Pipedreams, Bumpless Pipereams and Bruhat chains

Tianyi Yu
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Outline

- 3 Formulas of Schubert polynomials

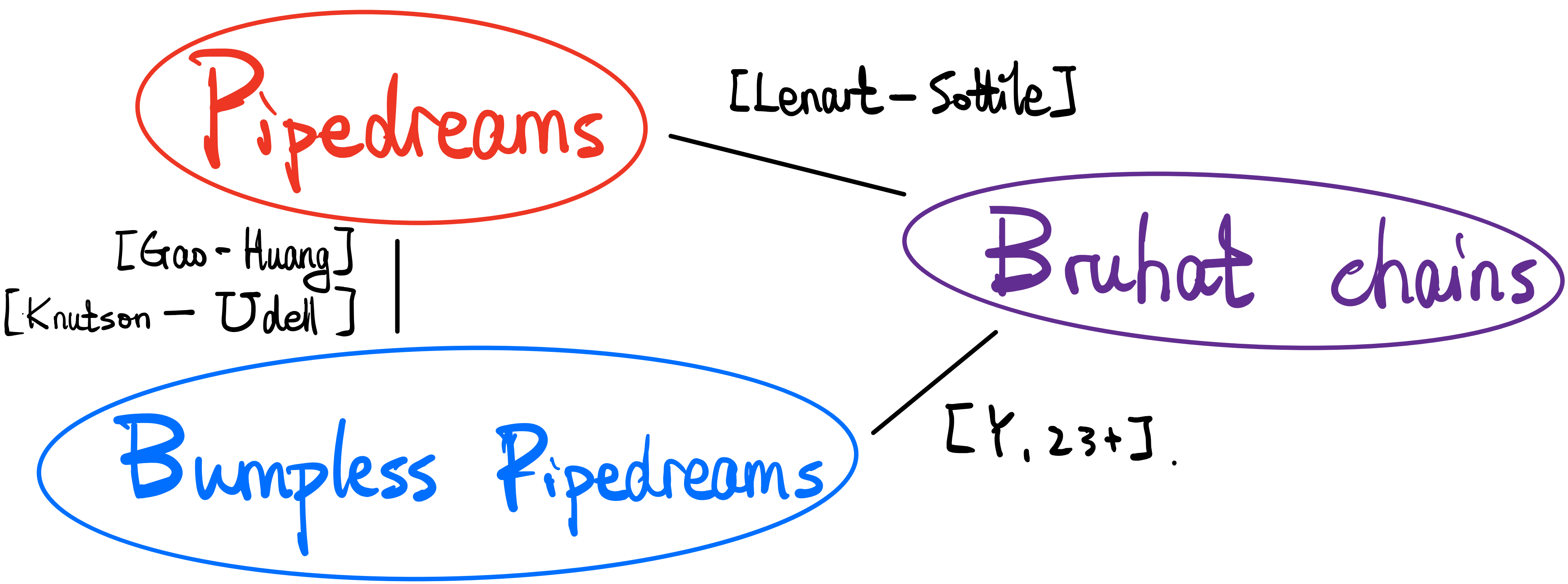
Pipedreams

Bruhat chains

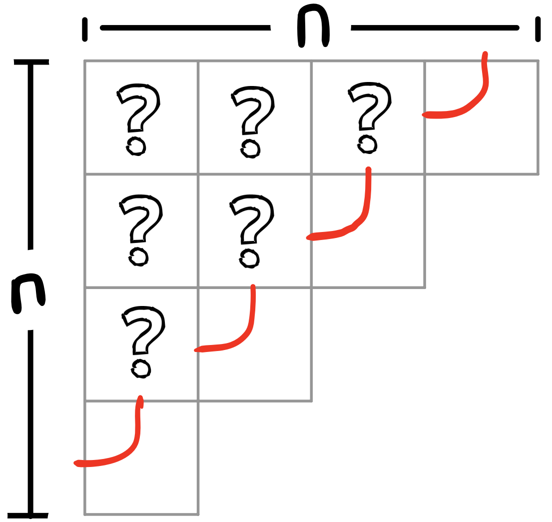
Bumpless Pipedreams

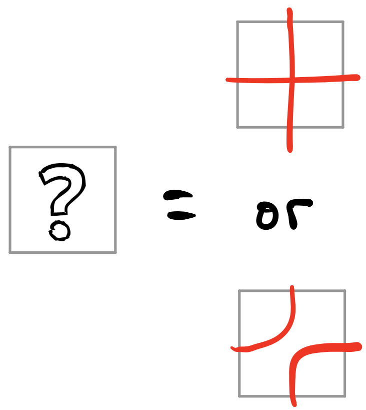
Outline

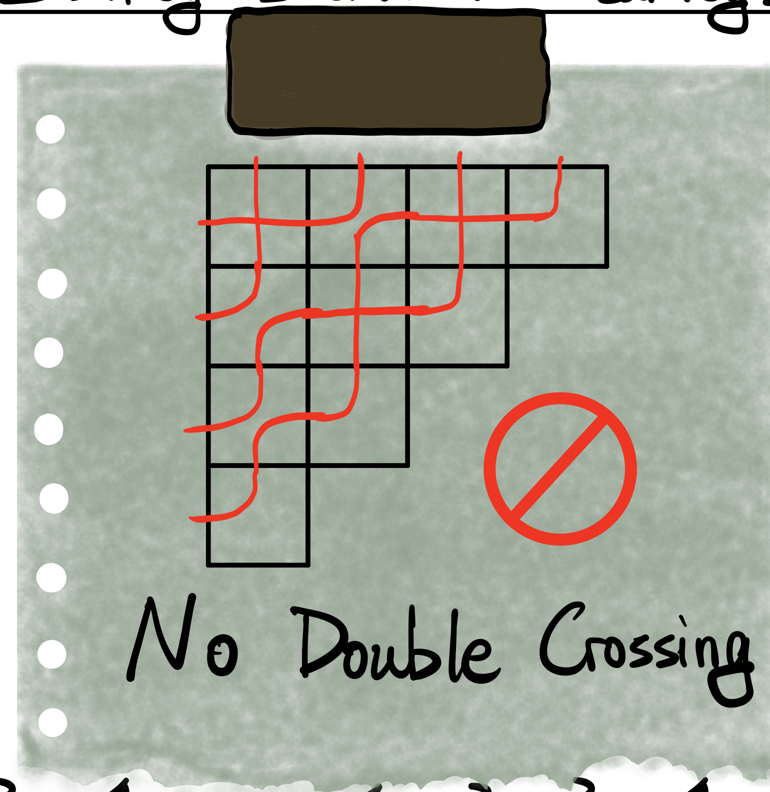
- 3 Formulas of Schubert polynomials
- and their connections

