Analogue of Fomin-Stanley Algebra on Bumpless Pipedreams

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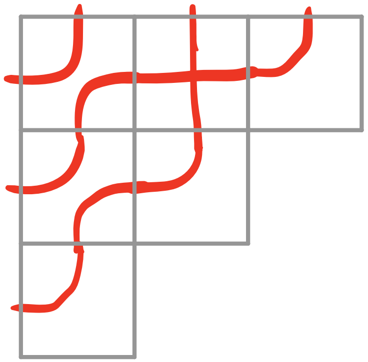
Pipedream (PD)



Fomin-Stanley Algebra

$$(1+x_1u_2)(1+x_1u_1)(1+x_2u_2)$$

Lenart-Sottile Chain

3 2 1

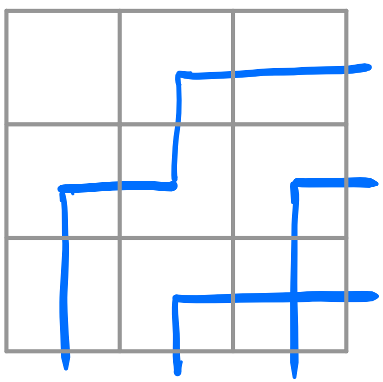
$\uparrow$ 2

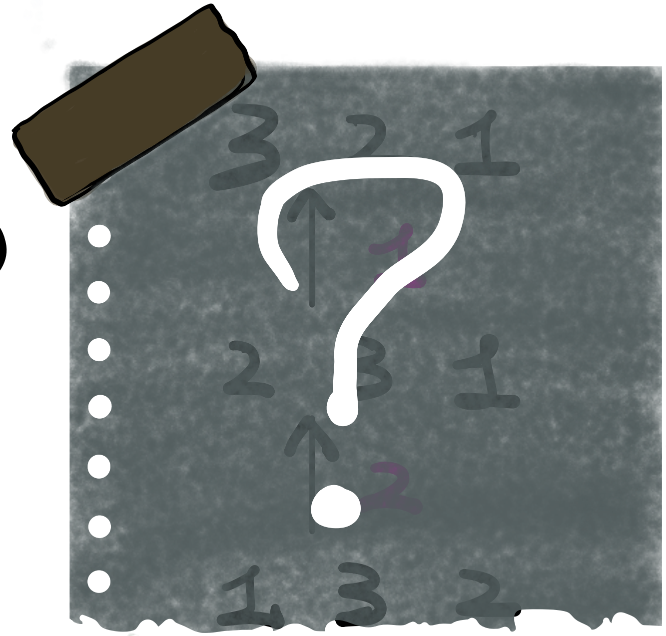
3 1 2

$\uparrow$ 1

1 3 2

Bumpless Pipedream (BPD)





Schubert Polynomial $\mathfrak{S}_w$.

Define ∂_i on polynomials : $\partial_i(f) = \dfrac{f - s_i f}{x_i - x_{i+1}}$

$$\partial_1 (x_1^2 x_2) = \frac{x_1^2 x_2 - x_1 x_2^2}{x_1 - x_2} = x_1 x_2 \,, \quad \partial_1 (x_1 x_2) = 0$$

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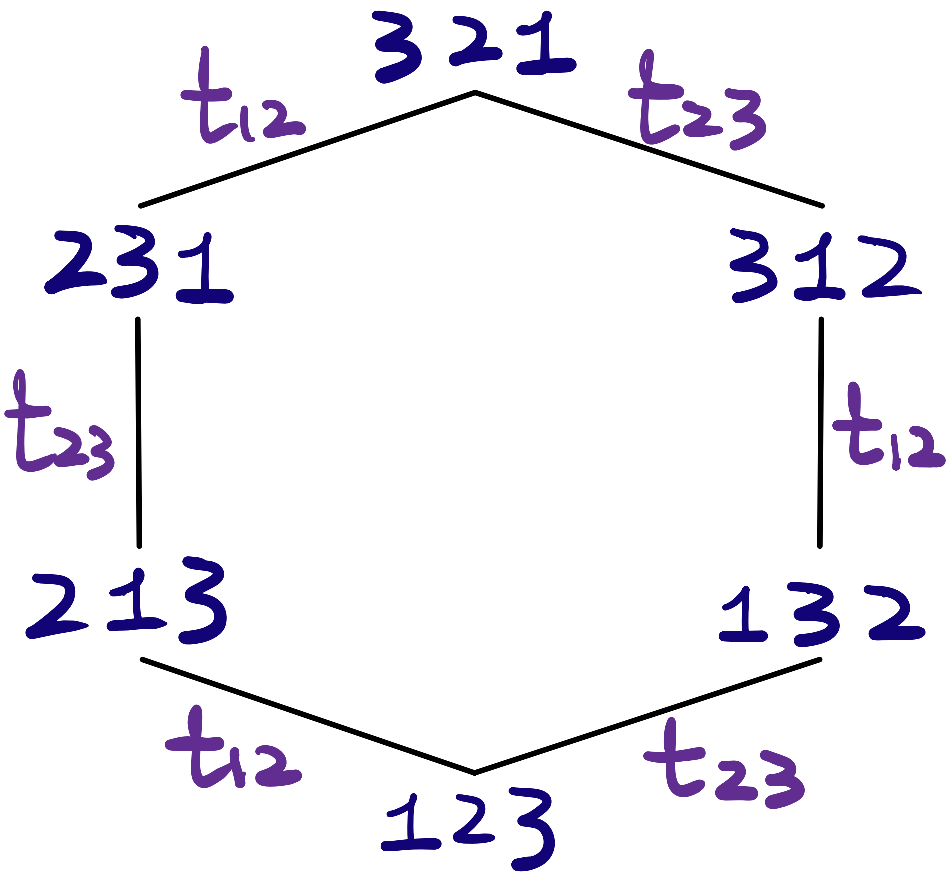
For $w \in S_n$, define $\mathfrak{S}_w$.

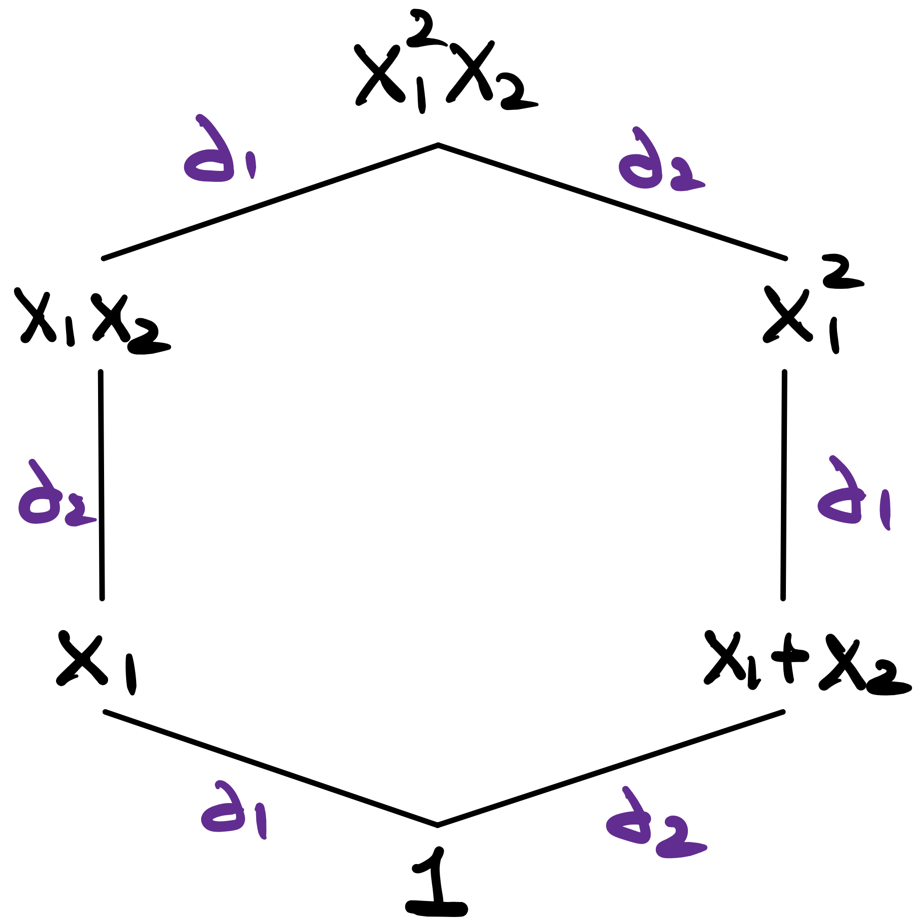
If $w_0 = [n, n-1, \dots, 2, 1]$,

$$\mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-2}^2 x_{n-1}$$

$$\partial_i(\mathfrak{S}_w) = \begin{cases} 0 & \text{if } w(i) < w(i+1) \\ \mathfrak{S}_{w t_{i,i+1}} & \text{if } w(i) > w(i+1) \end{cases}$$

Schubert Polynomial Examples

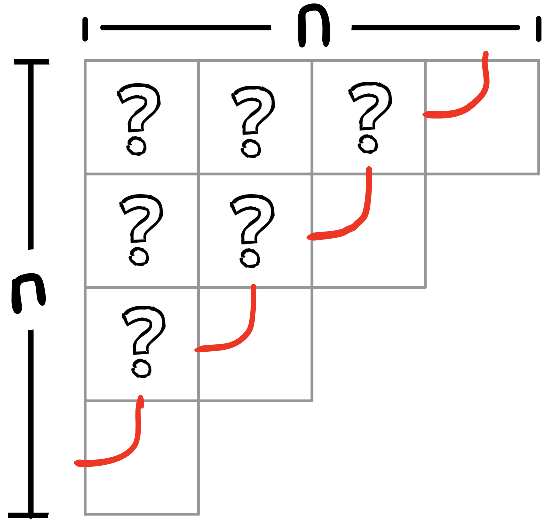


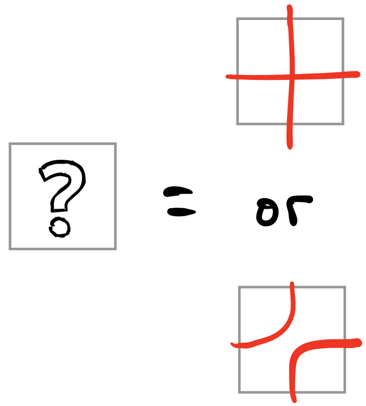


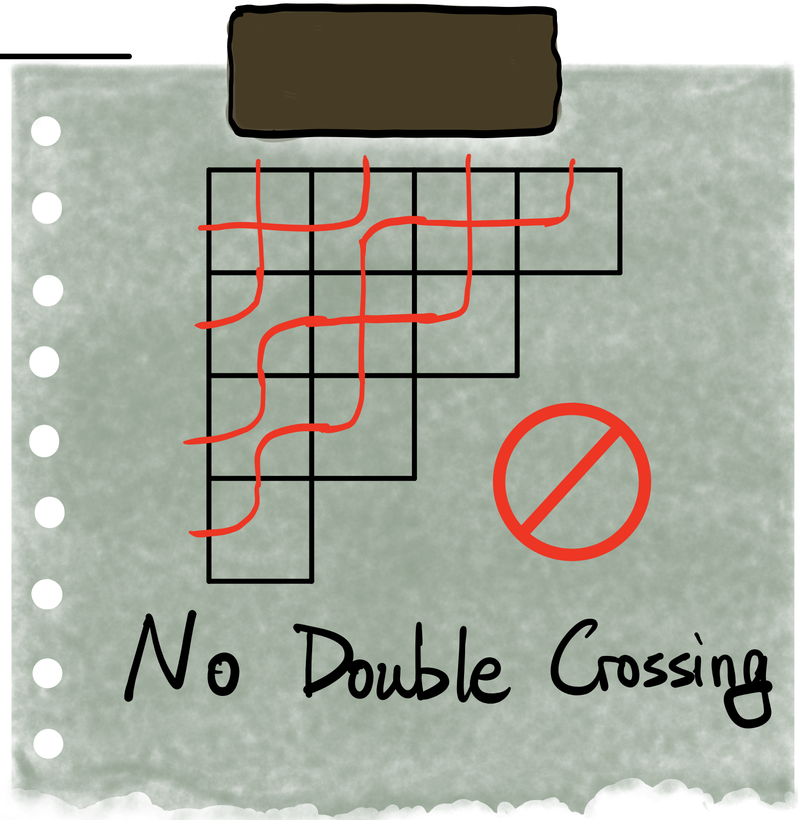
Running Example:

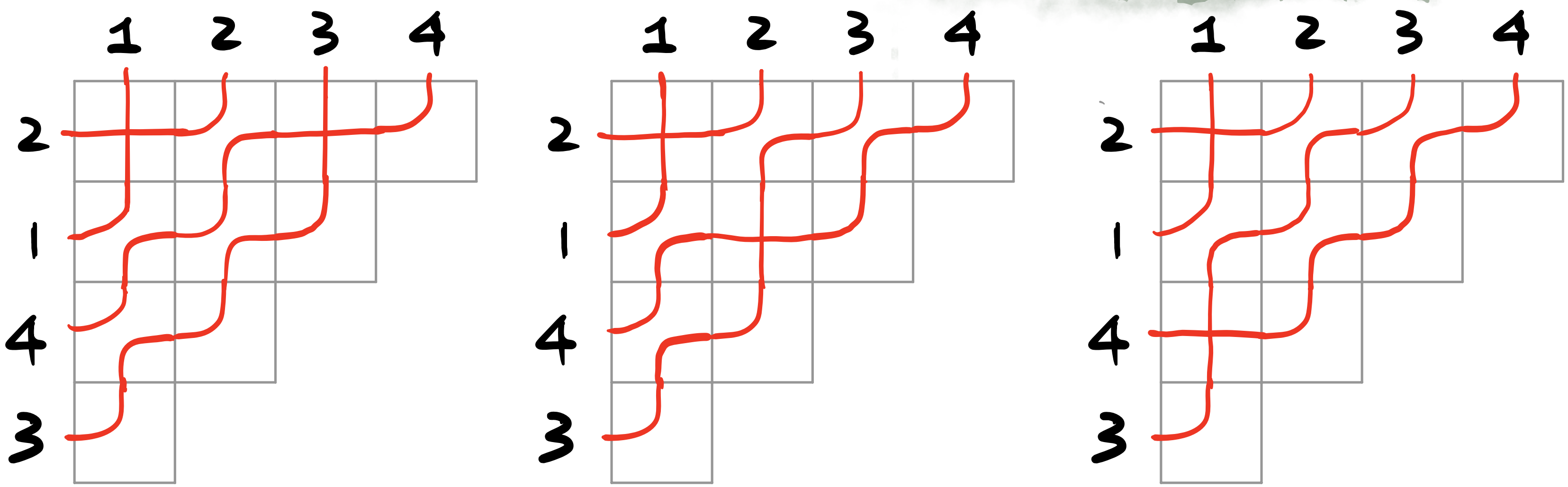
$$\mathfrak{S}_{2143} = x_1^2 + x_1x_2 + x_1x_3$$

Pipedream [Bergeron-Billey]



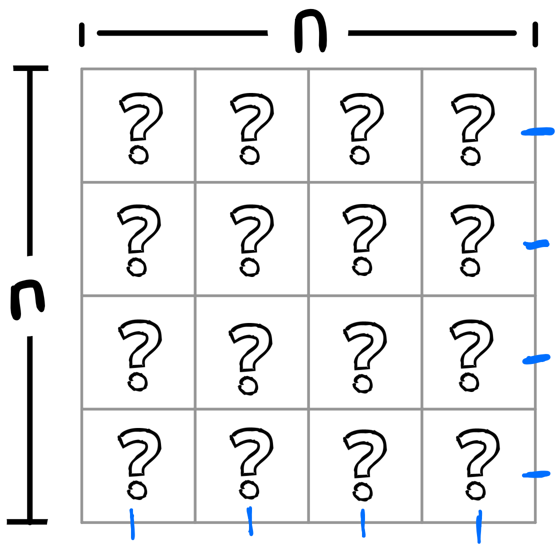


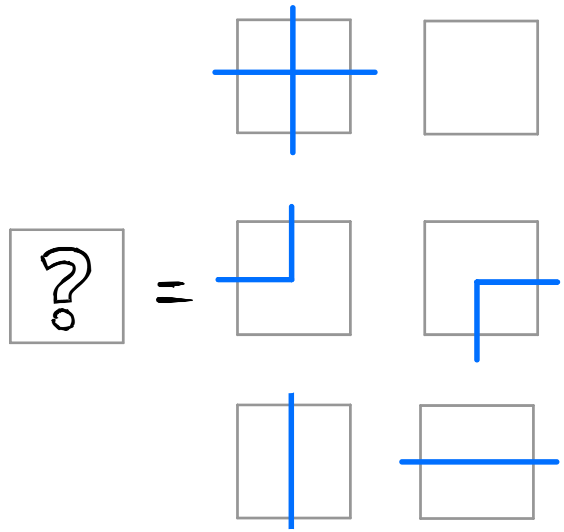


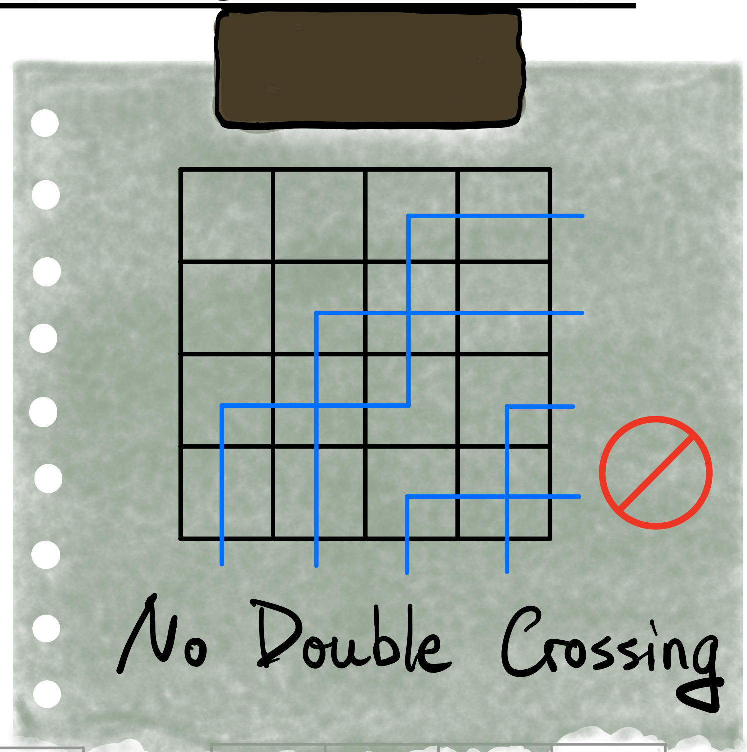


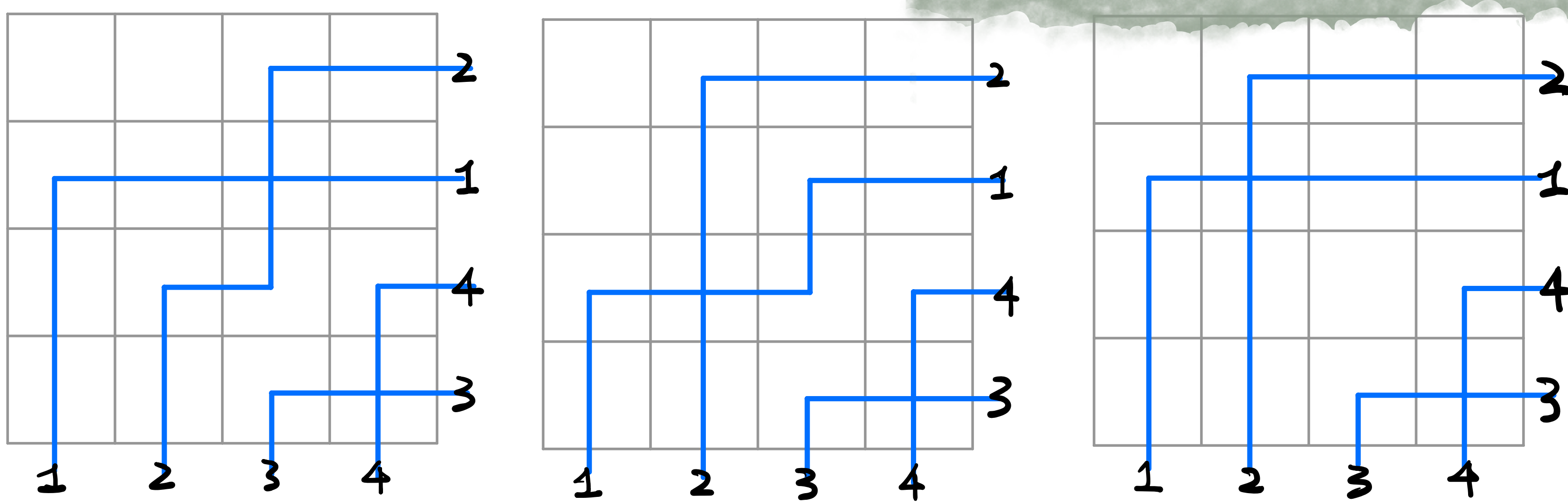
$$G_{2143} = x_1^2 + x_1 x_2 + x_1 x_3$$

Bumpless Pipedreams (BPD) [Lam-Lee-Shimozono]





