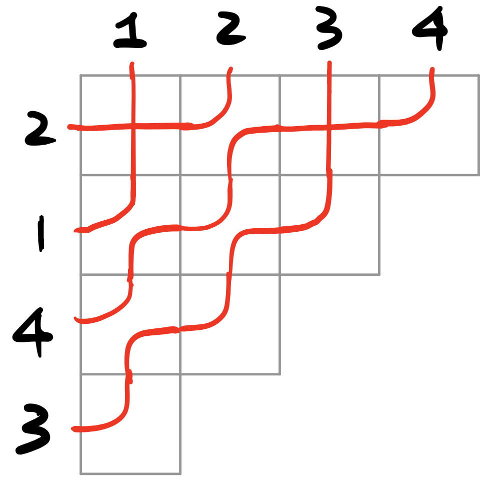
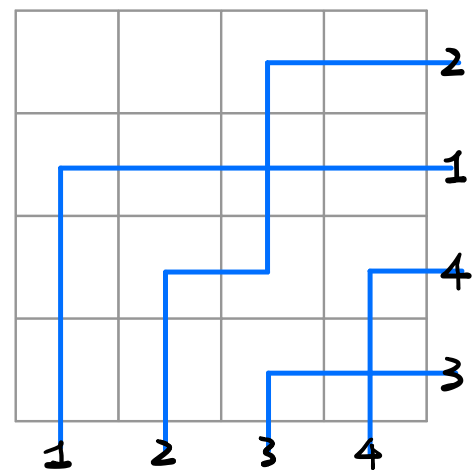
Analogue of Fomin-Stanley Algebra on Bumpless Pipedreams

Tianyi Yu (UC San Diego)

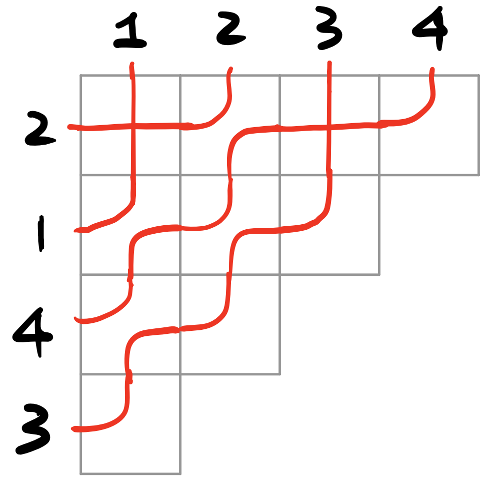
Combinatorial Algebra meets Algebraic Combinatorics



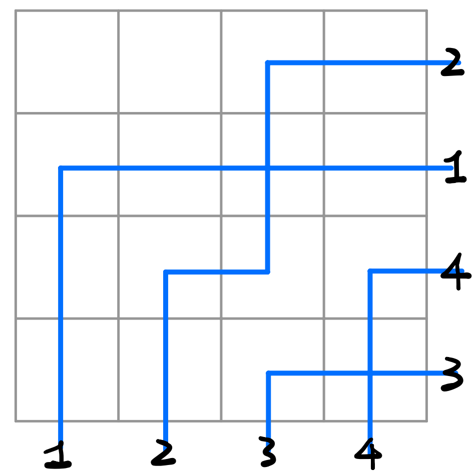


Combinatorial Algebra meets Algebraic Combinatorics

Nil-Coexter Algebra

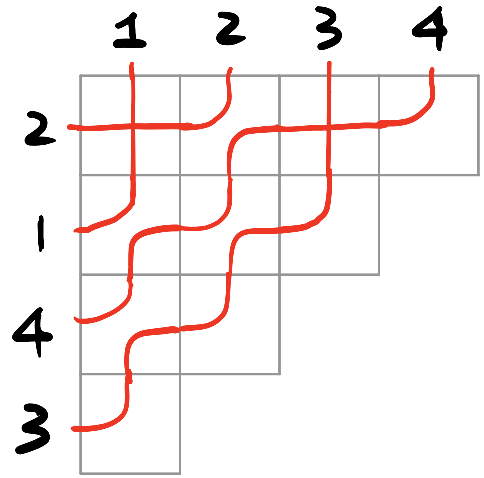


Fomin-Kirillov Algebra

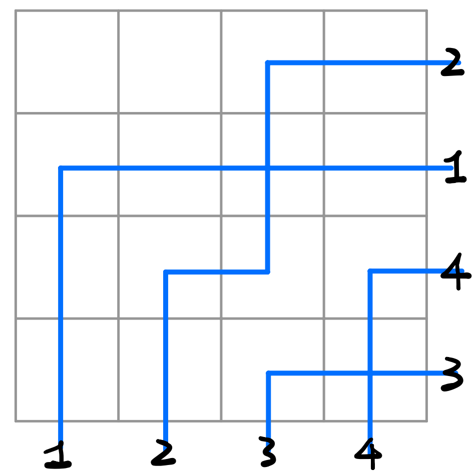


Combinatorial Algebra meets Algebraic Combinatorics

Nil-Coexter Algebra [Fomin-Stanley]



Fomin-Kirillov Algebra [Y, 24+]



Combinatorial Algebra meets Algebraic Combinatorics

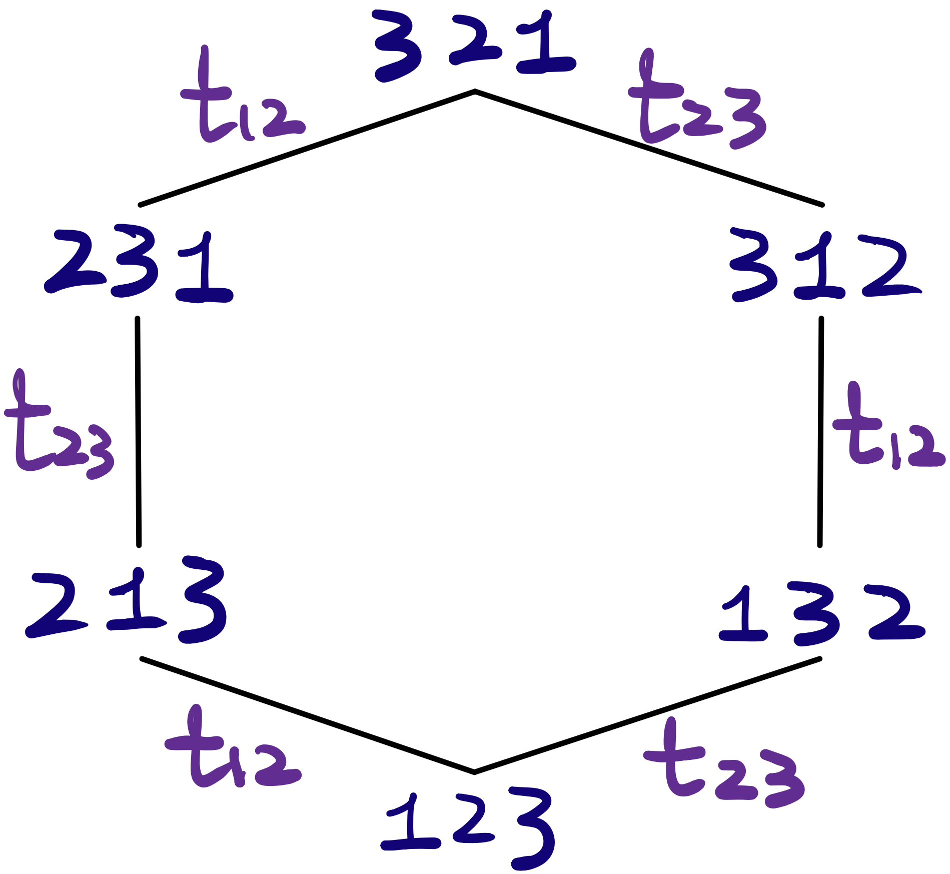
Schubert Polynomial $\mathfrak{S}_w$.

