Discrete Mathematics VE203
First Midterm - Fall 2018

Q1. Note that

$$Q \Rightarrow (R \wedge S) \equiv \neg Q \vee (R \wedge S) \qquad (\dagger)$$
$$\equiv (\neg Q \vee R) \wedge (\neg Q \vee S)$$

Note also that

$$\begin{array}{l} P \vee Q \\ \neg Q \vee R \\ \hline P \vee R \end{array} \qquad (\ddagger)$$

P	Q	R	$P \vee Q$	$\neg Q \vee R$	$P \vee R$
1	1	1	1	1	1
1	1	0	1	0	1
1	0	1	1	1	1
1	0	0	1	1	1
0	1	1	1	1	1
0	1	0	1	0	0
0	0	1	0	1	1
0	0	0	0	1	0

1. $Q \Rightarrow (R \wedge S)$ premise
2. $\neg Q \vee R$ by $(\dagger)$
3. $P \vee Q$ premise
4. $P \vee R$ by $(\ddagger)$
5. $P \vee R \Rightarrow U$ premise
6. U by Modus Ponens.

This is a valid argument.

$Q \Rightarrow (R \wedge S) \equiv \neg Q \vee (R \wedge S)$

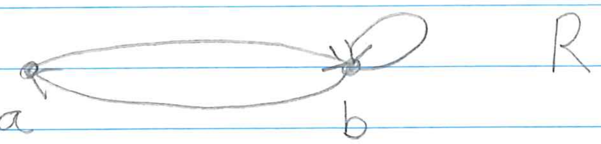
So, the argument is sound when Q is false, P is true and R and S are any of the four combinations of truth values.

Or, Q is true, R and S are true and P is either true or false.

This is 6 combinations of truth values.

In both cases U must be true.

ii) This should do it:



$R(a,b) \wedge R(b,b)$

$R(a,b) \wedge R(b,a)$

$\forall z\,(R(a,b) \wedge R(b,z))$

$\exists y\,\forall z\,(R(a,y) \wedge R(y,z))$

$R(b,b) \wedge R(b,b)$

$R(b,b) \wedge R(b,a)$

$\forall z\,(R(b,b) \wedge R(b,z))$

$\exists y\,\forall z\,(R(b,y) \wedge R(y,z))$

So, $\forall x\,\exists y\,\forall z\,(R(x,y) \wedge R(y,z))$

But $\neg R(a,a)$, so

$\forall u\,\forall v\, R(u,v)$ is false.

Q2. i) Let A, B, C and D be sets.

Let $\langle x,y\rangle \in (A\times B)\cup(C\times D)$

So, $\langle x,y\rangle \in A\times B$ or $\langle x,y\rangle \in C\times D$

So, $x\in A$ and $y\in B$ or

$x\in C$ and $y\in D$

So, $x\in A\cup C$ and $y\in B\cup D$

Therefore $\langle x,y\rangle \in (A\cup C)\times(B\cup D)$

This shows $(A\times B)\cup(C\times D)\subseteq(A\cup C)\times(B\cup D)$

ii) Now, $(\emptyset\times\{\emptyset\})\cup(\{\emptyset\}\times\emptyset) = \emptyset$

But $\emptyset\cup\{\emptyset\} = \{\emptyset\}$.

and $\{\emptyset\}\times\{\emptyset\} = \{\langle\emptyset,\emptyset\rangle\}$.

