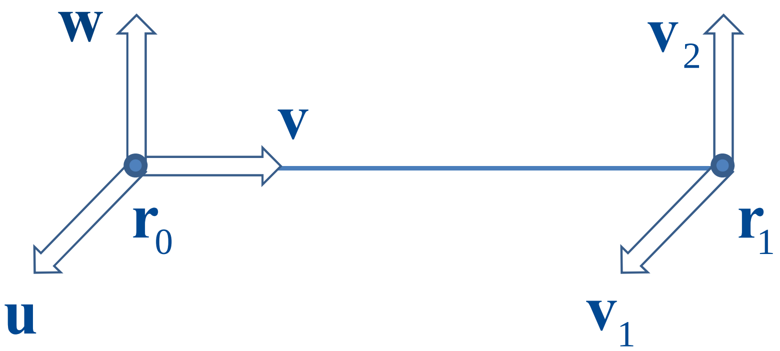
- J. García de Jalón and E. Bayo, *Kinematic and Dynamic Simulation of Multi-Body Systems: The Real-Time Challenge*, Springer-Verlag, New-York (1993).
- J. Cuadrado, J. Cardenal and J. García de Jalón, *Flexible Mechanisms Through Natural Coordinates and Component Synthesis: An Approach Fully Compatible with the Rigid Case*, International Journal for Numerical Methods in Engineering, Vol. 39, pp. 3535-3551, (1996).

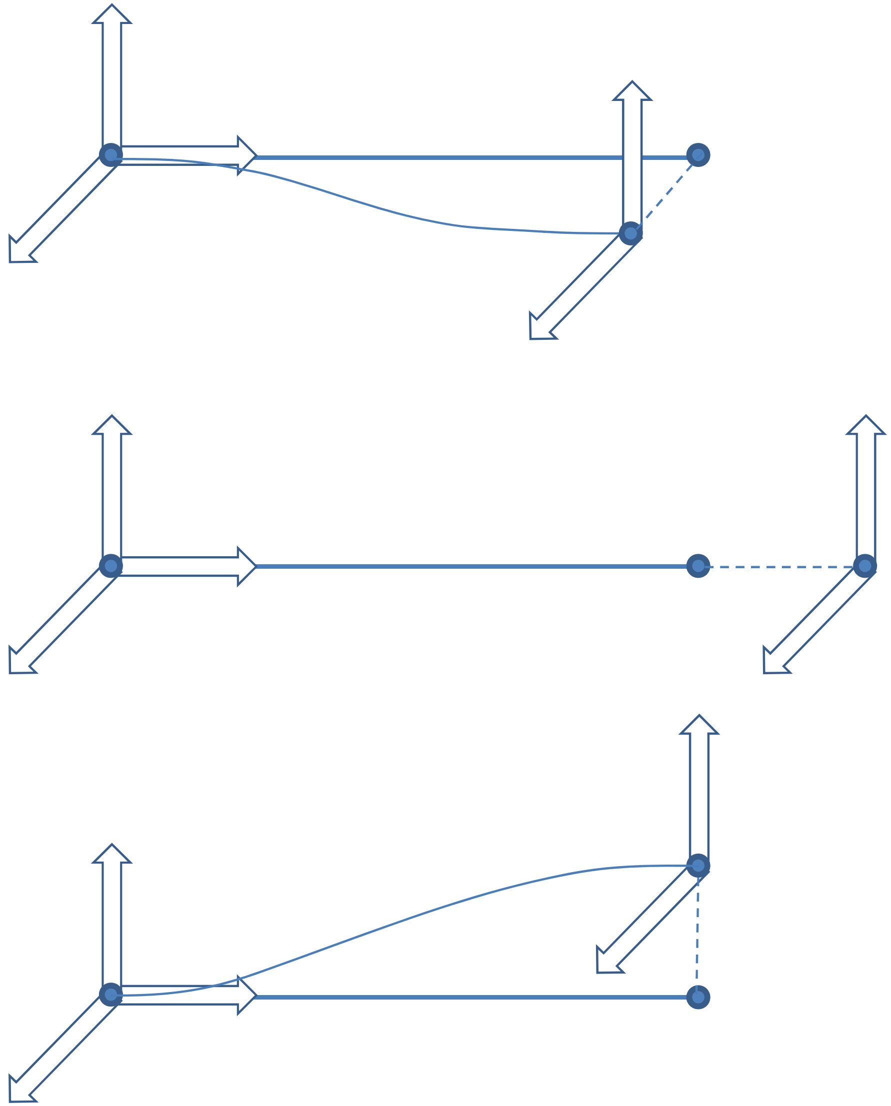
Moving frame approach with natural coordinates

- The classical moving frame approach separates the large rigid body motion and the small elastic deformations respect the moving frame
  - The elastic deformations are expressed as a linear combination of a reduced set of static and dynamic modes
- Natural coordinates provide a simple way to define boundary conditions for the static modes
  - The elastic static modes can be considered as the relaxation of one or more rigid body constraint equations
  - It is not necessary to consider all the possible static nodes. Those considered as irrelevant may be kept as constraint equations
  - It is not necessary to consider the amplitude of the static modes so defined: they can be computed from the natural coordinates

Static modes coming from a point displacement

- There are three static modes related to the displacement of point $\mathbf{r}_1$

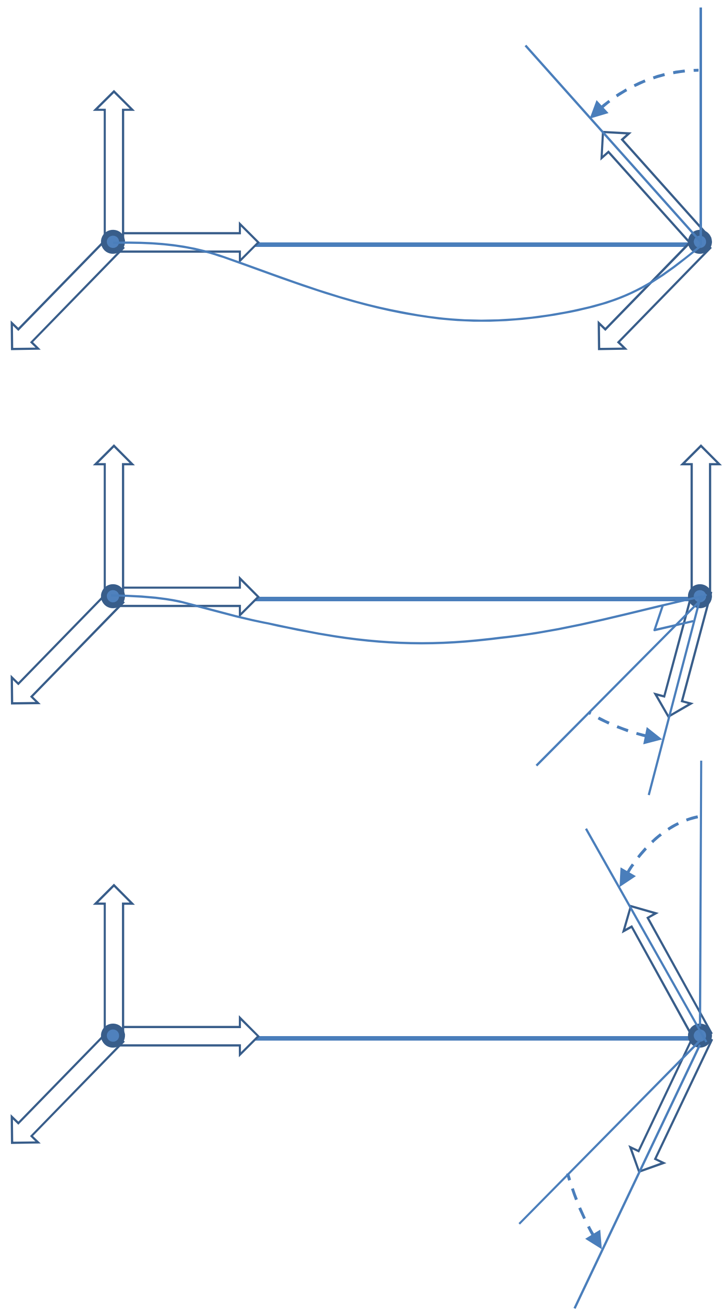




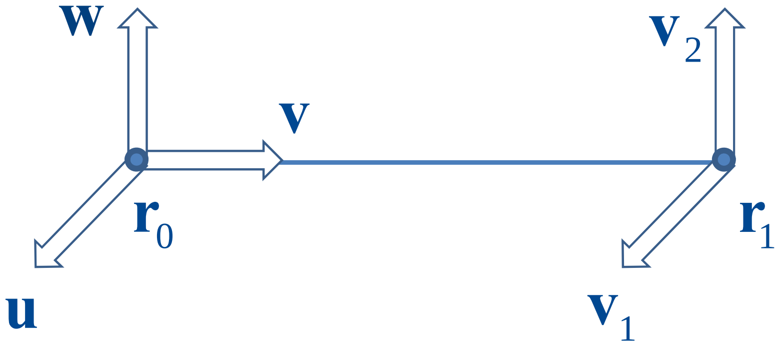
- The static modal amplitudes can be defined as

$$\boldsymbol{\eta}_r = \mathbf{A}^T\left(\mathbf{r}_1 - \mathbf{r}_0\right) - \overline{\mathbf{r}}_{1n}$$

Static modes coming from a point rotation



- There are three static modes related to the infinitesimal rotations of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$



- The modal amplitudes are obtained from

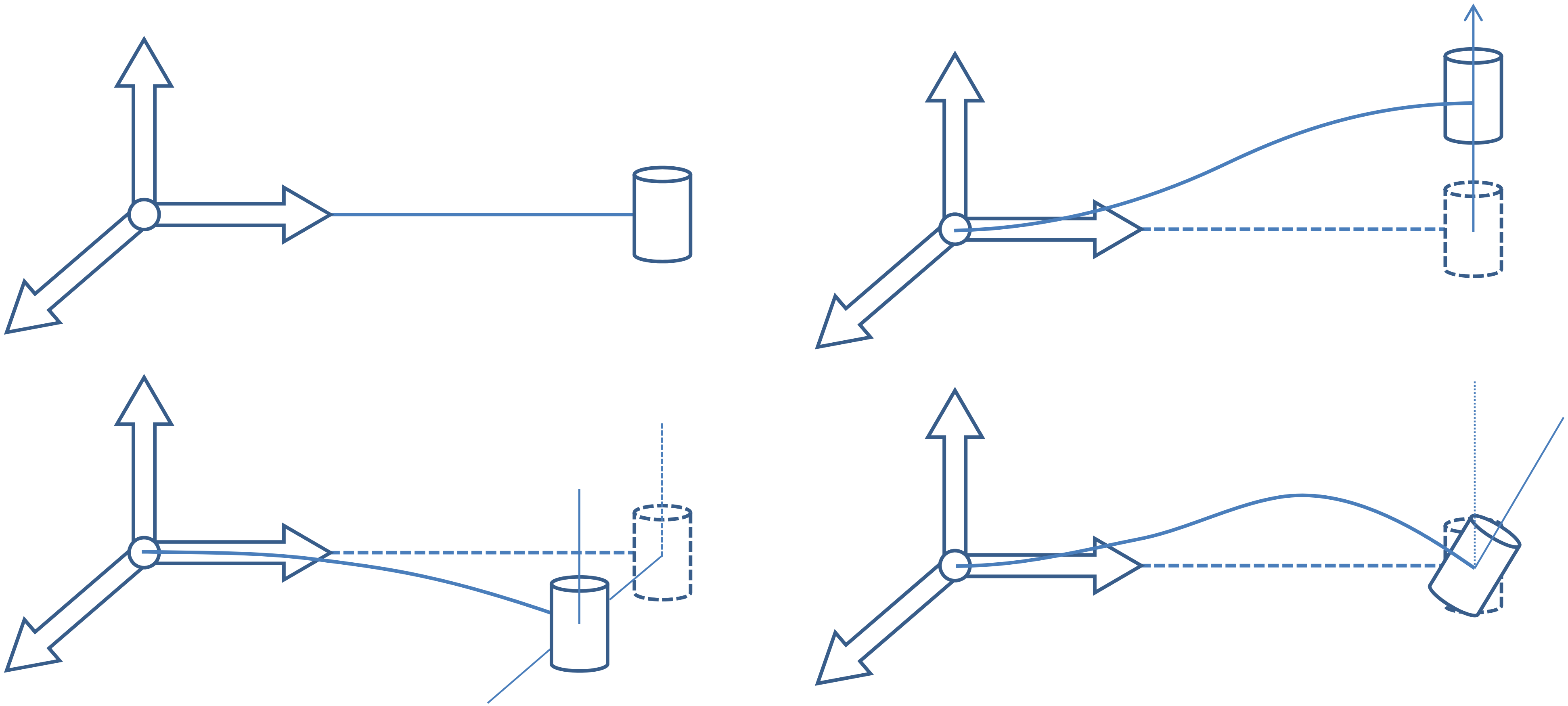
$$\mathbf{G} = \mathbf{I} + \tilde{\mathbf{u}}\,\eta_{rx} + \tilde{\mathbf{v}}\,\eta_{ry} + \tilde{\mathbf{w}}\,\eta_{rz}$$

$$\mathbf{A}^T\mathbf{v}_1 = \mathbf{G}\overline{\mathbf{v}}_{1n}$$

$$\mathbf{A}^T\mathbf{v}_2 = \mathbf{G}\overline{\mathbf{v}}_{2n}$$

A possible selection of static modes

- A revolute joint on the right hand side allows a free rotation respect to the joint axis
- Only two rotational and one translational static modes are considered



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Washington, USA, 28-31 Aug, 2011

Part III: Improving the efficiency throughout topological formulations

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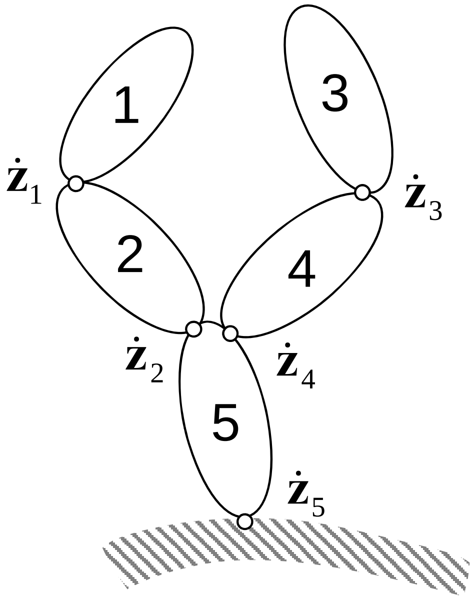
3.1 Semi-recursive formulation based on a double velocity transformation

- J.I. Rodríguez (2000), PhD Dissertation (in Spanish).
- J. I. Rodríguez et al., *Recursive and Residual Algorithms for the Efficient Numerical Integration of Multi-Body Systems*, Multibody System Dynamics, 11, pp. 295-320, (2004).
- J. Cuadrado et al., *A combined Penalty and Recursive Real-Time Formulation for Multibody Dynamics*, ASME Journal of Mechanical Design, pp. 602-608 (2004).

