$$\begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} & 0 \\ A_{10} & A_{11} & hE \\ h\mu & -hE^T & 0 \end{bmatrix} \begin{bmatrix} f_n \\ f_d \\ \lambda \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} f_n \\ f_d \\ \lambda \end{bmatrix} \geq 0$$

$$\begin{aligned}
&A_{00} f_n + A_{01} f_d + b_0 \geq 0 \\
&A_{10} f_n + A_{11} f_d + hE\lambda + b_1 \geq 0 \\
&h(\mu f_n - f_d) \geq 0 \\
&f_n(A_{00} f_n + A_{01} f_d + b_0) = 0 \\
&f_d(A_{10} f_n + A_{11} f_d + hE\lambda + b_1) = 0 \\
&\lambda h(\mu f_n - f_d) = 0
\end{aligned}$$

Here are a few concerns:
1. Can $h$ be arbitraily small?
No. We can think of h as 0. Then the whole LCP becomes

$$\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} \begin{bmatrix} f_n \\ f_d \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

In other words, the smaller h is, the looser the friciton cone constraint is. (Recalling the thrid row of LCP is about friction cone constraint $\mu f_n - f_d \geq 0$). If h is 0, the dimension of LCP degenerates and it become easier to solve. This is the reason why after scaling, the Lemke failure rate improves. (if h=1e-3, failure rate is 97/(2596-97)=.038815526; h=1e-4, Lemke failure rate is 52/(2551-52)=.020808323; h=1e-6, failure rate is 0). To keep a feasible friction cone constraint, a critical value of $h$ needs to be spotted. It can be found using static slopping test. Setting the angle as (45+1e-4), non-scaling method will have the cube sliding along the slope while scaling method (1e-6) will make it rolling forward. (The critical value is **4e-3.** If h is less than it, it starts rolling behavior). The friction cone constraint can be phrased as $h(\mu f_n - f_d) \geq -1e-6$ (1e-6 is used defined zero). Supposing $h = 1e-3$, then $(\mu f_n - f_d) \geq -1e-3$ The direct effect can be regarded that we have slightly larger $\mu$. In this sense, the rolling behavior is self-explained. At that angle, cube is supposed not to sliding. But if the angle is greater than 45, the gravity force deviate from diagonal and will make cube roll.

2. Does scaling affect pattern?

Scaling makes no change to $f_n$ and $f_d$. The only diff would be $\lambda$. After scaling, it gets $\frac{1}{h}$ times larger. Here are some simulation data for your consideration of the error it could make.

Size = 20, 2 contact points

Comparing for scaling effect

Before 0.00695303573089921418680 0.006953035730899323474 4 0 0.0010565512891130561697 0 0 0 0 0 0.0010565512891130008754 0.0014941891624063868838 0 0 0 0 0 0 0 0

After 0.006953035730899443170 3 0.006953035730898888058 8 0 0.0010565512891128605796 0 0 0 0 0 0.0010565512891133543253 0.0014941891624061763317 0 0 0 0 0 0 0 0

Comparing for scaling effect

Before 0.22085897063358178594 0.22085897063360832027 0 0.042040598555265977498 0 0 0 0 0 0.042040598555244189372 0.059454384647137363 54 0 0 0 0 0 0 0 0

After 0.22085897063359738457 0.22085897063359782866 0 0.042040598555260620672 0 0 0 0 0 0.042040598555260211278 0.059454384647132298147 0 0 0 0 0 0 0 0

Size = 40, 4 contact points.

Comparing for scaling effect

Before 0 0 0.0049049999999978641935 0.0049049999999974478598 -8.1817263274544688 91e-20 0 0 0 0 0 0 0 -1.6451063454269380151e-19 0 0 0 0 0 0 0 0.0049049999999978650608 0 0 0 0 0 0 0 0.0049049999999974487272 0 0 0 0 0 0 0 0.0027403200000000096231 0.0027403200000000113579 0.0027403200000000109242 0.0027403200000000096231

After 0 0 0.0049049999999976959253 0.0049049999999966195294 -2.8285966947455886656e-17 0 0 0 0 0 0 0 -5.3627388793833460519e-17 0 0 0 0 0 0 0 0.0049049999999977774573 0 0 0 0 0 0 0 0.0049049999999965978453 0 0 0 0 0 0 0 0.68508000000000235374 0.68508000000000279783 0.68508000000000224272 0.68508000000000290886

The patterns:

- fn=0, fd=0, lambda=0 contact break
- fn=0, fd=0, lambda>0 it has relative tangential velocities but no friction

Given the specific structure of A, both patterns are feasible for one single A. In other words, even scaling change the pattern, it doesn't matter.

- fn=0, fd>0, lambda=0
- fn=0, fd>0, lambda>0

Not possible at all

- fn>0, fd=0, lambda=0 static, no relative tangential velocities, no relative tangential acc
- fn>0, fd=0, lambda>0

This case could be the most dangerous one, but further analysis shows that it is safe. If $f_d$ is exactly 0, then it means no relative movement, therefore $\lambda = 0$. Simulation also reinforces this point. In conclusion, scaling doesn't have an impact.

Vector z 0.0049050000000000022402 0 0.0049050000000000013728 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

- fn>0, fd>0, lambda=0 static friction, no relative tangential velocities, relative tangential acc, in other words, pushing force is not enough to push the cube
- fn>0, fd>0, lambda>0 slide

This is the friction cone case, which is also dangerous. I verify this case by simulation because I want to see how large lambda could be considering all kinds of error. If 9.79N force is applied a 1kg cube, the cube will stay and there is a directional friction force. The lambda (after scaling) is around 1e-19, which is far away from the user-defined zero.

Vector z 0 1.0000000000002777392e-05 0.0048949999999999991782 0.0049050000000000022402 -1.4656042820026059291e-34 0 0 0 0 0 0 2.7864848896276937205e-18 0 0 0 0 0 0 0 0.0048949999999999983108 0 0 0 0 0 0 0.0048950000000000000455 0 0 0 0 0 0 0 2.1616814368153648224e-19 0 -5.1584586491299776805e-20 0

In conclusion, scaling doesn't affect pattern.

3. Scaling and Lemke tolerance

h is 4e-3, which is still a large number compared to Lemke piv_tol 1e-12. In this sense, after scaling, it shouldn't change $f_n$ and $f_d$ too much. BTW, it makes entries of A almost the same order of maginitude, so that Lemke becomes more stable.