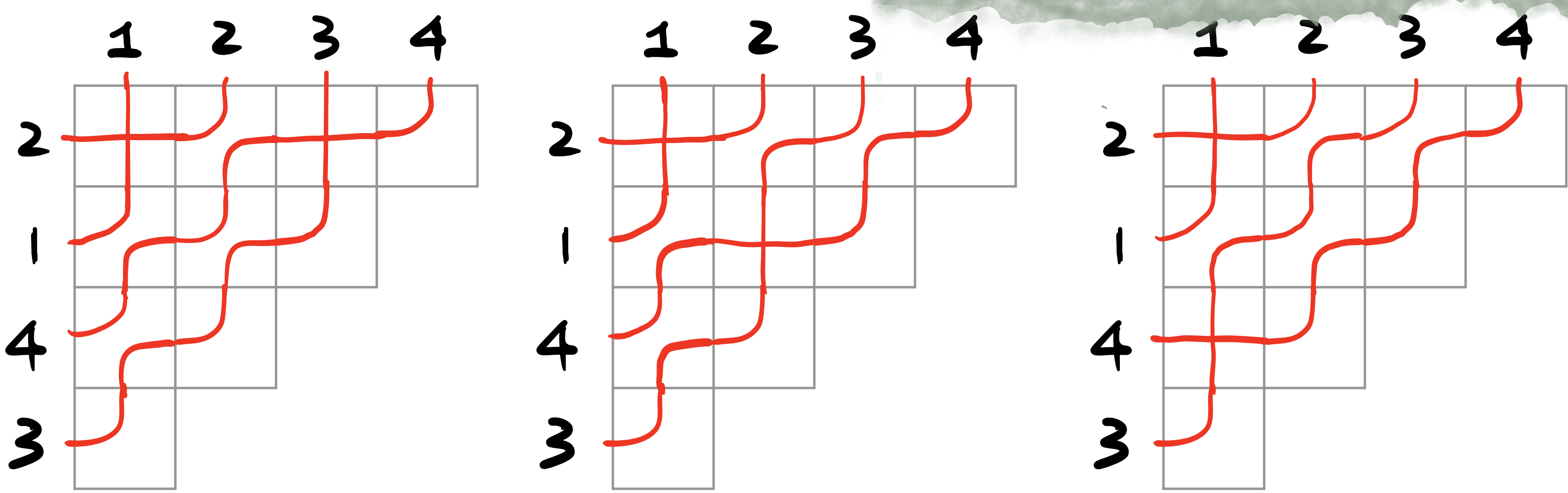


Pipedream [Bergeron-Billey] [Billey-Jockusch-Stanley]



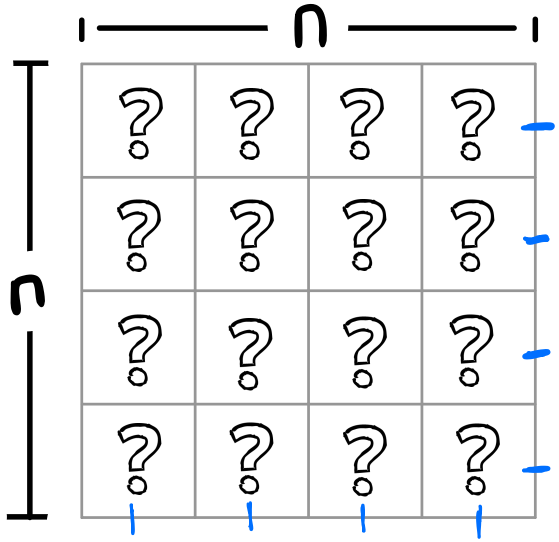


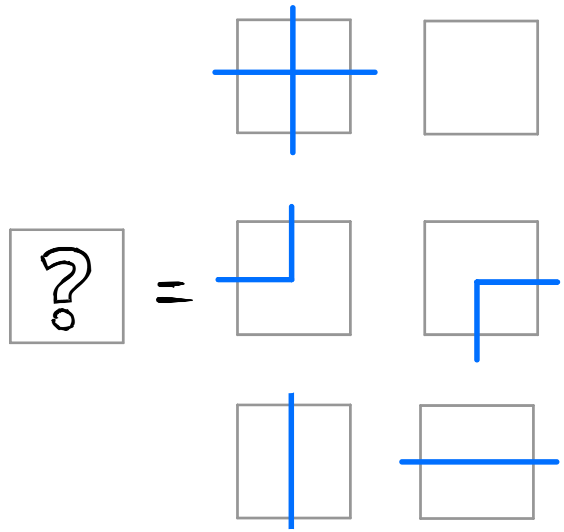


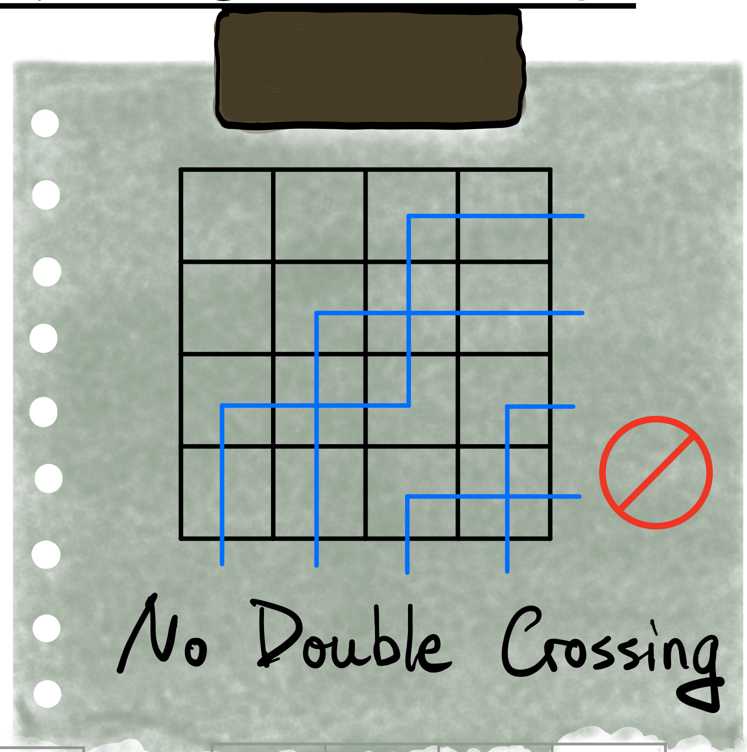


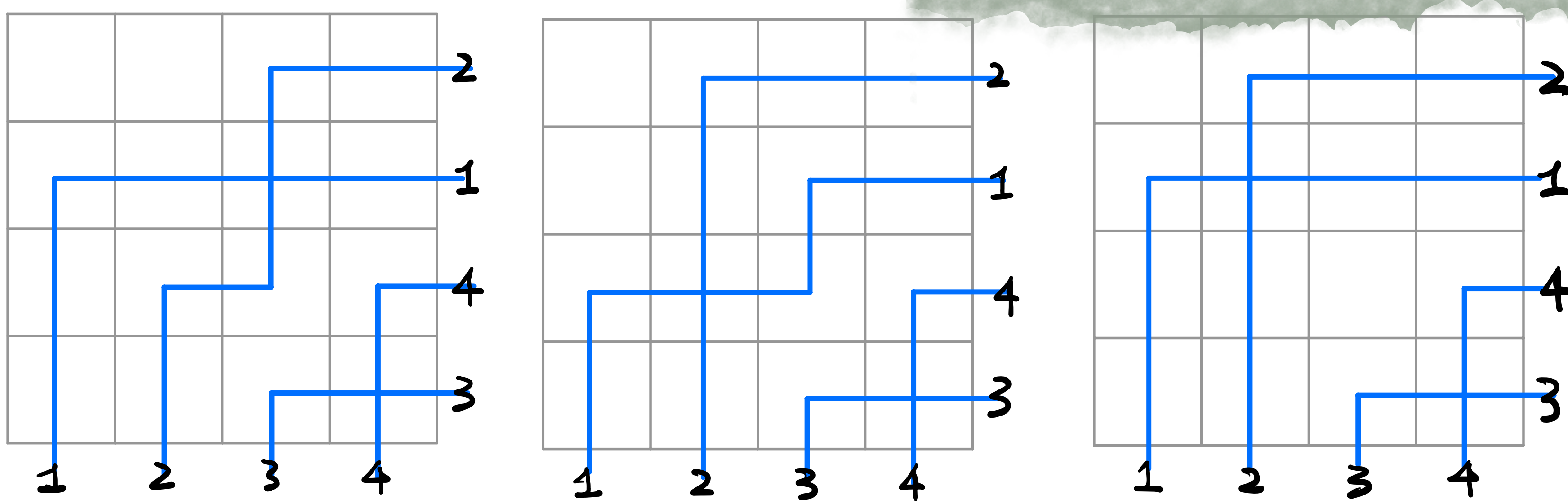
$$G_{2143} = x_1^2 + x_1 x_2 + x_1 x_3$$