Open-chain topology: path matrix

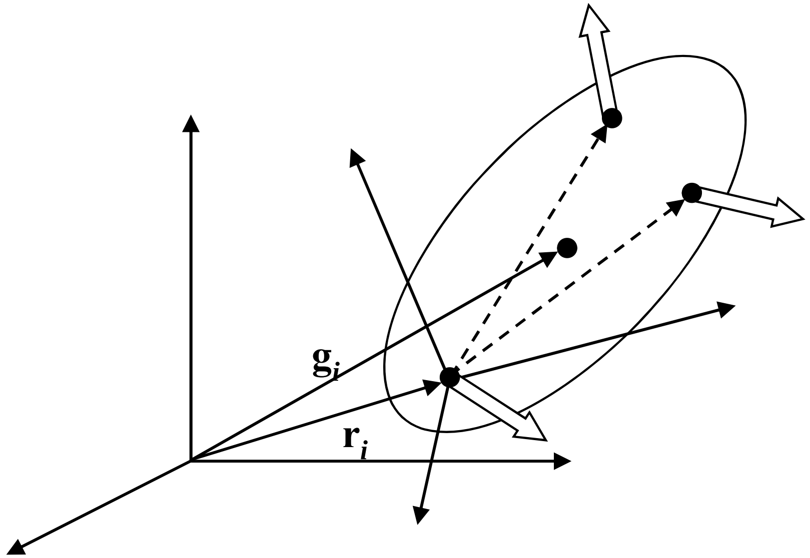
- Path matrix **T**
  - Rows correspond to rigid bodies.
  - Columns correspond to joints.
  - $T_{ij} = 1$ if body *i* is upwards of joint *j*. Otherwise $T_{ij} = 0$.
- Column *i* of **T**:
  - Defines the bodies that are upwards of joint *i*.
- Row *j* of **T**:
  - Defines the joints that are downwards of body *j*.
- Numbering of joints and bodies
  - Each rigid body has the same number as its input joint.
  - Bodies and joints are numbered from the leaves to the root.
  - Matrix **T** is upper triangular.

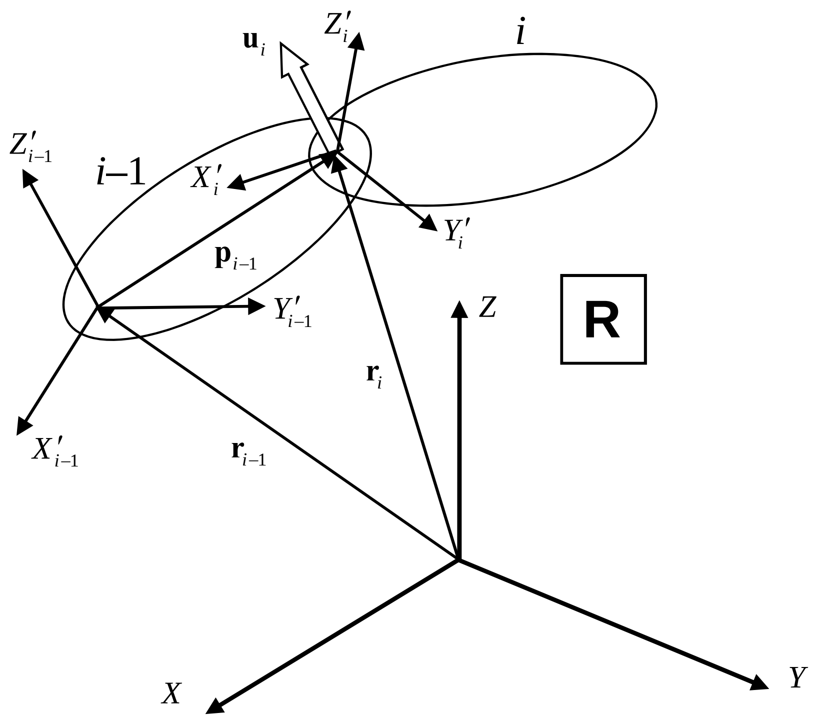


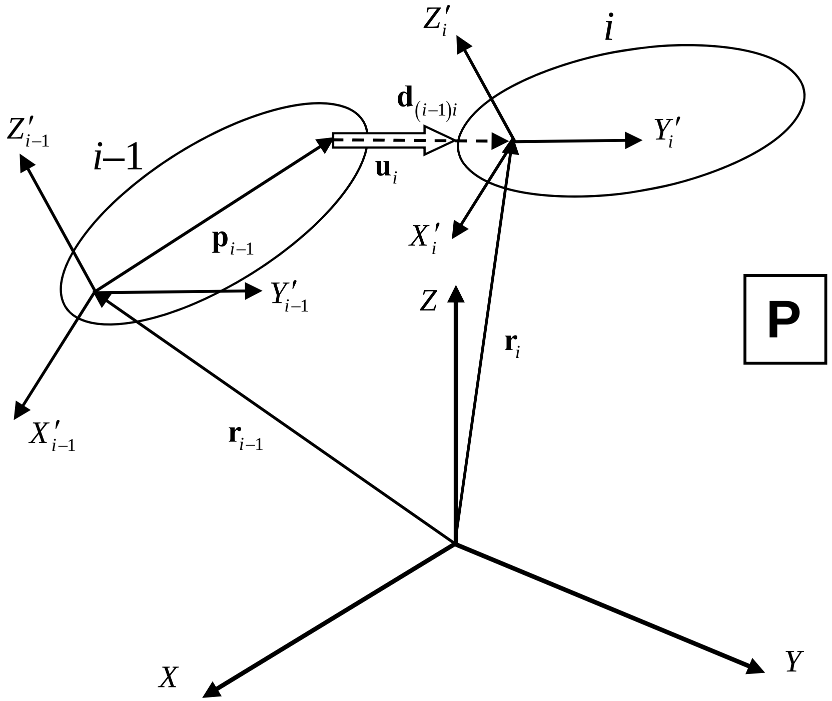
$$\mathbf{T} \equiv \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{T} \equiv \begin{bmatrix} \mathbf{I}_6 & \mathbf{I}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{I}_6 \\ \mathbf{0}_6 & \mathbf{I}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{I}_6 \\ \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{I}_6 & \mathbf{I}_6 & \mathbf{I}_6 \\ \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{I}_6 & \mathbf{I}_6 \\ \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{I}_6 \end{bmatrix}$$

Geometry and position

- The bodies' geometry is defined through **natural coordinates**.
- The position of each body is defined by the Cartesian coordinates of a **point** and the **rotation matrix**.
- All the joints are defined as a combination of **revolute** (R) and **prismatic** (P) joints.







Cartesian velocities and accelerations

- Cartesian velocities and accelerations:
  - According to the point of the solid chosen as reference:

Center of gravity

$$\mathbf{Y}_i \equiv \begin{Bmatrix} \dot{\mathbf{g}}_i \\ \boldsymbol{\omega}_i \end{Bmatrix}, \quad \dot{\mathbf{Y}}_i \equiv \begin{Bmatrix} \ddot{\mathbf{g}}_i \\ \dot{\boldsymbol{\omega}}_i \end{Bmatrix}$$

Point at the origin of global axis

$$\mathbf{Z}_i \equiv \begin{Bmatrix} \dot{\mathbf{s}}_i \\ \boldsymbol{\omega}_i \end{Bmatrix}, \quad \dot{\mathbf{Z}}_i \equiv \begin{Bmatrix} \ddot{\mathbf{s}}_i \\ \dot{\boldsymbol{\omega}}_i \end{Bmatrix}$$

  - Simple relationships between them:

$$\mathbf{Y}_i = \begin{Bmatrix} \dot{\mathbf{g}}_i \\ \boldsymbol{\omega}_i \end{Bmatrix} = \begin{bmatrix} \mathbf{I}_3 & -\tilde{\mathbf{g}}_i \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{s}}_i \\ \boldsymbol{\omega}_i \end{Bmatrix} = \mathbf{D}_i \mathbf{Z}_i, \quad \dot{\mathbf{Y}}_i = \begin{Bmatrix} \ddot{\mathbf{g}}_i \\ \dot{\boldsymbol{\omega}}_i \end{Bmatrix} = \begin{bmatrix} \mathbf{I}_3 & -\tilde{\mathbf{g}}_i \\ \mathbf{0} & \mathbf{I}_3 \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{s}}_i \\ \dot{\boldsymbol{\omega}}_i \end{Bmatrix} + \begin{Bmatrix} \tilde{\boldsymbol{\omega}}_i \tilde{\boldsymbol{\omega}}_i \mathbf{g}_i \\ \mathbf{0} \end{Bmatrix} = \mathbf{D}_i \dot{\mathbf{Z}}_i + \mathbf{e}_i$$

  - Recursive Cartesian velocities and accelerations (open-loop systems):

$$\mathbf{Y}_i = \bar{\mathbf{B}}_i \mathbf{Y}_{i-1} + \bar{\mathbf{b}}_i \dot{z}_i \qquad \mathbf{Z}_i = \mathbf{Z}_{i-1} + \mathbf{b}_i \dot{z}_i$$

$$\dot{\mathbf{Y}}_i = \bar{\mathbf{B}}_i \dot{\mathbf{Y}}_{i-1} + \bar{\mathbf{b}}_i \ddot{z}_i + \bar{\mathbf{d}}_i \qquad \dot{\mathbf{Z}}_i = \dot{\mathbf{Z}}_{i-1} + \mathbf{b}_i \ddot{z}_i + \mathbf{d}_i$$

  - Matrix $\bar{\mathbf{B}}_i$ and vectors $\bar{\mathbf{b}}_i, \bar{\mathbf{d}}_i, \mathbf{b}_i$ and $\mathbf{d}_i$ depend on the joint type between elements *i* and *i*−1. For instance, with **Z** velocities:

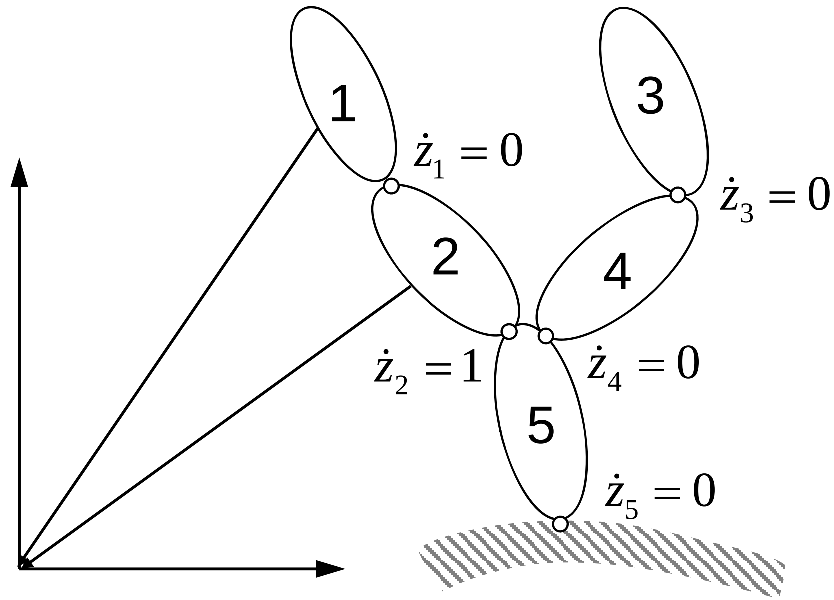
$$\mathbf{b}_j^R = \begin{Bmatrix} \mathbf{r}_i \times \mathbf{u}_i \\ \mathbf{u}_i \end{Bmatrix}, \quad \mathbf{d}_j^R = \begin{Bmatrix} (2\boldsymbol{\omega}_{i-1} + \mathbf{u}_i \dot{\varphi}_i) \times (\mathbf{r}_i \times \mathbf{u}_i \dot{\varphi}_i) \\ \boldsymbol{\omega}_{i-1} \times \mathbf{u}_i \dot{\varphi}_i \end{Bmatrix}; \quad \mathbf{b}_j^P = \begin{Bmatrix} \mathbf{u}_i \\ \mathbf{0} \end{Bmatrix}, \quad \mathbf{d}_j^P = \begin{Bmatrix} 2\boldsymbol{\omega}_{i-1} \times \mathbf{u}_i \dot{s}_i \\ \mathbf{0} \end{Bmatrix}$$

1st velocity transformation (open-chain)

- Transformation from Cartesian to relative coordinates
  - Cartesian velocities **Z** are dependent and can be obtained from the relative independent velocities:

$$\mathbf{Z} = \mathbf{R}\dot{\mathbf{z}} = \mathbf{R}_1\dot{z}_1 + \mathbf{R}_2\dot{z}_2 + \ldots + \mathbf{R}_n\dot{z}_n$$

  - Column *i* of matrix **R** contains the Cartesian velocities produced by a unit input velocity in joint *i* and null velocities in the rest of the joints.



  - All the bodies upwards of joint *i* have the same Cartesian velocity $\mathbf{b}_i$ (the same vector that appeared in $\mathbf{Z}_i = \mathbf{Z}_{i-1} + \mathbf{b}_i\dot{z}_i$).
  - For the whole open-chain system:

$$\mathbf{R} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & 0 & 0 & \mathbf{b}_5 \\ 0 & \mathbf{b}_2 & 0 & 0 & \mathbf{b}_5 \\ 0 & 0 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 \\ 0 & 0 & 0 & \mathbf{b}_4 & \mathbf{b}_5 \\ 0 & 0 & 0 & 0 & \mathbf{b}_5 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_6 & \mathbf{I}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{I}_6 \\ \mathbf{0}_6 & \mathbf{I}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{I}_6 \\ \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{I}_6 & \mathbf{I}_6 & \mathbf{I}_6 \\ \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{I}_6 & \mathbf{I}_6 \\ \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{0}_6 & \mathbf{I}_6 \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{b}_2 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{b}_3 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{b}_4 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{b}_5 \end{bmatrix} \equiv \mathbf{T}\mathbf{R}_d, \quad 0 \equiv \mathbf{0}_{6\times 1}$$

Open-chain motion differential equations 1/2

- Newton-Euler equations for each solid
  - Cartesian velocities and accelerations Y and Z can be used:

$$\mathbf{M}_i\dot{\mathbf{Y}}_i=\mathbf{Q}_i,\quad \mathbf{M}_i=\begin{bmatrix} m_i\mathbf{I}_3 & \mathbf{0}\\ \mathbf{0} & \mathbf{J}_i\end{bmatrix},\quad \mathbf{Q}_i=\begin{Bmatrix}\mathbf{F}_i\\ \mathbf{n}_i-\tilde{\boldsymbol{\omega}}_i\mathbf{J}_i\boldsymbol{\omega}_i\end{Bmatrix}$$

$$\bar{\mathbf{M}}_i\dot{\mathbf{Z}}_i=\bar{\mathbf{Q}}_i,\quad \bar{\mathbf{M}}_i=\mathbf{D}_i^T\mathbf{M}_i\mathbf{D}_i=\begin{bmatrix} m_i\mathbf{I}_3 & -m_i\tilde{\mathbf{g}}_i\\ m_i\tilde{\mathbf{g}}_i & \mathbf{J}_i-m_i\tilde{\mathbf{g}}_i\tilde{\mathbf{g}}_i\end{bmatrix},\quad \bar{\mathbf{Q}}_i=\mathbf{D}_i^T\left(\mathbf{Q}_i-\mathbf{M}_i\mathbf{e}_i\right)$$

  - By applying the **Virtual Power Principle** to the complete system:

$$\mathbf{Z}^{*T}\left(\bar{\mathbf{M}}\dot{\mathbf{Z}}-\bar{\mathbf{Q}}\right)=0$$

$$\bar{\mathbf{M}}\equiv \mathrm{diag}\left(\bar{\mathbf{M}}_1,\bar{\mathbf{M}}_2,\ldots,\bar{\mathbf{M}}_n\right),\quad \bar{\mathbf{Q}}^T=\left[\bar{\mathbf{Q}}_1^T,\bar{\mathbf{Q}}_2^T,\ldots,\bar{\mathbf{Q}}_n^T\right],\quad \dot{\mathbf{Z}}^T=\left[\dot{\mathbf{Z}}_1^T,\dot{\mathbf{Z}}_2^T,\ldots,\dot{\mathbf{Z}}_n^T\right]$$

  - Introducing now the **1st velocity transformation**:

$$\mathbf{Z}=\mathbf{R}\dot{\mathbf{z}}=\mathbf{T}\mathbf{R}_d\dot{\mathbf{z}},\qquad \dot{\mathbf{Z}}=\mathbf{T}\mathbf{R}_d\ddot{\mathbf{z}}+\mathbf{T}\dot{\mathbf{R}}_d\dot{\mathbf{z}}$$

    in the virtual power equation, the following ODE system is obtained:

$$\boxed{\mathbf{R}_d^T\left(\mathbf{T}^T\bar{\mathbf{M}}\mathbf{T}\right)\mathbf{R}_d\ddot{\mathbf{z}}=\boldsymbol{\tau}+\mathbf{R}_d^T\mathbf{T}^T\left(\bar{\mathbf{Q}}-\bar{\mathbf{M}}\mathbf{T}\dot{\mathbf{R}}_d\dot{\mathbf{z}}\right)}$$

    where τ are the forces/torques applied in the joints.
  - These equations are further developed in the next slide.