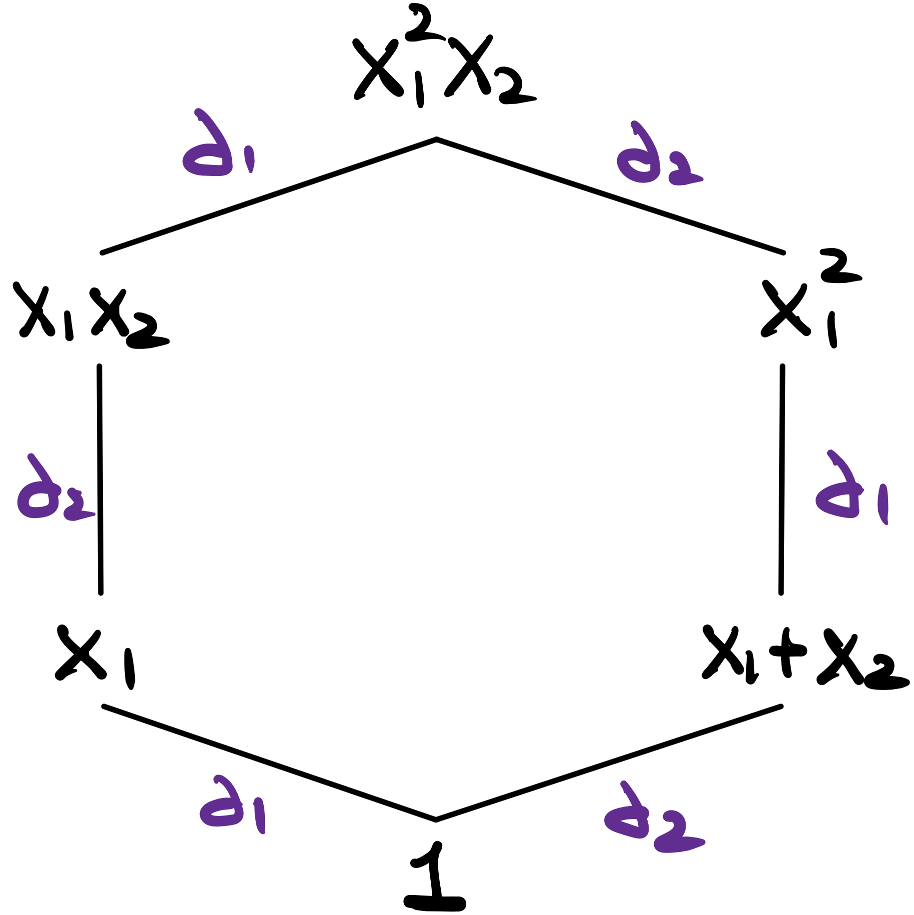
Define ∂_i on polynomials : $\partial_i(f) = \dfrac{f - s_i f}{x_i - x_{i+1}}$

For $w \in S_n$, define $\mathfrak{S}_w$.

If $w_0 = [n, n-1, \ldots, 2, 1]$,
$\mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-2}^{2} x_{n-1}$

$\partial_i(\mathfrak{S}_w) = \mathfrak{S}_{w t_{i,i+1}}$ if $w(i) > w(i+1)$

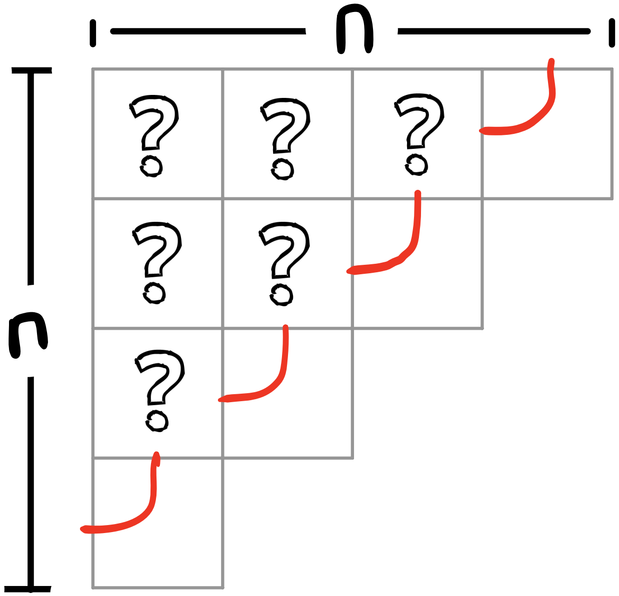


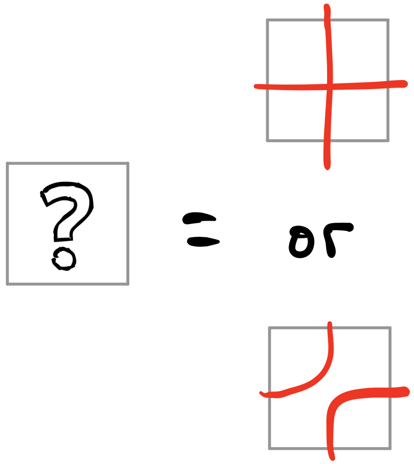


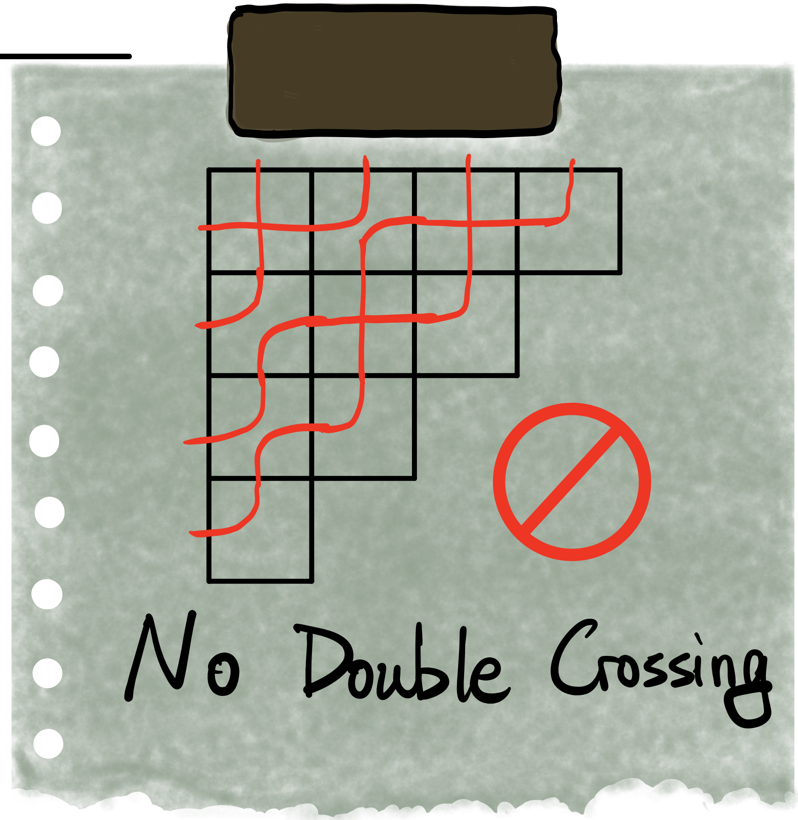
Running Example:

$$\mathfrak{G}_{2143} = x_1^2 + x_1 x_2 + x_1 x_3$$

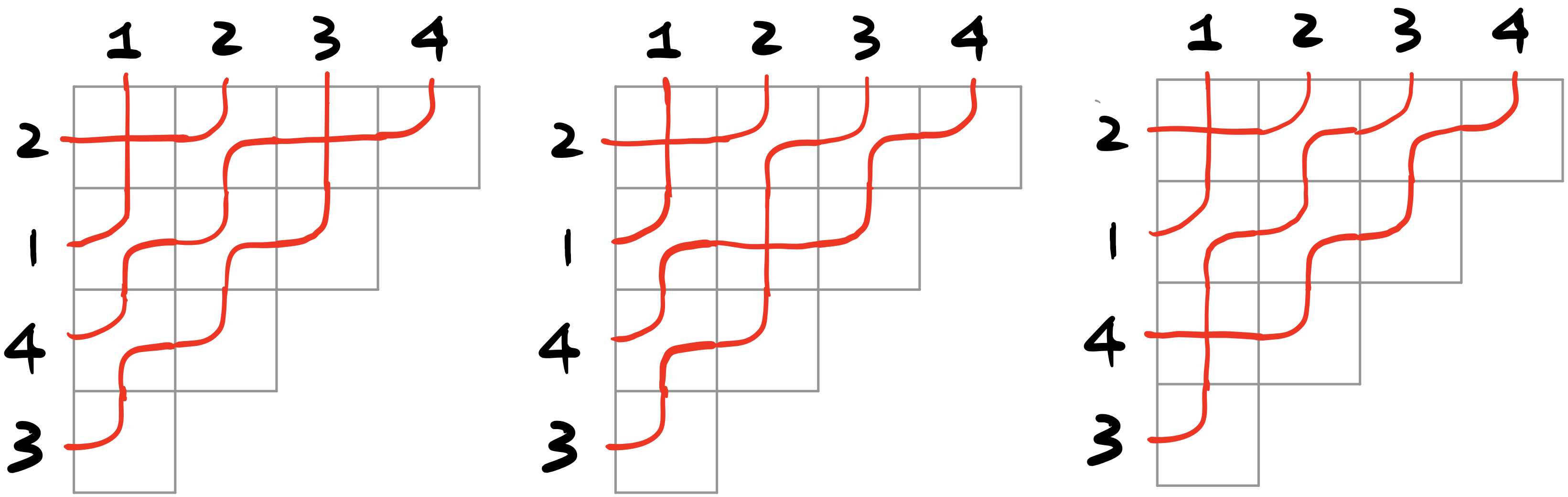
Pipedream [Bergeron-Billey]





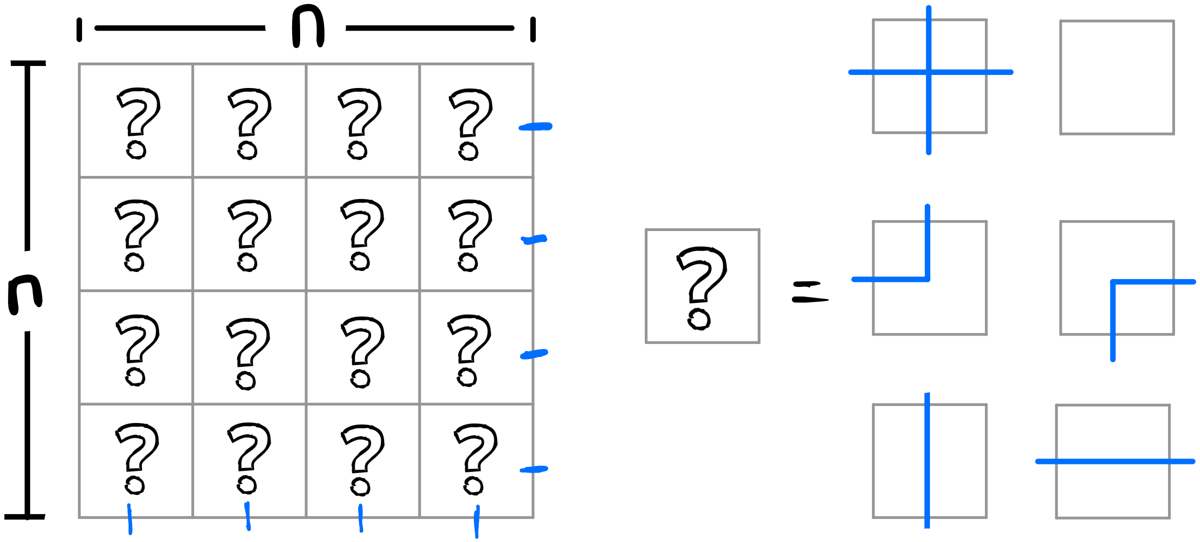


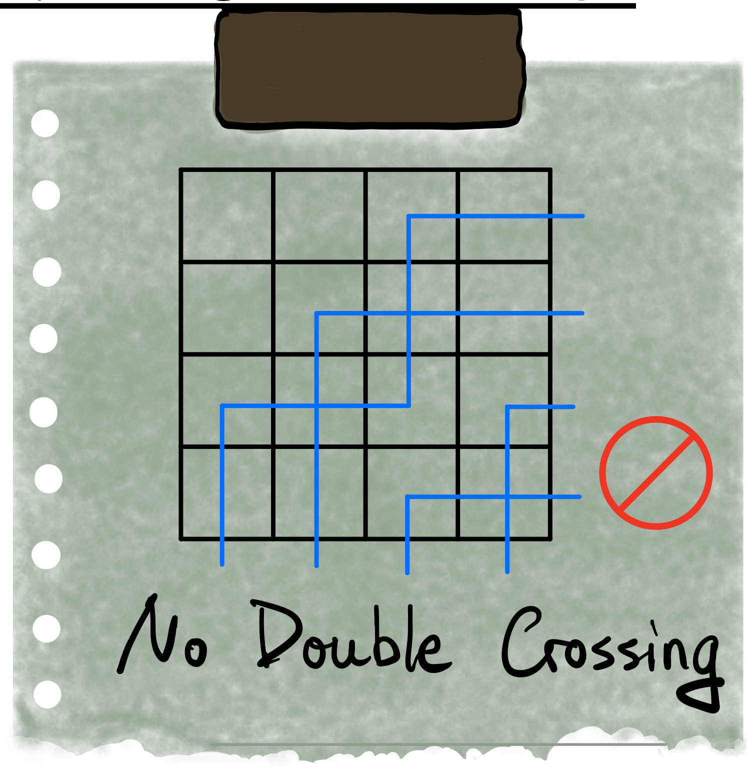
Pipedream [Bergeron - Billey]