Bumpless Pipedreams (BPD) [Lam-Lee-Shimozono]



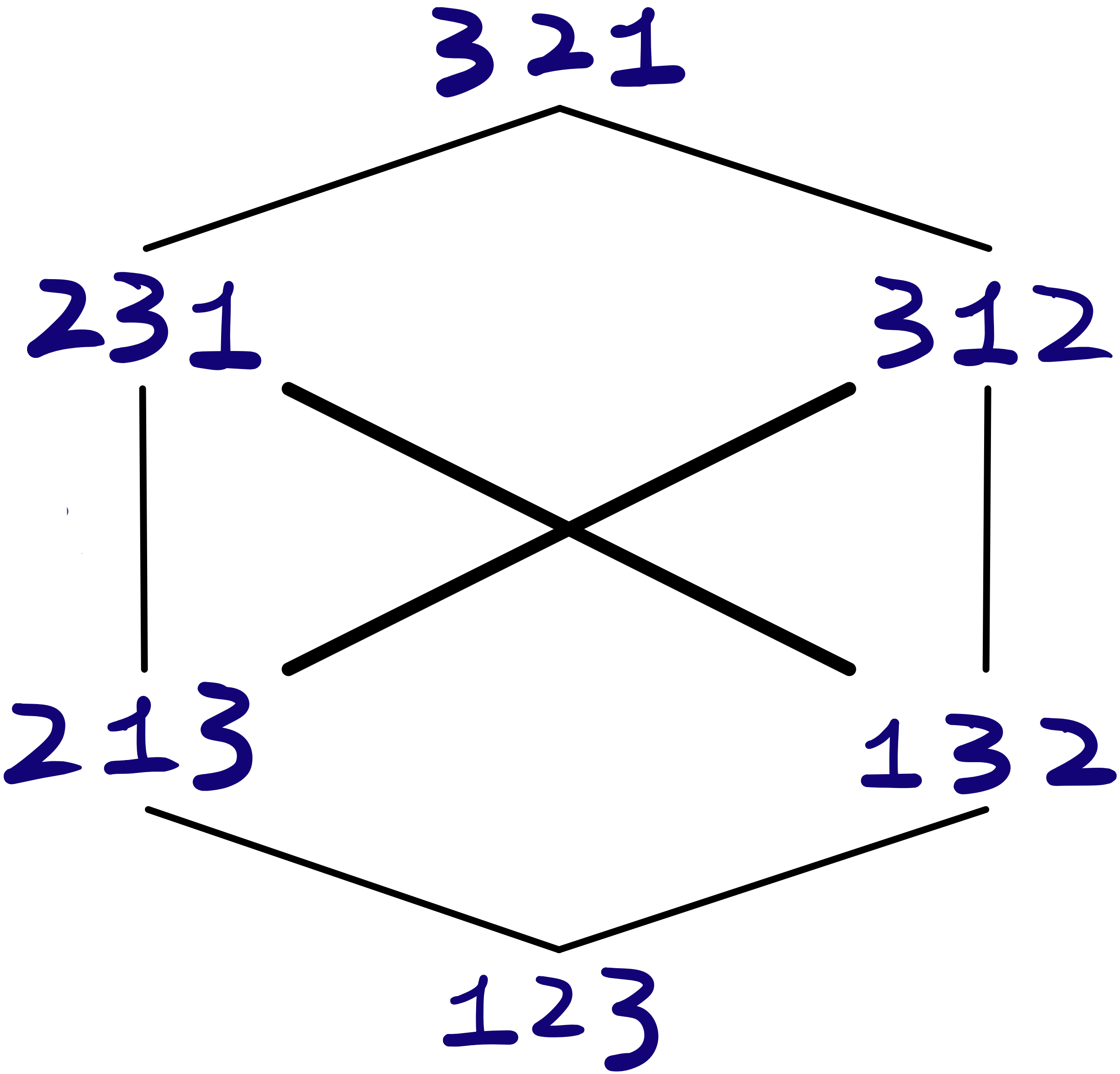






$$\mathfrak{S}_{2143} = x_1^2 + x_1 x_2 + x_1 x_3$$

Bruhat Order



$312 \lessdot 321$

$132 \lessdot 312$

$213 \lessdot 312$

. . . .

$w \lessdot w t_{ij}$ if

$i<j$, $w(i)<w(j)$

and #s in between are not in between

Labeled Bruhat chain

Suppose $w \lessdot w t_{ij}$.
We may label

$$w \xrightarrow{k} w t_{ij}$$

where $i \leq k \leq j-1$

A k-chain is :

$$w_1 \xrightarrow{k} w_2 \xrightarrow{k} w_3 \xrightarrow{k} w_4$$

$$21453 \xrightarrow{2} 24153 \xrightarrow{2} 34152 \xrightarrow{2} 35142$$

is a 2-chain of length 3.

Increasing k-chain

$$21|453 \xrightarrow{2} 24|153 \xrightarrow{2} 34|152 \xrightarrow{2} 35|142$$

is an increasing 2-chain.

(Small # being swapped increases.)

FACT:

For $u, w \in S_n$, $k \in [n-1]$, exists ≤ 1 increasing k-chain from u to w.

Bruhat Chain Formula [Bergeron - Sottile]

$$2143 \xrightarrow{\text{inc } 1} 4123 \xrightarrow{\text{inc } 2} 4312 \xrightarrow{\text{inc } 3} 4321$$

$$2143 \xrightarrow{\text{inc } 1} 4132 \xrightarrow{\text{inc } 2} 4312 \xrightarrow{\text{inc } 3} 4321$$

$$2143 \xrightarrow{\text{inc } 1} 4132 \xrightarrow{\text{inc } 2} 4321 \xrightarrow{\text{inc } 3} 4321$$