Open-chain motion differential equations 2/2

- Motion differential equations with relative coordinates:
  - The motion equations are: $\mathbf{R}_d^T\left(\mathbf{T}^T\bar{\mathbf{M}}\mathbf{T}\right)\mathbf{R}_d\ddot{\mathbf{z}} = \boldsymbol{\tau} + \mathbf{R}_d^T\mathbf{T}^T\left(\bar{\mathbf{Q}} - \bar{\mathbf{M}}\mathbf{T}\dot{\mathbf{R}}_d\dot{\mathbf{z}}\right)$
  - By substituting matrices $\mathbf{R}_d$ and $\mathbf{T}$ from the previous example:

$$\mathbf{R}_d^T\mathbf{T}^T\bar{\mathbf{M}}\mathbf{T}\mathbf{R}_d = \begin{bmatrix} \mathbf{b}_1^T\mathbf{M}_1^\Sigma\mathbf{b}_1 & \mathbf{b}_1^T\mathbf{M}_1^\Sigma\mathbf{b}_2 & \mathbf{0} & \mathbf{0} & \mathbf{b}_1^T\mathbf{M}_1^\Sigma\mathbf{b}_5 \\ \mathbf{b}_2^T\mathbf{M}_1^\Sigma\mathbf{b}_1 & \mathbf{b}_2^T\mathbf{M}_2^\Sigma\mathbf{b}_2 & \mathbf{0} & \mathbf{0} & \mathbf{b}_2^T\mathbf{M}_2^\Sigma\mathbf{b}_5 \\ \mathbf{0} & \mathbf{0} & \mathbf{b}_3^T\mathbf{M}_3^\Sigma\mathbf{b}_3 & \mathbf{b}_3^T\mathbf{M}_3^\Sigma\mathbf{b}_4 & \mathbf{b}_3^T\mathbf{M}_3^\Sigma\mathbf{b}_5 \\ \mathbf{0} & \mathbf{0} & \mathbf{b}_4^T\mathbf{M}_3^\Sigma\mathbf{b}_3 & \mathbf{b}_4^T\mathbf{M}_4^\Sigma\mathbf{b}_4 & \mathbf{b}_4^T\mathbf{M}_4^\Sigma\mathbf{b}_5 \\ \mathbf{b}_5^T\mathbf{M}_1^\Sigma\mathbf{b}_1 & \mathbf{b}_5^T\mathbf{M}_2^\Sigma\mathbf{b}_2 & \mathbf{b}_5^T\mathbf{M}_3^\Sigma\mathbf{b}_3 & \mathbf{b}_5^T\mathbf{M}_4^\Sigma\mathbf{b}_4 & \mathbf{b}_5^T\mathbf{M}_5^\Sigma\mathbf{b}_5 \end{bmatrix} \equiv \mathbf{R}_d^T\mathbf{M}^\Sigma\mathbf{R}_d$$

$$\begin{aligned} \mathbf{M}_1^\Sigma &= \bar{\mathbf{M}}_1 \\ \mathbf{M}_2^\Sigma &= \bar{\mathbf{M}}_2 + \mathbf{M}_1^\Sigma \\ \mathbf{M}_3^\Sigma &= \bar{\mathbf{M}}_3 \\ \mathbf{M}_4^\Sigma &= \bar{\mathbf{M}}_4 + \mathbf{M}_3^\Sigma \\ \mathbf{M}_5^\Sigma &= \bar{\mathbf{M}}_5 + \mathbf{M}_2^\Sigma + \mathbf{M}_4^\Sigma \end{aligned}$$

  - Notice that the LU factorization will not destroy the zeroes-pattern.
  - For external and velocity dependent inertia forces:

$$\mathbf{Q}^\Sigma \equiv \mathbf{T}^T\bar{\mathbf{Q}} = \begin{Bmatrix} \mathbf{Q}_1^\Sigma \\ \mathbf{Q}_2^\Sigma \\ \mathbf{Q}_3^\Sigma \\ \mathbf{Q}_4^\Sigma \\ \mathbf{Q}_5^\Sigma \end{Bmatrix}, \quad \mathbf{P}^\Sigma \equiv -\mathbf{T}^T\bar{\mathbf{M}}\mathbf{T}\dot{\mathbf{R}}_d\dot{\mathbf{z}} = \begin{Bmatrix} \mathbf{P}_1^\Sigma \\ \mathbf{P}_2^\Sigma \\ \mathbf{P}_3^\Sigma \\ \mathbf{P}_4^\Sigma \\ \mathbf{P}_5^\Sigma \end{Bmatrix}$$

$$\begin{aligned} \mathbf{Q}_1^\Sigma &= \bar{\mathbf{Q}}_1 \\ \mathbf{Q}_2^\Sigma &= \bar{\mathbf{Q}}_2 + \mathbf{Q}_1^\Sigma \\ \mathbf{Q}_3^\Sigma &= \bar{\mathbf{Q}}_3 \\ \mathbf{Q}_4^\Sigma &= \bar{\mathbf{Q}}_4 + \mathbf{Q}_3^\Sigma \\ \mathbf{Q}_5^\Sigma &= \bar{\mathbf{Q}}_5 + \mathbf{Q}_2^\Sigma + \mathbf{Q}_4^\Sigma \end{aligned}$$

$$\begin{aligned} \mathbf{P}_1^\Sigma &= -\bar{\mathbf{M}}_1\left(\mathbf{T}\dot{\mathbf{R}}_d\dot{\mathbf{z}}\right)_1 \\ \mathbf{P}_2^\Sigma &= -\bar{\mathbf{M}}_2\left(\mathbf{T}\dot{\mathbf{R}}_d\dot{\mathbf{z}}\right)_2 + \mathbf{P}_1^\Sigma \\ \mathbf{P}_3^\Sigma &= -\bar{\mathbf{M}}_3\left(\mathbf{T}\dot{\mathbf{R}}_d\dot{\mathbf{z}}\right)_3 \\ \mathbf{P}_4^\Sigma &= -\bar{\mathbf{M}}_4\left(\mathbf{T}\dot{\mathbf{R}}_d\dot{\mathbf{z}}\right)_4 + \mathbf{P}_3^\Sigma \\ \mathbf{P}_5^\Sigma &= -\bar{\mathbf{M}}_5\left(\mathbf{T}\dot{\mathbf{R}}_d\dot{\mathbf{z}}\right)_5 + \mathbf{P}_2^\Sigma + \mathbf{P}_4^\Sigma \end{aligned}$$

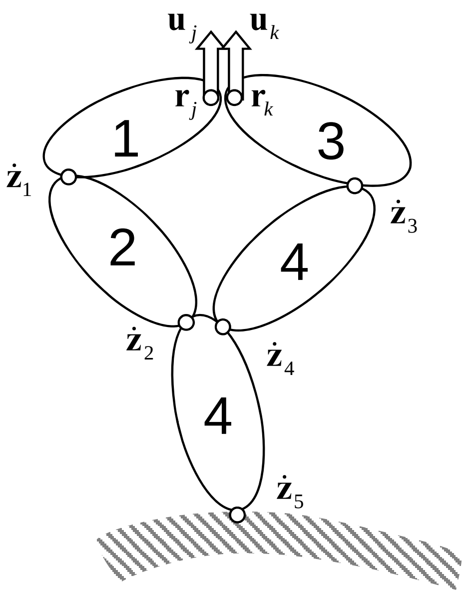
  - In both cases the accumulation of inertias and forces becomes very explicit. The path matrix **T** plays a fundamental role.

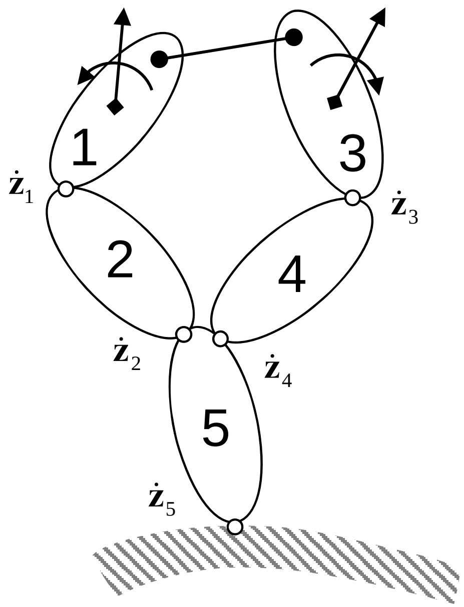
Closed-chain differential equations 1/2

- Closed-chain equations are obtained in two steps:
  - First, the loops are opened by removing some joints or special bodies (rods).
  - Second, the constraints are introduced by an appropriate method: Lagrange multipliers, penalties, augmented Lagrangian or a 2nd velocity transformation
- The constraint equations are formulated in the simplest way through natural coordinates:

$$\mathbf{r}_j - \mathbf{r}_k = \mathbf{0} \quad (3 \text{ independent equations})$$
$$\mathbf{u}_j - \mathbf{u}_k = \mathbf{0} \quad (\text{only } 2 \text{ independent equations})$$

$$\left(\mathbf{r}_j - \mathbf{r}_k\right)^T \left(\mathbf{r}_j - \mathbf{r}_k\right) - l_{jk}^2 = 0$$





Closed-chain differential equations 2/2

- Closure of the loop constraint equations
  - The Jacobian matrix in relative coordinates can be computed as:

$$\mathbf{\Phi}_{\mathbf{z}} = \mathbf{\Phi}_{\mathbf{r}_j} \frac{\partial \mathbf{r}_j}{\partial \mathbf{z}} + \mathbf{\Phi}_{\mathbf{r}_k} \frac{\partial \mathbf{r}_k}{\partial \mathbf{z}} = \mathbf{\Phi}_{\mathbf{r}_j} \frac{\partial \dot{\mathbf{r}}_j}{\partial \dot{\mathbf{z}}} + \mathbf{\Phi}_{\mathbf{r}_k} \frac{\partial \dot{\mathbf{r}}_k}{\partial \dot{\mathbf{z}}}; \qquad \mathbf{\Phi}_{\mathbf{r}_j} = 2\left(\mathbf{r}_j^T - \mathbf{r}_k^T\right), \quad \mathbf{\Phi}_{\mathbf{r}_k} = -2\left(\mathbf{r}_j^T - \mathbf{r}_k^T\right)$$

  - By using the coordinate partitioning method:

$$\begin{bmatrix} \mathbf{\Phi}_{\mathbf{z}}^d & \mathbf{\Phi}_{\mathbf{z}}^i \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{z}}^d \\ \dot{\mathbf{z}}^i \end{Bmatrix} = \mathbf{0}, \qquad \dot{\mathbf{z}}^d = -\left(\mathbf{\Phi}_{\mathbf{z}}^d\right)^{-1} \mathbf{\Phi}_{\mathbf{z}}^i \dot{\mathbf{z}}^i$$

$$\dot{\mathbf{z}} = \mathbf{R}_{\mathbf{z}} \dot{\mathbf{z}}^i, \qquad \begin{Bmatrix} \dot{\mathbf{z}}^d \\ \dot{\mathbf{z}}^i \end{Bmatrix} = \begin{bmatrix} -\left(\mathbf{\Phi}_{\mathbf{z}}^d\right)^{-1} \mathbf{\Phi}_{\mathbf{z}}^i \\ \mathbf{I} \end{bmatrix} \dot{\mathbf{z}}^i, \qquad \mathbf{R}_{\mathbf{z}} = \begin{bmatrix} -\left(\mathbf{\Phi}_{\mathbf{z}}^d\right)^{-1} \mathbf{\Phi}_{\mathbf{z}}^i \\ \mathbf{I} \end{bmatrix}$$

  - Introducing $\ddot{\mathbf{z}} = \mathbf{R}_{\mathbf{z}} \ddot{\mathbf{z}}^i + \dot{\mathbf{R}}_{\mathbf{z}} \dot{\mathbf{z}}^i$ in the motion equations:

$$\boxed{\mathbf{R}_{\mathbf{z}}^T \mathbf{R}_d^T \mathbf{M}^{\Sigma} \mathbf{R}_d \mathbf{R}_{\mathbf{z}} \ddot{\mathbf{z}}^i = \mathbf{R}_{\mathbf{z}}^T \boldsymbol{\tau} + \mathbf{R}_{\mathbf{z}}^T \mathbf{R}_d^T \mathbf{Q}^{\Sigma} - \mathbf{R}_{\mathbf{z}}^T \mathbf{R}_d^T \mathbf{T}^T \bar{\mathbf{M}} \mathbf{T} \left(\dot{\mathbf{R}}_d \dot{\mathbf{z}} + \mathbf{R}_d \dot{\mathbf{R}}_{\mathbf{z}} \dot{\mathbf{z}}^i\right)}$$

  - The parenthesis of this expression can be written as:

$$\dot{\mathbf{R}}_d \dot{\mathbf{z}} + \mathbf{R}_d \dot{\mathbf{R}}_{\mathbf{z}} \dot{\mathbf{z}}^i = \dot{\mathbf{R}}_d \mathbf{R}_{\mathbf{z}} \dot{\mathbf{z}}^i + \mathbf{R}_d \dot{\mathbf{R}}_{\mathbf{z}} \dot{\mathbf{z}}^i = \left(\dot{\mathbf{R}}_d \mathbf{R}_{\mathbf{z}} + \mathbf{R}_d \dot{\mathbf{R}}_{\mathbf{z}}\right) \dot{\mathbf{z}}^i = \frac{d\left(\mathbf{R}_d \mathbf{R}_{\mathbf{z}}\right)}{dt} \dot{\mathbf{z}}^i$$

$$\mathbf{Z} = \mathbf{R}\dot{\mathbf{z}} = \mathbf{T}\mathbf{R}_d \dot{\mathbf{z}} = \mathbf{T}\mathbf{R}_d \mathbf{R}_{\mathbf{z}} \dot{\mathbf{z}}^i, \qquad \dot{\mathbf{Z}} = \mathbf{T}\mathbf{R}_d \mathbf{R}_{\mathbf{z}} \ddot{\mathbf{z}}^i + \mathbf{T} \frac{d\left(\mathbf{R}_d \mathbf{R}_{\mathbf{z}}\right)}{dt} \dot{\mathbf{z}}^i, \qquad \boxed{\mathbf{T} \frac{d\left(\mathbf{R}_d \mathbf{R}_{\mathbf{z}}\right)}{dt} \dot{\mathbf{z}}^i = \dot{\mathbf{Z}}\Big|_{\ddot{\mathbf{z}}^i = 0}}$$

