ME751
Advanced Computational Multibody Dynamics

Section 9.2

January 28, 2010

"Age is an issue of mind over matter. If you don't mind, it doesn't matter." Mark Twain

Before we get started…

- Last Time:
  - Finished Calculus review
  - Introduced the concept of Geometric Vector
    - Definition and five basic operations you can do with G. Vectors
    - Combined simple operations: intuitive but tricky to prove
    - Introduced reference frames to simplify handling of G. Vectors

- Today:
  - Introduce Algebraic Vectors (the algebraic counterpart of Geometric Vectors)
  - Understand what it takes to change a RF
  - Hopefully start talking about angular velocity of a rigid body

- HW assigned today, available at class website
  - Due on Feb. 4

- I'll be out on Th Feb. 4
  - Justin and Makarand will present an overview of ADAMS

Representing a G. Vector in a RF
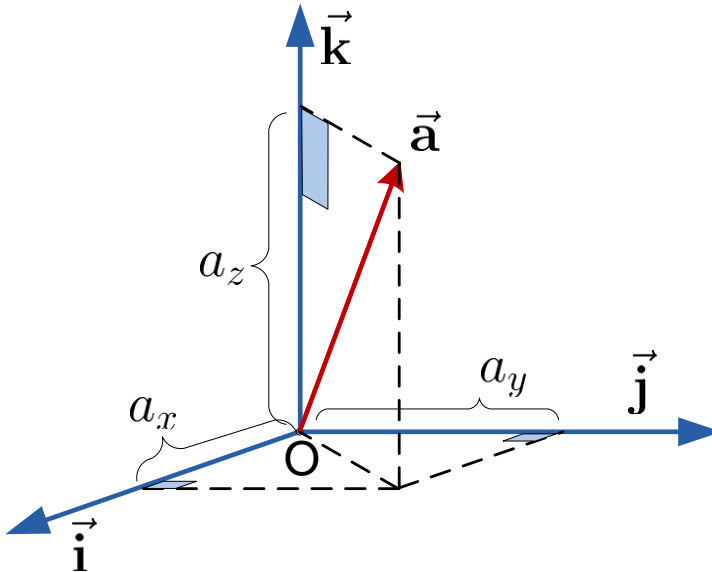
$$\vec{\mathbf{a}} = a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}} + a_z \vec{\mathbf{k}}$$

- Inner product of two g. vectors, recall: $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a\,b\cos\theta(\vec{\mathbf{a}}, \vec{\mathbf{b}})$

- Since the angle between basis vectors is $\pi/2$:

$$\vec{\mathbf{i}} \cdot \vec{\mathbf{i}} = \vec{\mathbf{j}} \cdot \vec{\mathbf{j}} = \vec{\mathbf{k}} \cdot \vec{\mathbf{k}} = 1 \qquad\qquad \vec{\mathbf{i}} \cdot \vec{\mathbf{j}} = \vec{\mathbf{j}} \cdot \vec{\mathbf{k}} = \vec{\mathbf{k}} \cdot \vec{\mathbf{i}} = 0$$

  - Therefore, the Cartesian coordinates are computed as

$$a_x = \vec{\mathbf{a}} \cdot \vec{\mathbf{i}} \qquad\qquad a_y = \vec{\mathbf{a}} \cdot \vec{\mathbf{j}} \qquad\qquad a_z = \vec{\mathbf{a}} \cdot \vec{\mathbf{k}}$$

Geometric Vectors and RFs: Revisiting the Basic Operations

- Assume that $\alpha \in \mathbb{R}$, and we work with two arbitrary vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$:

$$\vec{\mathbf{a}} = a_x\vec{\mathbf{i}} + a_y\vec{\mathbf{j}} + a_z\vec{\mathbf{k}} \qquad \& \qquad \vec{\mathbf{b}} = b_x\vec{\mathbf{i}} + b_y\vec{\mathbf{j}} + b_z\vec{\mathbf{k}}$$

- Sum of two geometric vectors can be shown to be computed as (HOME-WORK):

$$\vec{\mathbf{a}}+\vec{\mathbf{b}} = (a_x\vec{\mathbf{i}}+a_y\vec{\mathbf{j}}+a_z\vec{\mathbf{k}})+(b_x\vec{\mathbf{i}}+b_y\vec{\mathbf{j}}+b_z\vec{\mathbf{k}}) = (a_x+b_x)\vec{\mathbf{i}}+(a_y+b_y)\vec{\mathbf{j}}+(a_z+b_z)\vec{\mathbf{k}}$$

- Multiplication by a real number (scaling) of a geometric vector – the Cartesian coordiantes of the resultsing vector are αa_x, αa_y, and αa_z (HOME-WORK):

$$\alpha\vec{\mathbf{a}} = \alpha \cdot (a_x\vec{\mathbf{i}} + a_y\vec{\mathbf{j}} + a_z\vec{\mathbf{k}}) = (\alpha a_x)\vec{\mathbf{i}} + (\alpha a_y)\vec{\mathbf{j}} + (\alpha a_z)\vec{\mathbf{k}}$$

- Inner product of two geometric vectors can be shown to be computed as (HOMEWORK):

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = (a_x\vec{\mathbf{i}} + a_y\vec{\mathbf{j}} + a_z\vec{\mathbf{k}}) \cdot (b_x\vec{\mathbf{i}} + b_y\vec{\mathbf{j}} + b_z\vec{\mathbf{k}}) = a_xb_x + a_yb_y + a_zb_z$$