Bruhat Chain Formula [Bergeron - Sottile]

$$\prod_i x_i^{|\#\text{s on the right fixed by the } i\text{-chain}|}$$

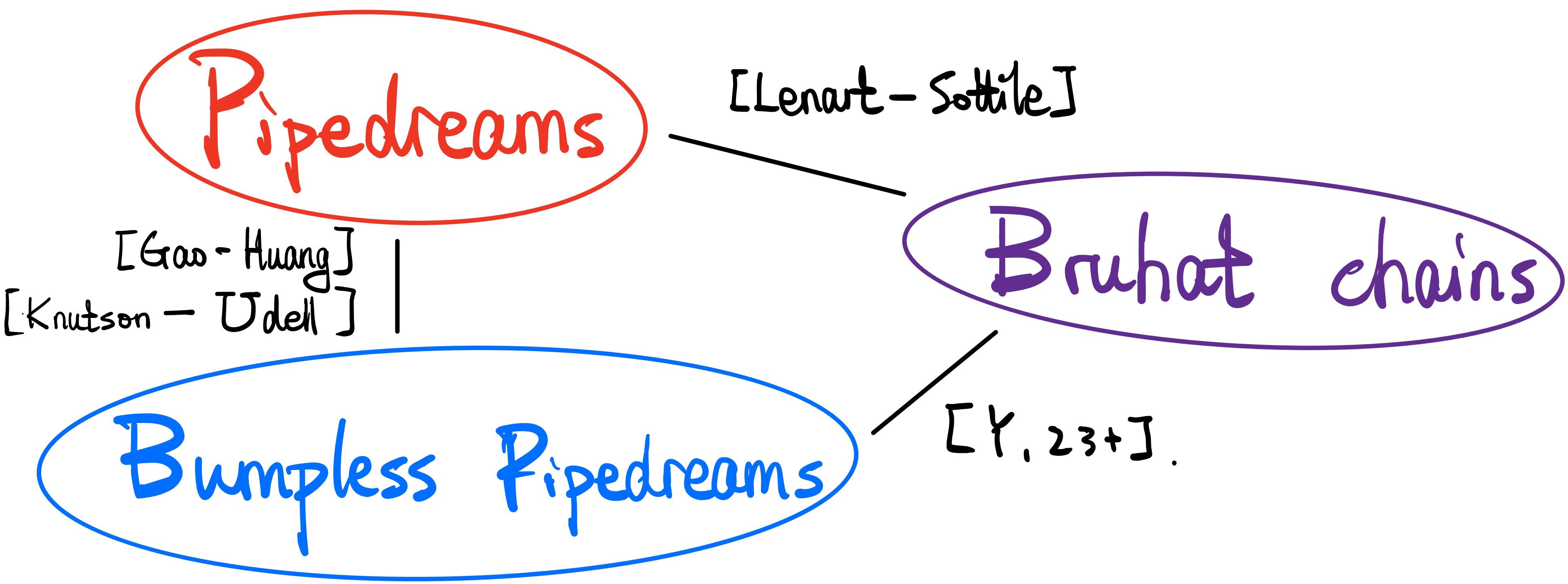
$$2|143 \xrightarrow[x_1^2]{\text{inc } 1} 41|23 \xrightarrow{\text{inc } 2} 431|2 \xrightarrow{\text{inc } 3} 4321$$

$$2|143 \xrightarrow[x_1]{\text{inc } 1} 41|32 \xrightarrow[x_2]{\text{inc } 2} 431|2 \xrightarrow{\text{inc } 3} 4321$$

$$2|143 \xrightarrow[x_1]{\text{inc } 1} 41|32 \xrightarrow{\text{inc } 2} 432|1 \xrightarrow[x_3]{\text{inc } 3} 4321$$

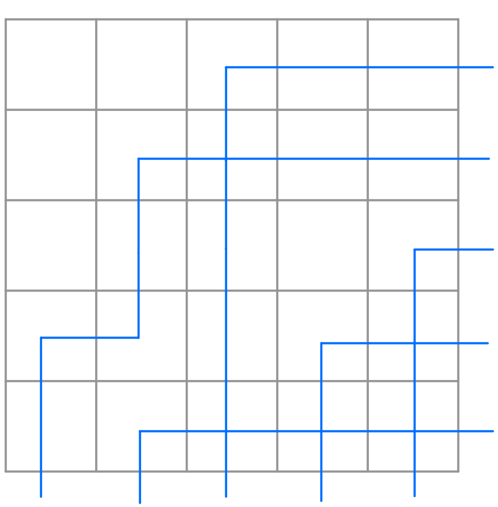
Outline

- 3 Formulas of Schubert polynomials
- and their connections



Bijection between PDs and BPDs

- [Gao - Huang]

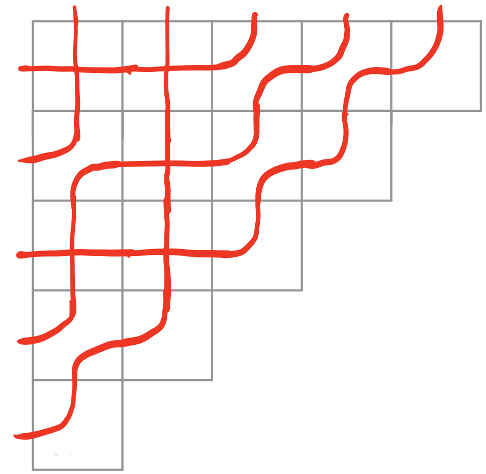


$\rightleftharpoons$

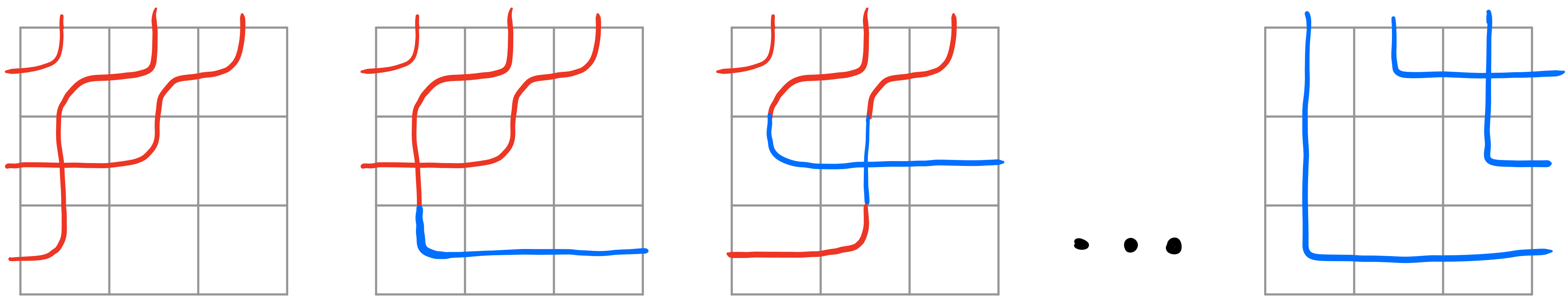
2 1 3 4 3

1 1 2 3 3

$\rightleftharpoons$

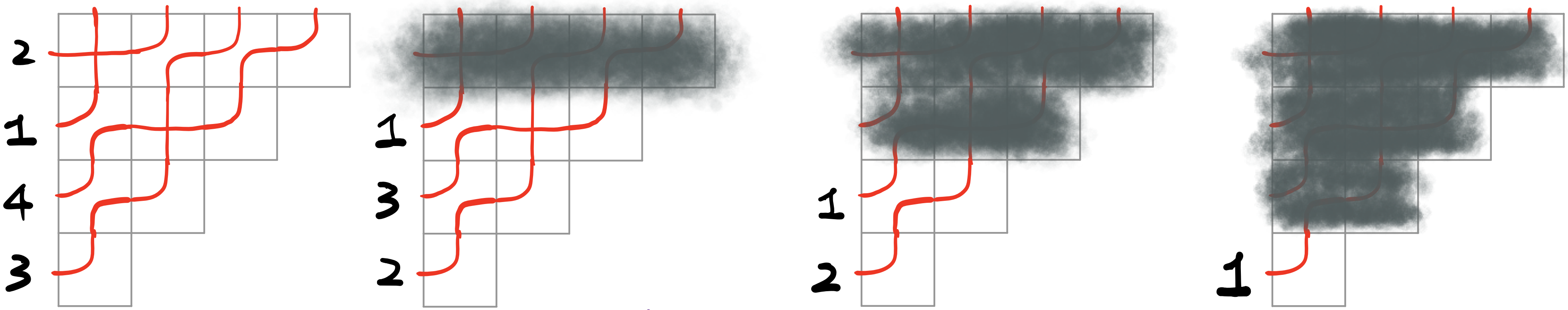


- [Knutson – Udell]



Thm [Knutson – Udell] The two bijections agree.