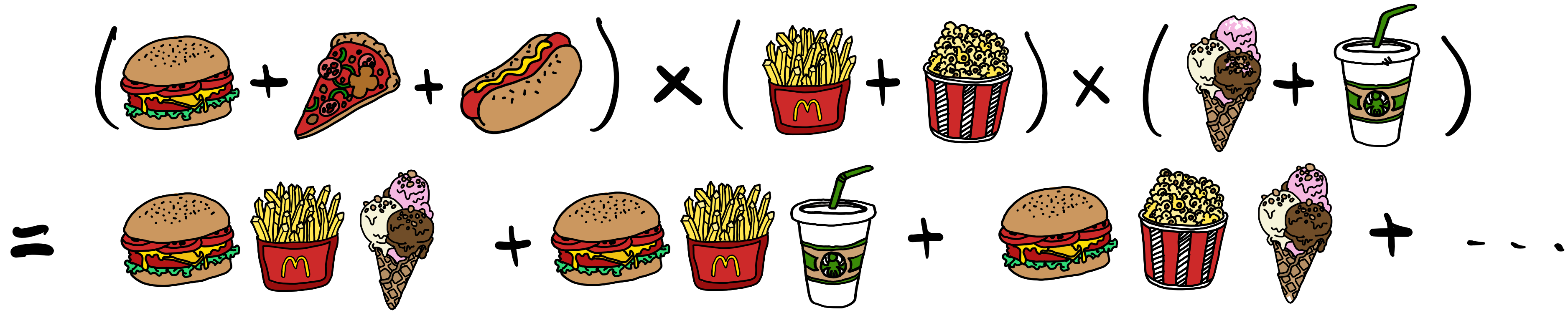


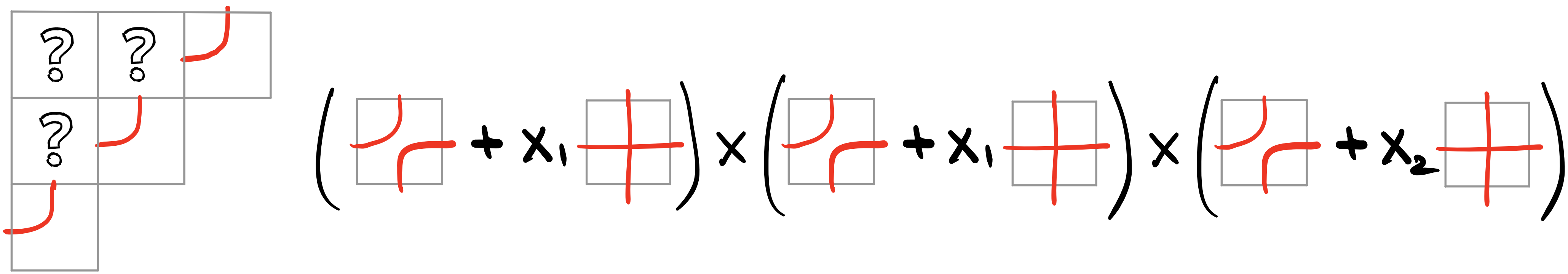


$$G_{2143} = x_1^2 + x_1 x_2 + x_1 x_3$$

Fomin - Stanley Algebra

- Use one expression to include all possible pipedreams.





- How to read permutations ?
- How to make sure no double crossings ?

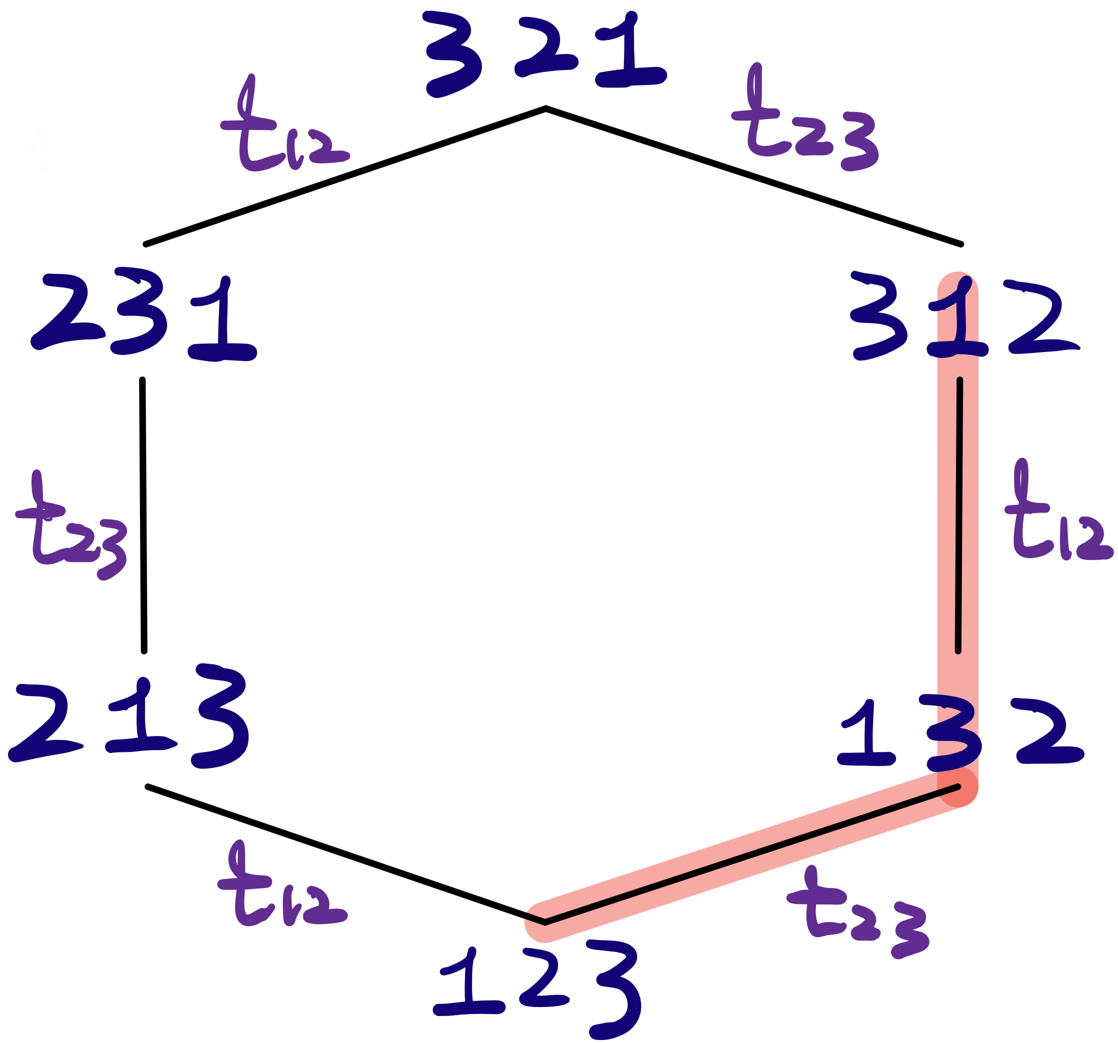
[Food stickers by gabby_scarba11]

Nil-Coxeter Algebra [Fomin-Stanley]

Generated by $u_1, u_2, \dots, u_{n-1}$

- $u_i u_i = 0$
- $u_i u_j = u_j u_i$ if $|i-j| > 1$.
- $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$

Each $u_{i_1} u_{i_2} \dots u_{i_m}$ is zero or corresponds to some $w \in S_n$

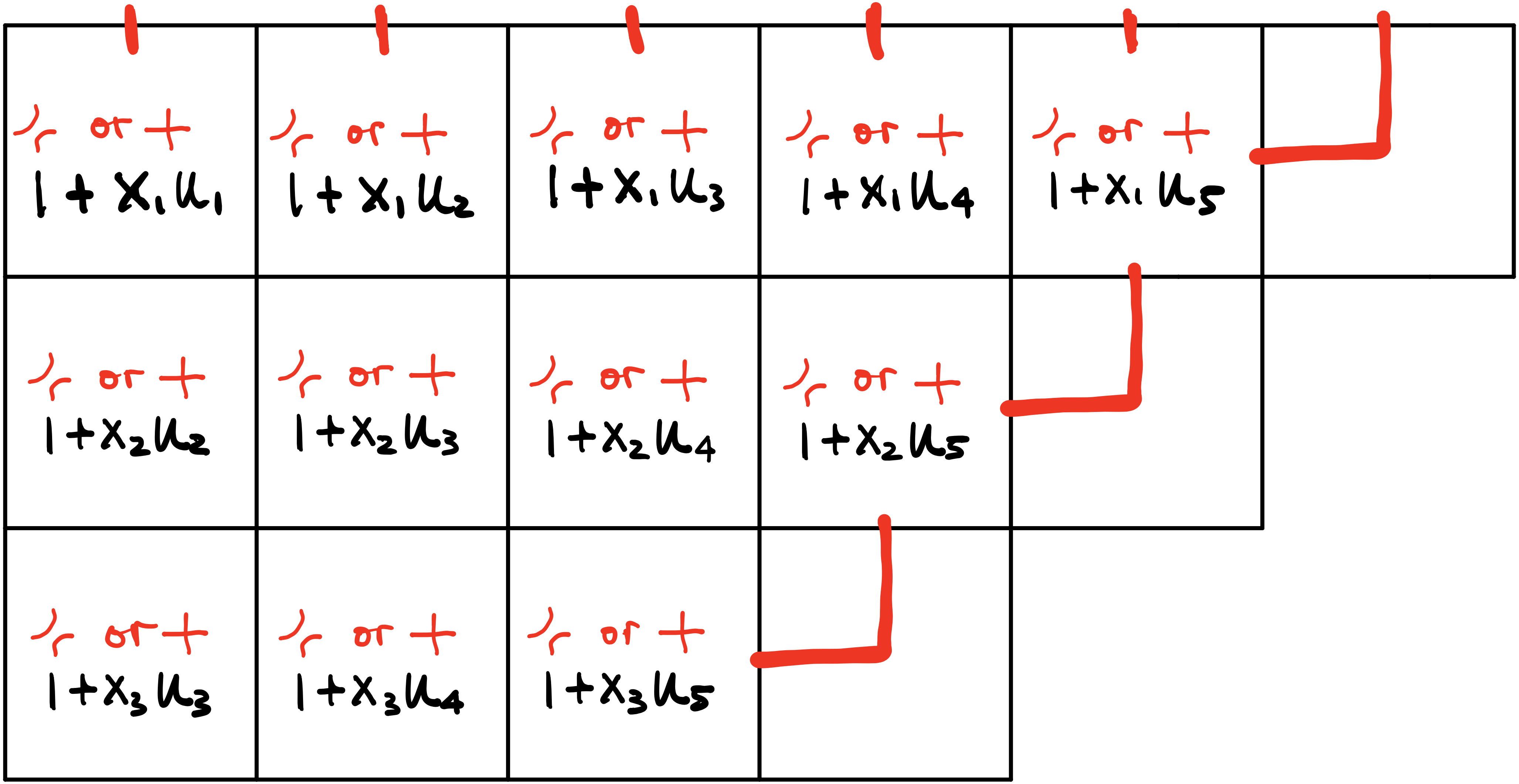


$u_2 u_1 \sim 312$

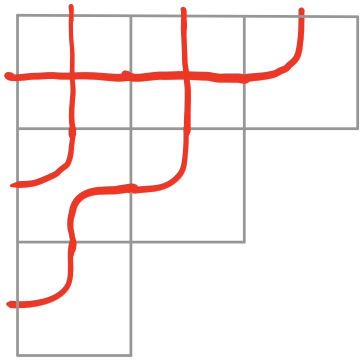
$u_2 u_2 = 0$

Fomin-Stanley Algebra

read this way ←

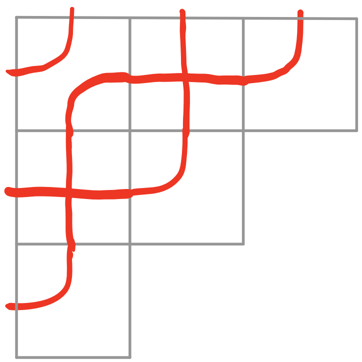


Fomin - Stanley Algebra



$$(1 + x_1 u_2) \times (1 + x_1 u_1) \times (1 + x_2 u_2)$$

$$x_1^2 u_2 u_1$$

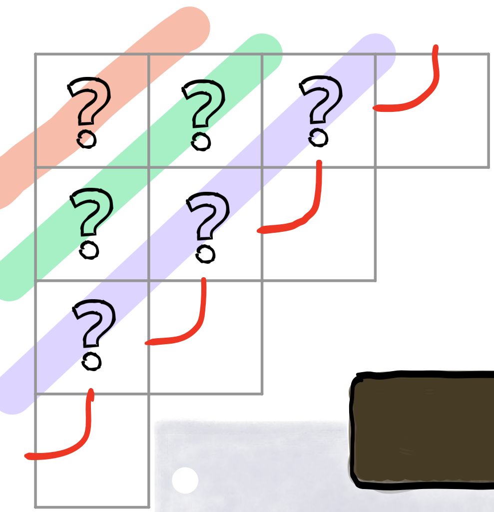


$$(1 + x_1 u_2) \times (1 + x_1 u_1) \times (1 + x_2 u_2)$$

$$x_1 x_2 u_2 u_2 = 0$$

Fomin - Stanley Algebra

$$G := (1+x_1u_3)(1+x_1u_2)(1+x_1u_1)(1+x_2u_2)(1+x_2u_3)(1+x_3u_3)$$



Observation :