- Outer product of two geometric vectors can be shown to be computed as (HOMEWORK):

$$\vec{\mathbf{a}}\times\vec{\mathbf{b}} = (a_x\vec{\mathbf{i}}+a_y\vec{\mathbf{j}}+a_z\vec{\mathbf{k}})\times(b_x\vec{\mathbf{i}}+b_y\vec{\mathbf{j}}+b_z\vec{\mathbf{k}}) = (a_yb_z-a_zb_y)\vec{\mathbf{i}}+(a_zb_x-a_xb_z)\vec{\mathbf{j}}+(a_xb_y-a_yb_x)\vec{\mathbf{k}}$$

New Concept: Algebraic Vectors

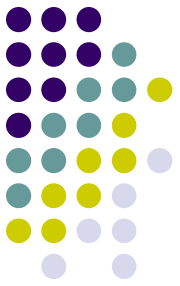- Given a RF, each vector can be represented by a triplet

$$\vec{\mathbf{a}} = a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}} + a_z \vec{\mathbf{k}} \qquad \Leftrightarrow \qquad \vec{\mathbf{a}} \mapsto (a_x, a_y, a_z)$$

- It doesn't take too much imagination to associate to each <u>geometric</u> vector a tridimensional <u>algebraic</u> vector:

$$\vec{\mathbf{a}} = a_x \vec{\mathbf{i}} + a_y \vec{\mathbf{j}} + a_z \vec{\mathbf{k}} \qquad \Leftrightarrow \qquad \mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

- Note that I dropped the arrow on **a** to indicate that we are talking about an algebraic vector

Putting Things in Perspective…

- Step 1: I started with geometric vectors

- Step 2: I introduced a reference frame

- Step 3: Relative to that reference frame each geometric vector is uniquely represented as a triplet (the Cartesian coordinates)

- Step 4: I generate an algebraic vector whose entries are provided by the triplet above
  - This vector is the algebraic representation of the geometric vector

- Note that the algebraic representations of the basis vectors are

$$\vec{\mathbf{i}} \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{\mathbf{j}} \mapsto \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{\mathbf{k}} \mapsto \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Revisiting the Basic Vector Operations
[An algebraic perspective]

- Based on conclusions drawn in slide "Geometric Vectors and RFs: Revisiting the Basic Operations" it's easy to see that:

  - If you scale a geometric vector, the algebraic representation of the result is obtained by scaling of the original algebraic representation

  $$\vec{\mathbf{a}} \mapsto \alpha\vec{\mathbf{a}} \qquad \Leftrightarrow \qquad \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \mapsto \begin{bmatrix} \alpha a_x \\ \alpha a_y \\ \alpha a_z \end{bmatrix}$$

  - If you add two geometric vectors and are curious about the algebraic representation of the result, you simply have to add the two algebraic representations of the original vectors

  $$\vec{\mathbf{c}} = \vec{\mathbf{a}} + \vec{\mathbf{b}} \qquad \Leftrightarrow \qquad \mathbf{c} = \begin{bmatrix} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{bmatrix} = \mathbf{a} + \mathbf{b}$$

Revisiting the Basic Operations
[An algebraic perspective, Cntd.]

- Based on conclusions drawn in slide "Geometric Vectors and RFs: Revisiting the Basic Operations" it's easy to see that:

  - If you take an inner product of two geometric vectors you get the same results if you compute the dot product of their algebraic counterparts

  $$c = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_x b_x + a_y b_y + a_z b_z \qquad \Leftrightarrow \qquad c = \mathbf{a}^T \mathbf{b}$$
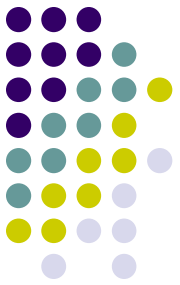
  - Dealing with the outer product of two geometric vectors is slightly less intuitive
    - Requires the concept of "cross product matrix" of a given algebraic vector **a**
      - A 3 X 3 matrix defined as :

  $$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \quad \mapsto \quad \tilde{\mathbf{a}} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

  Note the slight inconsistency: I promised I'd have all the matrices in this class in bold upper case. This is the only exception.

Revisiting the Basic Operations
[An algebraic perspective, Cntd.]

- Based on conclusions drawn in slide "Geometric Vectors: Revisiting the Basic Operations" it's easy to see that:

  - If you take the outer product of two geometric vectors, then the algebraic vector representation of the result is obtained by left multiplying the second vector by the cross product matrix of the first vector:

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = (a_y b_z - a_z b_y)\vec{\mathbf{i}} + (a_z b_x - a_x b_z)\vec{\mathbf{j}} + (a_x b_y - a_y b_x)\vec{\mathbf{k}} \quad \mapsto \quad \tilde{\mathbf{a}} \cdot \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

  - Note that the cross product matrix of a vector is a skew-symmetric matrix:

$$\tilde{\mathbf{a}}^T = -\tilde{\mathbf{a}}$$

Reference Frames:
Nomenclature & Notation

- G-RF: Global Reference Frame (the "world" reference frame)
  - This RF is unique
  - This RF is fixed; that is, its location & orientation don't change in time