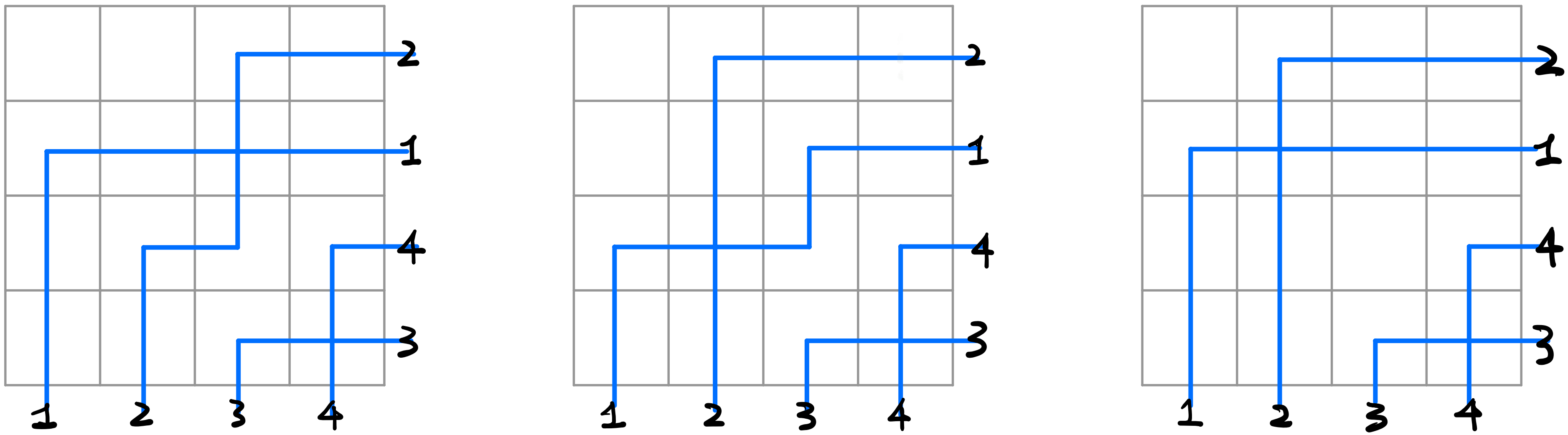
$$\mathfrak{S}_{2143} = x_1^2 + x_1 x_2 + x_1 x_3$$

Bumpless Pipedreams (BPD) [Lam-Lee-Shimozono]





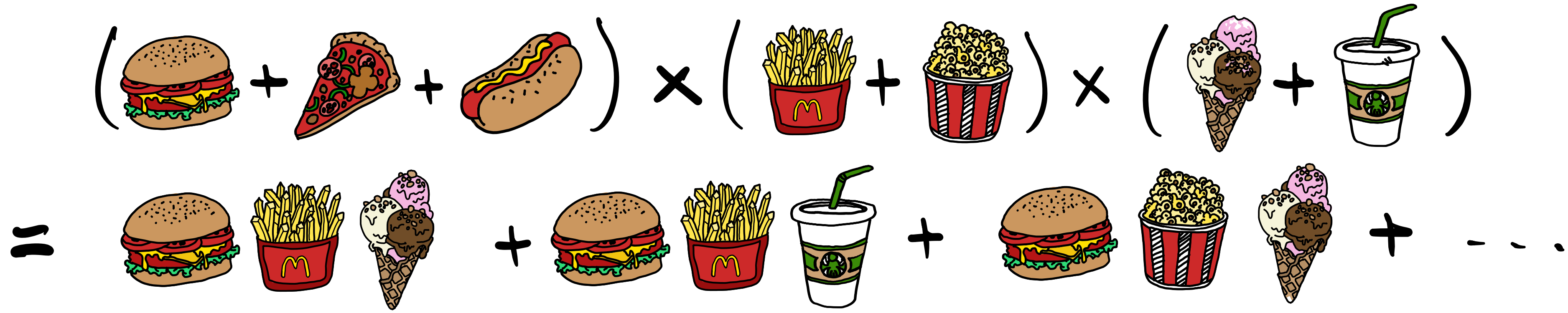
Bumpless Pipedreams (BPD) [Lam-Lee-Shimozono]

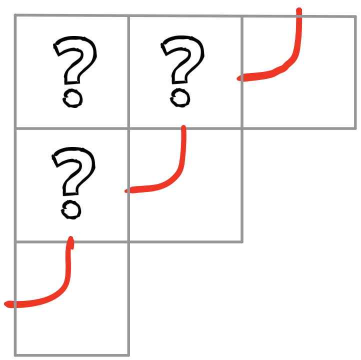


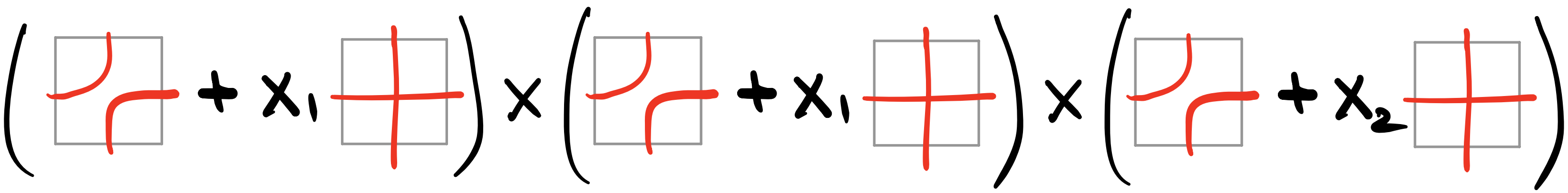
$$\mathfrak{S}_{2143} = x_1^2 + x_1x_2 + x_1x_3$$

Fomin-Stanley Algebra

- Use one expression to include all possible pipedreams.