$$G = \sum_P x^{wt(P)} \, w(P)$$

over all pipedreams

To prove :

$$G_w = \sum_{P \in PD(w)} x^{wt(P)}$$

It is enough to show

$$G = \sum_{w \in S_n} G_w \, w$$

Fomin - Stanley Algebra

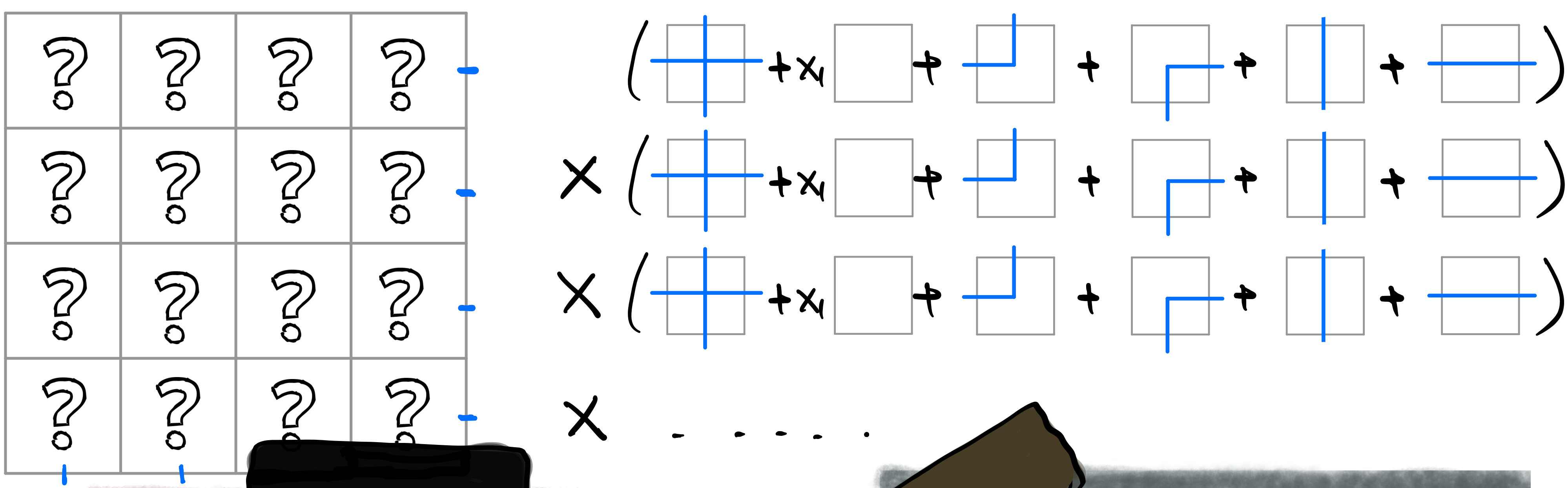
$$\mathfrak{S} = \sum_{w \in S_n} \mathfrak{S}_w \, w \quad \text{reduces to}$$

$$\text{[Fomin - Stanley]} \qquad \partial_i(\mathfrak{S}) = \mathfrak{S}\, u_i$$

$$\sum_{w \in S_3} \mathfrak{S}_w \, w$$

$$\begin{array}{lcl}
\mathfrak{S}_{231}\,[321] & & \mathfrak{S}_{321}\,[321] & & \mathfrak{S}_{321}\,[321] \\
+\,\mathfrak{S}_{231}\,[231] & & +\,\mathfrak{S}_{231}\,[231] & & +\,\mathfrak{S}_{231}\,[321] \\
+\,\mathfrak{S}_{132}\,[312] & \xleftarrow{\ \partial_1\ } & +\,\mathfrak{S}_{312}\,[312] & \xrightarrow{\ u_1\ } & +\,\mathfrak{S}_{312}\,[312] \\
+\,\mathfrak{S}_{132}\,[132] & & +\,\mathfrak{S}_{132}\,[132] & & +\,\mathfrak{S}_{132}\,[312] \\
+\,\mathfrak{S}_{123}\,[213] & & +\,\mathfrak{S}_{213}\,[213] & & +\,\mathfrak{S}_{213}\,[213] \\
+\,\mathfrak{S}_{123}\,[123] & & +\,\mathfrak{S}_{123}\,[123] & & +\,\mathfrak{S}_{123}\,[213]
\end{array}$$

Fomin-Stanley Algebra on BPD?



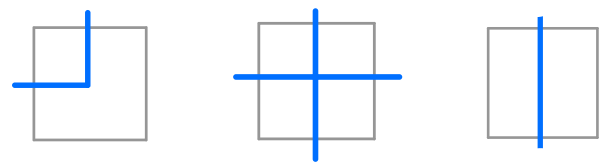
Challenges:

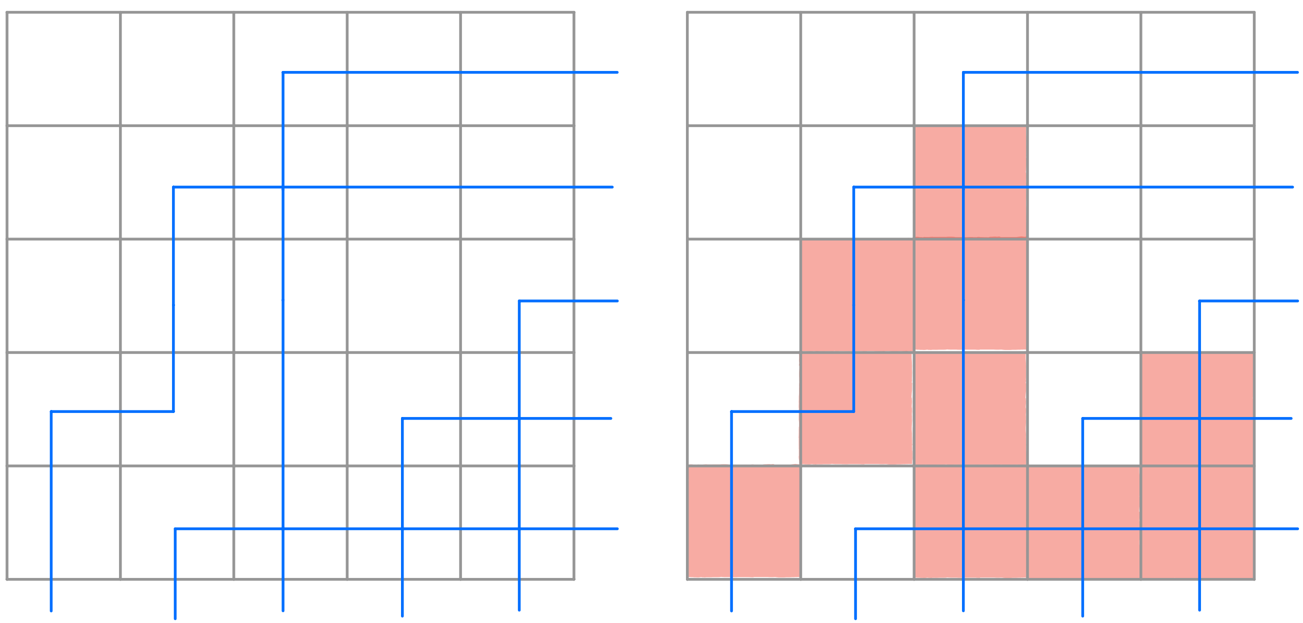
- Too many terms
- Too many "bad" terms
- Hard to read permutation

Solution:

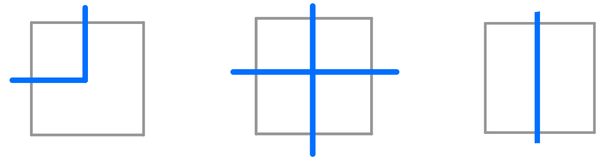
Encode BPD in a more compact way!