- L-RF: Local Reference Frame
  - It typically represents a RF that is *rigidly* attached to a moving rigid body
  - Notation used
    - An algebraic vector represented in an L-RF has either a prime, as in s', or it has an overbar, like in $\bar{s}$
    - The book *always* uses a prime, I will use both of these notations

- A-RF: Arbitrary Reference Frame
  - Notation used: See "Notation used" for L-RF

Differentiation of Vectors

(pp.315, Haug book)
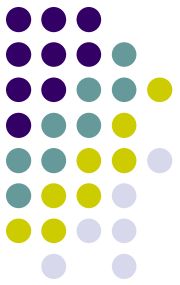
- Assumption: for the sake of this discussion on vector differentiation, the geometric vectors are assumed to be represented in a G-RF. Therefore:

$$\dot{\vec{\mathbf{i}}} = \dot{\vec{\mathbf{j}}} = \dot{\vec{\mathbf{k}}} = \mathbf{0}$$
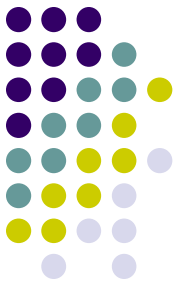
- Due to the assumption above, one has:

$$
\begin{aligned}
\dot{\vec{\mathbf{a}}} &\equiv \frac{d}{dt}\vec{\mathbf{a}}(t) = \frac{d}{dt}[a_x(t)\vec{\mathbf{i}} + a_y(t)\vec{\mathbf{j}} + a_z(t)\vec{\mathbf{k}}] \\
&= \left[\frac{d}{dt}a_x(t)\right]\vec{\mathbf{i}} + \left[\frac{d}{dt}a_y(t)\right]\vec{\mathbf{j}} + \left[\frac{d}{dt}a_z(t)\right]\vec{\mathbf{k}} \\
&= \dot{a}_x(t)\vec{\mathbf{i}} + \dot{a}_y(t)\vec{\mathbf{j}} + \dot{a}_z(t)\vec{\mathbf{k}}
\end{aligned}
$$

- Therefore, the algebraic representation of the derivative of $\vec{a}$ is

$$
\dot{\mathbf{a}} \equiv \frac{d}{dt}\mathbf{a}(t) = \left[\frac{d}{dt}a_x(t), \frac{d}{dt}a_y(t), \frac{d}{dt}a_z(t)\right]^T = [\dot{a}_x(t), \dot{a}_y(t), \dot{a}_z(t)]^T
$$

Differentiation of Vectors

(pp.315)

- Similarly, by taking one more time derivative, it is easy to see that the second time derivative of a geometric vector has the following algebraic vector representation
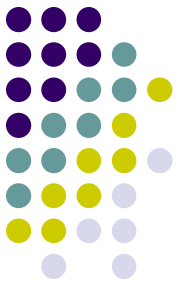
$$\ddot{\mathbf{a}} \equiv \frac{d}{dt}(\dot{\mathbf{a}}(t)) = \left[\frac{d^2}{dt^2}a_x(t), \frac{d^2}{dt^2}a_y(t), \frac{d^2}{dt^2}a_z(t)\right]^T = [\ddot{a}_x(t), \ddot{a}_y(t), \ddot{a}_z(t)]^T$$

- Likewise, consider the only operation introduced so far involving two geometric vectors that leads to a real number: the inner product

$$\frac{d}{dt}[\vec{\mathbf{a}}(t) \cdot \vec{\mathbf{b}}(t)] = \frac{d}{dt}[a_x(t)b_x(t) + a_y(t)b_y(t) + a_z(t)b_z(t)] = \frac{d}{dt}[\mathbf{a}^T(t) \cdot \mathbf{b}(t)]$$

Differentiation of Vectors
(pp.315)

- The concluding remark is that as long as we are working in a G-RF, the time derivative of a geometric vector has an algebraic representation that comes in line with our expectations. Specifically:
  - Simply take the time derivative of the components of the algebraic representation

- This means that the time derivative of any basic operation that involves two geometric vectors to produce a third one (scaling, summation, outer product) boils down to taking the time derivative of the algebraic representation of the third geometric vector

- Note that we just saw that this extends also to the inner product, so we covered all the basic operations of interest

- It becomes apparent that I need to know how to take time derivative of operations that involve algebraic vectors

Differentiation of Algebraic Vectors: Rules

- Assume that $\alpha \in \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^3$, $\mathbf{b} \in \mathbb{R}^3$ are all functions of time. Then the following hold (HOMEWORK):

$$\frac{d}{dt}(\mathbf{a}(t) + \mathbf{b}(t)) = \dot{\mathbf{a}} + \dot{\mathbf{b}}$$
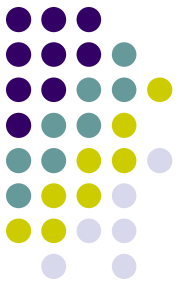
$$\frac{d}{dt}(\alpha\mathbf{a}) = \dot{\alpha}\mathbf{a} + \alpha\dot{\mathbf{a}}$$

$$\frac{d}{dt}(\mathbf{a}^T\mathbf{b}) = \dot{\mathbf{a}}^T\mathbf{b} + \mathbf{a}^T\dot{\mathbf{b}}$$

$$\frac{d}{dt}(\tilde{\mathbf{a}}\mathbf{b}) = \dot{\tilde{\mathbf{a}}}\mathbf{b} + \tilde{\mathbf{a}}\dot{\mathbf{b}}$$