Closed-chain equations with penalties 1/2

- The closed-chain constraints are applied for the open-chain relative coordinates as in Cuadrado and Dopico
  - By using the penalty method for positions the following system of differential equations is obtained

$$\mathbf{M}(\mathbf{z})\ddot{\mathbf{z}} = \mathbf{Q}(\mathbf{z},\dot{\mathbf{z}}) - \mathbf{\Phi}_{\mathbf{z}}^{T}(\mathbf{z})\boldsymbol{\alpha}\mathbf{\Phi}(\mathbf{z}) \equiv \bar{\mathbf{Q}}(\mathbf{z},\dot{\mathbf{z}})$$

  - These equations are integrated by the HHT implicit method

$$\mathbf{M}_{n+1}\ddot{\mathbf{z}}_{n+1} - \left(1-\delta_f\right)\bar{\mathbf{Q}}(\mathbf{z},\dot{\mathbf{z}})_{n+1} - \delta_f\bar{\mathbf{Q}}(\mathbf{z},\dot{\mathbf{z}})_n = \mathbf{0}$$

  - The Newmark's expressions are assumed for velocities and accelerations

$$\dot{\mathbf{z}}_{n+1} = \frac{\gamma}{\beta h}\mathbf{z}_{n+1} + \hat{\dot{\mathbf{z}}}_n; \qquad \hat{\dot{\mathbf{z}}}_n = -\left(\frac{\gamma}{\beta h}\mathbf{z}_n + \left(\frac{\gamma}{\beta}-1\right)\dot{\mathbf{z}}_n + \left(\frac{\gamma}{2\beta}-1\right)h\ddot{\mathbf{z}}_n\right)$$

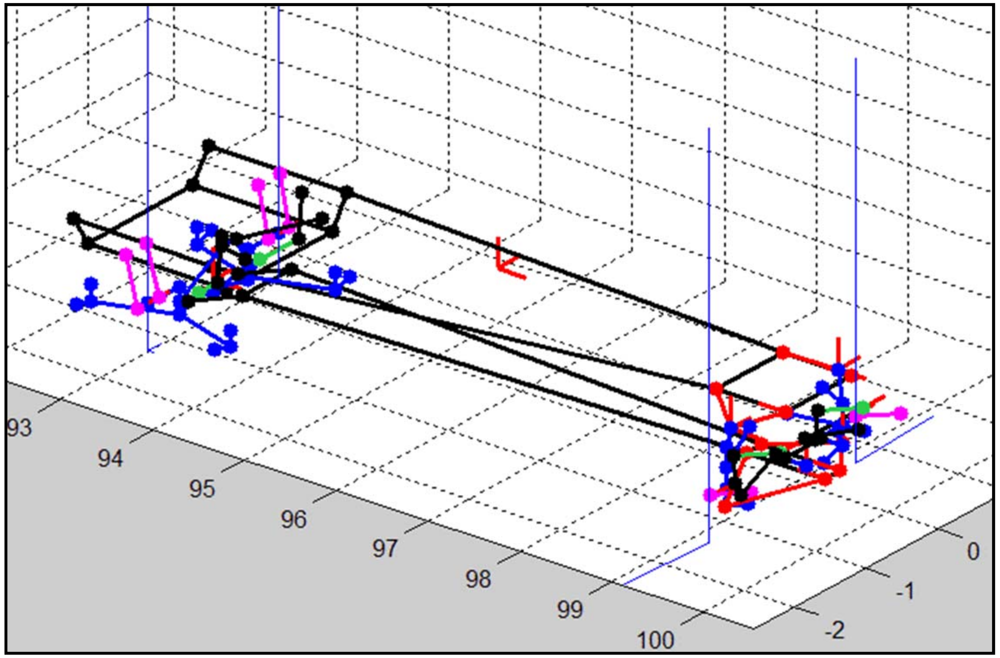
$$\ddot{\mathbf{z}}_{n+1} = \frac{1}{\beta h^2}\mathbf{z}_{n+1} + \hat{\ddot{\mathbf{z}}}_n; \qquad \hat{\ddot{\mathbf{z}}}_n = -\left(\frac{1}{\beta h^2}\mathbf{z}_n + \frac{1}{\beta h}\dot{\mathbf{z}}_n + \left(\frac{1}{2\beta}-1\right)\ddot{\mathbf{z}}_n\right)$$

  - By substituting in the HHT equations, the following nonlinear system of equations is obtained

$$\mathbf{f}(\mathbf{z}_{n+1}) = \mathbf{M}_{n+1}\mathbf{z}_{n+1} + \beta h^2\mathbf{M}_{n+1}\hat{\ddot{\mathbf{z}}}_n - (1-\delta_f)\beta h^2\bar{\mathbf{Q}}_{n+1} - \delta_f\beta h^2\bar{\mathbf{Q}}_n = \mathbf{0}$$

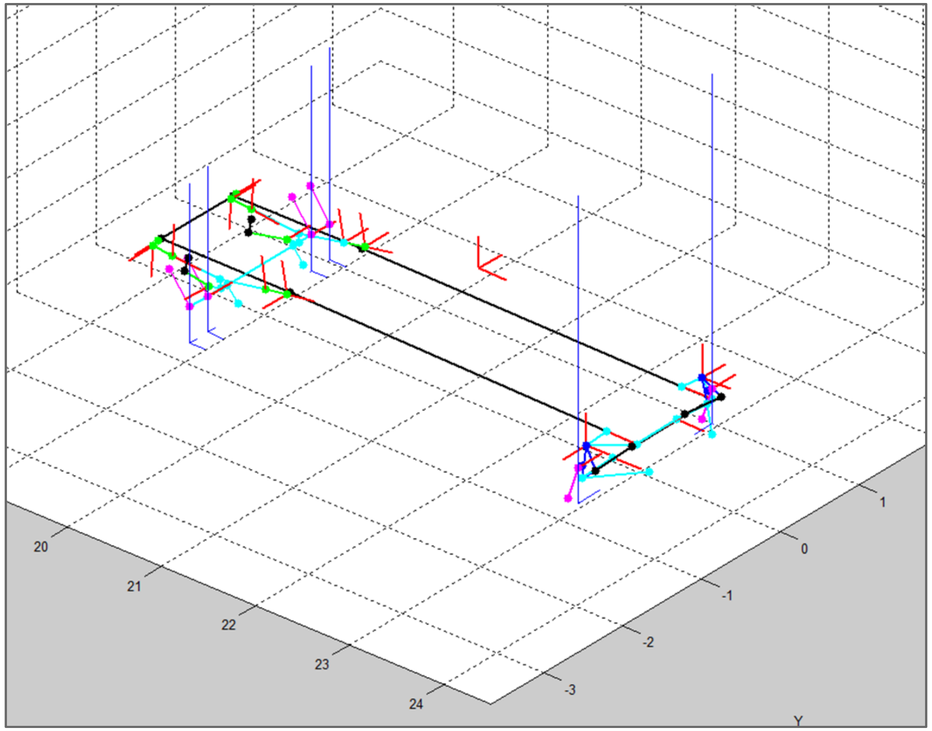
3.2 Numerical examples (semi-recursive formulation)

Coach



- 15 dof
- 33 bodies (14 auxiliary)
- 34 joints
- 11 rods
- Closed loops opened by removing:
  - 2 spherical joints
  - 11 rods

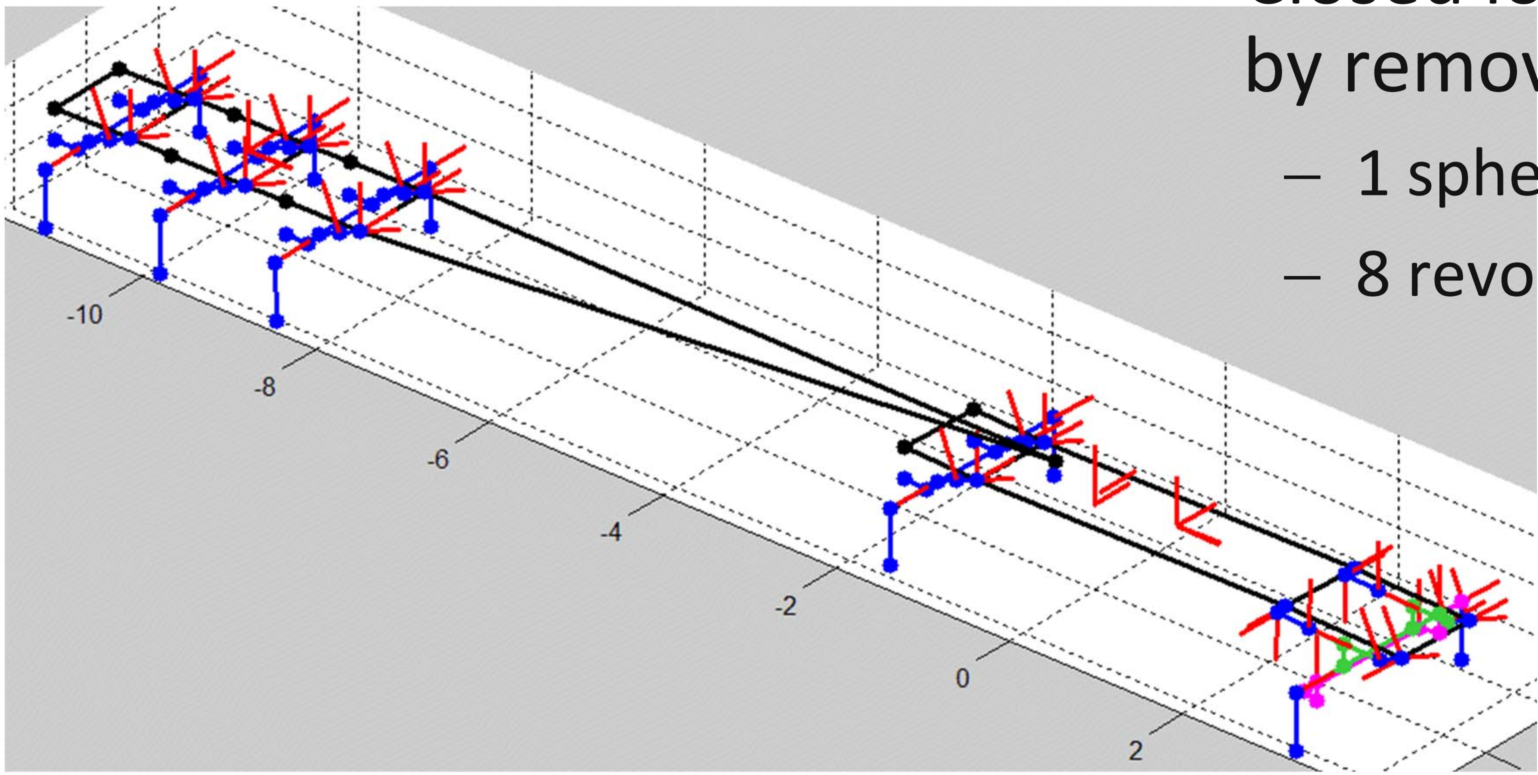
IVECO van



- 17 dof
- 43 bodies (17 auxiliary)
- 47 joints
- 4 rods
- Closed loops opened by removing:
  - 4 rods
  - 2 spherical joints
  - 3 revolute joints

Semi-trailer truck

INSIA-ETSII-UPM



- 40 dof
- 5 axles
- 81 bodies (34 auxiliary)
- 89 joints
- Closed loops opened by removing:
  - 1 spherical joints
  - 8 revolute joints

Computation times (explicit RK4 integrator) [s]

INSIA-ETSII-UPM

Algorithm Step	Coach			Semi-trailer truck		
	Time step: $1\text{x}10^{-3}$ s			Time step: $2.5\text{x}10^{-5}$ s		
	C++	BLAS	SPARSE	C++	BLAS	SPARSE
$\mathbf{b}_j,\ \mathbf{A}_i,\ \mathbf{J}_i,\ \bar{\mathbf{M}}_i,\ \mathbf{M}_i^{\Sigma} = \bar{\mathbf{M}}_i + \sum_{j<i} \mathbf{M}_i^{\Sigma}$	0.276	0.279	0.276	26.410	26.361	26.436
$\begin{bmatrix} \boldsymbol{\Phi}_{\mathbf{z}}^d & \boldsymbol{\Phi}_{\mathbf{z}}^i \end{bmatrix},\quad [\mathbf{L},\mathbf{U}] = \mathrm{lu}\left(\boldsymbol{\Phi}_{\mathbf{z}}^d\right)$	0.120	0.215	0.408	18.071	32.926	28.763
$\mathbf{R}_{\mathbf{z}} = \begin{bmatrix} -\left(\boldsymbol{\Phi}_{\mathbf{z}}^d\right)^{-1} \boldsymbol{\Phi}_{\mathbf{z}}^i \\ \mathbf{I} \end{bmatrix}$	0.355	0.183	0.089	223.253	29.733	28.228
$\dot{\mathbf{z}}^d = -\left(\boldsymbol{\Phi}_{\mathbf{z}}^d\right)^{-1} \boldsymbol{\Phi}_{\mathbf{z}}^i \dot{\mathbf{z}}^i,\quad \text{recVel: } \mathbf{Z}_i,\ \mathbf{d}_i,\ \mathbf{d}_i^{\Sigma}$	0.122	0.133	0.200	13.360	14.647	7.671
$\mathbf{Q}_i,\ \bar{\mathbf{Q}}_i,\ \boldsymbol{\tau}_i$	0.193	0.192	0.182	13.861	13.849	14.126
$\mathbf{c} = -\dot{\boldsymbol{\Phi}}_{\mathbf{z}} \dot{\mathbf{z}},\quad \dot{\mathbf{R}}_{\mathbf{z}} \dot{\mathbf{z}} = -\boldsymbol{\Phi}_{\mathbf{z}}^d \backslash \dot{\boldsymbol{\Phi}}_{\mathbf{z}} \dot{\mathbf{z}}$	0.151	0.143	0.100	12.972	9.925	9.371
$\text{recAccel: } \mathbf{Z}_i,\ \bar{\mathbf{Q}},\ \mathbf{Q}^{\Sigma}$	0.087	0.086	0.637	8.638	8.543	9.080
$\mathbf{M} = \mathbf{R}_{\mathbf{z}}^T \mathbf{R}_d^T \mathbf{M}^{\Sigma} \mathbf{R}_d \mathbf{R}_{\mathbf{z}}$ $\mathbf{b} = \mathbf{R}_{\mathbf{z}}^T \left(\mathbf{R}_d^T \mathbf{Q}^{\Sigma} - \mathbf{R}_d^T \mathbf{M}^{\Sigma} \left(\dot{\mathbf{R}}_d \dot{\mathbf{z}} + \mathbf{R}_d \dot{\mathbf{R}}_{\mathbf{z}} \dot{\mathbf{z}}^i \right) \right)$	0.649	0.606	0.637	151.284	73.521	73.715
$\ddot{\mathbf{z}}_d = \mathbf{M}_{d,d} \backslash \left(\mathbf{M}_{d,g} \ddot{\mathbf{z}}_g \right),\quad \mathbf{b}_g = \mathbf{M}_{g,d} \ddot{\mathbf{z}}_d + \mathbf{M}_{g,g} \ddot{\mathbf{z}}_g$	0.073	0.074	0.083	13.696	13.765	13.927
Elapsed time [s]	**2.026**	**1.912**	**2.461**	**481.543**	**223.27**	**211.316**