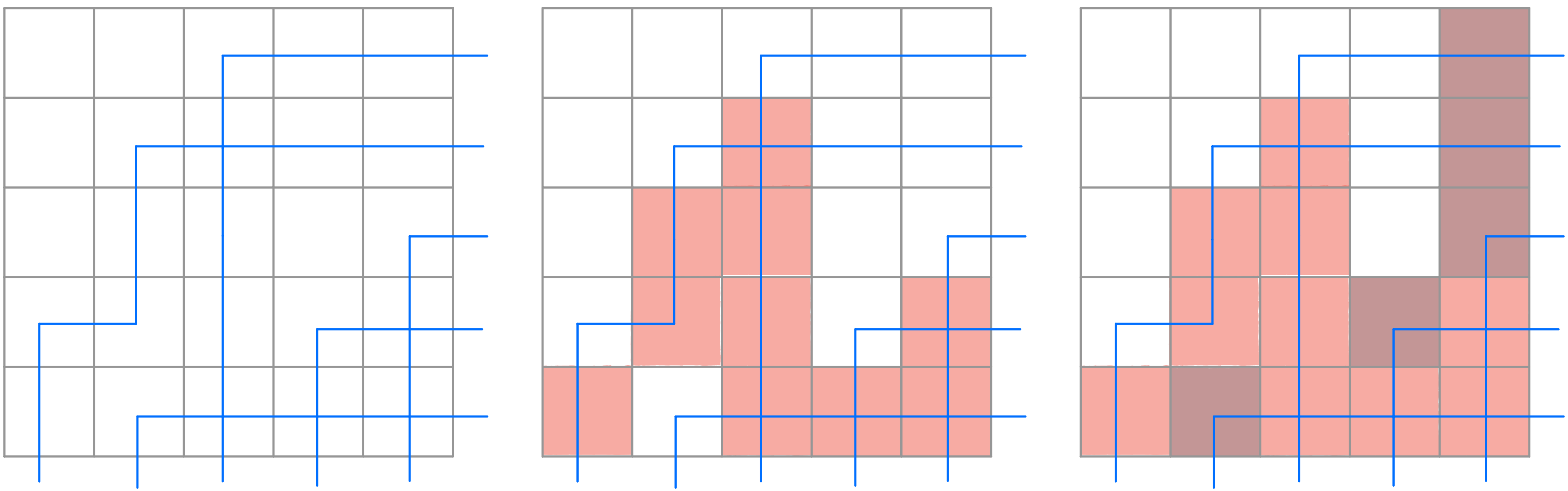
Encoding a BPD

- Step 1: KILL ALL 

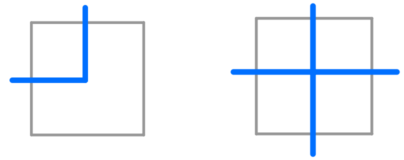


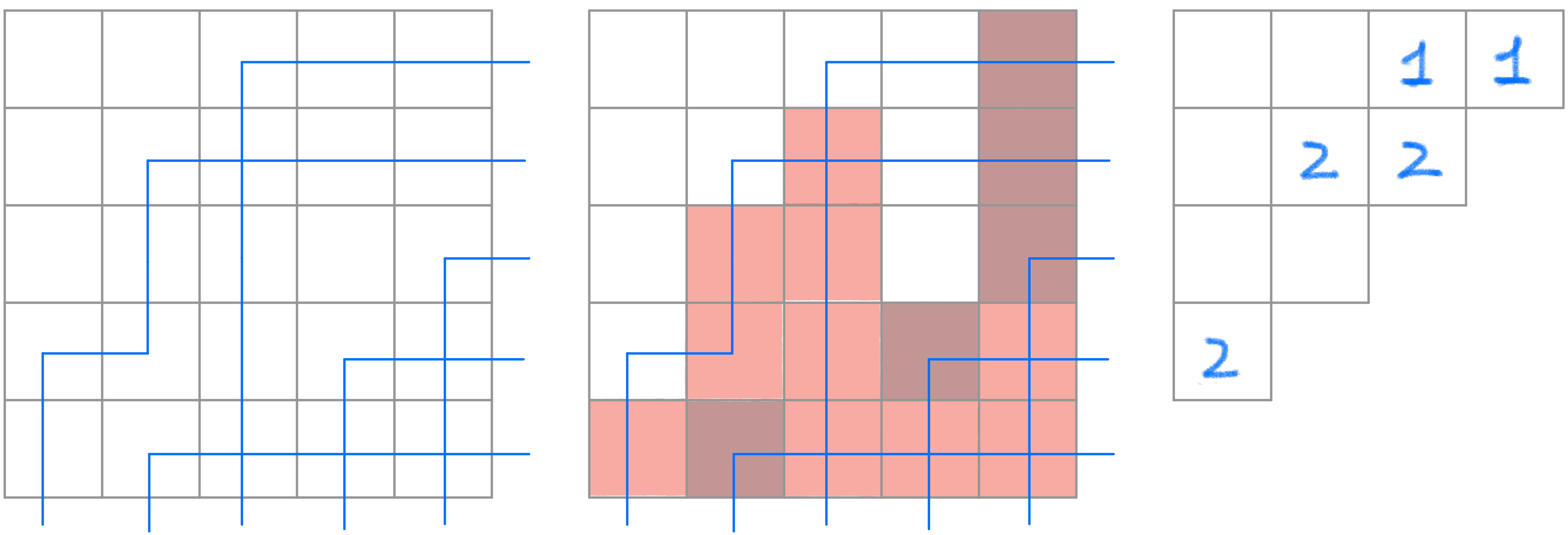
Encoding a BPD

- Step 1: KILL ALL 
- Step 2: KILL the rightmost survivor on each row.



Encoding a BPD

- Step 1: KILL ALL 
- Step 2: KILL the rightmost survivor on each row.
- Step 3: Record the pipe in each survivor.

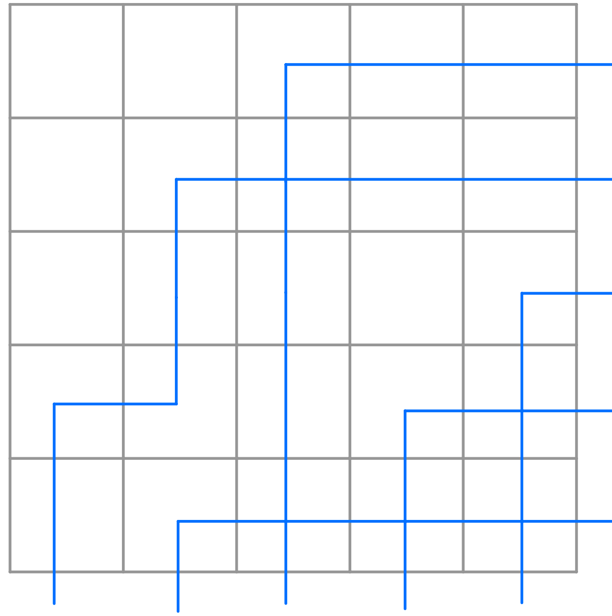


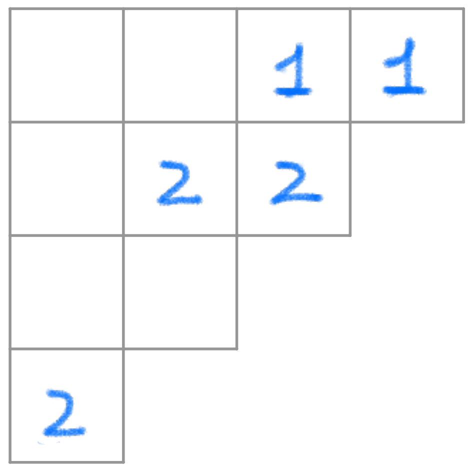
Encoding a BPD

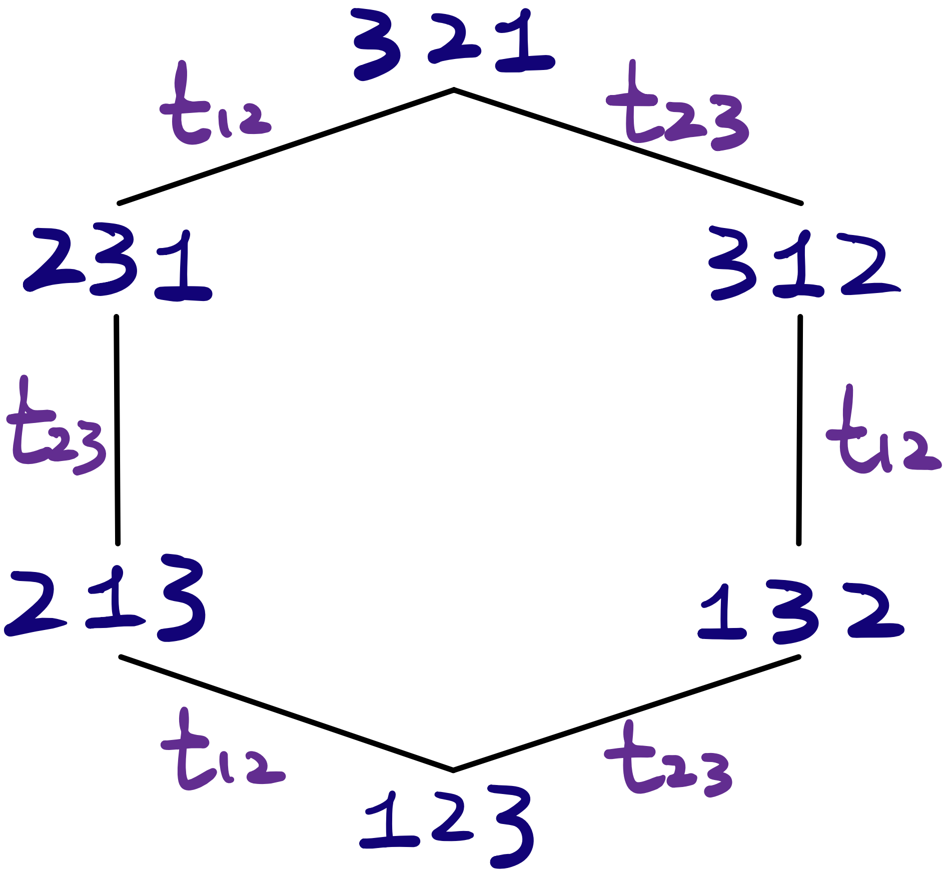
Prop [Y, 23+]

This is a bijection between BPD of w and fillings of the "staircase" such that

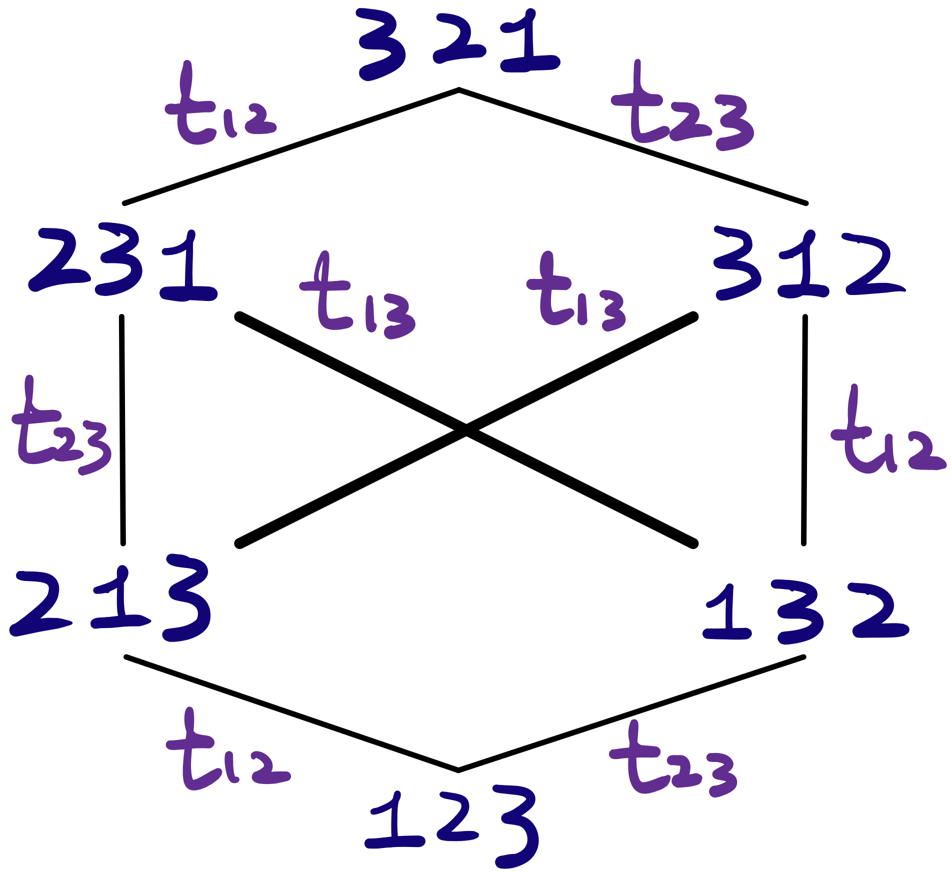
- Row i filled w/ #s $\leq i$
- Correspond to a chain from w to w_0 in Bruhat order.