Take a minute to reflect on this, specifically, on what its geometric counterpart is

$$\dot{\tilde{\mathbf{a}}} = \tilde{\dot{\mathbf{a}}}$$

14

Algebraic Vectors
and
Reference Frames

- Recall that an algebraic vector was introduced as a representation of a geometric vector in a particular reference frame (RF)

- Question: What if I now want to represent the same geometric vector in a different RF_{new} that is rotated relative to the original RF?
  - This is one of the three tricky question of Computational Dynamics

Problem Setup
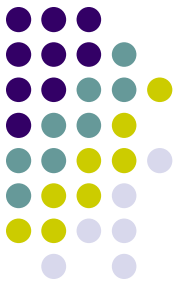
- A rigid body is provided and fixed at point O

- G-RF is attached at O

- P is some point of the body

- Geometric vector in red assumes different algebraic representations in the blue and black RFs.

- Question of Interest:

  - What's the relationship between these two representations?
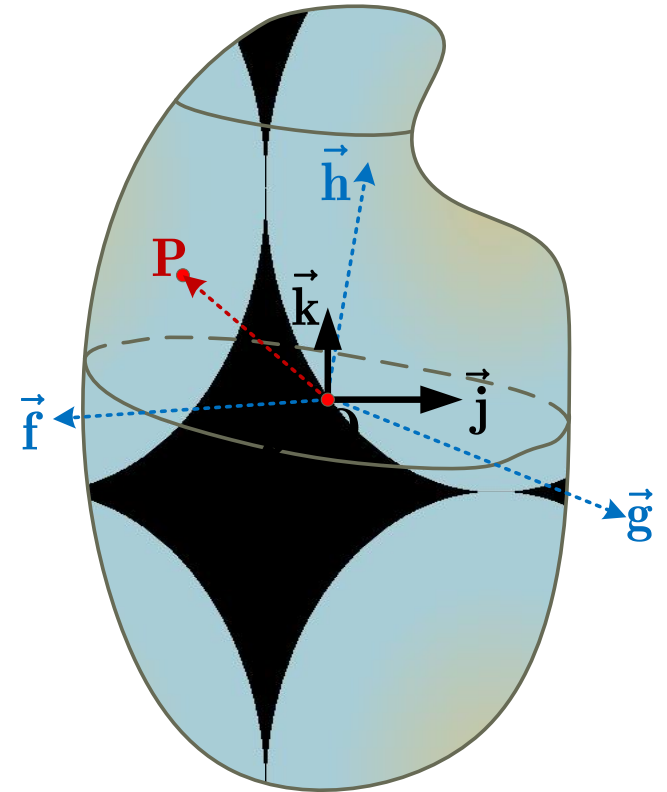
Algebraic Vectors and Reference Frames

- Let $\vec{\mathbf{s}} = \overrightarrow{\mathbf{OP}}$ be a geometric vector (see figure on previous slide)

- In the RF defined by $(\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}})$, the geometric vector $\vec{\mathbf{s}}$ is represented as

$$\vec{\mathbf{s}} = s_x \vec{\mathbf{i}} + s_y \vec{\mathbf{j}} + s_z \vec{\mathbf{k}}$$

- If I consider a different RF defined by $(\vec{\mathbf{f}}, \vec{\mathbf{g}}, \vec{\mathbf{h}})$, the geometric vector $\vec{\mathbf{s}}$ is represented as

$$\vec{\mathbf{s}} = s_{\bar{x}} \vec{\mathbf{f}} + s_{\bar{y}} \vec{\mathbf{g}} + s_{\bar{z}} \vec{\mathbf{h}}$$

- The QUESTION: how are (s_x, s_y, s_z) and $(s_{\bar{x}}, s_{\bar{y}}, s_{\bar{z}})$ related?

- NOTE: The vectors $(\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}})$ define the global ('world') RF, and therefore

$$\dot{\vec{\mathbf{i}}} = \dot{\vec{\mathbf{j}}} = \dot{\vec{\mathbf{k}}} = \mathbf{0}$$

Relationship Between ARF Vectors and GRF Vectors

$$\overrightarrow{\mathbf{f}} = a_{11}\overrightarrow{\mathbf{i}} + a_{21}\overrightarrow{\mathbf{j}} + a_{31}\overrightarrow{\mathbf{k}}$$

$$\overrightarrow{\mathbf{g}} = a_{12}\overrightarrow{\mathbf{i}} + a_{22}\overrightarrow{\mathbf{j}} + a_{32}\overrightarrow{\mathbf{k}}$$

$$\overrightarrow{\mathbf{h}} = a_{13}\overrightarrow{\mathbf{i}} + a_{23}\overrightarrow{\mathbf{j}} + a_{33}\overrightarrow{\mathbf{k}}$$