Computation times (Implicit method) [s]

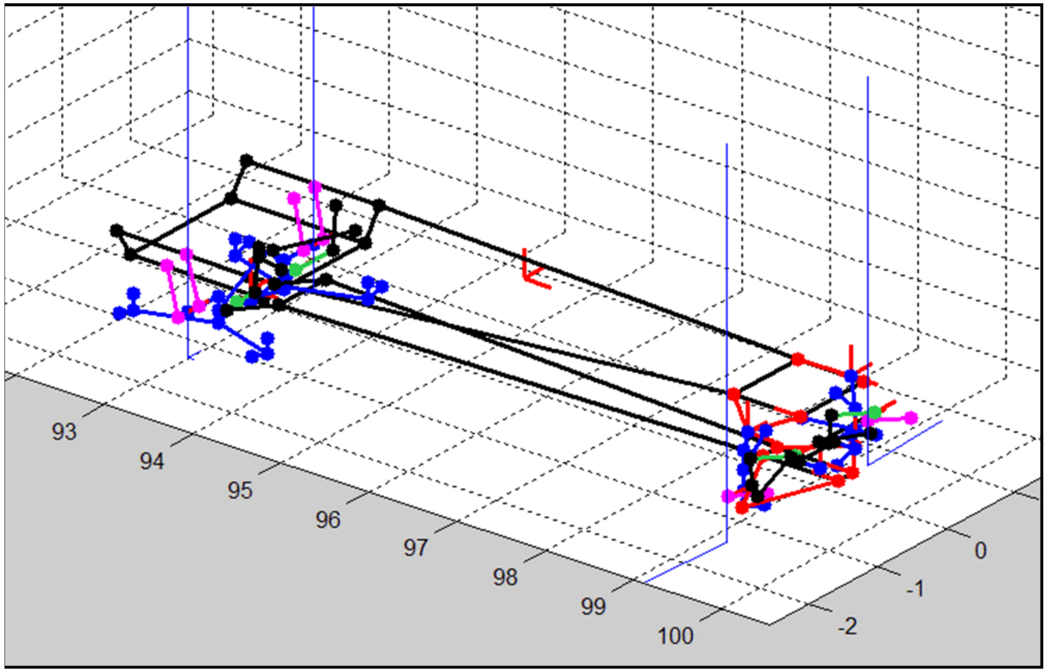
Algorithm Step	Iveco van		Semi-trailer truck	
	Time step: 5×10^{-3} s		Time step: 5×10^{-3} s	
	Serial	TBB	Serial	TBB
$\ddot{\mathbf{z}}_0$	0.001	0.001	0.001	0.001
$\mathbf{M}(\mathbf{z}),\ \mathbf{\Phi}(\mathbf{z}),\ \mathbf{\Phi}_\mathbf{z}(\mathbf{z}),\ \mathbf{Q}(\mathbf{z},\dot{\mathbf{z}}),\ \mathbf{f}(\mathbf{z})$	0.153	0.194	0.193	0.236
$\mathbf{\Phi}_\mathbf{z}^T \alpha \mathbf{\Phi}_\mathbf{z}$	0.333	0.115	1.748	0.243
$\mathbf{K} = -\left(\partial \mathbf{Q}/\partial \mathbf{z}\right), \quad \mathbf{C} = -\left(\partial \mathbf{Q}/\partial \dot{\mathbf{z}}\right)$	4.437	1.581	9.199	2.802
$\left[\partial \mathbf{f}(\mathbf{z})/\partial \mathbf{z}\right]_{n+1}^{k}$	0.028	0.033	0.081	0.096
$\Delta \mathbf{z}_{(n+1)}^{k} = -\left[\partial \mathbf{f}(\mathbf{z})/\partial \mathbf{z}\right]_{n+1}^{k} \backslash \left[\mathbf{f}(\mathbf{z})\right]_{n+1}^{k}$	0.179	0.164	0.674	0.706
Convergence evaluation	0.001	0.001	0.002	0.002
Velocities projections	0.019	0.020	0.072	0.073
Accelerations projections	0.013	0.014	0.030	0.031
Elapsed time [s]	**5.162**	**2.122**	**11.999**	**4.191**

Computation times

INSIA-ETSII-UPM

- 15 s simulation time
- Intel Core 2 Duo 1.8 GHz, GeForce 8

Slalom manoeuvre (15 s)	**Without graphics**	**MATLAB graphics**	**3D graphics**
Coach	16.31	1440	19.17
FSAE racing car	31.12	1419	43.86



3.2 Semi-recursive formulation for flexible bodies: method of the 24 constant matrices

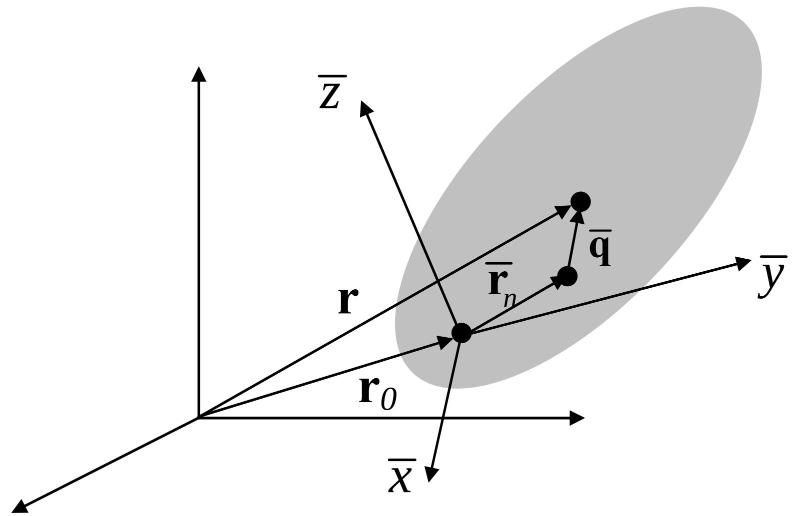
- F.J. Funes and J. García de Jalón, *Simulation of Flexible Multibody Systems Based on Recursive Topological Methods and Modal Synthesis*, ECCOMAS Multibody Dynamics 2009, Warsaw, Poland.
- F.J. Funes and J. García de Jalón, *Efficient Simulation of Complex Flexible Multibody Systems Based on Recursive Methods and Implicit Integrators*, ECCOMAS Multibody Dynamics 2011, Brussels, Belgium.

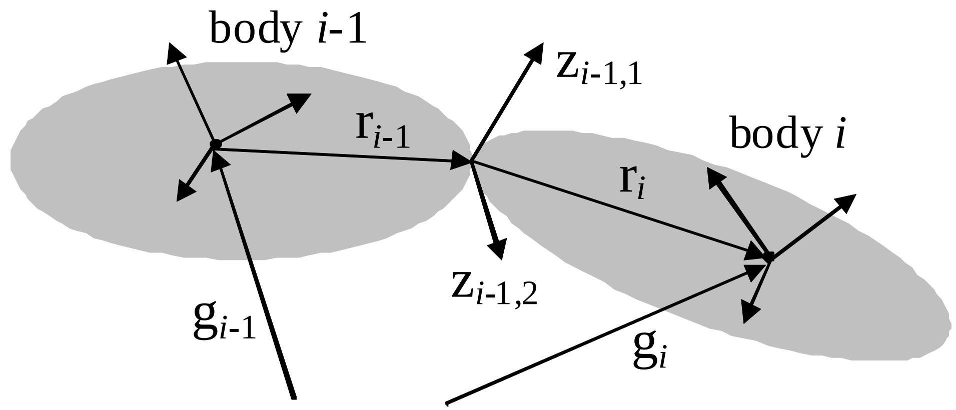
Objectives

- Motivation and objectives
  - To present a work in progress to analyze large size, real-life flexible multibody systems
  - This formulation uses two efficient methods for implicit integration of the dynamic equations of systems with flexible bodies (not described here)
- Outline
  - Semi-recursive formulation for flexible bodies: a review
    - Kinematics and dynamics of the flexible body
    - Method of the “24 constant matrices” (ECCOMAS 2009, Warsaw)
    - Dynamic equations of the multibody system
  - Definition of the reduced model of a flexible body
  - Recursive computation of the dynamic equations

Kinematics of a flexible body 1/2

- Combines large rigid body motions with small deformations





  - Based on the classical Moving Frame Approach
  - Deformed position of a point

$$\mathbf{r} = \mathbf{g} + \mathbf{A}\cdot\bar{\mathbf{r}} = \mathbf{g} + \mathbf{A}\left(\bar{\mathbf{r}}_n + \bar{\mathbf{q}}\right), \qquad \bar{\mathbf{q}} = \sum_{i=1}^{n_s} \bar{\boldsymbol{\varphi}}_i \eta_i + \sum_{j=1}^{n_d} \bar{\boldsymbol{\psi}}_j \varepsilon_j = \bar{\boldsymbol{\Phi}}\boldsymbol{\eta} + \bar{\boldsymbol{\Psi}}\boldsymbol{\varepsilon}$$

  - Relative position and velocity between two contiguous bodies

$$\mathbf{g}_i = \mathbf{g}_{i-1} + \mathbf{A}_{i-1}\left(\bar{\mathbf{r}}_{i-1} + \sum_{k1} \boldsymbol{\varphi}^f_{t,i-1}\eta^f_{i-1,k1}\right) + \sum_j \mathbf{u}_j \cdot z_j - \mathbf{A}_i\left(\bar{\mathbf{r}}_i + \sum_{k2} \boldsymbol{\varphi}^f_{t,i}\eta^f_{i,k2}\right)$$

$$\begin{Bmatrix} \dot{\mathbf{s}}_i \\ \boldsymbol{\omega}_i \end{Bmatrix} = \begin{Bmatrix} \dot{\mathbf{s}}_{i-1} \\ \boldsymbol{\omega}_{i-1} \end{Bmatrix} + \sum_{k1} \begin{bmatrix} \boldsymbol{\varphi}_t \\ \boldsymbol{\varphi}_r \end{bmatrix}_{i-1} \dot{\eta}_{i-1,k1} + \sum_j \mathbf{b}_j \cdot \dot{z}_j - \sum_{k2} \begin{bmatrix} \boldsymbol{\varphi}_t \\ \boldsymbol{\varphi}_r \end{bmatrix}_i \dot{\eta}_{i,k2}$$

Kinematics of a flexible body 2/2

- Transformation between Cartesian/FEM and Cartesian/modal coordinates
  - Cartesian linear and angular velocity of a FEM node

$$\dot{\mathbf{r}}_i = \dot{\mathbf{s}} + \boldsymbol{\omega} \times \mathbf{r}_i + \mathbf{A}\left(\sum_{j=1}^{n_s} \bar{\boldsymbol{\varphi}}_{t,j} \dot{\eta}_j + \sum_{k=1}^{n_d} \bar{\boldsymbol{\psi}}_{t,k} \dot{\varepsilon}_k \right)$$

$$\boldsymbol{\omega}_i = \boldsymbol{\omega} + \mathbf{A}\left(\sum_{j=1}^{n_s} \bar{\boldsymbol{\varphi}}_{r,j} \dot{\eta}_j + \sum_{k=1}^{n_d} \bar{\boldsymbol{\psi}}_{r,k} \dot{\varepsilon}_k \right)$$

  - For the *n* nodes of a flexible body:

$$\mathbf{v}_f = \begin{Bmatrix} \dot{\mathbf{r}}_1 \\ \dot{\boldsymbol{\theta}}_1 \\ \dot{\mathbf{r}}_2 \\ \dot{\boldsymbol{\theta}}_2 \\ \vdots \\ \dot{\mathbf{r}}_n \\ \dot{\boldsymbol{\theta}}_n \end{Bmatrix} = \left[\begin{array}{cc|ccc|ccc} \mathbf{I}_3 & -\tilde{\mathbf{r}}_1 & \mathbf{A}\bar{\boldsymbol{\varphi}}^1_{t,1} & \cdots & \mathbf{A}\bar{\boldsymbol{\varphi}}^1_{t,n_f} & \mathbf{A}\bar{\boldsymbol{\psi}}^1_{t,1} & \cdots & \mathbf{A}\bar{\boldsymbol{\psi}}^1_{t,n_i} \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{A}\bar{\boldsymbol{\varphi}}^1_{r,1} & \cdots & \mathbf{A}\bar{\boldsymbol{\varphi}}^1_{r,n_f} & \mathbf{A}\bar{\boldsymbol{\psi}}^1_{r,1} & \cdots & \mathbf{A}\bar{\boldsymbol{\psi}}^1_{r,n_i} \\ \mathbf{I}_3 & -\tilde{\mathbf{r}}_2 & \mathbf{A}\bar{\boldsymbol{\varphi}}^2_{t,1} & \cdots & \mathbf{A}\bar{\boldsymbol{\varphi}}^2_{t,n_f} & \mathbf{A}\bar{\boldsymbol{\psi}}^2_{t,1} & \cdots & \mathbf{A}\bar{\boldsymbol{\psi}}^2_{t,n_i} \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{A}\bar{\boldsymbol{\varphi}}^2_{r,1} & \cdots & \mathbf{A}\bar{\boldsymbol{\varphi}}^2_{r,n_f} & \mathbf{A}\bar{\boldsymbol{\psi}}^2_{r,1} & \cdots & \mathbf{A}\bar{\boldsymbol{\psi}}^2_{r,n_i} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{I}_3 & -\tilde{\mathbf{r}}_n & \mathbf{A}\bar{\boldsymbol{\varphi}}^n_{t,1} & \cdots & \mathbf{A}\bar{\boldsymbol{\varphi}}^n_{t,n_f} & \mathbf{A}\bar{\boldsymbol{\psi}}^n_{t,1} & \cdots & \mathbf{A}\bar{\boldsymbol{\psi}}^n_{t,n_i} \\ \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{A}\bar{\boldsymbol{\varphi}}^n_{r,1} & \cdots & \mathbf{A}\bar{\boldsymbol{\varphi}}^n_{r,n_f} & \mathbf{A}\bar{\boldsymbol{\psi}}^n_{r,1} & \cdots & \mathbf{A}\bar{\boldsymbol{\psi}}^n_{r,n_i} \end{array} \right] \begin{Bmatrix} \dot{\mathbf{s}} \\ \boldsymbol{\omega} \\ \hline \dot{\eta}_1 \\ \vdots \\ \dot{\eta}_{n_s} \\ \hline \dot{\varepsilon}_1 \\ \vdots \\ \dot{\varepsilon}_{n_d} \end{Bmatrix} \equiv \left[\mathbf{D}_{RB} \mid \mathbf{D}_{\eta} \mid \mathbf{D}_{\xi} \right] \begin{Bmatrix} \dot{\mathbf{s}} \\ \boldsymbol{\omega} \\ \hline \dot{\eta}_1 \\ \vdots \\ \dot{\eta}_{n_s} \\ \hline \dot{\varepsilon}_1 \\ \vdots \\ \dot{\varepsilon}_{n_d} \end{Bmatrix} \equiv \mathbf{D} \begin{Bmatrix} \dot{\mathbf{s}} \\ \boldsymbol{\omega} \\ \hline \dot{\eta}_1 \\ \vdots \\ \dot{\eta}_{n_s} \\ \hline \dot{\varepsilon}_1 \\ \vdots \\ \dot{\varepsilon}_{n_d} \end{Bmatrix} = \mathbf{D}\dot{\mathbf{q}}_f$$

$$\dot{\mathbf{v}}_f = \mathbf{D}\ddot{\mathbf{q}}_f + \dot{\mathbf{D}}\dot{\mathbf{q}}_f$$