Q3. i) $\phi(A, B, C) \equiv$

$\neg((A \wedge \neg B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge C) \vee (\neg A \wedge \neg B \wedge \neg C))$

ii) $\phi(A, B, C) \equiv$

$\neg(A \wedge \neg B \wedge \neg C) \wedge \neg(\neg A \wedge \neg B \wedge C)$

$\wedge \neg(\neg A \wedge \neg B \wedge \neg C)$

Now, $\neg(A \wedge \neg B \wedge \neg C) \equiv \neg A \vee B \vee C$

$\equiv \neg A \vee (B \vee C)$

$\equiv \neg A \vee (\neg B \Rightarrow C)$

$\equiv A \Rightarrow (\neg B \Rightarrow C)$

$\neg(\neg A \wedge \neg B \wedge C) \equiv A \vee B \vee \neg C$

$\equiv A \vee (B \vee \neg C)$

$\equiv A \vee (C \Rightarrow B)$

$\equiv \neg A \Rightarrow (C \Rightarrow B)$

$\neg(\neg A \wedge \neg B \wedge \neg C) \equiv A \vee B \vee C$

$\equiv A \vee (B \vee C) \equiv A \vee (\neg B \Rightarrow C)$

$\equiv \neg A \Rightarrow (\neg B \Rightarrow C).$

Now, $P \wedge Q \equiv \neg(\neg P \vee \neg Q)$

$\equiv \neg(P \Rightarrow \neg Q)$

So, $\phi(A, B, C) \equiv$

$(A \Rightarrow (\neg B \Rightarrow C)) \wedge$

$((\neg A \Rightarrow (C \Rightarrow B) \wedge (\neg A \Rightarrow (\neg B \Rightarrow C)))$

$\equiv (A \Rightarrow (\neg B \Rightarrow C)) \wedge$

$\neg((\neg A \Rightarrow (C \Rightarrow B)) \Rightarrow \neg(\neg A \Rightarrow (\neg B \Rightarrow C)))$

$\equiv (A \Rightarrow (\neg B \Rightarrow C)) \Rightarrow$

$((\neg A \Rightarrow (C \Rightarrow B)) \Rightarrow \neg(\neg A \Rightarrow (\neg B \Rightarrow C)))$

iii)

A	B	$\phi(A, A, B)$	$\phi(A, B, A)$
1	1	1	1
1	0	1	1
0	1	$\ominus$	1
0	0	$\ominus$	0

$\phi(A, B, B)$
1
$\ominus$
1
0

So, $A \Rightarrow B \equiv \neg A \vee B$

$$\equiv \phi(\neg A, B, \neg A)$$

And, $A \vee B \equiv \phi(A, B, A)$.

iv) No, the argument is not valid.
When A is false, and B and C are both true:

$\phi(A,B,C)$, $\phi(C,A,B)$ and $\phi(B,C,A)$ are true, and A is false.

Q4. i) Prove that if $|A| \geqslant 2$ and $|A \times A| = |A|$, then A is Dedekind infinite.

Assume that $|A| \geqslant 2$ and $|A| = |A \times A|$.

Let $g : A \times A \longrightarrow A$ be a bijection.

Let $a, b \in A$ with $a \neq b$.

Define $f : A \longrightarrow A$ by: for all $x \in A$,

$$f(x) = g((x, x))$$

Now, f is an injection and $g((a, b)) \notin ran(f)$, so f is not a surjection.

Therefore A is Dedekind infinite.

ii) Let $n \in \mathbb{N}$ with $n \geqslant 2$. Let $X = \{k \in \mathbb{N} \mid k < n\}$.

Let $R \subseteq X \times X$ be irreflexive and symmetric.

Define $f: X \longrightarrow \mathbb{N}$ by: for all $x \in X$,

$$f(x) = |\{y \in \mathbb{N} \mid (y,x) \in R\}|$$

Note that $|X| = n$ and

$$\{y \in \mathbb{N} \mid (y,x) \in R\} \subseteq X.$$

Therefore, let $Y = \{i \in \mathbb{N} \mid i \leq n\}$.

We have $ran(f) \subseteq Y$.

For all $x \in X$, $x \notin \{y \in \mathbb{N} \mid (y,x) \in R\}$, because R is irreflexive.

Therefore $f(x) < n$ and $ran(f) \subseteq X$.

Claim: $0 \notin ran(f)$ or $n-1 \notin ran(f)$.

Proof: Suppose $n-1 \in ran(f)$.