$$\mathbf{f} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \qquad \mathbf{h} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$
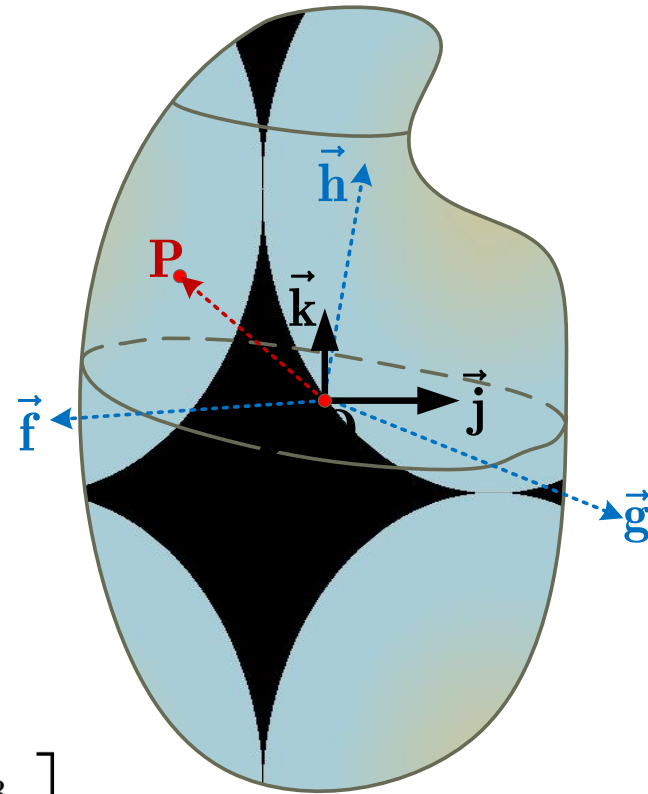
$$a_{11} = \overrightarrow{\mathbf{i}} \cdot \overrightarrow{\mathbf{f}} = \cos\theta(\overrightarrow{\mathbf{i}}, \overrightarrow{\mathbf{f}})$$
$$a_{12} = \overrightarrow{\mathbf{i}} \cdot \overrightarrow{\mathbf{g}} = \cos\theta(\overrightarrow{\mathbf{i}}, \overrightarrow{\mathbf{g}})$$
$$a_{13} = \overrightarrow{\mathbf{i}} \cdot \overrightarrow{\mathbf{h}} = \cos\theta(\overrightarrow{\mathbf{i}}, \overrightarrow{\mathbf{h}})$$

$$a_{21} = \overrightarrow{\mathbf{j}} \cdot \overrightarrow{\mathbf{f}} = \cos\theta(\overrightarrow{\mathbf{j}}, \overrightarrow{\mathbf{f}})$$
$$a_{22} = \overrightarrow{\mathbf{j}} \cdot \overrightarrow{\mathbf{g}} = \cos\theta(\overrightarrow{\mathbf{j}}, \overrightarrow{\mathbf{g}})$$
$$a_{23} = \overrightarrow{\mathbf{j}} \cdot \overrightarrow{\mathbf{h}} = \cos\theta(\overrightarrow{\mathbf{j}}, \overrightarrow{\mathbf{h}})$$

$$a_{31} = \overrightarrow{\mathbf{k}} \cdot \overrightarrow{\mathbf{f}} = \cos\theta(\overrightarrow{\mathbf{k}}, \overrightarrow{\mathbf{f}})$$
$$a_{32} = \overrightarrow{\mathbf{k}} \cdot \overrightarrow{\mathbf{g}} = \cos\theta(\overrightarrow{\mathbf{k}}, \overrightarrow{\mathbf{g}})$$
$$a_{33} = \overrightarrow{\mathbf{k}} \cdot \overrightarrow{\mathbf{h}} = \cos\theta(\overrightarrow{\mathbf{k}}, \overrightarrow{\mathbf{h}})$$

Relationship Between ARF and GRF Representations

$$\begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} s_{\bar{x}} \\ s_{\bar{y}} \\ s_{\bar{z}} \end{bmatrix}$$
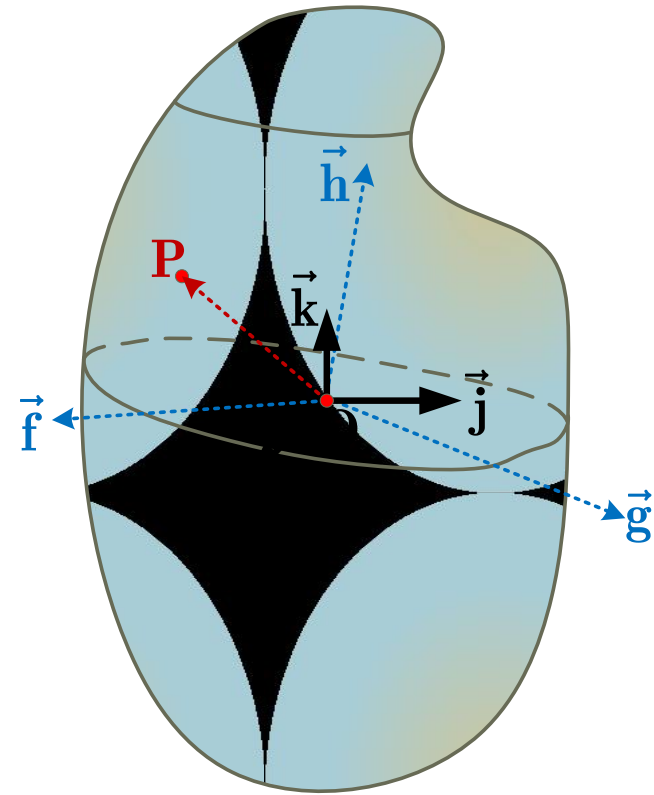
$$\mathbf{s} = \mathbf{A}\bar{\mathbf{s}}$$

This is important (see pp. 321)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{f} & \mathbf{g} & \mathbf{h} \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \qquad \mathbf{g} = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \qquad \mathbf{h} = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