Dynamics of a flexible body

- By using the Principle of Virtual Power and a non-consistent interpolation for velocities, the dynamic equations can be obtained
  - Motion equations of a flexible body in Cartesian coordinates

$$\mathbf{D}^T\mathbf{M}^{FEM}\mathbf{D}\ddot{\mathbf{q}}+\mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda}=\mathbf{Q}_{ext}-\mathbf{D}^T\mathbf{M}^{FEM}\dot{\mathbf{D}}\dot{\mathbf{q}}-\mathbf{K}\mathbf{q}\equiv\mathbf{Q}_{ext}-\mathbf{Q}_{vel}-\mathbf{Q}_{elastic}$$

$$\dot{\mathbf{q}}^T\equiv\left\{\dot{\mathbf{s}}^T\quad\boldsymbol{\omega}^T\;\middle|\;\dot{\eta}_1\;\cdots\;\dot{\eta}_{n_s}\;\middle|\;\dot{\varepsilon}_1\;\cdots\;\dot{\varepsilon}_{n_d}\right\}=\left\{\mathbf{Z}^T\;\middle|\;\boldsymbol{\eta}^T\;\middle|\;\boldsymbol{\varepsilon}^T\right\}$$

  - The mass matrix $\mathbf{D}^T\mathbf{M}^{FEM}\mathbf{D}$ can be expressed in partitioned form:

$$\begin{bmatrix}\mathbf{M}_{RB} & \mathbf{M}_{RB,\eta} & \mathbf{M}_{RB,\varepsilon}\\ & \mathbf{M}_{\eta} & \mathbf{M}_{\eta,\varepsilon}\\ sym & & \mathbf{M}_{\varepsilon}\end{bmatrix}\begin{Bmatrix}\dot{\mathbf{Z}}\\ \ddot{\boldsymbol{\eta}}\\ \ddot{\boldsymbol{\varepsilon}}\end{Bmatrix}+\mathbf{\Phi}_{\mathbf{q}}^T\boldsymbol{\lambda}=\mathbf{Q}_{ext}+\begin{Bmatrix}\mathbf{Q}_{RB}\\ \mathbf{Q}_{\eta}\\ \mathbf{Q}_{\varepsilon}\end{Bmatrix}_{vel}+\begin{Bmatrix}\mathbf{0}\\ \mathbf{Q}_{\eta}\\ \mathbf{Q}_{\varepsilon}\end{Bmatrix}_{elast}$$

- The inertia matrix coming from FEM has been computed in the global reference frame and in the initial position:

$$\mathbf{D}^T\mathbf{M}^{FEM}\mathbf{D}=\mathbf{D}^T\mathbf{A}_n\bar{\mathbf{M}}^{FEM}\mathbf{A}_n^T\mathbf{D}=\bar{\mathbf{D}}^T\bar{\mathbf{M}}^{FEM}\bar{\mathbf{D}},\qquad\mathbf{A}_n\equiv\operatorname{diag}\left(\mathbf{A},\mathbf{A},\cdots,\mathbf{A}\right)$$

(constant matrices/vectors are typed in **blue** color)

- The evaluation of terms $\mathbf{D}^T\mathbf{M}^{FEM}\mathbf{D}$ and $\mathbf{D}^T\mathbf{M}^{FEM}\dot{\mathbf{D}}\dot{\mathbf{q}}$ can be very expensive.

Efficient computation of the term $\mathbf{D}^T\mathbf{M}^{FEM}\mathbf{D}$ 1/3

- Computation of matrix $\bar{\mathbf{D}}$

$$\bar{\mathbf{D}} \equiv \mathbf{A}_n^T\mathbf{D} = \left[\begin{array}{cc:ccc:ccc}
\mathbf{A}^T & -\mathbf{A}^T\tilde{\mathbf{r}}_1 & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\varphi}}^1_{t,1} & \cdots & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\varphi}}^1_{t,n_f} & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\psi}}^1_{t,1} & \cdots & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\psi}}^1_{t,n_i} \\
\mathbf{0}_3 & \mathbf{A}^T & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\varphi}}^1_{r,1} & \cdots & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\varphi}}^1_{r,n_f} & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\psi}}^1_{r,1} & \cdots & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\psi}}^1_{r,n_i} \\
\mathbf{A}^T & -\mathbf{A}^T\tilde{\mathbf{r}}_2 & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\varphi}}^2_{t,1} & \cdots & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\varphi}}^2_{t,n_f} & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\psi}}^2_{t,1} & \cdots & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\psi}}^2_{t,n_i} \\
\mathbf{0}_3 & \mathbf{A}^T & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\varphi}}^2_{r,1} & \cdots & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\varphi}}^2_{r,n_f} & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\psi}}^2_{r,1} & \cdots & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\psi}}^2_{r,n_i} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{A}^T & -\mathbf{A}^T\tilde{\mathbf{r}}_n & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\varphi}}^n_{t,1} & \cdots & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\varphi}}^n_{t,n_f} & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\psi}}^n_{t,1} & \cdots & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\psi}}^n_{t,n_i} \\
\mathbf{0}_3 & \mathbf{A}^T & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\varphi}}^n_{r,1} & \cdots & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\varphi}}^n_{r,n_f} & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\psi}}^n_{r,1} & \cdots & \mathbf{A}^T\mathbf{A}\bar{\boldsymbol{\psi}}^n_{r,n_i}
\end{array}\right] \equiv \left[\begin{array}{c:c:c} \bar{\mathbf{D}}_{RB} & \bar{\mathbf{D}}_\eta & \bar{\mathbf{D}}_\xi \end{array}\right]$$

constant

– By taking into account that $\mathbf{A}^T\tilde{\mathbf{r}}_i\mathbf{A} = \tilde{\bar{\mathbf{r}}}_i$

$$\bar{\mathbf{D}}_{RB} = \begin{bmatrix} \mathbf{A}^T & -\mathbf{A}^T\tilde{\mathbf{r}}_1 \\ \mathbf{0}_3 & \mathbf{A}^T \\ \mathbf{A}^T & -\mathbf{A}^T\tilde{\mathbf{r}}_2 \\ \mathbf{0}_3 & \mathbf{A}^T \\ \cdots & \cdots \\ \mathbf{A}^T & -\mathbf{A}^T\tilde{\mathbf{r}}_n \\ \mathbf{0}_3 & \mathbf{A}^T \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 \\ \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 \\ \cdots & \cdots \\ \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{A}^T & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{A}^T \end{bmatrix} + \begin{bmatrix} -\mathbf{A}^T\tilde{\mathbf{r}}_1 \\ \mathbf{0}_3 \\ -\mathbf{A}^T\tilde{\mathbf{r}}_2 \\ \mathbf{0}_3 \\ \cdots \\ -\mathbf{A}^T\tilde{\mathbf{r}}_n \\ \mathbf{0}_3 \end{bmatrix} \begin{bmatrix} \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 \\ \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 \\ \cdots & \cdots \\ \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{A}^T & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{A}^T \end{bmatrix} + \begin{bmatrix} -\tilde{\bar{\mathbf{r}}}_1 \\ \mathbf{0}_3 \\ -\tilde{\bar{\mathbf{r}}}_2 \\ \mathbf{0}_3 \\ \cdots \\ -\tilde{\bar{\mathbf{r}}}_n \\ \mathbf{0}_3 \end{bmatrix} \begin{bmatrix} \mathbf{0}_3 & \mathbf{A}^T \end{bmatrix} \equiv \bar{\mathbf{D}}^1_{RB} + \bar{\mathbf{D}}^2_{RB}$$

Efficient computation of the term $\mathbf{D}^T \mathbf{M}^{FEM} \mathbf{D}$ 2/3

- The global position of a node, expressed in the moving frame is

$$\mathbf{A}^T \cdot \tilde{\mathbf{r}}_i \cdot \mathbf{A} \equiv \tilde{\bar{\mathbf{r}}}_i \qquad \Rightarrow \qquad \bar{\mathbf{r}}_i = \mathbf{A}^T \cdot \mathbf{r}_i = \mathbf{A}^T \cdot \mathbf{r}_0 + \bar{\mathbf{r}}_{0i} + \sum_{j=1}^{n_s} \bar{\boldsymbol{\varphi}}_{ij}^t \eta_j + \sum_{j=1}^{n_d} \bar{\boldsymbol{\psi}}_{ij}^t \varepsilon_j$$

- Substituting this equation in the expression of $\bar{\mathbf{D}}_{RB}^2$

$$\bar{\mathbf{D}}_{RB}^2 = \begin{bmatrix} -\tilde{\bar{\mathbf{r}}}_1 \\ \mathbf{0}_3 \\ -\tilde{\bar{\mathbf{r}}}_2 \\ \mathbf{0}_3 \\ \cdots \\ -\tilde{\bar{\mathbf{r}}}_n \\ \mathbf{0}_3 \end{bmatrix} \begin{bmatrix} \mathbf{0}_3 & \mathbf{A}^T \end{bmatrix} = - \left[\begin{array}{cc|ccc|ccc} \mathbf{I}_3 & \tilde{\bar{\mathbf{r}}}_{01} & \tilde{\bar{\boldsymbol{\varphi}}}_{11}^t & \cdots & \tilde{\bar{\boldsymbol{\varphi}}}_{1nf}^t & \tilde{\bar{\boldsymbol{\psi}}}_{11}^t & \cdots & \tilde{\bar{\boldsymbol{\psi}}}_{1ni}^t \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \cdots & \mathbf{0}_3 & \mathbf{0}_3 & \cdots & \mathbf{0}_3 \\ \mathbf{I}_3 & \tilde{\bar{\mathbf{r}}}_{02} & \tilde{\bar{\boldsymbol{\varphi}}}_{21}^t & \cdots & \tilde{\bar{\boldsymbol{\varphi}}}_{2nf}^t & \tilde{\bar{\boldsymbol{\psi}}}_{21}^t & \cdots & \tilde{\bar{\boldsymbol{\psi}}}_{2ni}^t \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \cdots & \mathbf{0}_3 & \mathbf{0}_3 & \cdots & \mathbf{0}_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{I}_3 & \tilde{\bar{\mathbf{r}}}_{0n} & \tilde{\bar{\boldsymbol{\varphi}}}_{n1}^t & \cdots & \tilde{\bar{\boldsymbol{\varphi}}}_{nnf}^t & \tilde{\bar{\boldsymbol{\psi}}}_{n1}^t & \cdots & \tilde{\bar{\boldsymbol{\psi}}}_{nni}^t \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \cdots & \mathbf{0}_3 & \mathbf{0}_3 & \cdots & \mathbf{0}_3 \end{array} \right] \begin{bmatrix} \tilde{\bar{\mathbf{r}}}_0 \\ \mathbf{I}_3 \\ \hline \eta_1 \mathbf{I}_3 \\ \cdots \\ \eta_{nf} \mathbf{I}_3 \\ \hline \varepsilon_1 \mathbf{I}_3 \\ \cdots \\ \varepsilon_{ni} \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{0}_3 & \mathbf{A}^T \end{bmatrix} \equiv \bar{\mathbf{D}}_{RB\Gamma}^2 \begin{bmatrix} \mathbf{0}_{3,m} & \boldsymbol{\Gamma}_1 \end{bmatrix}$$

Efficient computation of the term $\mathbf{D}^T\mathbf{M}^{FEM}\mathbf{D}$ 3/3

- The matrix to compute is $\mathbf{D}^T\mathbf{M}^{FEM}\mathbf{D} = \begin{bmatrix} \mathbf{M}_{RB} & \mathbf{M}_{RB\eta\varepsilon} \\ \mathbf{M}_{RB\eta\varepsilon}^T & \mathbf{M}_{\eta\varepsilon} \end{bmatrix}$
  - It depends on 10 precomputed, constant matrices
  - Sub-matrix $\mathbf{M}_{RB}$

$$\mathbf{M}_{RB} = \left(\bar{\mathbf{D}}_{RB}^{1T} + \bar{\mathbf{D}}_{RB}^{2T}\right)\bar{\mathbf{M}}^{FEM}\left(\bar{\mathbf{D}}_{RB}^{1} + \bar{\mathbf{D}}_{RB}^{2}\right) =$$

$$= \begin{bmatrix} \mathbf{A} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{A} \end{bmatrix}\begin{bmatrix} \bar{\mathbf{M}}^1 & \bar{\mathbf{M}}^2 \\ \bar{\mathbf{M}}^{2T} & \bar{\mathbf{M}}^3 \end{bmatrix}\begin{bmatrix} \mathbf{A}^T & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{A}^T \end{bmatrix} + \begin{bmatrix} \mathbf{A} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{A} \end{bmatrix}\begin{bmatrix} \mathbf{0}_3 & \bar{\mathbf{M}}^4 \\ \mathbf{0}_3 & \bar{\mathbf{M}}^5 \end{bmatrix}\begin{bmatrix} \mathbf{0}_{m,3} & \mathbf{\Gamma}_1 \end{bmatrix} +$$

$$+ \begin{bmatrix} \mathbf{0}_{3,m} \\ \mathbf{\Gamma}_1^T \end{bmatrix}\begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 \\ \bar{\mathbf{M}}^{4T} & \bar{\mathbf{M}}^{5T} \end{bmatrix}\begin{bmatrix} \mathbf{A}^T & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{A}^T \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{3,m} \\ \mathbf{\Gamma}_1^T \end{bmatrix}\begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_{3,m} \\ \mathbf{0}_{3,m} & \bar{\mathbf{M}}^{6T} \end{bmatrix}\begin{bmatrix} \mathbf{0}_{m,3} & \mathbf{\Gamma}_1 \end{bmatrix}$$

  - Sub-matrix $\mathbf{M}_{RB\eta\varepsilon}$

$$\mathbf{M}_{RB\eta\xi} = \begin{bmatrix} \mathbf{M}_{RB\eta} & \mathbf{M}_{RB\xi} \end{bmatrix} = \left(\bar{\mathbf{D}}_{RB}^{1T} + \bar{\mathbf{D}}_{RB}^{2T}\right)\bar{\mathbf{M}}^{FEM}\begin{bmatrix} \bar{\mathbf{D}}_\eta & \bar{\mathbf{D}}_\xi \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{A} \end{bmatrix}\begin{bmatrix} \bar{\mathbf{M}}^7 \\ \bar{\mathbf{M}}^8 \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{3,m} \\ \mathbf{\Gamma}_1^T \end{bmatrix}\begin{bmatrix} \bar{\mathbf{M}}^9 \end{bmatrix}$$