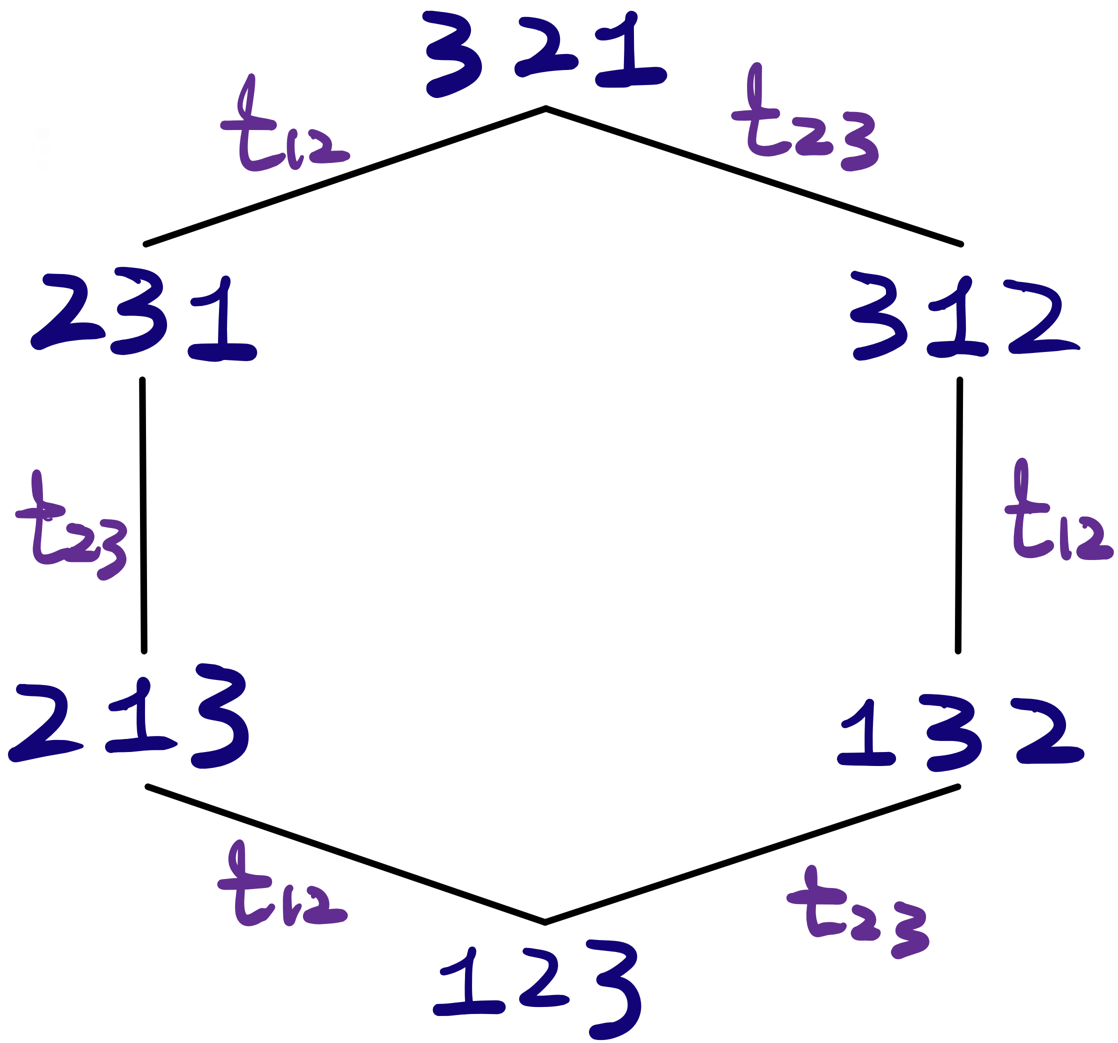


[Food stickers by gabby_scarba11]

Nil-Coxeter Algebra [Fomin-Stanley]

Generated by $u_1, u_2, \ldots, u_{n-1}$

- $u_i u_i = 0$
- $u_i u_j = u_j u_i$ if $|i-j| > 1$.
- $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$

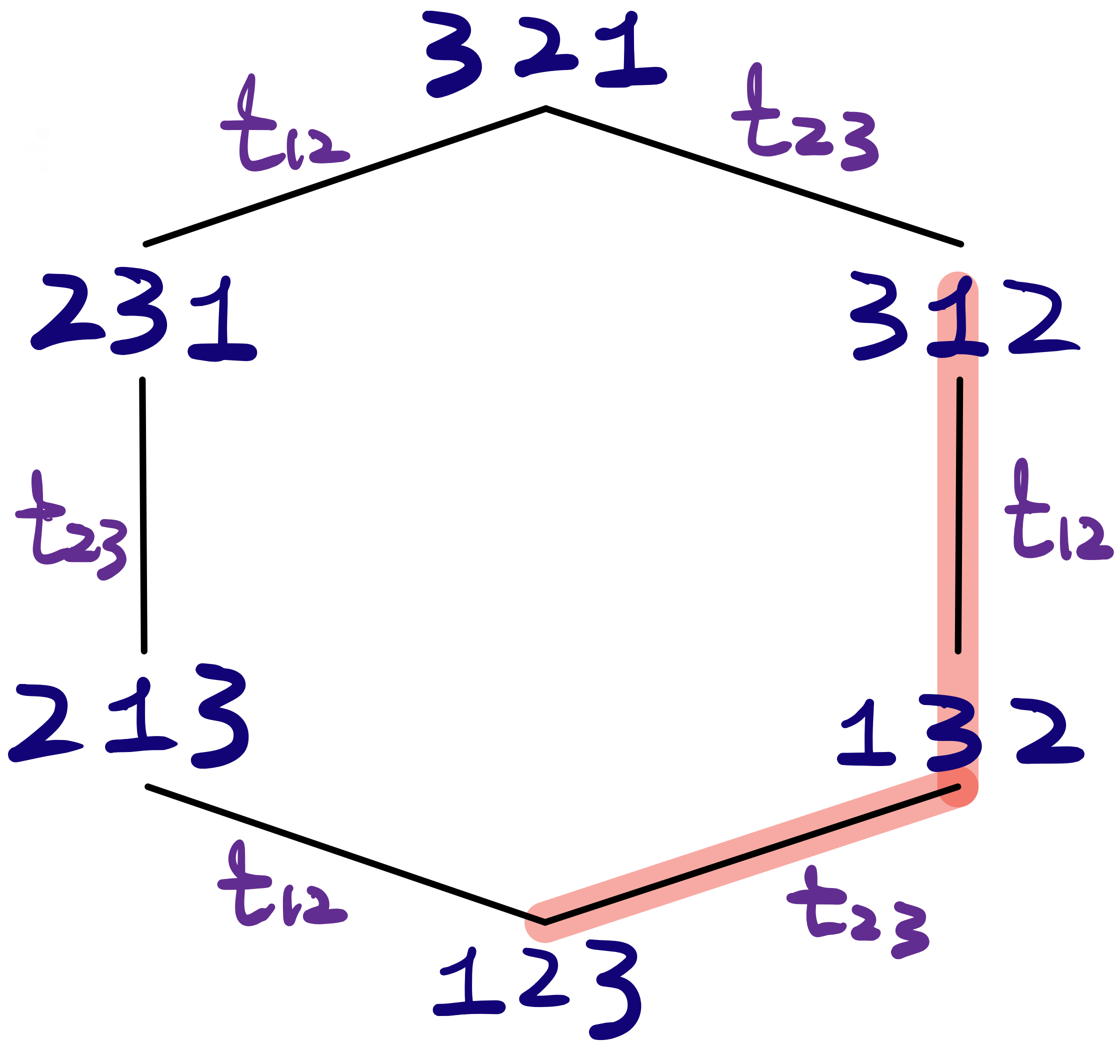


Nil-Coxeter Algebra [Fomin-Stanley]

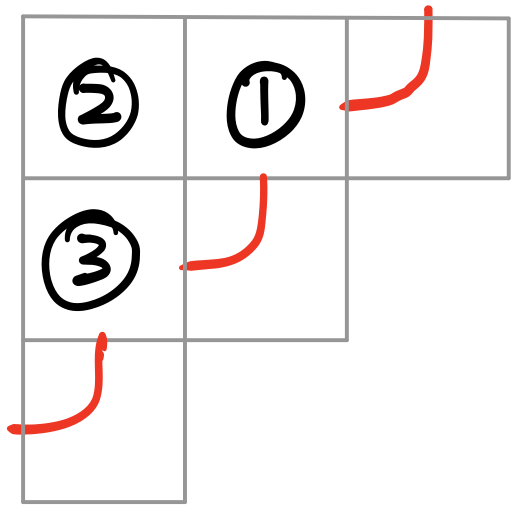
Each $u_{i_1} u_{i_2} \dots u_{i_m}$ is zero or corresponds to some $w \in S_n$