Therefore, there exists $x \in X$ such that

$$f(x) = |\{y \in \mathbb{N} \mid (y,x) \in R\}| = n-1$$

Let $A = \{y \in \mathbb{N} \mid (y,x) \in R\}$.

Now, $A \subseteq X$.

And, $x \notin A$ because R is irreflexive.

Therefore, for all $y \in X$ with $y \neq x$, $y \in A$.

Suppose $0 \in ran(f)$.

Let $z \in X$ such that

$$f(z) = |\{y \in \mathbb{N} \mid (y,z) \in R\}| = 0$$

Since $n-1 > 0$, $z \neq x$ and so

$(z, x) \in R$.

But then $(x, z) \in R$ by symmetry.

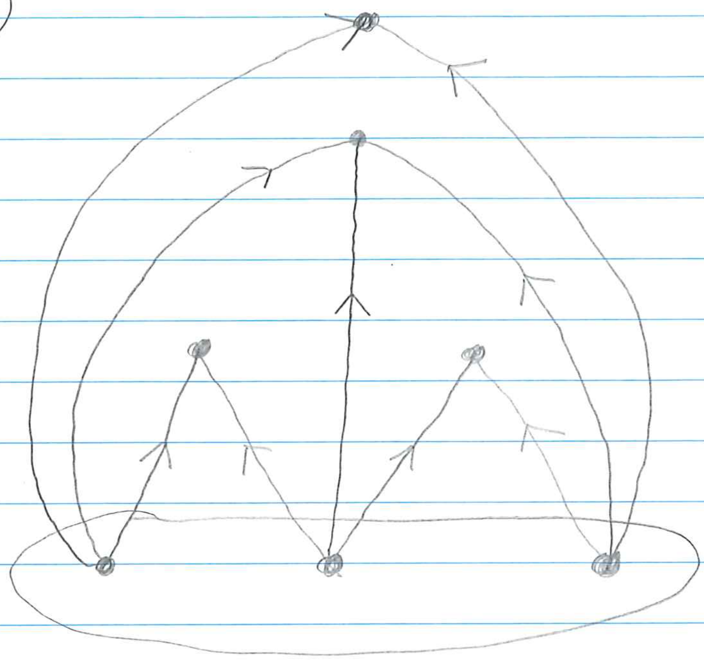
But this contradicts the fact that

$|\{y \in \mathbb{N} \mid (y,z) \in R\}| = 0$.

Therefore $0 \notin ran(f)$ or $n-1 \notin ran(f)$. and completes the proof of the claim.

Now, this shows that $f : X \to X$ is not surjective. Therefore f can not be injective, because X is not Dedekind infinite. Therefore there exists $x, y \in X$ with $x \neq y$ such that $f(x) = f(y)$.

Q5. i)



Does this do the trick?

Σ

ii) Let $(L, \leq)$ be a lattice.

Let $a, b, c \in L$.

Now, $(a \wedge b) \vee (a \wedge c) \geq (a \wedge b) \vee (a \wedge c)$

$\geq a \wedge ((a \wedge b) \vee (a \wedge c))$

Conversely, $(a \wedge b) \vee (a \wedge c) \leq (a \wedge b) \vee (a \wedge c)$.

And, $a \wedge b \leq a$ and $a \wedge c \leq a$.

So, $(a \wedge b) \vee (a \wedge c) \leq a$, because it is least upper bound.

Therefore $(a \wedge b) \vee (a \wedge c)$ is a lower bound on $\{a, (a \wedge c) \vee (a \wedge c)\}$.

So, $(a \wedge b) \vee (a \wedge c) \leqslant a \wedge ((a \wedge b) \vee (a \wedge c))$

Therefore $(a \wedge c) \vee (a \wedge c) = a \wedge ((a \wedge b) \vee (a \wedge c))$
by antisymmetry.

iii) Let $a, b, c \in L$.