  - Sub-matrix $\mathbf{M}_{\eta\varepsilon}$

$$\mathbf{M}_{\eta\xi} \equiv \begin{bmatrix} \mathbf{M}_\eta & \mathbf{M}_{\eta\xi} \\ \mathbf{M}_{\xi\eta} & \mathbf{M}_\xi \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{D}}_\eta^T \\ \bar{\mathbf{D}}_\xi^T \end{bmatrix}\bar{\mathbf{M}}^{FEM}\begin{bmatrix} \bar{\mathbf{D}}_\eta & \bar{\mathbf{D}}_\xi \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{D}}_\eta^T\bar{\mathbf{M}}^{FEM}\bar{\mathbf{D}}_\eta & \bar{\mathbf{D}}_\eta^T\bar{\mathbf{M}}^{FEM}\bar{\mathbf{D}}_\xi \\ \bar{\mathbf{D}}_\xi^T\bar{\mathbf{M}}^{FEM}\bar{\mathbf{D}}_\eta & \bar{\mathbf{D}}_\xi^T\bar{\mathbf{M}}^{FEM}\bar{\mathbf{D}}_\xi \end{bmatrix} \equiv \begin{bmatrix} \bar{\mathbf{M}}^{10} & \bar{\mathbf{M}}^{11} \\ \bar{\mathbf{M}}^{11T} & \bar{\mathbf{M}}^{12} \end{bmatrix}$$

Efficient computation of the term $\mathbf{D}^T\mathbf{M}^{FEM}\dot{\mathbf{D}}\dot{\mathbf{q}}$ 1/3

- Velocity dependent inertia forces

$$\mathbf{Q}_{vel} = \mathbf{D}^T\mathbf{M}^{FEM}\dot{\mathbf{D}}\dot{\mathbf{q}} = \mathbf{D}^T\mathbf{A}_n\overline{\mathbf{M}}^{FEM}\mathbf{A}_n^T\dot{\mathbf{D}}\dot{\mathbf{q}}$$

- The term $\dot{\mathbf{D}}\dot{\mathbf{q}}$ can be computed as follows:

$$\dot{\mathbf{D}}\dot{\mathbf{q}} = \left[\begin{array}{cc|ccc|ccc}
\mathbf{0}_3 & -\left(\tilde{\dot{\mathbf{r}}}_1 - \tilde{\dot{\mathbf{s}}}\right) & \dot{\mathbf{A}}\overline{\boldsymbol{\varphi}}^1_{t,1} & \cdots & \dot{\mathbf{A}}\overline{\boldsymbol{\varphi}}^1_{t,n_f} & \dot{\mathbf{A}}\overline{\boldsymbol{\psi}}^1_{t,1} & \cdots & \dot{\mathbf{A}}\overline{\boldsymbol{\psi}}^1_{t,n_i} \\
\mathbf{0}_3 & \mathbf{0}_3 & \dot{\mathbf{A}}\overline{\boldsymbol{\varphi}}^1_{r,1} & \cdots & \dot{\mathbf{A}}\overline{\boldsymbol{\varphi}}^1_{r,n_f} & \dot{\mathbf{A}}\overline{\boldsymbol{\psi}}^1_{r,1} & \cdots & \dot{\mathbf{A}}\overline{\boldsymbol{\psi}}^1_{r,n_i} \\
\mathbf{0}_3 & -\left(\tilde{\dot{\mathbf{r}}}_2 - \tilde{\dot{\mathbf{s}}}\right) & \dot{\mathbf{A}}\overline{\boldsymbol{\varphi}}^2_{t,1} & \cdots & \dot{\mathbf{A}}\overline{\boldsymbol{\varphi}}^2_{t,n_f} & \dot{\mathbf{A}}\overline{\boldsymbol{\psi}}^2_{t,1} & \cdots & \dot{\mathbf{A}}\overline{\boldsymbol{\psi}}^2_{t,n_i} \\
\mathbf{0}_3 & \mathbf{0}_3 & \dot{\mathbf{A}}\overline{\boldsymbol{\varphi}}^2_{r,1} & \cdots & \dot{\mathbf{A}}\overline{\boldsymbol{\varphi}}^2_{r,n_f} & \dot{\mathbf{A}}\overline{\boldsymbol{\psi}}^2_{r,1} & \cdots & \dot{\mathbf{A}}\overline{\boldsymbol{\psi}}^2_{r,n_i} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdot\cdot\dot{\mathbf{D}}\dot{\mathbf{q}} & \cdots & \cdots \\
\mathbf{0}_3 & -\left(\tilde{\dot{\mathbf{r}}}_n - \tilde{\dot{\mathbf{s}}}\right) & \dot{\mathbf{A}}\overline{\boldsymbol{\varphi}}^n_{t,1} & \cdots & \dot{\mathbf{A}}\overline{\boldsymbol{\varphi}}^n_{t,n_f} & \dot{\mathbf{A}}\overline{\boldsymbol{\psi}}^n_{t,1} & \cdots & \dot{\mathbf{A}}\overline{\boldsymbol{\psi}}^n_{t,n_i} \\
\mathbf{0}_3 & \mathbf{0}_3 & \dot{\mathbf{A}}\overline{\boldsymbol{\varphi}}^n_{r,1} & \cdots & \dot{\mathbf{A}}\overline{\boldsymbol{\varphi}}^n_{r,n_f} & \dot{\mathbf{A}}\overline{\boldsymbol{\psi}}^n_{r,1} & \cdots & \dot{\mathbf{A}}\overline{\boldsymbol{\psi}}^n_{r,n_i}
\end{array}\right]
\left\{\begin{matrix} \dot{\mathbf{s}} \\ \boldsymbol{\omega} \\ \dot{\eta}_1 \\ \cdots \\ \dot{\eta}_{n_s} \\ \dot{\varepsilon}_1 \\ \cdots \\ \dot{\varepsilon}_{n_d} \end{matrix}\right\}
= \left\{\begin{matrix}
\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}_1 + 2\left(\boldsymbol{\omega}\times\mathbf{v}_1^r\right) \\
\boldsymbol{\omega}\times\boldsymbol{\omega}_1^r \\
\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}_2 + 2\left(\boldsymbol{\omega}\times\mathbf{v}_2^r\right) \\
\boldsymbol{\omega}\times\boldsymbol{\omega}_2^r \\
\cdots \\
\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}_1 + 2\left(\boldsymbol{\omega}\times\mathbf{v}_n^r\right) \\
\boldsymbol{\omega}\times\boldsymbol{\omega}_n^r
\end{matrix}\right\}$$

with $\quad \mathbf{v}^r = \sum_{i=1}^{n_s}\boldsymbol{\varphi}_{t,i}\dot{\eta}_i + \sum_{j=1}^{n_s}\boldsymbol{\psi}_{t,j}\dot{\varepsilon}_j, \qquad \boldsymbol{\omega}^r = \sum_{i=1}^{n_s}\boldsymbol{\varphi}_{r,i}\dot{\eta}_i + \sum_{j=1}^{n_s}\boldsymbol{\psi}_{r,j}\dot{\varepsilon}_j$

- For each node there are two kind of terms: $\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}_j$ and $\left\{\begin{matrix} 2\left(\boldsymbol{\omega}\times\mathbf{v}_j^r\right) \\ \boldsymbol{\omega}\times\boldsymbol{\omega}_j^r \end{matrix}\right.$

Efficient computation of the term $\mathbf{D}^T\mathbf{M}^{FEM}\dot{\mathbf{D}}\dot{\mathbf{q}}$ 2/3

- Computation of the centrifugal term at each node:

$$\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}_i=\mathbf{A}\cdot\tilde{\bar{\boldsymbol{\omega}}}^2\cdot\mathbf{A}^T\cdot\mathbf{r}_i=\mathbf{A}\cdot\tilde{\bar{\boldsymbol{\omega}}}^2\cdot\bar{\mathbf{r}}_i=\mathbf{A}\cdot\left[\tilde{\bar{\boldsymbol{\omega}}}_1^2\;\middle|\;\tilde{\bar{\boldsymbol{\omega}}}_2^2\;\middle|\;\tilde{\bar{\boldsymbol{\omega}}}_3^2\right]\cdot\bar{\mathbf{r}}_i=\mathbf{A}\cdot\left(\begin{bmatrix}\bar{\mathbf{r}}_i^T\\\mathbf{0}_{1,3}\\\mathbf{0}_{1,3}\end{bmatrix}\tilde{\bar{\boldsymbol{\omega}}}_1^2+\begin{bmatrix}\mathbf{0}_{1,3}\\\bar{\mathbf{r}}_i^T\\\mathbf{0}_{1,3}\end{bmatrix}\tilde{\bar{\boldsymbol{\omega}}}_2^2+\begin{bmatrix}\mathbf{0}_{1,3}\\\mathbf{0}_{1,3}\\\bar{\mathbf{r}}_i^T\end{bmatrix}\tilde{\bar{\boldsymbol{\omega}}}_3^2\right)$$

- For each one of the three adding terms

$$\mathbf{A}\cdot\begin{bmatrix}\bar{\mathbf{r}}_i^T\\\mathbf{0}_{1,3}\\\mathbf{0}_{1,3}\end{bmatrix}\tilde{\bar{\boldsymbol{\omega}}}_1^2=\mathbf{A}\cdot\left[\begin{array}{ccc|ccc|ccc}\mathbf{0}_{1,3} & \mathbf{e}_{123}^T & \bar{\mathbf{r}}_{0i}^{\;T} & \bar{\boldsymbol{\varphi}}_{t,1}^{i\;T} & \cdots & \bar{\boldsymbol{\varphi}}_{t,n_s}^{i\;T} & \bar{\boldsymbol{\psi}}_{t,1}^{i\;T} & \cdots & \bar{\boldsymbol{\psi}}_{t,n_d}^{i\;T}\\\mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \cdots & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \cdots & \mathbf{0}_{1,3}\\\mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \cdots & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} & \cdots & \mathbf{0}_{1,3}\end{array}\right]\boldsymbol{\Gamma}_2\cdot\tilde{\bar{\boldsymbol{\omega}}}_1^2,\qquad\boldsymbol{\Gamma}_2\equiv\begin{bmatrix}\mathbf{0}_3\\\bar{\mathbf{r}}_0\mathbf{I}_3\\\mathbf{I}_3\\\hline\eta_1\mathbf{I}_3\\\cdots\\\eta_{nf}\mathbf{I}_3\\\hline\varepsilon_1\mathbf{I}_3\\\cdots\\\varepsilon_{ni}\mathbf{I}_3\end{bmatrix}$$

with:

$$\bar{\mathbf{r}}_0\mathbf{I}\equiv\begin{bmatrix}\bar{\mathbf{r}}_{0x} & \mathbf{0} & \mathbf{0}\\\mathbf{0} & \bar{\mathbf{r}}_{0y} & \mathbf{0}\\\mathbf{0} & \mathbf{0} & \bar{\mathbf{r}}_{0z}\end{bmatrix},\qquad\mathbf{e}_{123}^T\equiv\begin{bmatrix}1 & 1 & 1\end{bmatrix}$$

Efficient computation of the term $\mathbf{D}^T \mathbf{M}^{FEM} \dot{\mathbf{D}} \dot{\mathbf{q}}$ 3/3

- Computation of the Coriolis acceleration term

$$\begin{Bmatrix} 2\left(\boldsymbol{\omega}\times\mathbf{v}_i^r\right) \\ \boldsymbol{\omega}\times\boldsymbol{\omega}_i^r \end{Bmatrix} = -\begin{bmatrix} 2\cdot\tilde{\mathbf{v}}_i^r \\ \tilde{\boldsymbol{\omega}}_i^r \end{bmatrix}\boldsymbol{\omega} = -\begin{bmatrix} 2\cdot\mathbf{A}\cdot\tilde{\bar{\mathbf{v}}}_i^r\cdot\mathbf{A}^T \\ \mathbf{A}\cdot\tilde{\bar{\boldsymbol{\omega}}}_i^r\mathbf{A}^T \end{bmatrix}\boldsymbol{\omega} = -\begin{bmatrix} \mathbf{A} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{A} \end{bmatrix}\left[\begin{array}{ccc|ccc} 2\cdot\tilde{\bar{\boldsymbol{\varphi}}}_{t,1}^i & \cdots & 2\cdot\tilde{\bar{\boldsymbol{\varphi}}}_{t,n_s}^i & 2\cdot\tilde{\bar{\boldsymbol{\psi}}}_{t,1}^i & \cdots & 2\cdot\tilde{\bar{\boldsymbol{\psi}}}_{t,n_d}^i \\ \tilde{\bar{\boldsymbol{\varphi}}}_{r,1}^i & \cdots & \tilde{\bar{\boldsymbol{\varphi}}}_{r,n_s}^i & \tilde{\bar{\boldsymbol{\psi}}}_{r,1}^i & \cdots & \tilde{\bar{\boldsymbol{\psi}}}_{r,n_d}^i \end{array}\right]\begin{Bmatrix} \dot{\eta}_1\mathbf{A}^T \\ \cdots \\ \dot{\eta}_{n_s}\mathbf{A}^T \\ \dot{\varepsilon}_1\mathbf{A}^T \\ \cdots \\ \dot{\varepsilon}_{n_d}\mathbf{A}^T \end{Bmatrix}\cdot\boldsymbol{\omega}$$

- Finally, the velocity dependent inertia forces are:

$$\mathbf{Q}_{vel} = \begin{Bmatrix} \mathbf{Q}_{vel,RB} \\ \mathbf{Q}_{vel,\eta\varepsilon} \end{Bmatrix}$$

$$\mathbf{Q}_{vel,RB} = \begin{bmatrix} \mathbf{A} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{A} \end{bmatrix}\left(\bar{\mathbf{M}}^{13}\cdot\boldsymbol{\Gamma}_2\cdot\tilde{\boldsymbol{\omega}}_1^2 + \bar{\mathbf{M}}^{14}\cdot\boldsymbol{\Gamma}_2\cdot\tilde{\boldsymbol{\omega}}_2^2 + \bar{\mathbf{M}}^{15}\cdot\boldsymbol{\Gamma}_2\cdot\tilde{\boldsymbol{\omega}}_3^2 - \bar{\mathbf{M}}^{16}\cdot\boldsymbol{\Gamma}_3\cdot\boldsymbol{\omega}\right) +$$

$$+\begin{bmatrix} \mathbf{0}_{3,m} \\ \boldsymbol{\Gamma}_1^T \end{bmatrix}\left(\bar{\mathbf{M}}^{17}\cdot\boldsymbol{\Gamma}_2\cdot\tilde{\boldsymbol{\omega}}_1^2 + \bar{\mathbf{M}}^{18}\cdot\boldsymbol{\Gamma}_2\cdot\tilde{\boldsymbol{\omega}}_2^2 + \bar{\mathbf{M}}^{19}\cdot\boldsymbol{\Gamma}_2\cdot\tilde{\boldsymbol{\omega}}_3^2 - \bar{\mathbf{M}}^{20}\cdot\boldsymbol{\Gamma}_3\cdot\boldsymbol{\omega}\right)$$