19

Algebraic Vectors and Reference Frames

- Representing the same geometric vector in a different RF leads to the important concept of Rotation Matrix **A**:

  - Getting the new coordinates, that is, representation of the **<u>same</u>** geometric vector in the new RF is as simple as multiplying the coordinates by the rotation matrix **A**:

$$\mathbf{s} = \mathbf{A}\bar{\mathbf{s}}$$

- NOTE 1: what is changed is the RF used for representing the vector, and not the underlying geometric vector

- NOTE 2: rotation matrix **A** is sometimes called "orientation matrix"

On the Orthonormality of A
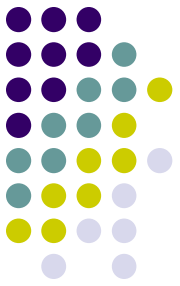
- Recall that $\vec{\mathbf{f}}$, $\vec{\mathbf{g}}$, and $\vec{\mathbf{h}}$ are mutually orthogonal

- Recall that $\vec{\mathbf{f}}$, $\vec{\mathbf{g}}$, and $\vec{\mathbf{h}}$ are are unit vectors

- Therefore, the following hold:

$$\mathbf{f}^T\mathbf{f} = \mathbf{g}^T\mathbf{g} = \mathbf{h}^T\mathbf{h} = 1$$

$$\mathbf{f}^T\mathbf{g} = \mathbf{g}^T\mathbf{h} = \mathbf{h}^T\mathbf{f} = 0$$

- Consequently, the rotation matrix **A** is orthonormal:

$$\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T = \mathbf{I}_{3\times 3}$$

The Transformation Matrix A: Further Comments
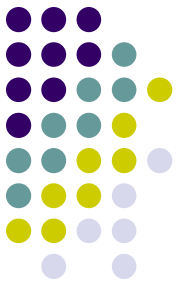
- The nine entries of matrix **A** are called direction cosines
  - The first column are the direction cosines of **f**, the second contains the direction cosines of **g**, etc.

- Found the representation in G-RF given the one in an A-RF
  - Found A-RF $\rightarrow$ G-RF

- Since **A** is orthonormal, it's easy to find the transformation in the other direction: G-RF $\rightarrow$ A-RF

$$\bar{\mathbf{s}} = \mathbf{A}^T \mathbf{s}$$

<u>Summarizing the Key Points</u>

- Linking two algebraic vector representations of the same geometric vector

$$\mathbf{s} = \mathbf{A}\bar{\mathbf{s}}$$

- Sometimes called a change of base or reference frame

- Recall that **A** (its columns) are made up of the representation of **f**, **g**, and **h** in the new RF
  - The algebraic vectors **f**, **g**, and **h** define the "old", "local", "initial" RF, that is, that reference frame in which where $\bar{\mathrm{s}}$ is expressed
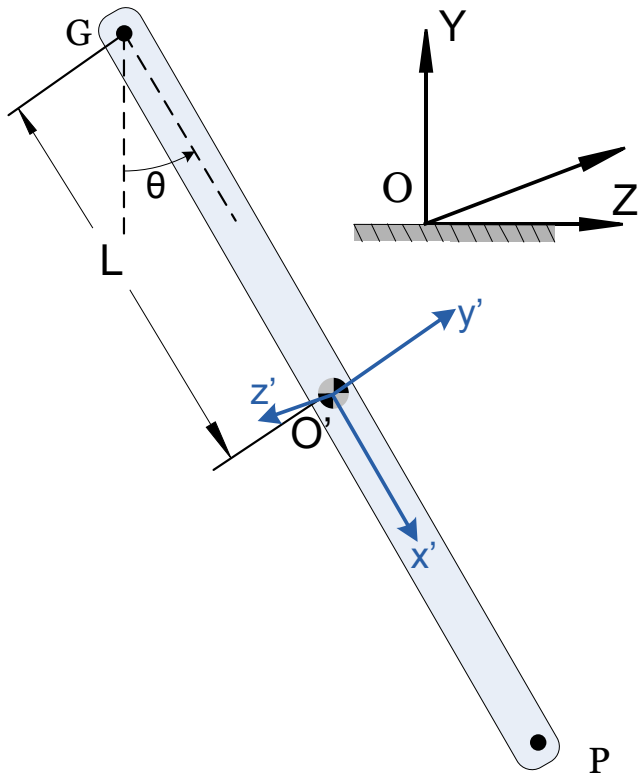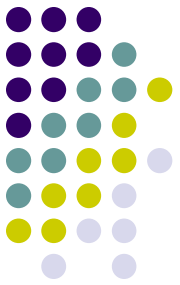
Example: Assembling Matrix A
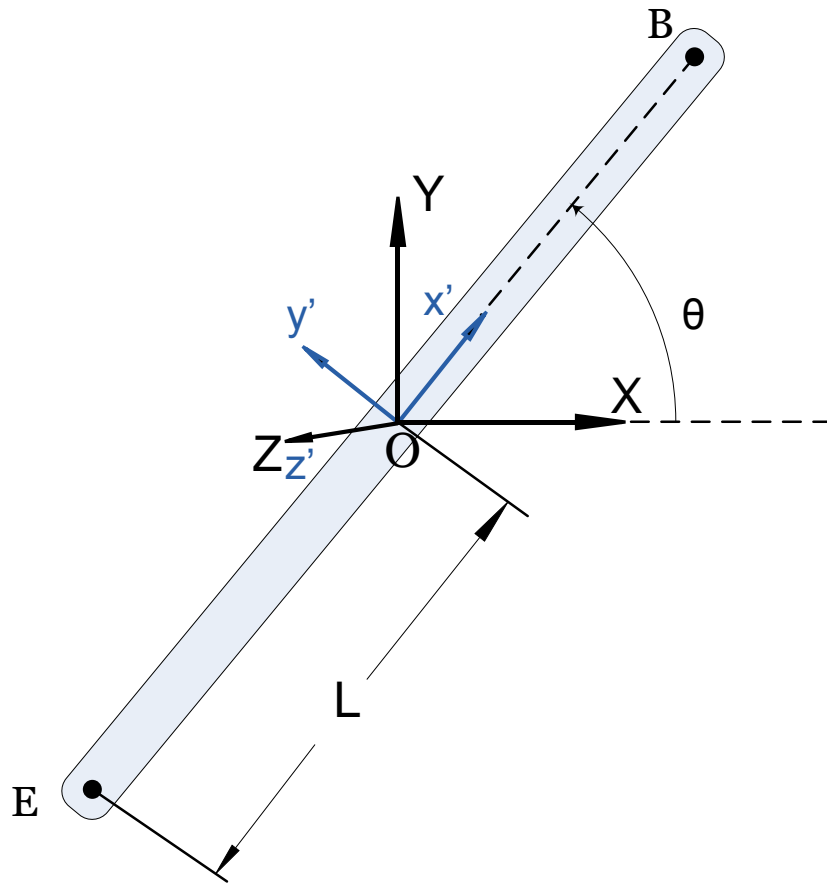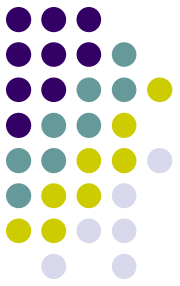
- Express the geometric vector $\overrightarrow{OB}$ in the local reference frame OX'Y'.