Weak Bruhat Order



(Strong) Bruhat Order

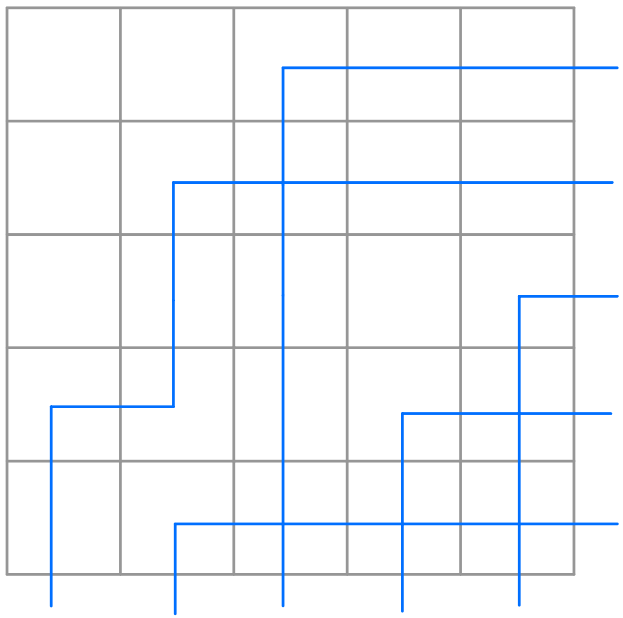
Bruhat Order : wt_{ij}

w

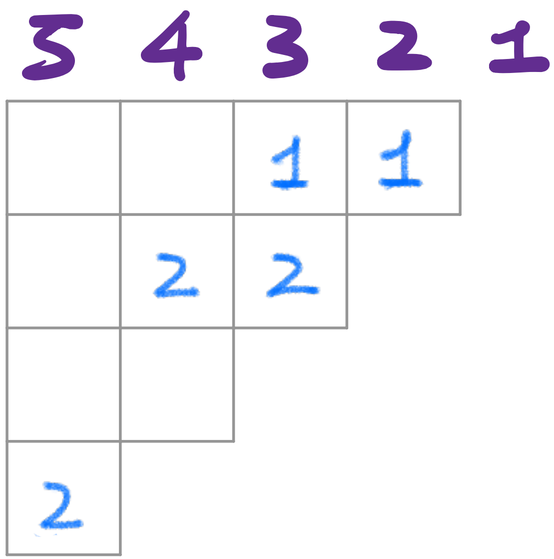
$w < wt_{ij}$ if $w(i) < w(j)$, $i < j$

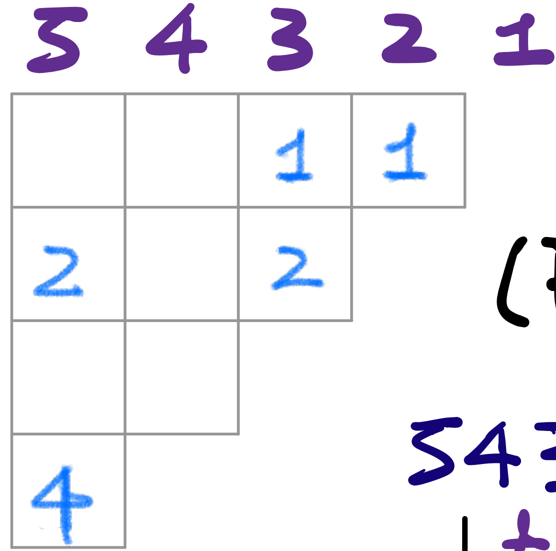
and #s in between are not in between.

Read the permutation



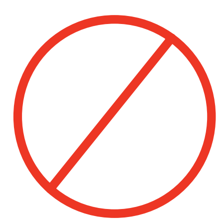
$$54321 \xrightarrow{t_{12}} 45321 \xrightarrow{t_{13}} 35421 \xrightarrow{t_{23}} 34521 \xrightarrow{t_{24}} 32541 \xrightarrow{t_{25}} 31542$$





(BAD)

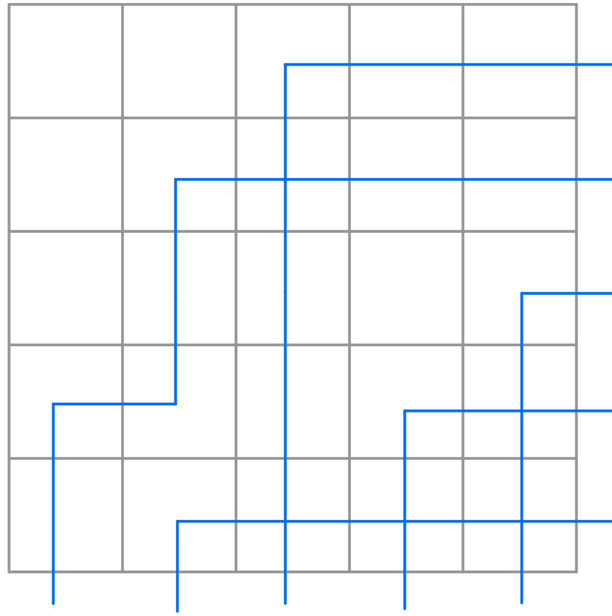
$$54321 \xrightarrow{t_{12}} 45321 \xrightarrow{t_{13}} 35421 \xrightarrow{t_{23}} 34521 \xrightarrow{t_{25}} 31524$$



Encoding a BPD

Prop [Y, 23+]

This is a bijection between BPD of w and fillings of the "staircase" such that



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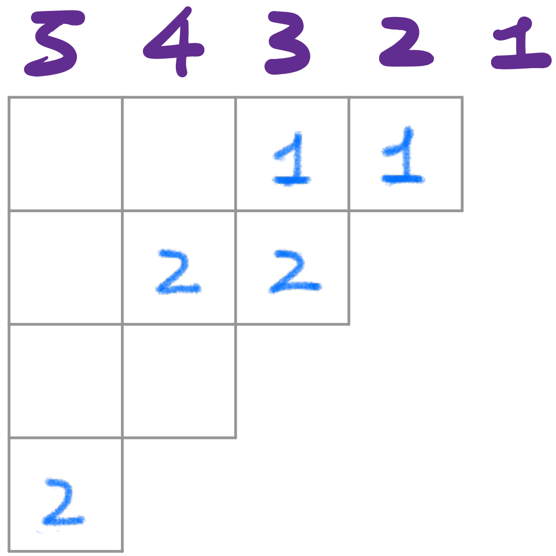


Fomin - Kirillov Algebra

$\mathcal{E}_n$ is generated by d_{ij}, $1 \le i < j \le n$.

- $d_{ij} d_{ij} = 0$
- $d_{ij} d_{jk} = d_{ik} d_{ij} + d_{jk} d_{ik}$
- $d_{jk} d_{ij} = d_{ij} d_{ik} + d_{ik} d_{jk}$
- $d_{ij} d_{kl} = d_{kl} d_{ij}$

  if i, j, k, l distinct.

Fomin-Kirillov Algebra

$\mathcal{E}_n$ is generated by d_{ij}, $1 \le i < j \le n$.

$\mathcal{E}_n$ acts on $\mathbb{Q}[S_n]$

$$w \odot d_{ij} = \begin{cases} w t_{ij} & \text{if } w > w t_{ij} \\ 0 & \text{otherwise.} \end{cases}$$

BPD Analogue of Fomin – Stanley

5	4	3	2	1
		1	1	
	2	2		
2				

$$w_0 \odot d_{12}\, d_{13}\, d_{23}\, d_{24}\, d_{25}$$

$$= [31524]$$

5	4	3	2	1
		1	1	
2		2		
4				

(BAD)

$$w_0 \odot d_{12}\, d_{13}\, d_{23}\, d_{25}\, d_{45}$$

$$= 0$$

BPD Analogue of Fomin – Stanley

4	3	2	1
?	?	?	
?	?		
?			

$$(x_1+\overset{2}{\boxed{1}})(x_1+\overset{3}{\boxed{1}})(x_1+\overset{4}{\boxed{1}})$$
$$=(x_1+d_{12})(x_1+d_{13})(x_1+d_{14})$$

$$(x_2+\overset{3}{\boxed{1}}+\overset{3}{\boxed{2}})(x_2+\overset{4}{\boxed{1}}+\overset{4}{\boxed{2}})$$
$$=(x_2+d_{13}+d_{23})(x_2+d_{14}+d_{24})$$

$$(x_3+\overset{4}{\boxed{1}}+\overset{4}{\boxed{2}}+\overset{4}{\boxed{3}})$$
$$=(x_3+d_{14}+d_{24}+d_{34})$$

BPD Analogue of Fomin – Stanley

Definition [Y, 23+]

4	3	2	1
?	?	?	
?	?		
?			

$$\mathcal{G} = w_0 \odot (x_1+d_{12})(x_1+d_{13})(x_1+d_{14})$$
$$(x_2+d_{13}+d_{23})(x_2+d_{14}+d_{24})$$
$$(x_3+d_{14}+d_{24}+d_{34})$$

To show:

$$\mathcal{G}_w = \sum_{D \in BPD(w)} x^{wt(D)}$$

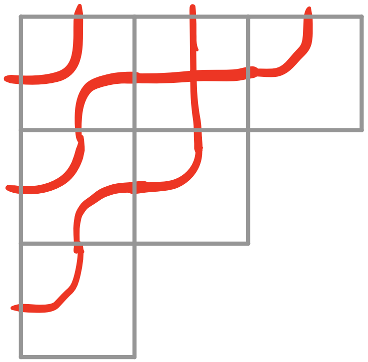
Just need:

$$\mathcal{G} = \sum_{w \in S_n} \mathcal{G}_w \, w$$

As before, just need [Y, 23+]

$$\partial_i(\mathcal{G}) = \mathcal{G} u_i$$

Pipedream (PD)



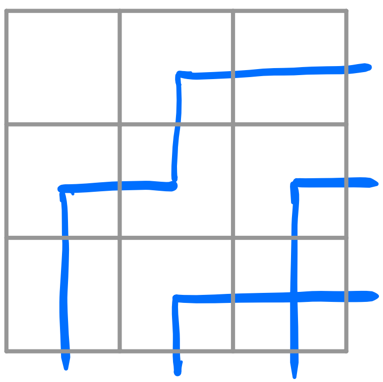
Fomin-Stanley Algebra

$$(1+x_1u_2)(1+x_1u_1)(1+x_2u_2)$$

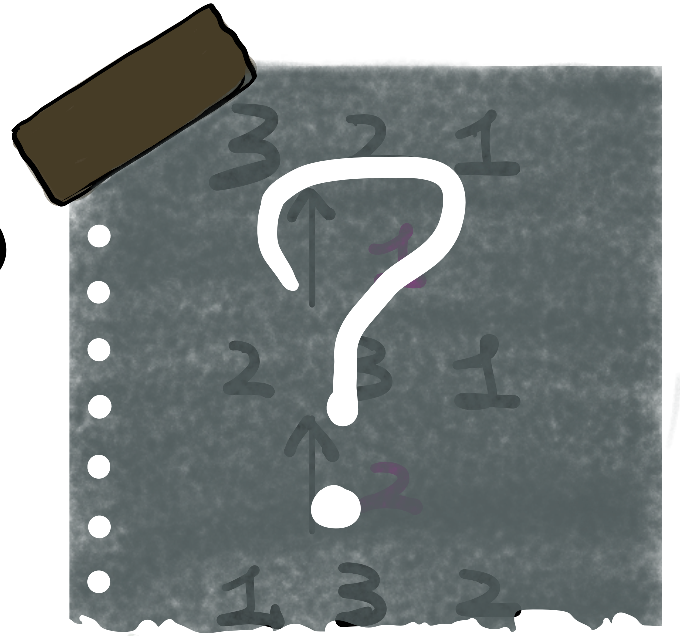
Lenart-Sottile Chain

$$\begin{array}{c} 3\ 2\ 1 \\ \uparrow 2 \\ 3\ 1\ 2 \\ \uparrow 1 \\ 1\ 3\ 2 \end{array}$$

Bumpless Pipedream (BPD)



$$(x_1+d_{12})(x_1+d_{13})(x_2+d_{13}+d_{23})$$



Labeled Bruhat chain

Suppose $w < wt_{ij}$.
We may label

$$w \xrightarrow{k} wt_{ij}$$

where $i \leq k < j$.