Bijection between PDs and chains



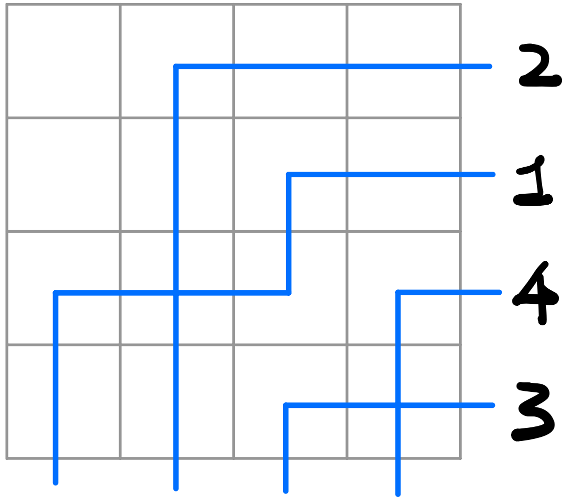
$$2143 \xrightarrow{\text{inc } 1} \underline{4}132 \xrightarrow{\text{inc } 2} \underline{43}12 \xrightarrow{\text{inc } 3} \underline{432}1 = w_0$$

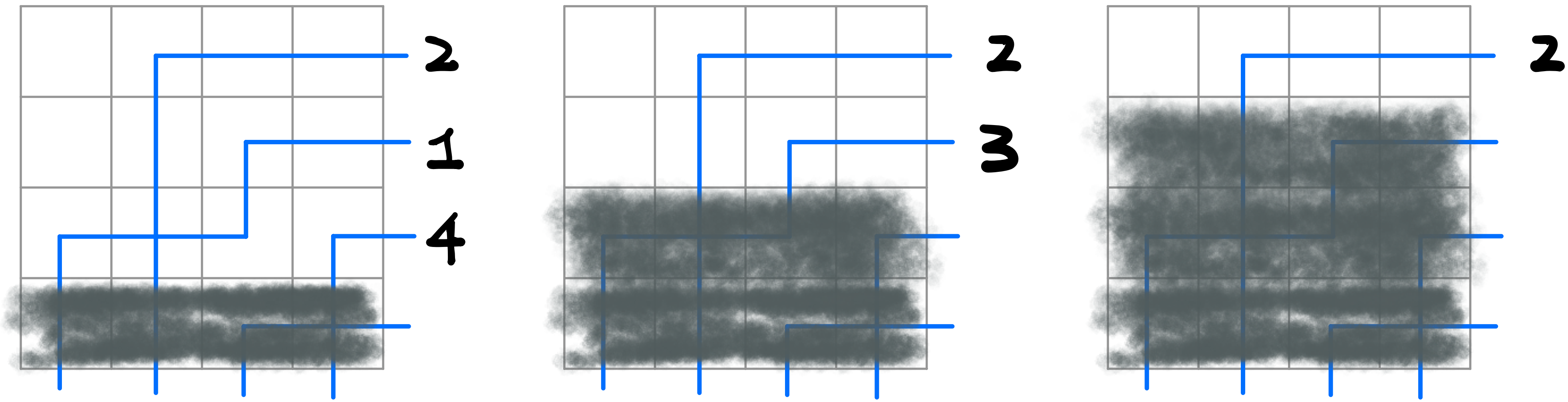
Thm [Lenart – Sottile]

This is a wt-preserving bijection between $PD(w)$

and $w \xrightarrow{\text{inc } 1} \bullet \xrightarrow{\text{inc } 2} \cdots \xrightarrow{\text{inc } n-2} \bullet \xrightarrow{\text{inc } n-1} w_0$

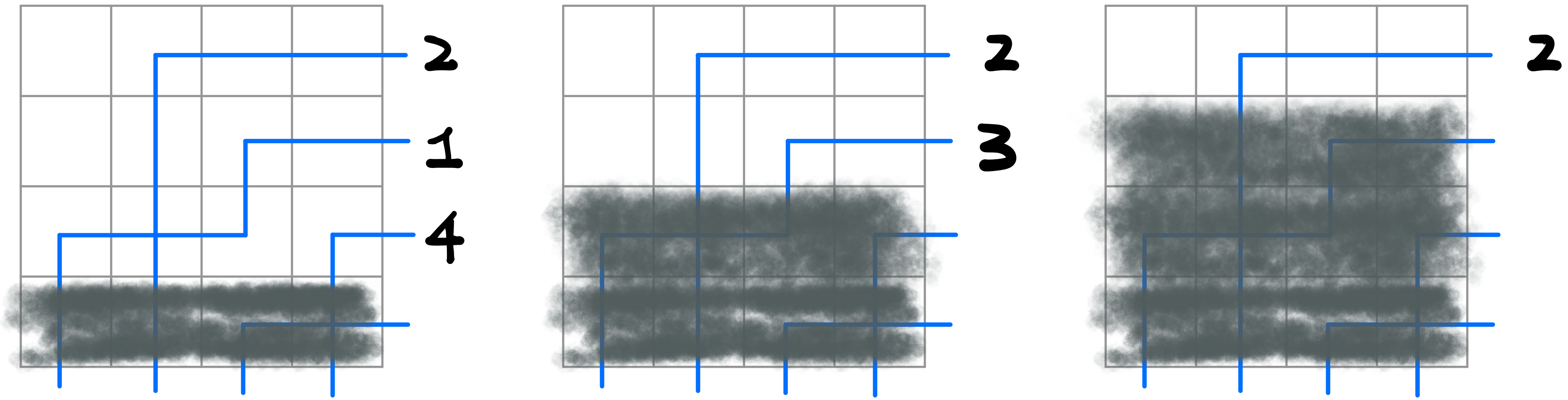
BPDs to chains?





BPDs to chains?

Append missing #s in decreasing order?

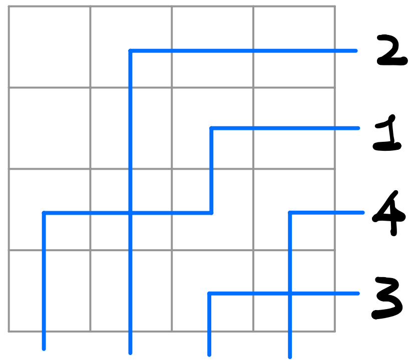


2143   2341   2431

We get :

$2143 \longrightarrow 2341 \longrightarrow 2431 \longrightarrow 4321$

Wait! What did we get?