Now, $a \wedge b \leqslant a$ and $a \wedge c \leqslant a$.

and $a \wedge b \leqslant b \leqslant b \vee c$

$a \wedge c \leqslant c \leqslant b \vee c$

So, both $a \wedge b$ and $a \wedge c$ are lower bounds on $\{a, b \vee c\}$.

So, $a \wedge b \leqslant a \wedge (b \vee c)$

and $a \wedge c \leqslant a \wedge (b \vee c)$

So, $a \wedge (b \vee c)$ is an upper bound on $\{a \wedge b, a \wedge c\}$.

So, $(a \wedge b) \vee (a \wedge c) \leqslant a \wedge (b \vee c)$.

Q6. i) Prove that if $f: L_1 \longrightarrow L_2$ is a lattice homomorphism, then f is order preserving.

Claim: Let $(L, \leq)$ be a lattice. For all $a, b \in L$, $a \leq b$ iff $a \vee b = b$.

Prf of claim: Let $a, b \in L$.

If $a \leq b$, then $a \vee b = b$.

Conversely, if $a \vee b = b$, then

$$a \leq a \vee b = b.$$

$\square$ - Claim.

Let $f: L_1 \longrightarrow L_2$ be a lattice homomorphism.

Let $x, y \in L_1$ be such that $x \leq_1 y$.

Therefore $x \vee y = y$

And, $f(x \vee y) = \bigvee f''\{x, y\}$

$$= f(x) \vee f(y)$$

So, $f(y) = f(x) \vee f(y)$ and

$$f(x) \leq_2 f(y).$$

Therefore f is order-preserving.

ii) Define $\leq^* \subseteq \mathbb{N} \times \mathbb{N}$ by

$\leq^* = \{(n,m) \mid n \neq 0 \land n \leq m\}$

$\cup \{(n,0) \mid n \in \mathbb{N}\}$

It should be clear that $\leq^*$ is a linear order on $\mathbb{N}$ and so $(\mathbb{N}, \leq^*)$ is a lattice.

Define $f : (\mathbb{N}, |) \longrightarrow (\mathbb{N}, \leq^*)$ by:

for all $n \in \mathbb{N}$, $f(n) = n$.

Let $n, m \in \mathbb{N}$ with $n | m$.

If $m \neq 0$, then $n \leq m$

Therefore $n \leq^* m$.

This shows that f is order preserving.

But, $f(2 \vee 3) = f(6) = 6$

$\neq f(2) \vee f(3) = 3$.

So this is not a lattice homomorphism.

iii) Let $(L_1, \leq_1)$ and $(L_2, \leq_2)$ be lattices. Let $f: L_1 \to L_2$ be an order preserving bijection such that f^{-1} is order preserving.

Claim: $f(x \vee y) = f(x) \vee f(y)$

Proof of Claim: Let $x, y \in L_1$.

Now, $x \leq_1 x \vee y$

So, $f(x) \leq_2 f(x \vee y)$

and $y \leq_1 x \vee y$

So, $f(y) \leq_2 f(x \vee y)$

So, $f(x \vee y)$ is an upper bound on $\{f(x), f(y)\}$ and

$f(x) \vee f(y) \leq_2 f(x \vee y)$

Now, $f(x) \leq_2 f(x) \vee f(y)$

and $f(y) \leq_2 f(x) \vee f(y)$

Therefore, $x \leq_1 f^{-1}(f(x) \vee f(y))$

and $y \leq_1 f^{-1}(f(x) \vee f(y))$

So, $x \vee y \leq_1 f^{-1}(f(x) \vee f(y))$

Therefore $f(x \vee y) \leq_2 f(x) \vee f(y)$

This shows that

$$f(x \vee y) = f(x) \vee f(y).$$

$\square$ - Claim

~~For all~~

For all $X \subseteq L_1$ finite with $|X| \geq 2$ can then be proved by induction.

Q7. Let $X = \{0,1\} \times \mathbb{N}$

Define $\leq^* \subseteq X \times X$ by:

For all $(a,x), (b,y) \in X$,

$(a,x) \leq^* (b,y)$ iff $a < b$

or $a = b = 0$ and $x \leq y$

or $a = b = 1$ and $y \leq x$

This is clearly a linear ordering.