$u_2 u_1 \sim 312$

$u_2 u_2 = 0$



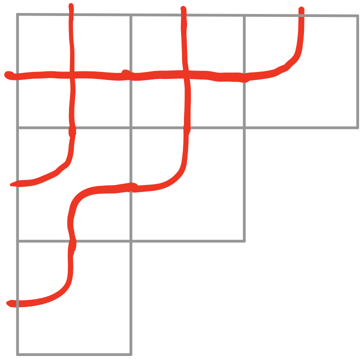
Fomin - Stanley Algebra



$$(1 + x_1 u_2) \times (1 + x_1 u_1) \times (1 + x_2 u_2)$$

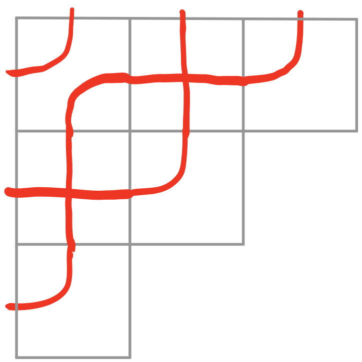
1 2 3

Fomin-Stanley Algebra



$$(1 + x_1 u_2) \times (1 + x_1 u_1) \times (1 + x_2 u_2)$$

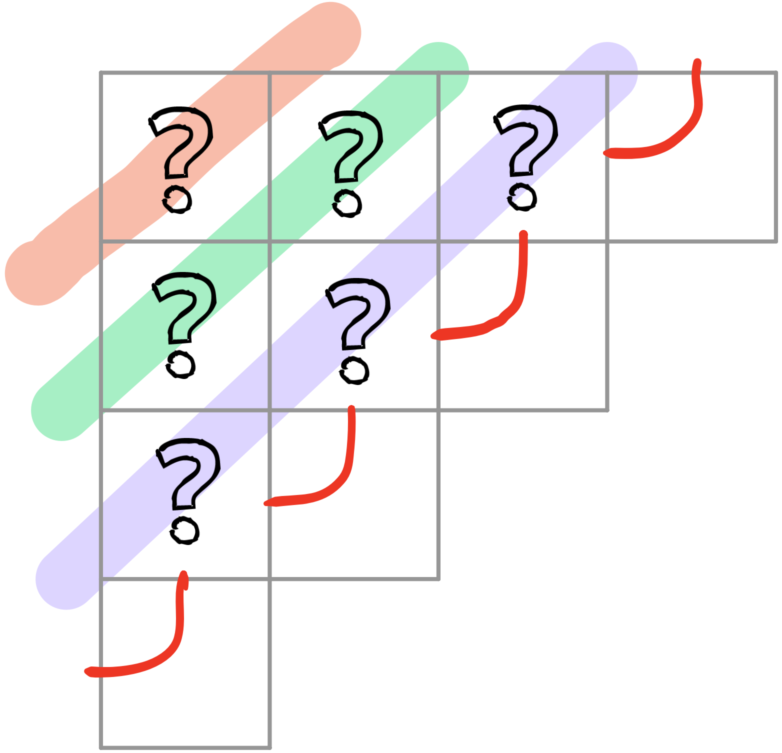
$$x_1^2 u_2 u_1$$



$$(1 + x_1 u_2) \times (1 + x_1 u_1) \times (1 + x_2 u_2)$$

$$x_1 x_2 u_2 u_2 = 0$$

Fomin-Stanley Algebra



$\leftarrow$ start this way

$G :=$

$(1+x_1u_3)(1+x_1u_2)(1+x_1u_1)(1+x_2u_2)(1+x_2u_3)(1+x_3u_3)$

Fomin - Stanley Algebra

$$\mathfrak{G} = (1+x_1u_3)(1+x_1u_2)(1+x_1u_1)(1+x_2u_2)(1+x_2u_3)(1+x_3u_3)$$

Observation :

$$\mathfrak{G} = \sum_P x^{wt(P)} w(P)$$

over all pipedreams

To prove :

$$\mathfrak{G}_w = \sum_{P \in PD(w)} x^{wt(P)}$$

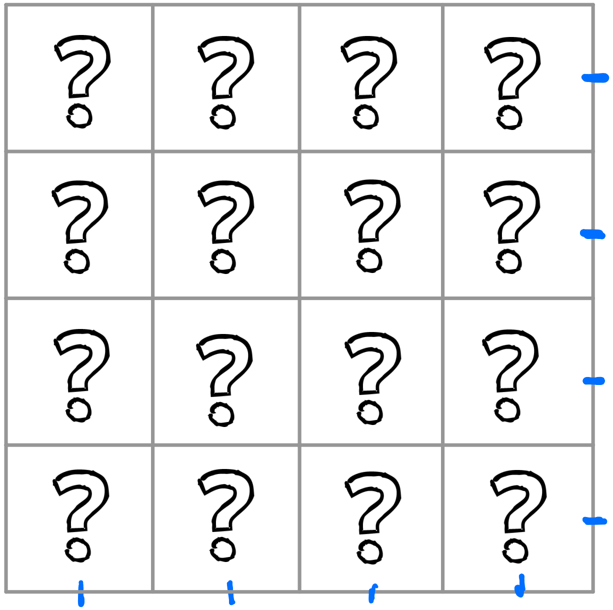
It is enough to show

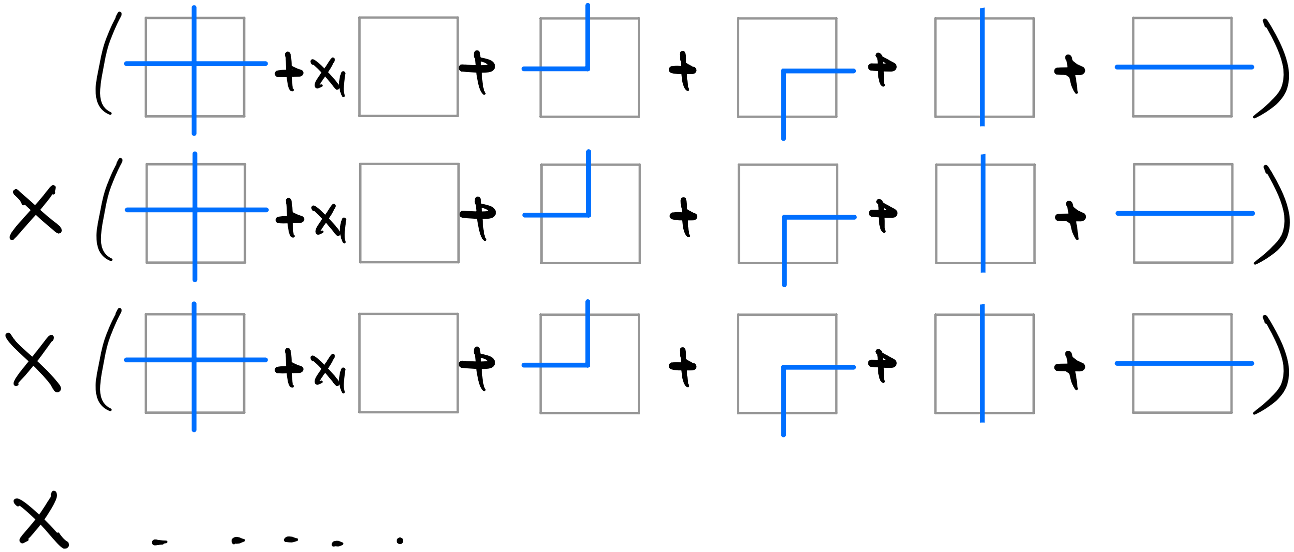
$$\mathfrak{G} = \sum_{w \in S_n} \mathfrak{G}_w \, w$$

Then it reduces to :

[Fomin - Stanley] $\quad \partial_i(\mathfrak{G}) = \mathfrak{G}\, u_i$

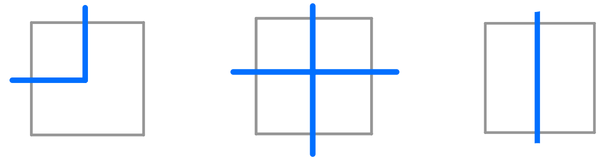
Fomin-Stanley Algebra on BPD?