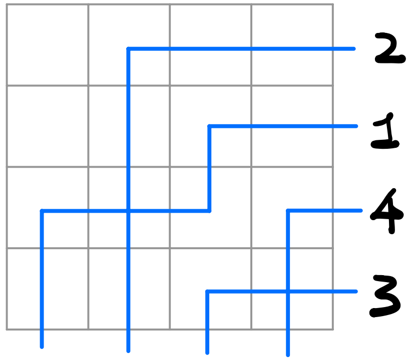


$$2143 \xrightarrow{\text{inc } 3} 2341 \xrightarrow{\text{inc } 2} 2431 \xrightarrow{\text{inc } 1} 4321$$

Thm [Y, 23+] It is a bijection between

BPD(w) and $w \xrightarrow{\text{inc } n-1} \bullet \xrightarrow{\text{inc } n-2} \cdots \xrightarrow{\text{inc } 2} \bullet \xrightarrow{\text{inc } 1} w_0$

Wait! What did we get?



$\longrightarrow x_1 x_2$

$$2143 \xrightarrow{\text{inc } 3} 2341 \xrightarrow{\text{inc } 2} 2431 \xrightarrow{\text{inc } 1} 4321$$

x_2 $\qquad$ x_1

Thm [Y, 23+] It is a $\overbrace{\text{bijection}}^{\text{weight-preserving}}$ between

BPD(w) and $w \xrightarrow{\text{inc } n-1} \cdot \xrightarrow{\text{inc } n-2} \cdots \xrightarrow{\text{inc } 2} \cdot \xrightarrow{\text{inc } 1} w_0$

FACT

Fix $u, w \in S_n$, $k_1, k_2 \in [n-1]$, $a, b \in \mathbb{Z}_{\geq 0}$,

\# of $v \in S_n$ such that

$$u \xrightarrow[\text{length } a]{\text{inc. } k_1} v \xrightarrow[\text{length } b]{\text{inc. } k_2} w$$

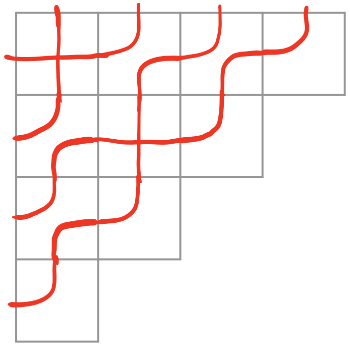
$=$ \# of $v' \in S_n$ such that

$$u \xrightarrow[\text{length } b]{\text{inc. } k_2} v' \xrightarrow[\text{length } a]{\text{inc. } k_1} w$$

(By Sottile's Pieri rule of $\mathfrak{S}_u$).

OPEN: Find a bijection?

2+(n-1)! Formulas of Schubert polynomials



[Lenart-Sottile] ⟷

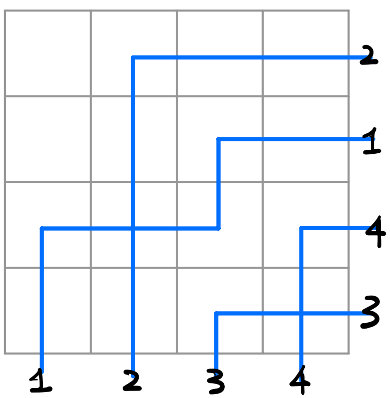
$$2143 \xrightarrow{\text{inc } 1} 4132 \xrightarrow{\text{inc } 2} 4312 \xrightarrow{\text{inc } 3} 4321$$

$$2143 \xrightarrow{\text{inc } 1} 4132 \xrightarrow{\text{inc } 3} 4231 \xrightarrow{\text{inc } 2} 4321$$

$$2143 \xrightarrow{\text{inc } 2} 2413 \xrightarrow{\text{inc } 1} 4312 \xrightarrow{\text{inc } 3} 4321$$

$$2143 \xrightarrow{\text{inc } 2} 2413 \xrightarrow{\text{inc } 3} 2431 \xrightarrow{\text{inc } 1} 4321$$

$$2143 \xrightarrow{\text{inc } 3} 2341 \xrightarrow{\text{inc } 1} 4231 \xrightarrow{\text{inc } 2} 4321$$



[Y] ⟷

$$2143 \xrightarrow{\text{inc } 3} 2341 \xrightarrow{\text{inc } 2} 2431 \xrightarrow{\text{inc } 1} 4321$$

New bijection between PDs and BPDs

Prop [Y, 23+]

Lenart's growth diagram gives a bijection

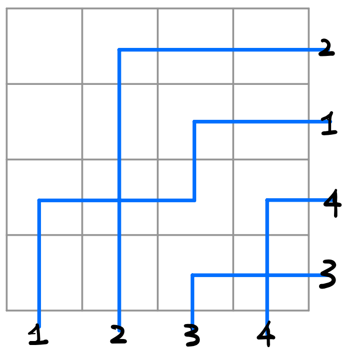
$$\left\{ u \xrightarrow[\text{length } a]{\text{inc. } k} v \xrightarrow[\text{length } b]{\text{inc. } l} w \right\}$$

$$\left\{ u \xrightarrow[\text{length } b]{\text{inc. } l} v' \xrightarrow[\text{length } a]{\text{inc. } k} w \right\}$$

if $w(k+1) > w(k+2) > \cdots$

$$\begin{tikzcd}
v' \arrow[r, dashed] \arrow[dr, phantom, "\boxed{\text{LENART}}"] & w \\
u \arrow[u, dashed] \arrow[r, "\text{inc } k"'] & v \arrow[u, "\text{inc } 1"']
\end{tikzcd}$$

New bijection between PDs and BPDs

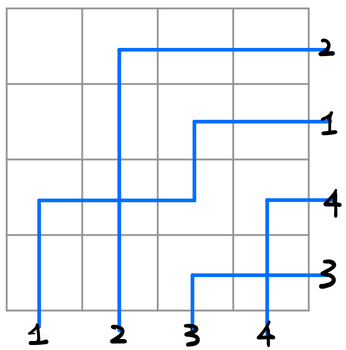


$$2143 \xrightarrow{\text{inc } 3} 2341 \xrightarrow{\text{inc } 2} 2431 \xrightarrow{\text{inc } 1} 4321$$

$$2143 \xrightarrow{\text{inc } 3} 2341 \xrightarrow{\text{inc } 1} 4231 \xrightarrow{\text{inc } 2} 4321$$

$$2143 \xrightarrow{\text{inc } 1} 4132 \xrightarrow{\text{inc } 3} 4231 \xrightarrow{\text{inc } 2} 4321$$

New bijection between PDs and BPDs

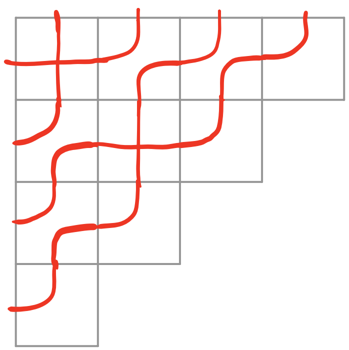


$$2143 \xrightarrow{\text{inc } 3} 2341 \xrightarrow{\text{inc } 2} 2431 \xrightarrow{\text{inc } 1} 4321$$

Conjecture: It agrees w/ [Gao-Huang].