- Express the same geometric vector in the global reference frame OXY

- Do the same for the geometric vector $\overrightarrow{OE}$

Assembling A

- Express the geometric vector
$$\overrightarrow{O'P}$$
in the local reference frame O'X'Y'Z'.

- Express the same geometric vector in the global reference frame OXYZ

- Do the same for the geometric vector
$$\overrightarrow{O'G}$$

- Note that the plane (**O'X'Y'**) is parallel to the (**OYZ**) plane
- Note that **O** and **O'** should have been coincident; avoided to do that to prevent clutter of the figure (you should work under this assumption though)
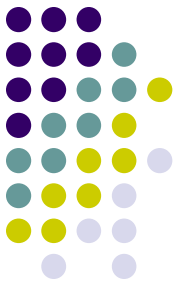
RF Change:
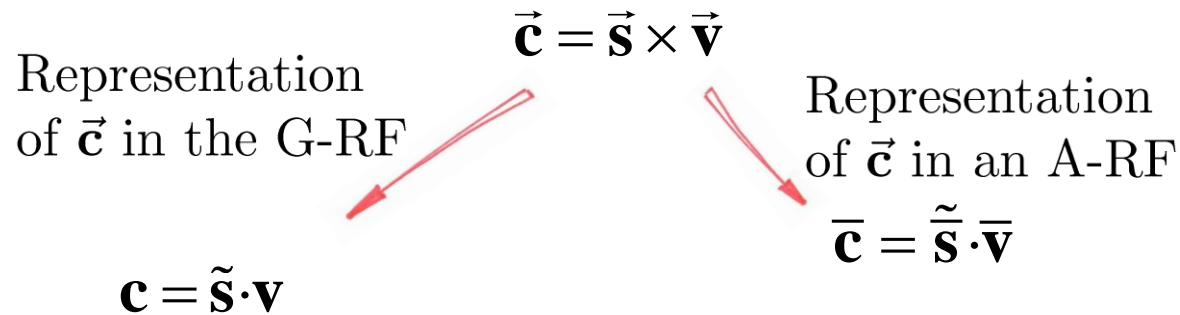# The Outer Product and Cross Product Matrix

- Problem Setup:
    - We saw how to switch between A-RF and G-RF when it comes to the algebraic representation of a geometric vector
        - Boils down to multiplication by the rotation matrix **A**

    - Recall that associated with each algebraic vector there is a cross product matrix

- **Question**: How do you have to change the *cross product matrix* when you go from an A-RF to the G-RF ?

$$\tilde{s} \quad \overset{?}{\longleftrightarrow} \quad \tilde{\tilde{s}}$$

- The geometric vector representation: I have two geometric vectors, $\vec{\mathbf{s}}, \vec{\mathbf{v}}$ and care about their outer product,

$$\vec{\mathbf{c}} = \vec{\mathbf{s}} \times \vec{\mathbf{v}}$$

Representation of $\vec{c}$ in the G-RF

Representation of $\vec{c}$ in an A-RF

$$\mathbf{c} = \tilde{\mathbf{s}} \cdot \mathbf{v}$$

$$\overline{\mathbf{c}} = \tilde{\overline{\mathbf{s}}} \cdot \overline{\mathbf{v}}$$