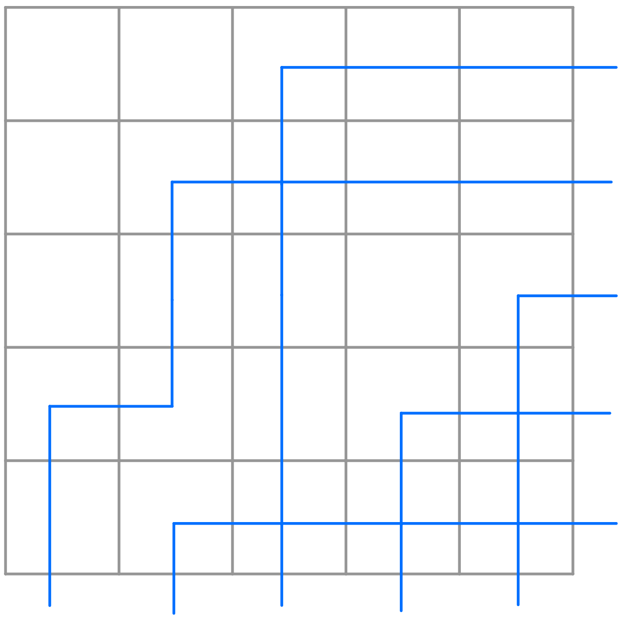
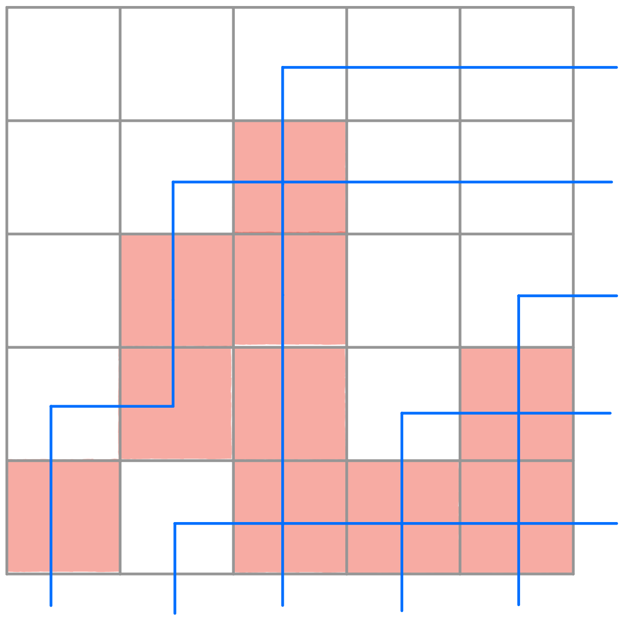




Too many terms!

Encoding a BPD

- Step 1: KILL ALL 

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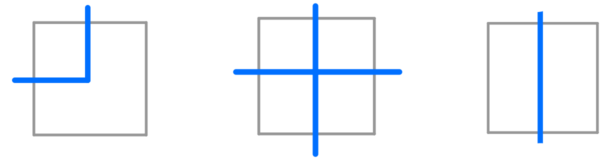
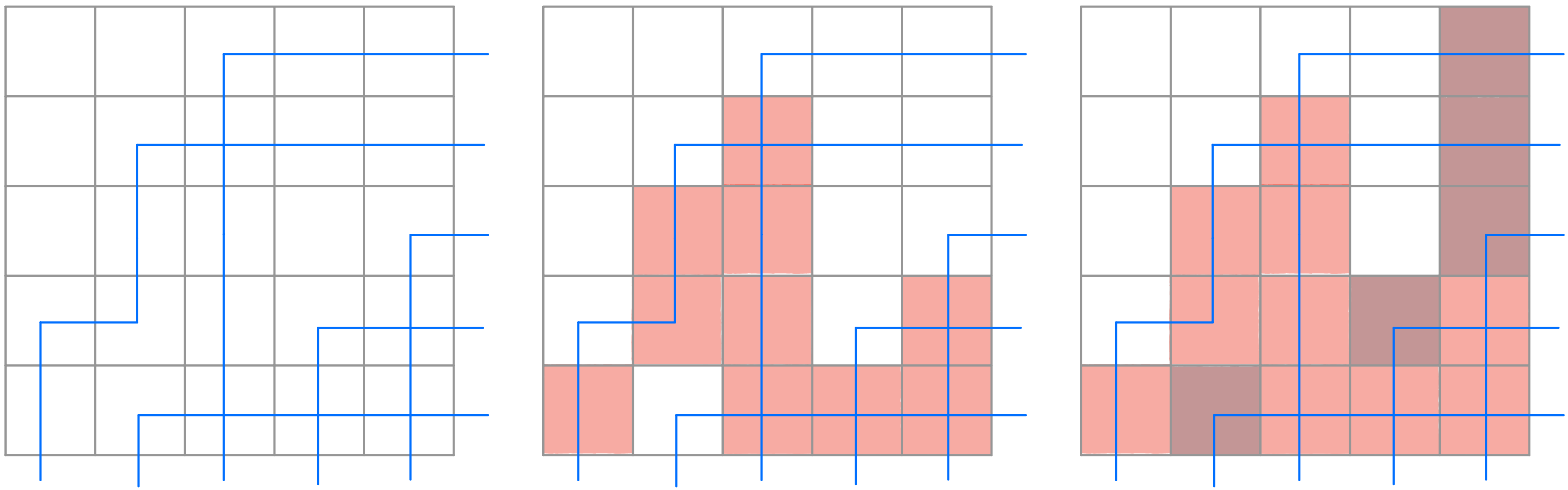
- Step 1: KILL ALL 
- Step 2: KILL the rightmost survivor on each row.