A k-chain is :

$$w_1 \xrightarrow{k} w_2 \xrightarrow{k} w_3 \xrightarrow{k} w_4$$

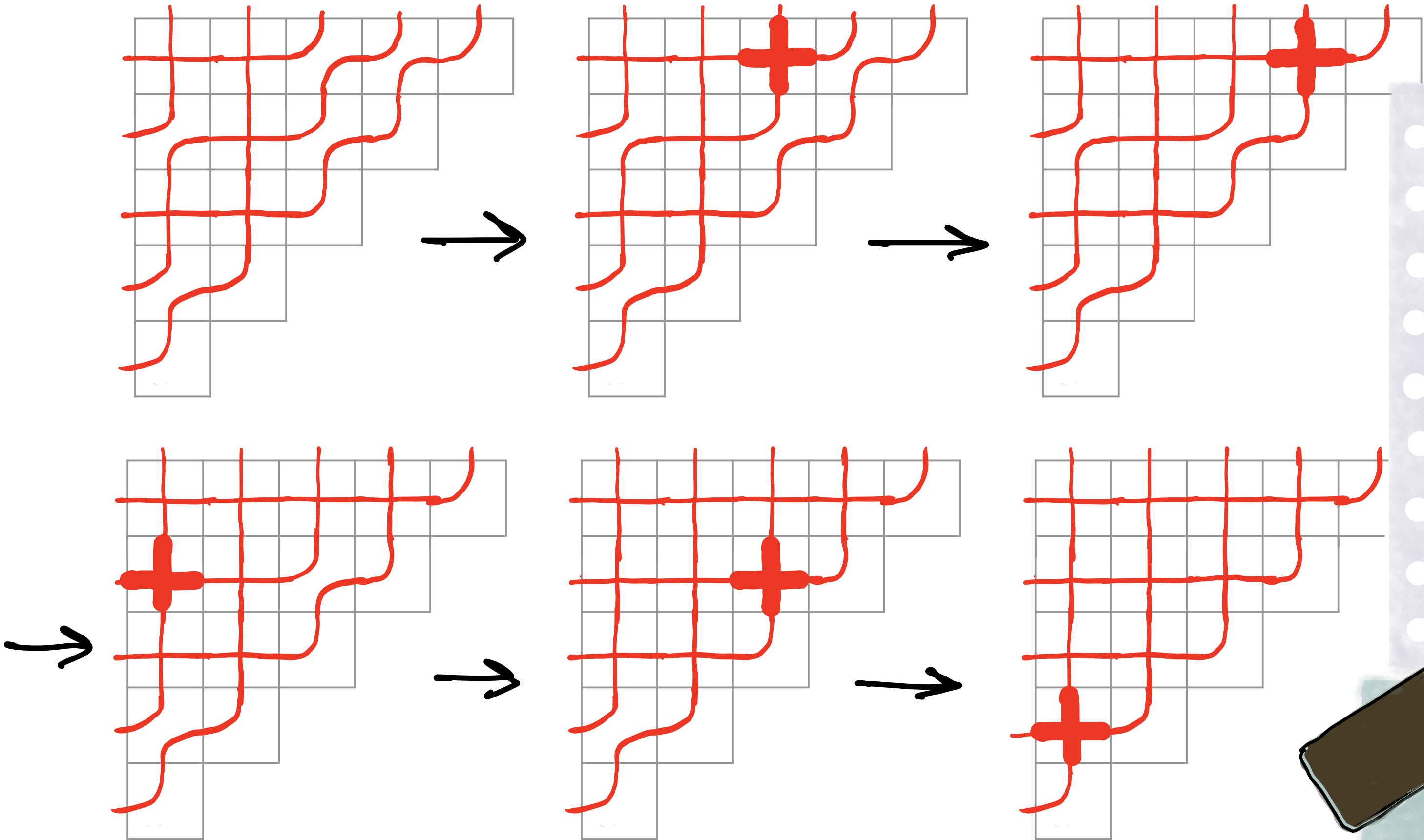
E.g. $31542 \xrightarrow{1} 41532 \xrightarrow{1} 51432$

is a 1-chain.

$51432 \xrightarrow{2} 53412 \xrightarrow{2} 54312$

is a 2-chain.

Lenart - Sottile Chain

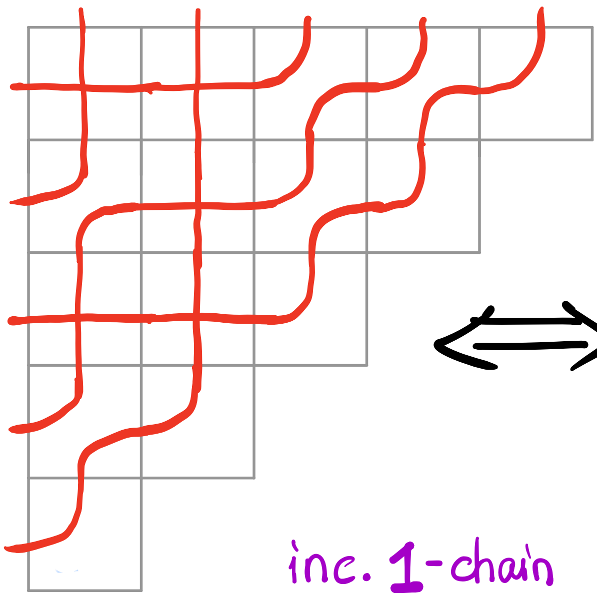


Change into + from top to bottom, left to right on each row.

Labels given by the number of the

$$31542 \xrightarrow{1} 41532 \xrightarrow{1} 51432 \xrightarrow{2} 53412 \xrightarrow{2} 54312 \xrightarrow{4} 54321$$

Lenart - Sottile Chain



$\Longleftrightarrow$

inc. 1-chain : A 1-chain where smaller # being swapped is increasing

$31542 \xrightarrow{1} 41532 \xrightarrow{1} 51432$

$\xrightarrow{2} 53412 \xrightarrow{2} 54312 \xrightarrow{4} 54321$

$31542 \xrightarrow{\text{inc. 1-chain}} 51432 \xrightarrow{\text{inc. 2-chain}} 54312 \xrightarrow{\text{inc. 3-chain}} 54312 \xrightarrow{\text{inc. 4-chain}} 54321$

Thm [Lenart – Sottile, 02]

This is a bijection between PD(w) and

$$w \xrightarrow{\text{inc. 1-chain}} \bullet \xrightarrow{\text{inc. 2-chain}} \cdots \xrightarrow{\text{inc. } n\text{-2-chain}} \bullet \xrightarrow{\text{inc. } n\text{-1-chain}} w_0$$

Lenart - Sottile Chain

Thm [Lenart – Sottile, 02]

This is a bijection between PD(w) and

$$w \xrightarrow{\text{inc. 1-chain}} \bullet \xrightarrow{\text{inc. 2-chain}} \cdots \xrightarrow{\text{inc. n-2-chain}} \bullet \xrightarrow{\text{inc. n-1-chain}} w_0$$

This bijection recovers

[Bergeron - Sottile]

$$\mathfrak{S}_w = \sum_{w \xrightarrow{\text{inc } 1} \cdots \xrightarrow{\text{inc } n-1} w_0} \prod_{i=1}^{n-1} x_i^{\,n-i-(\text{length of the } i \text{ chain})}$$

Analogue of Lenart-Sottile chain

5	4	3	2	1
		1	1	
	2	2		
2				

$\Longleftrightarrow$

$$31542 \xrightarrow{4} 32541 \xrightarrow{2} 34521 \xrightarrow{2} 35421 \xrightarrow{1} 45321 \xrightarrow{1} 54321$$

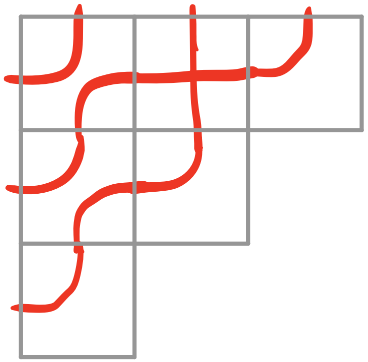
$$31542 \xrightarrow{\text{inc. 4-chain}} 32541 \xrightarrow{\text{inc. 3-chain}} 32541 \xrightarrow{\text{inc. 2-chain}} 35421 \xrightarrow{\text{inc. 1-chain}} 54321$$

Thm [Y, 23+]

This is a bijection between BPD(w) and

$$w \xrightarrow{\text{inc. } n\text{-1-chain}} \bullet \xrightarrow{\text{inc. } n\text{-2-chain}} \cdots \xrightarrow{\text{inc. 2-chain}} \bullet \xrightarrow{\text{inc. 1-chain}} w_0$$

Pipedream (PD)	Fomin-Stanley Algebra	Lenart-Sottile Chain



$$(1+x_1u_2)(1+x_1u_1)(1+x_2u_2)$$

3 2 1

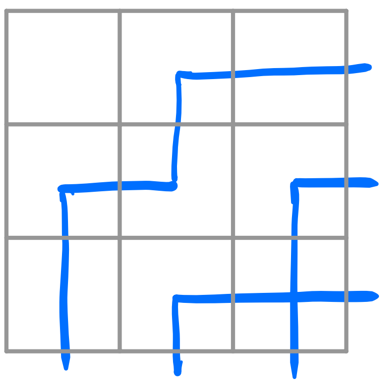
$\uparrow$ 2

3 1 2

$\uparrow$ 1

1 3 2

Bumpless Pipedream (BPD)



$$(x_1+d_{12})(x_1+d_{13})(x_2+d_{13}+d_{23})$$

3 2 1

$\uparrow$ 1

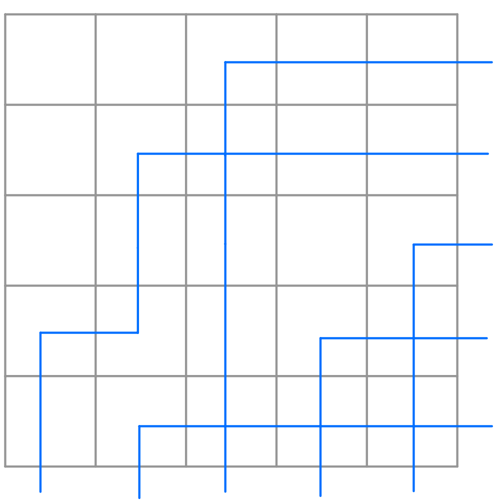
2 3 1

$\uparrow$ 2

1 3 2

Bijection between PDs and BPDs

- [Gao-Huang]