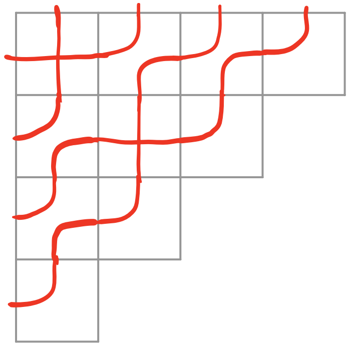
$$2143 \xrightarrow{\text{inc } 3} 2341 \xrightarrow{\text{inc } 1} 4231 \xrightarrow{\text{inc } 2} 4321$$

$$2143 \xrightarrow{\text{inc } 1} 4132 \xrightarrow{\text{inc } 3} 4231 \xrightarrow{\text{inc } 2} 4321$$



$$2143 \xrightarrow{\text{inc } 1} 4132 \xrightarrow{\text{inc } 2} 4312 \xrightarrow{\text{inc } 3} 4321$$

Future Direction 1: More pipe formulas?



[Lenart-Sottile] $\longleftrightarrow$

$2143 \xrightarrow{\text{inc } 1} 4132 \xrightarrow{\text{inc } 2} 4312 \xrightarrow{\text{inc } 3} 4321$

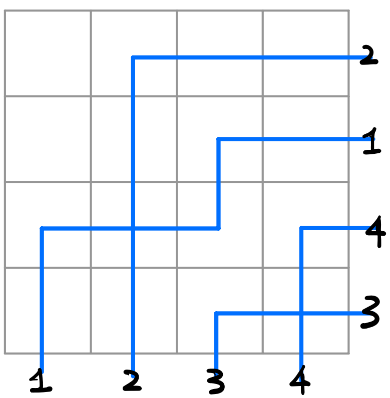
$2143 \xrightarrow{\text{inc } 1} 4132 \xrightarrow{\text{inc } 3} 4231 \xrightarrow{\text{inc } 2} 4321$

Can they be represented as pipes?

$2143 \xrightarrow{\text{inc } 2} 2413 \xrightarrow{\text{inc } 1} 4312 \xrightarrow{\text{inc } 3} 4321$

$2143 \xrightarrow{\text{inc } 2} 2413 \xrightarrow{\text{inc } 3} 2431 \xrightarrow{\text{inc } 1} 4321$

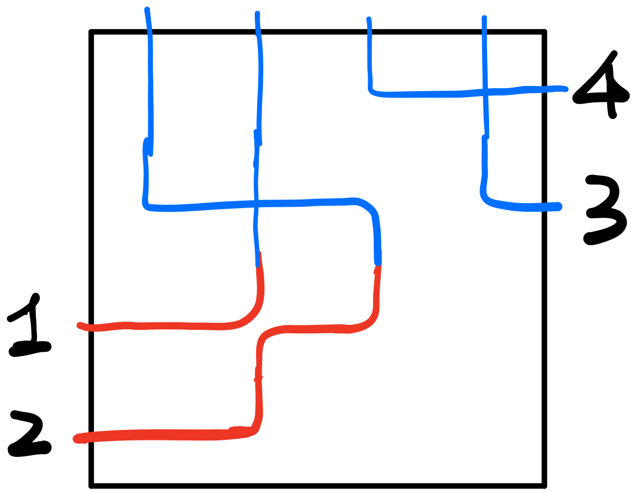
$2143 \xrightarrow{\text{inc } 3} 2341 \xrightarrow{\text{inc } 1} 4231 \xrightarrow{\text{inc } 2} 4321$



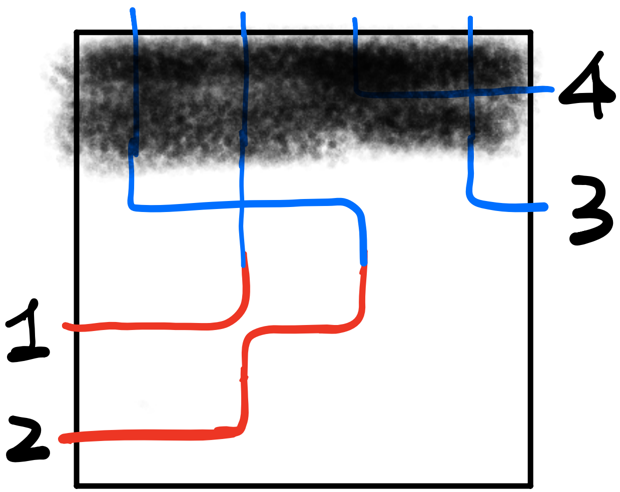
[Y] $\longleftrightarrow$

$2143 \xrightarrow{\text{inc } 3} 2341 \xrightarrow{\text{inc } 2} 2431 \xrightarrow{\text{inc } 1} 4321$

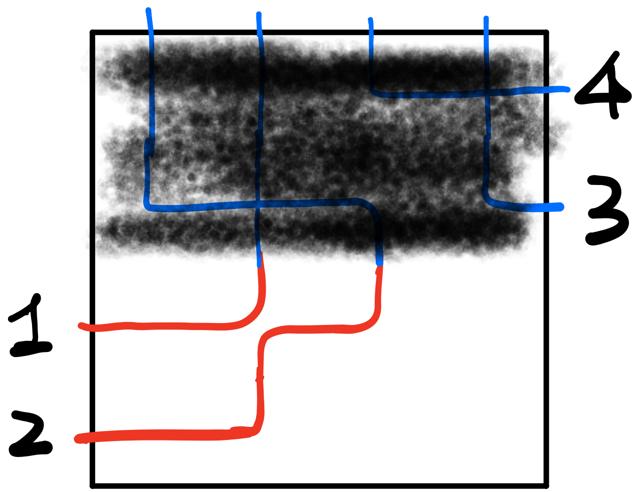
Hybrid Pipedreams [Knutson - Udell]



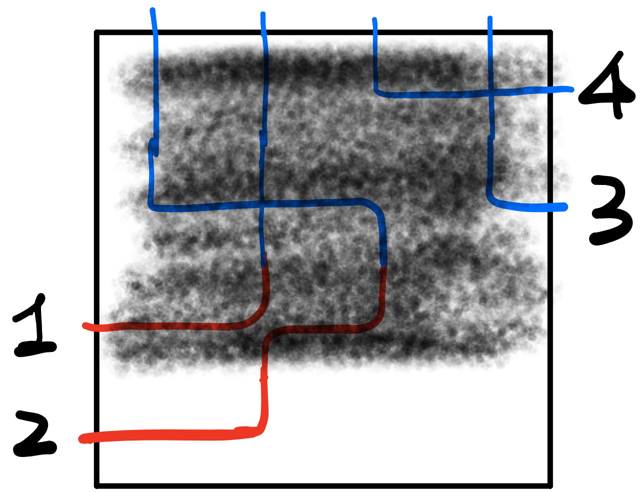
Hybrid Pipedreams correspond to 2^{n-2} chain formulas.