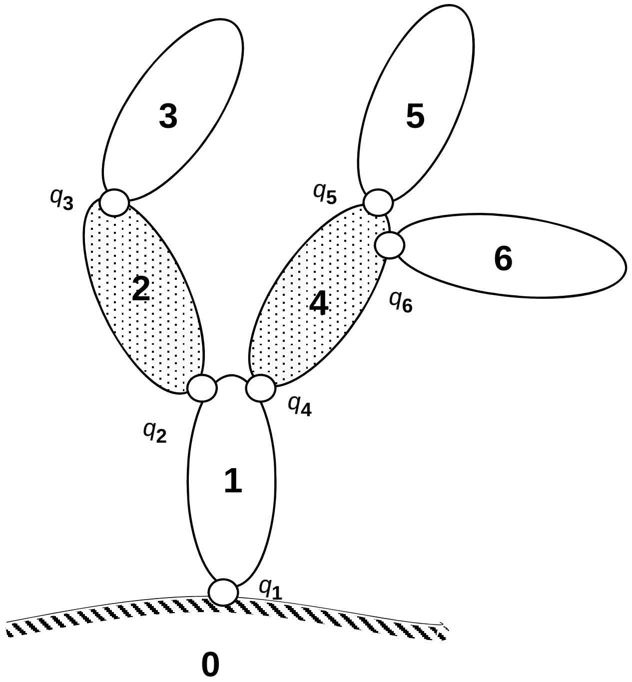
$$\mathbf{Q}_{vel,\eta\varepsilon} = \left(\bar{\mathbf{M}}^{21}\cdot\boldsymbol{\Gamma}_2\cdot\tilde{\boldsymbol{\omega}}_1^2 + \bar{\mathbf{M}}^{22}\cdot\boldsymbol{\Gamma}_2\cdot\tilde{\boldsymbol{\omega}}_2^2 + \bar{\mathbf{M}}^{23}\cdot\boldsymbol{\Gamma}_2\cdot\tilde{\boldsymbol{\omega}}_3^2 - \bar{\mathbf{M}}^{24}\cdot\boldsymbol{\Gamma}_3\cdot\boldsymbol{\omega}\right)$$

$$\boldsymbol{\Gamma}_3 = \begin{Bmatrix} \dot{\eta}_1\mathbf{A}^T \\ \cdots \\ \dot{\eta}_{n_s}\mathbf{A}^T \\ \dot{\varepsilon}_1\mathbf{A}^T \\ \cdots \\ \dot{\varepsilon}_{n_d}\mathbf{A}^T \end{Bmatrix}$$

- There 12 constant matrices that can be pre-computed

1st velocity transformation 1/4

- First, the closed loops are open. The Cartesian velocities are expressed in terms of the relative velocities and the static and dynamic modal amplitudes (computations carried out recursively):



$$\left\{\begin{array}{c} \mathbf{Z}_1 \\ \mathbf{Z}_2 \\ \mathbf{Z}_3 \\ \mathbf{Z}_4 \\ \mathbf{Z}_5 \\ \mathbf{Z}_6 \\ \hline \dot{\boldsymbol{\eta}}_2 \\ \dot{\boldsymbol{\eta}}_3 \\ \dot{\boldsymbol{\eta}}_4 \\ \dot{\boldsymbol{\eta}}_5 \\ \dot{\boldsymbol{\eta}}_6 \\ \hline \dot{\boldsymbol{\varepsilon}}_2 \\ \dot{\boldsymbol{\varepsilon}}_4 \end{array}\right\} = \left[\begin{array}{cccccc:ccccc:cc} \mathbf{b}_1 & & & & & & & & & & & & \\ \mathbf{b}_1 & \mathbf{b}_2 & & & & & -\boldsymbol{\varphi}_{2,1} & & & & & & \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & & & & -\boldsymbol{\varphi}_{2,1} & \boldsymbol{\varphi}_{2,3} & & & & & \\ \mathbf{b}_1 & & & \mathbf{b}_4 & & & & & -\boldsymbol{\varphi}_{4,1} & & & & \\ \mathbf{b}_1 & & & \mathbf{b}_4 & \mathbf{b}_5 & & & & -\boldsymbol{\varphi}_{4,1} & \boldsymbol{\varphi}_{5,6} & & & \\ \mathbf{b}_1 & & & \mathbf{b}_4 & & \mathbf{b}_6 & & & -\boldsymbol{\varphi}_{4,1} & & \boldsymbol{\varphi}_{5,6} & & \\ \hdashline & & & & & & \mathbf{I} & & & & & & \\ & & & & & & & \mathbf{I} & & & & & \\ & & & & & & & & \mathbf{I} & & & & \\ & & & & & & & & & \mathbf{I} & & & \\ & & & & & & & & & & \mathbf{I} & & \\ \hdashline & & & & & & & & & & & \mathbf{I} & \\ & & & & & & & & & & & & \mathbf{I} \end{array}\right] \left\{\begin{array}{c} \dot{\mathbf{z}}_1 \\ \dot{\mathbf{z}}_2 \\ \dot{\mathbf{z}}_3 \\ \dot{\mathbf{z}}_4 \\ \dot{\mathbf{z}}_5 \\ \dot{\mathbf{z}}_6 \\ \hline \dot{\boldsymbol{\eta}}_2 \\ \dot{\boldsymbol{\eta}}_3 \\ \dot{\boldsymbol{\eta}}_4 \\ \dot{\boldsymbol{\eta}}_5 \\ \dot{\boldsymbol{\eta}}_6 \\ \hline \dot{\boldsymbol{\varepsilon}}_2 \\ \dot{\boldsymbol{\varepsilon}}_4 \end{array}\right\}$$

- The transformation matrix **R** can be partitioned in the form:

$$\left\{\begin{array}{c} \mathbf{Z} \\ \dot{\boldsymbol{\eta}} \\ \dot{\boldsymbol{\varepsilon}} \end{array}\right\} = \mathbf{R} \left\{\begin{array}{c} \dot{\mathbf{q}} \\ \dot{\boldsymbol{\eta}} \\ \dot{\boldsymbol{\varepsilon}} \end{array}\right\}, \qquad \mathbf{R} = \begin{bmatrix} \mathbf{R}_{RB} & \mathbf{R}_{\eta} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

1st velocity transformation 2/4

- The transformation matrix **R** relate Cartesian and relative coordinates for velocities and accelerations:

$$\begin{Bmatrix} \mathbf{Z} \\ \dot{\boldsymbol{\eta}} \\ \dot{\boldsymbol{\varepsilon}} \end{Bmatrix} = \mathbf{R} \begin{Bmatrix} \dot{\mathbf{q}} \\ \dot{\boldsymbol{\eta}} \\ \dot{\boldsymbol{\varepsilon}} \end{Bmatrix}, \qquad \begin{Bmatrix} \dot{\mathbf{Z}} \\ \ddot{\boldsymbol{\eta}} \\ \ddot{\boldsymbol{\varepsilon}} \end{Bmatrix} = \mathbf{R} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \ddot{\boldsymbol{\eta}} \\ \ddot{\boldsymbol{\varepsilon}} \end{Bmatrix} + \dot{\mathbf{R}} \begin{Bmatrix} \dot{\mathbf{q}} \\ \dot{\boldsymbol{\eta}} \\ \dot{\boldsymbol{\varepsilon}} \end{Bmatrix}$$

- By assembling the dynamic equations of each body and setting them in terms of the relative and modal coordinates, the final expression for the open-chain dynamic equations is:

$$\mathbf{R}^T \mathbf{M} \mathbf{R} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \ddot{\boldsymbol{\eta}} \\ \ddot{\boldsymbol{\varepsilon}} \end{Bmatrix} + \mathbf{R}^T \mathbf{M} \dot{\mathbf{R}} \begin{Bmatrix} \dot{\mathbf{q}} \\ \dot{\boldsymbol{\eta}} \\ \dot{\boldsymbol{\varepsilon}} \end{Bmatrix} - \mathbf{R}^T \mathbf{Q} = \mathbf{0}$$

- All the terms in this equation can be evaluated efficiently, using a recursive algorithm.

1st velocity transformation 3/4

- By substituting matrix **R** partitioned in the expression $\mathbf{R}^T\mathbf{M}\mathbf{R}$:

$$\mathbf{R}^T\mathbf{M}\mathbf{R} = \begin{bmatrix} \mathbf{R}_{RB}^T\mathbf{M}_{RB}\mathbf{R}_{RB} & \mathbf{R}_{RB}^T\mathbf{M}_{RB}\mathbf{R}_{\eta} & \mathbf{0} \\ \mathbf{R}_{\eta}^T\mathbf{M}_{RB}\mathbf{R}_{RB} & \mathbf{R}_{\eta}^T\mathbf{M}_{RB}\mathbf{R}_{\eta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\eta} & \mathbf{M}_{\eta,\varepsilon} \\ \mathbf{0} & \mathbf{M}_{\varepsilon,\eta} & \mathbf{M}_{\varepsilon} \end{bmatrix} +$$

$$+ \begin{bmatrix} \mathbf{0} & \mathbf{R}_{RB}^T\mathbf{M}_{RB,\eta} & \mathbf{R}_{RB}^T\mathbf{M}_{RB,\varepsilon} \\ \mathbf{M}_{\eta,RB}\mathbf{R}_{RB} & \mathbf{R}_{\eta}^T\mathbf{M}_{RB,\eta} + \mathbf{M}_{\eta,RB}\mathbf{R}_{\eta} & \mathbf{R}_{\eta}^T\mathbf{M}_{RB,\varepsilon} \\ \mathbf{M}_{RB,\varepsilon}\mathbf{R}_{RB} & \mathbf{M}_{\varepsilon,RB}\mathbf{R}_{\eta} & \mathbf{0} \end{bmatrix}$$

- The first term of this expression is the mass accumulation in the relative coordinates and in the amplitudes of the boundary nodes.
  - A single recursive calculation can evaluate all expressions.

$$\left(\mathbf{R}_{RB}^T\mathbf{M}_{RB}\mathbf{R}_{RB}\right)_{i,j} = \mathbf{b}_i^T\left(\sum_{k=i}^{n}\mathbf{M}_{RB}^k\right)\mathbf{b}_j, \quad \left(\mathbf{R}_{\eta}^T\mathbf{M}_{RB}\mathbf{R}_{\eta}\right)_{i,j} = \boldsymbol{\varphi}_i^T\left(\sum_{k=i}^{n}\mathbf{M}_{RB}^k\right)\boldsymbol{\varphi}_j, \quad \left(\mathbf{R}_{\eta}^T\mathbf{M}_{RB}\mathbf{R}_{RB}\right)_{i,j} = \boldsymbol{\varphi}_i^T\left(\sum_{k=i}^{n}\mathbf{M}_{RB}^k\right)\mathbf{b}_j$$

- The second and third terms have analogous meanings.

1st velocity transformation 4/4

- The projection of the forces $\mathbf{R}^T\mathbf{Q}$ can be evaluated similarly:

$$\mathbf{R}^T\mathbf{Q} = \begin{bmatrix} \mathbf{R}_{RB}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{R}_{\eta}^T & \mathbf{I} & \mathbf{0} \\ \mathbf{R}_{\varepsilon}^T & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{Q}_{RB} \\ \mathbf{Q}_{\eta} \\ \mathbf{Q}_{\varepsilon} \end{Bmatrix} = \begin{Bmatrix} \mathbf{R}_{RB}^T\mathbf{Q}_{RB} \\ \mathbf{R}_{\eta}^T\mathbf{Q}_{RB} + \mathbf{Q}_{\eta} \\ \mathbf{R}_{\varepsilon}^T\mathbf{Q}_{RB} + \mathbf{Q}_{\varepsilon} \end{Bmatrix}$$

- The evaluation is similar to that corresponding to the mass matrix:
  - All the terms are evaluated recursively following the open-chain tree structure.
  - The forces applied to the relative coordinates and modal amplitudes are added.

$$\left(\mathbf{R}_{RB}^T\mathbf{Q}_{RB} + \mathbf{Q}_{z_i}\right)_{z_i} = \mathbf{b}_i^T \sum_{k=i}^{n} \mathbf{Q}_k + \mathbf{Q}_{z_i}$$

$$\left(\mathbf{R}_{\eta}^T\mathbf{Q}_{RB} + \mathbf{Q}_{\eta_i}\right)_{\eta_i} = \boldsymbol{\varphi}_{\eta_i} \sum_{k=i}^{n} \mathbf{Q}_k + \mathbf{Q}_{\eta_i}$$

$$\left(\mathbf{Q}_{\varepsilon_i}\right)_{\varepsilon_i} = \mathbf{Q}_{\varepsilon_i}$$

- The remaining expressions also can be evaluated recursively.

2nd velocity transformation

- Closed-chain systems:
  - First the closed-loops are open by removing some joints
  - A subset of independent relative coordinates is chosen

$$\mathbf{z}^i = \mathbf{B} \cdot \mathbf{z}$$

  - The constraint equations are enforced by a 2nd velocity transformation:

$$\boldsymbol{\Phi}(\mathbf{z}, \boldsymbol{\eta}, \boldsymbol{\varepsilon}) = \mathbf{0} \quad \Rightarrow \quad \bar{\boldsymbol{\Phi}}_{\mathbf{z}} = \begin{bmatrix} \boldsymbol{\Phi}_{\mathbf{z}} \\ \mathbf{B} \end{bmatrix} \text{ and } \bar{\boldsymbol{\Phi}}_{\boldsymbol{\eta}} = \begin{bmatrix} \boldsymbol{\Phi}_{\boldsymbol{\eta}} \\ \mathbf{0} \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \bar{\boldsymbol{\Phi}}_{\mathbf{z}} & \bar{\boldsymbol{\Phi}}_{\boldsymbol{\eta}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{R}_{\mathbf{z}} = \mathbf{0}$$

$$\mathbf{z} = \mathbf{R}_{\mathbf{z}} \mathbf{z}^i$$

  - Substituting in the open-chain differential equations:

$$\mathbf{R}_{\mathbf{z}}^T \mathbf{R}^T \mathbf{M} \mathbf{R} \mathbf{R}_{\mathbf{z}} \ddot{\mathbf{z}}^i = \mathbf{R}_{\mathbf{z}}^T \mathbf{R}^T \left(\mathbf{Q} - \mathbf{M} \frac{d(\mathbf{R}\mathbf{R}_{\mathbf{z}})}{dt} \dot{\mathbf{z}}^i \right)$$

4. Live demos and concluding remarks

Mbs3d: Implementation details

- Program mbs3d
  - Computer program that implements the semi-recursive formulation previously described
  - Written in MATLAB
  - Available at http://mec21.etsii.upm.es/mbs/
- mbs3d/MEX
  - The core functions for computations and graphics written in C++ and called from MATLAB
  - Speed up of two orders of magnitude: real-time attainable!
  - Use of BLAS and sparse matrix libraries
- Some advantages of the MATLAB/C++ combination
  - MATLAB is faster for the first implementation and testing
  - Afterwards, C++ functions may be implemented faster

Concluding remarks

- Natural coordinates provide a very simple way to define multibody systems.
- Some of their characteristics (constant mass matrices and not velocity dependent inertia forces, quadratic constraint equations and linear Jacobians, …) allow fast global formulations.
- For faster simulations topological methods with relative coordinates are preferable. Even in this case, natural coordinates may help in the definition of the system geometry and in the closure of the loop constraint equations, keeping the formulation simpler.
- Natural coordinates and MATLAB may be useful for education in multibody systems under severe time limitations.
- Algorithm implementation and testing are very easy with MATLAB, but the numerical performance is very poor. The combined implementation MATLAB/C++ is fast, reliable and efficient.
- Keeping things as simple as possible always has an added value.

Thank you very much!

(Slides available in mec21.etsii.upm.es/mbs

in a few days)