Define $f : X \longrightarrow X$ by

$f((a,x)) = (a, x+1)$

Let $(a,x), (b,y) \in X$.

Suppose $(a,x) \leq^* (b,y)$

Case I: $a < b$, then $(a, x+1) \leq^* (b, y+1)$

Case II: $a = b = 0$

Then $x+1 \leq y+1$ and $(a, x+1) \leq^* (b, y+1)$

Case III: $a = b = 1$, then $y+1 \leq x+1$ and $(a, x+1) \leq^* (b, y+1)$

f is order preserving with no f.p.

ii) See the proof of the Tarski-Knaster Theorem on slide 113 ff.

iii) Let $\mathbb{P}$ be a set and let $R \subseteq \mathbb{P} \times \mathbb{P}$

Let $\mathbb{X} = \{ S \in \mathcal{P}(\mathbb{P} \times \mathbb{P}) \mid R \subseteq S \}$.

Note that $(\mathbb{X}, \subseteq)$ is clearly a partial order - I don't feel that this needs any justification.

Let $\mathcal{A} \subseteq \mathbb{X}$.

Assume that $\mathcal{A} \neq \emptyset$.

Claim: $\bigwedge \mathcal{A} = \bigcap \mathcal{A} \in \mathbb{X}$.

Proof of Claim: Note that $\bigcap \mathcal{A} \in \mathbb{X}$

because $\mathcal{A} \neq \emptyset$ and for all $B \in \mathcal{A}$, $R \subseteq B$.

Now, for all $B \in \mathcal{A}$, $\bigcap \mathcal{A} \subseteq B$.

Therefore $\bigcap \mathcal{A}$ is a lower bound of $\mathcal{A}$.

Suppose that C is such that for all $B \in \mathcal{A}$, $C \subseteq B$. Therefore $C \subseteq \bigcap \mathcal{A}$

It follows that $\bigwedge \mathcal{A} = \bigcap \mathcal{A}$. $\square$ - Claim.

Claim: $\bigvee \mathcal{A} = \bigcup \mathcal{A} \in \mathbb{X}$

Proof of Claim: It is clear that $\bigcup \mathcal{A} \in \mathbb{X}$.

Moreover, for all $B \in \mathcal{A}$, $B \subseteq \bigcup \mathcal{A}$, so $\bigcup \mathcal{A}$ is an upper bound.

Suppose that C is such that for all $B \in \mathcal{A}$, $B \subseteq C$.

Then $\bigcup \mathcal{A} \subseteq C$.

Therefore $\bigvee \mathcal{A} = \bigcup \mathcal{A}$ $\square$-Claim.

Now, if $\mathcal{A} = \emptyset$, then $\bigwedge \mathcal{A} = P \times P$ and $\bigvee \mathcal{A} = R$.

So, $(\mathbb{X}, \subseteq)$ is a complete lattice.

iv) Define $F : X \longrightarrow X$ by

$F(S) = S \cup (S \circ S)$

Let $S' \in X$ be such that $F(S') = S'$.

Let $(x, y), (y, z) \in S'$.

Suppose $(x, z) \notin S'$.

But $(x, z) \in S' \circ S' \subseteq F(S') = S'$,

which is a contradiction.

Therefore S' is transitive.

v) Claim: F is order-preserving.

Let $S, U \in X$ with $S \subseteq U$.

Now, let $(x, y) \in S \circ S$.

Therefore, there exists $z \in X$ s.t.

$(x, z), (z, y) \in S$.

So, $(x, z), (z, y) \in U$ and $(x, y) \in U \circ U$

This shows $S \circ S \subseteq U \circ U$

And $F(S) \subseteq F(U)$

Now, by the Tarski-Knaster Theorem, F has a least f.p. and this least f.p. is the $\subseteq$-least transitive relation required.