2143



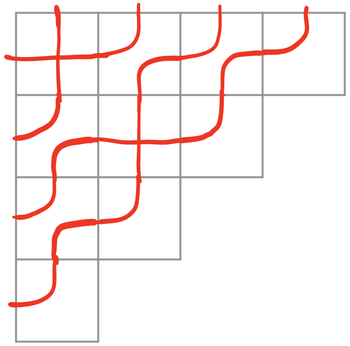
2341



4231

$2\,1\,4\,|\,3 \xrightarrow{\text{inc } 3} 2\,|\,3\,4\,1 \xrightarrow{\text{inc } 1} 4\,2\,3\,1 \xrightarrow{\text{inc } 2} 4\,3\,2\,1$

Future Direction 1: More pipe formulas?



[Lenart-Sottile]

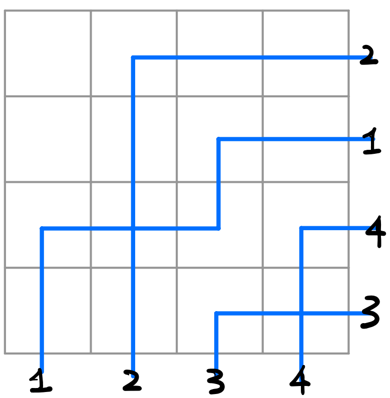
$2143 \xrightarrow{\text{inc } 1} 4132 \xrightarrow{\text{inc } 2} 4312 \xrightarrow{\text{inc } 3} 4321$

$2143 \xrightarrow{\text{inc } 1} 4132 \xrightarrow{\text{inc } 3} 4231 \xrightarrow{\text{inc } 2} 4321$

$2143 \xrightarrow{\text{inc } 2} 2413 \xrightarrow{\text{inc } 1} 4312 \xrightarrow{\text{inc } 3} 4321$

$2143 \xrightarrow{\text{inc } 2} 2413 \xrightarrow{\text{inc } 3} 2431 \xrightarrow{\text{inc } 1} 4321$

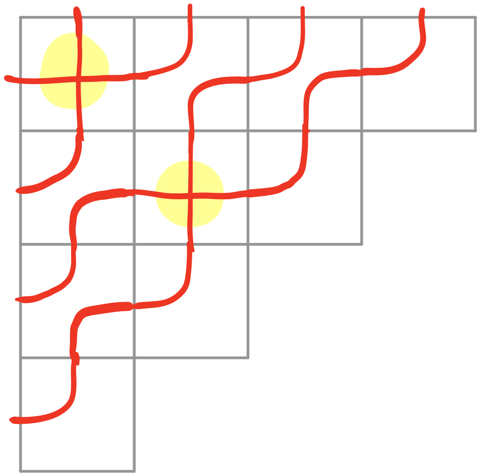
$2143 \xrightarrow{\text{inc } 3} 2341 \xrightarrow{\text{inc } 1} 4231 \xrightarrow{\text{inc } 2} 4321$



[Y]

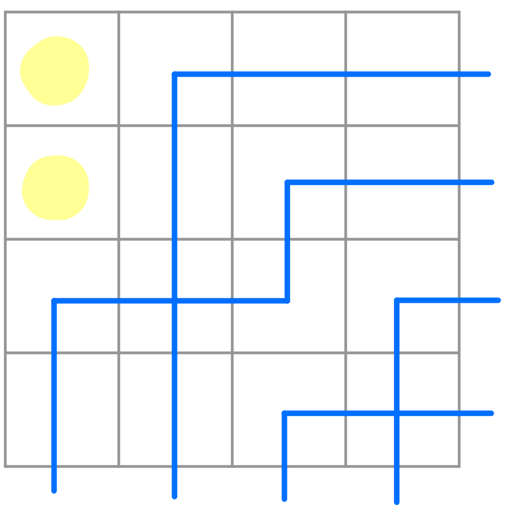
$2143 \xrightarrow{\text{inc } 3} 2341 \xrightarrow{\text{inc } 2} 2431 \xrightarrow{\text{inc } 1} 4321$

Future Direction 2: Double Schubert ?



$$(x_1 - y_1)(x_2 - y_2)$$

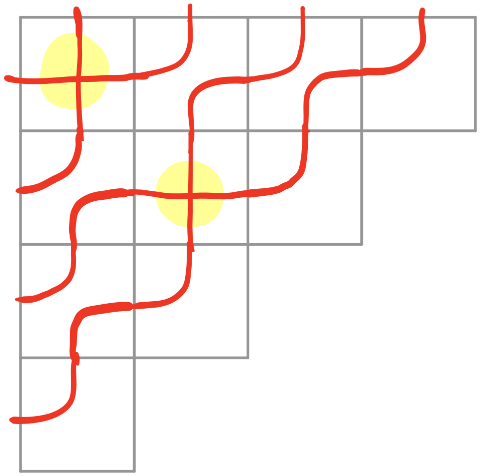
[Knutson - Miller]



$$(x_1 - y_1)(x_2 - y_1)$$

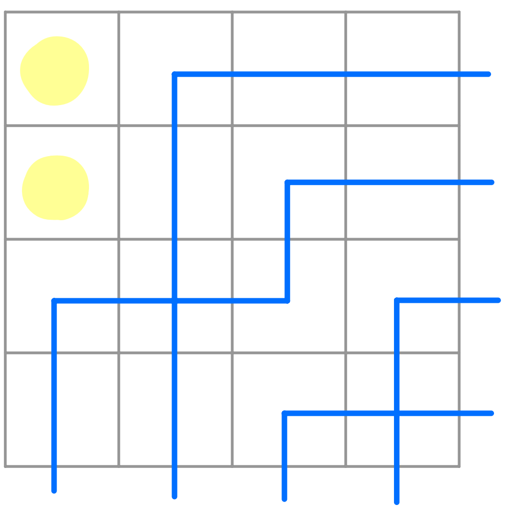
[Weigandt]

Future Direction 2: Double Schubert ?



$$(x_1 - y_1)(x_2 - y_2)$$

$2 \,|\, 1 43 \xrightarrow{\text{inc } 1} 41 \,|\, 32 \xrightarrow{\text{inc } 2} 431 \,|\, 2 \xrightarrow{\text{inc } 3} 4321$



$$(x_1 - y_1)(x_2 - y_1)$$

$214 \,|\, 3 \xrightarrow{\text{inc } 3} 23 \,|\, 41 \xrightarrow{\text{inc } 2} 2 \,|\, 431 \xrightarrow{\text{inc } 1} 4321$

Future Direction 2: Double Schubert ?

Conj. Those $(n-1)!$ formulas extend to double Schubert polynomials.

$$2143 \xrightarrow{\text{inc } 1} 4132 \xrightarrow{\text{inc } 3} 4231 \xrightarrow{\text{inc } 2} 4321 \qquad (x_1-y_1)(x_2-y_1)$$

$$2143 \xrightarrow{\text{inc } 2} 2413 \xrightarrow{\text{inc } 1} 4312 \xrightarrow{\text{inc } 3} 4321 \qquad (x_2-y_3)(x_1-y_1)$$

$$2143 \xrightarrow{\text{inc } 2} 2413 \xrightarrow{\text{inc } 3} 2431 \xrightarrow{\text{inc } 1} 4321 \qquad (x_2-y_3)(x_1-y_1)$$

$$2143 \xrightarrow{\text{inc } 3} 2341 \xrightarrow{\text{inc } 1} 4231 \xrightarrow{\text{inc } 2} 4321 \qquad (x_1-y_1)(x_2-y_1)$$

Thank Yibo Gao and Zachary Hamaker for valuable guidance.

Thank you for listening!