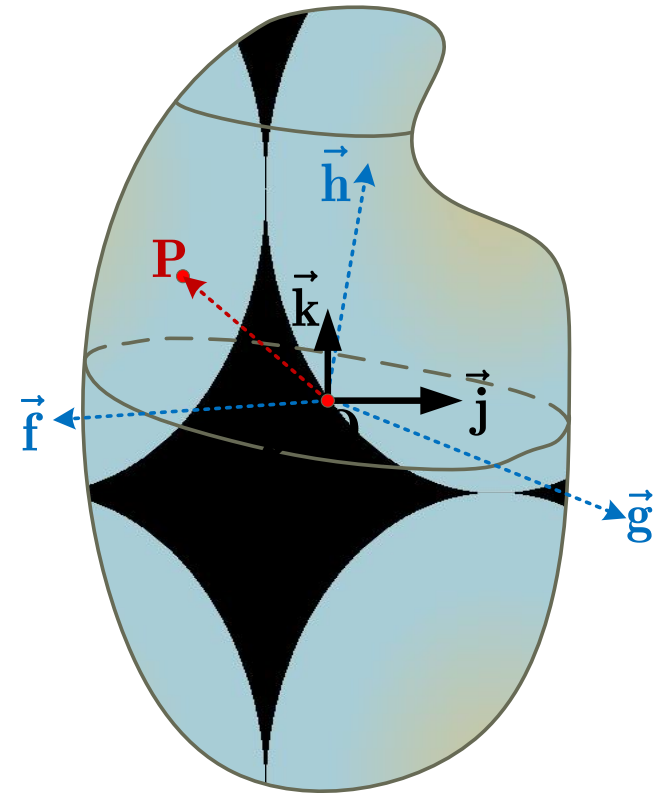$$\mathbf{c} = \mathbf{A}\overline{\mathbf{c}}$$

$$\Downarrow$$

$$\tilde{\mathbf{s}} = \widetilde{(\mathbf{A}\overline{\mathbf{s}})} = \mathbf{A}\tilde{\overline{\mathbf{s}}}\mathbf{A}^T$$

$$\tilde{\overline{\mathbf{s}}} = \widetilde{(\mathbf{A}^T\mathbf{s})} = \mathbf{A}^T\tilde{\mathbf{s}}\mathbf{A}$$

Angular Velocity: Intro
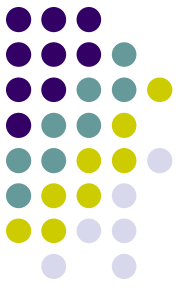
- The motivating question: How does the orientation matrix **A** change in time?

- Matrix **A** changes whenever the representation of **f**, **g**, or **h** in the G-RF changes

- Example: Assume blue RF is attached to the body (the L-RF) and the black is the G-RF, fixed to ground
  - A ball joint (spherical joint) present between the body and ground at point **O**

Angular Velocity: Intro

- Note that if **f**, **g**, and **h** change, then a_{11}, a_{21},…, a_{33} change
  - In other words, $\mathbf{A} = \mathbf{A}(t)$

- Recall how the orientation matrix **A** was defined:

$$\vec{\mathbf{f}}(t) = a_{11}(t)\,\vec{\mathbf{i}} + a_{21}(t)\,\vec{\mathbf{j}} + a_{31}(t)\,\vec{\mathbf{k}}$$

$$\vec{\mathbf{g}}(t) = a_{12}(t)\,\vec{\mathbf{i}} + a_{22}(t)\,\vec{\mathbf{j}} + a_{32}(t)\,\vec{\mathbf{k}}$$

$$\vec{\mathbf{h}}(t) = a_{13}(t)\,\vec{\mathbf{i}} + a_{23}(t)\,\vec{\mathbf{j}} + a_{33}(t)\,\vec{\mathbf{k}}$$

Note that $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, $\vec{\mathbf{k}}$ do not depend on time (G-RF is fixed).

$$\mathbf{f}(t) = \begin{bmatrix} a_{11}(t) \\ a_{21}(t) \\ a_{31}(t) \end{bmatrix} \qquad \mathbf{g}(t) = \begin{bmatrix} a_{12}(t) \\ a_{22}(t) \\ a_{32}(t) \end{bmatrix} \qquad \mathbf{h}(t) = \begin{bmatrix} a_{13}(t) \\ a_{23}(t) \\ a_{33}(t) \end{bmatrix}$$

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{f}(t) & \mathbf{g}(t) & \mathbf{h}(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{bmatrix}$$

Angular Velocity: Getting There…

- Recall that $\mathbf{A}\mathbf{A}^T=\mathbf{I}_3$. Take time derivative to get:

$$\dot{\mathbf{A}}\mathbf{A}^T + \mathbf{A}\dot{\mathbf{A}}^T = \mathbf{0}_{3\times 3} \qquad \Rightarrow \qquad \dot{\mathbf{A}}\mathbf{A}^T = -\mathbf{A}\dot{\mathbf{A}}^T$$

- Notice the following:

  - The matrix $\dot{\mathbf{A}}\mathbf{A}^T$ is a 3×3 matrix

  - The matrix $\dot{\mathbf{A}}\mathbf{A}^T$ is skew-symmetric

- CONCLUSION: there must be a vector, ω, whose cross product matrix is equal to the 3×3 skew symmetric matrix $\dot{\mathbf{A}}\mathbf{A}^T$:

$$\tilde{\omega} = \dot{\mathbf{A}}\mathbf{A}^T$$

- This vector ω is called the angular velocity of the L-RF with respect to the G-RF.

30