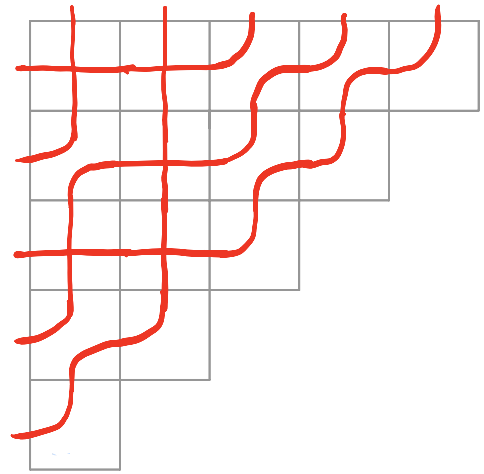


$\rightleftharpoons$

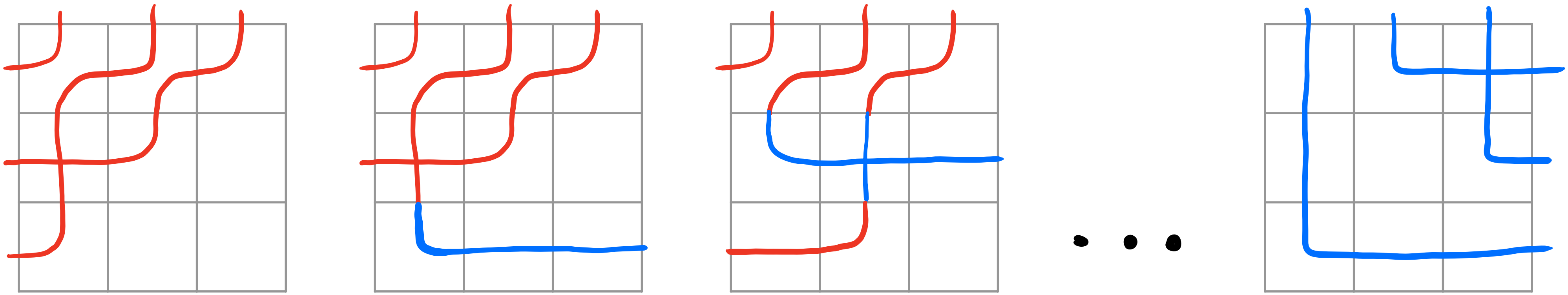
2 1 3 4 3

1 1 2 3 3

$\rightleftharpoons$



- [Knutson-Udell]



Thm [Knutson-Udell] The two bijections agree.

New bijection between PDs and BPDs ?

$$31542 \xrightarrow{\text{inc. 1-chain}} 51432 \xrightarrow{\text{inc. 2-chain}} 54312 \xrightarrow{\text{inc. 3-chain}} 54312 \xrightarrow{\text{inc. 4-chain}} 54321$$

PD

IDEA:
Can we swap two chains?

BPD

$$31542 \xrightarrow[\text{inc. 4-chain}]{} 32541 \xrightarrow[\text{inc. 3-chain}]{} 32541 \xrightarrow[\text{inc. 2-chain}]{} 35421 \xrightarrow[\text{inc. 1-chain}]{} 54321$$

Open Problem

Fix $u, w \in S_n$, $k_1, k_2 \in [n-1]$, $a, b \in \mathbb{Z}_{\geq 0}$,

The # of chains:

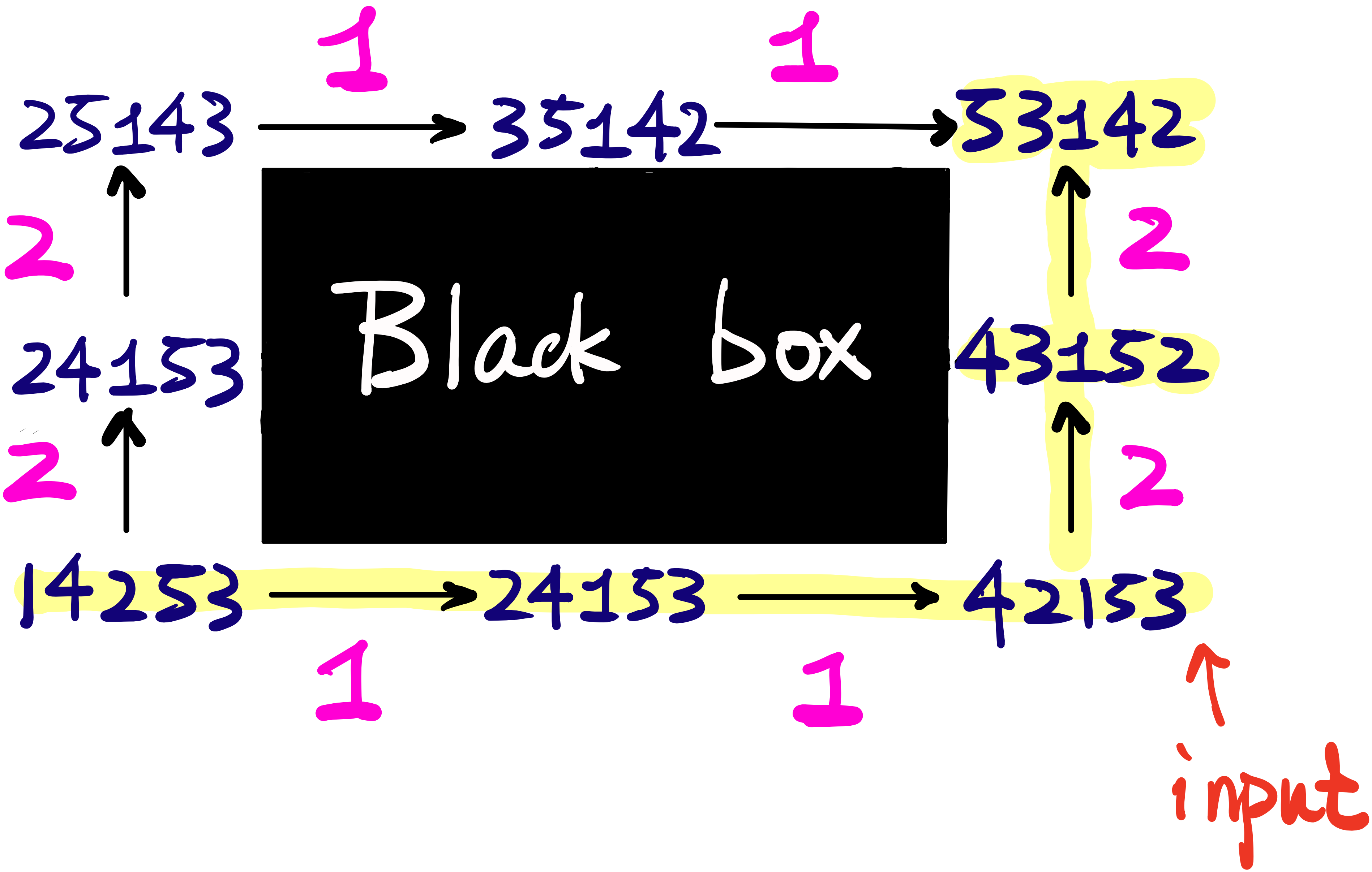
$$u \xrightarrow[\text{length } a,]{\text{inc. } k_1} v \xrightarrow[\text{length } b,]{\text{inc. } k_2} w$$

is the same as the # of chains:

$$u \xrightarrow[\text{length } b,]{\text{inc. } k_2} v' \xrightarrow[\text{length } a,]{\text{inc. } k_1} w$$

Find a bijection?

Lenart's Growth Diagram



Lenart's Growth Diagram

Even if two input chains are inc., output chains might not.

NOT an inc. 2-chain

$$\begin{tikzcd}
2341 \arrow[r, "2"] & 2431 \arrow[r, "2"] & 3421 \\
2143 \arrow[u, "3"] \arrow[r, "2"'] & 2413 \arrow[r, "2"'] & 3412 \arrow[u, "3"']
\end{tikzcd}$$

New bijection between PDs and BPDs

Prop [Y, 23+]

Lenart's growth diagram gives us a bijection between:

$$\left\{ u \xrightarrow[\text{length } a,]{\text{inc. } k} v \xrightarrow[\text{length } b,]{\text{inc. } l} w \right\}$$

$$\left\{ u \xrightarrow[\text{length } b,]{\text{inc. } l} v' \xrightarrow[\text{length } a,]{\text{inc. } k} w \right\}$$

if $w(k+1) > w(k+2) > \cdots$

New bijection between PDs and BPDs

$$\text{BPD} \Rightarrow w \xrightarrow{4} \cdot \xrightarrow{3} \cdot \xrightarrow{2} \cdot \xrightarrow{1} w_0$$

$$\Rightarrow w \xrightarrow{1} \cdot \xrightarrow{4} \cdot \xrightarrow{3} \cdot \xrightarrow{2} w_0$$

All start with 5

$$\Rightarrow w \xrightarrow{1} \cdot \xrightarrow{2} \cdot \xrightarrow{4} \cdot \xrightarrow{3} w_0$$

$$\Rightarrow w \xrightarrow{1} \cdot \xrightarrow{2} \cdot \xrightarrow{3} \cdot \xrightarrow{4} w_0$$

$$\Rightarrow \text{PD}.$$

Is this the same as Gao-Huang bijections?

Open Problem

PD: $w \xrightarrow[\text{inc.}]{1\text{-chain}} \xrightarrow[\text{inc.}]{2\text{-chain}} \xrightarrow[\text{inc.}]{3\text{-chain}} w_0$

$w \xrightarrow[\text{inc.}]{1\text{-chain}} \xrightarrow[\text{inc.}]{3\text{-chain}} \xrightarrow[\text{inc.}]{2\text{-chain}} w_0$

$w \xrightarrow[\text{inc.}]{2\text{-chain}} \xrightarrow[\text{inc.}]{1\text{-chain}} \xrightarrow[\text{inc.}]{3\text{-chain}} w_0$

$w \xrightarrow[\text{inc.}]{2\text{-chain}} \xrightarrow[\text{inc.}]{3\text{-chain}} \xrightarrow[\text{inc.}]{1\text{-chain}} w_0$

$w \xrightarrow[\text{inc.}]{3\text{-chain}} \xrightarrow[\text{inc.}]{1\text{-chain}} \xrightarrow[\text{inc.}]{2\text{-chain}} w_0$

BPD: $w \xrightarrow[\text{inc.}]{3\text{-chain}} \xrightarrow[\text{inc.}]{2\text{-chain}} \xrightarrow[\text{inc.}]{1\text{-chain}} w_0$

Just finished a practice talk on the CN tower.

That's quite the place to do a practice talk!

Facing everyone in Toronto

HA HA

Now they all know what is the BPD analogue of Fomin Stanley algebra.

Delivered

Quite the audience

That's way more interesting than most Canadian media

Thank Yibo Gao for telling me this problem.

Thank Yibo Gao and Zachary Hamaker for valuable guidances.

Thank you for listening!