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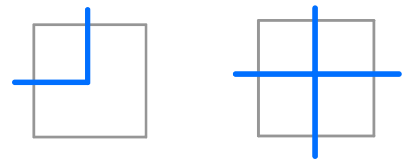
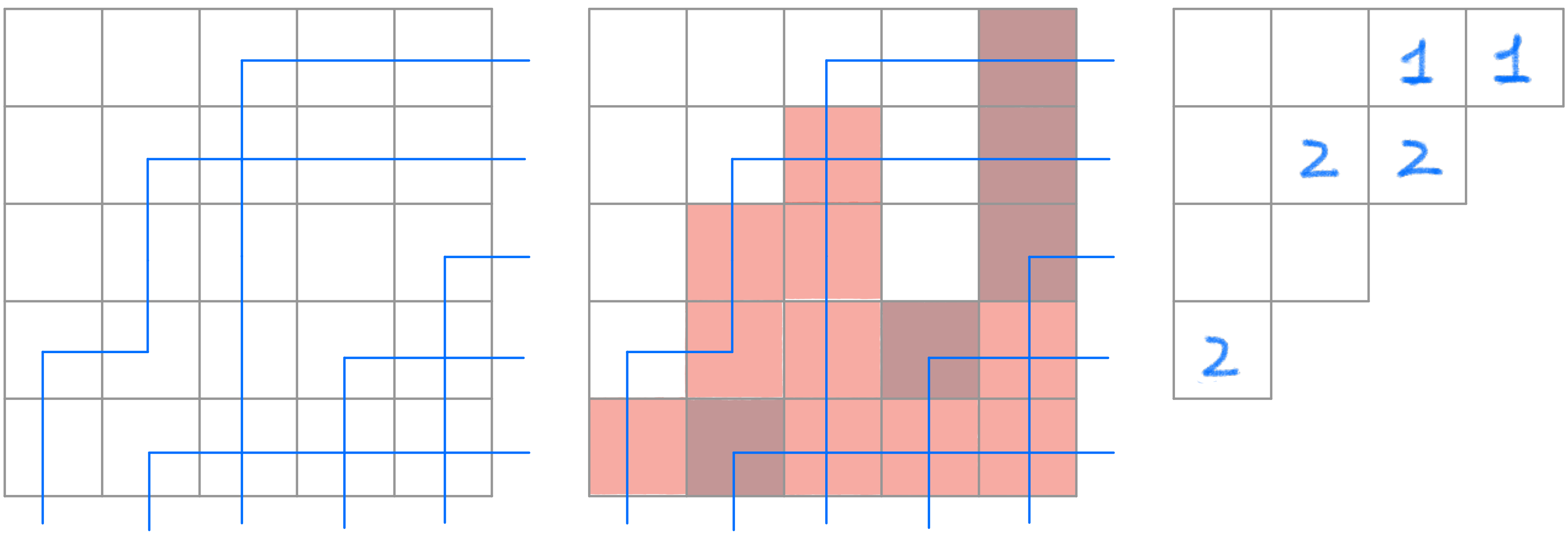
- Step 1: KILL ALL 
- Step 2: KILL the rightmost survivor on each row.
- Step 3: Record the pipe in each survivor.

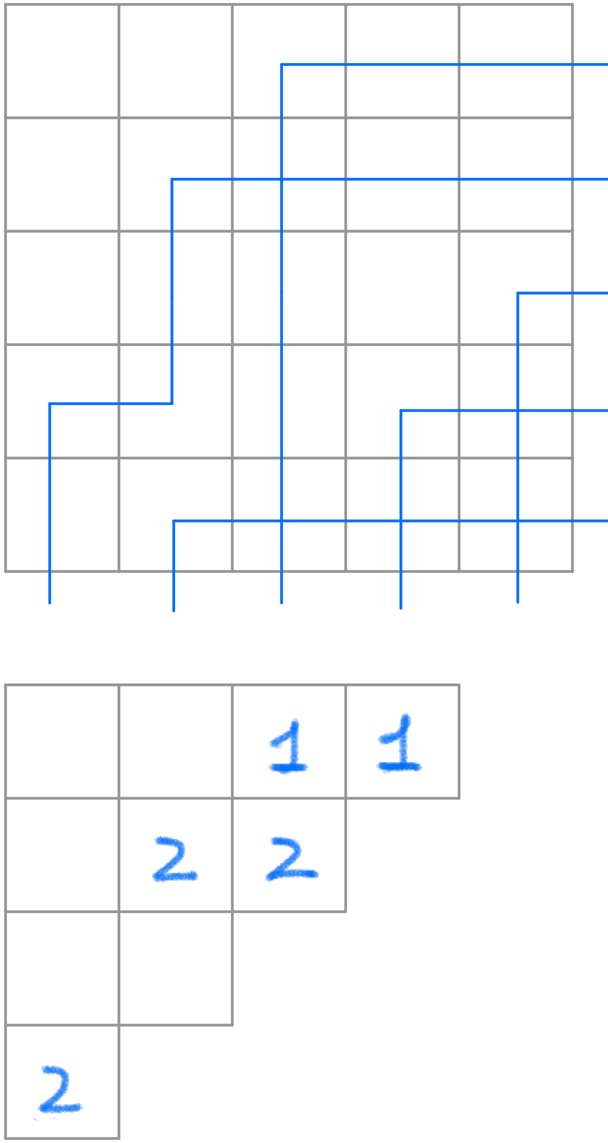


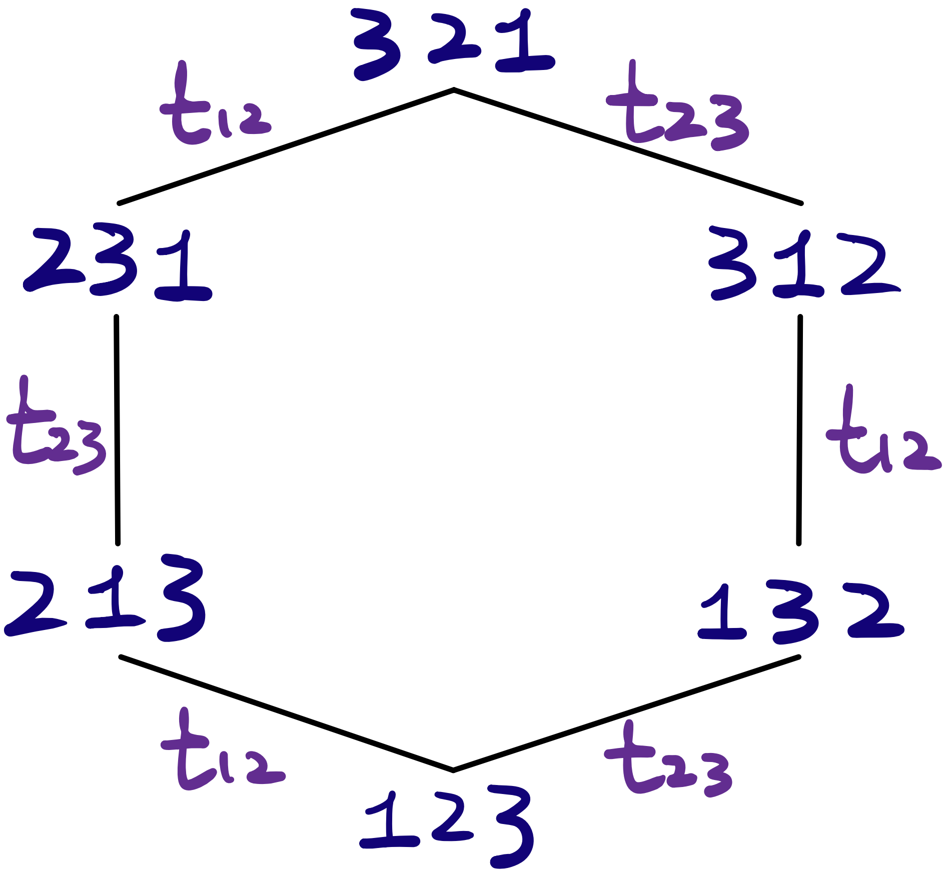
Encoding a BPD

Prop [Y, 24+]

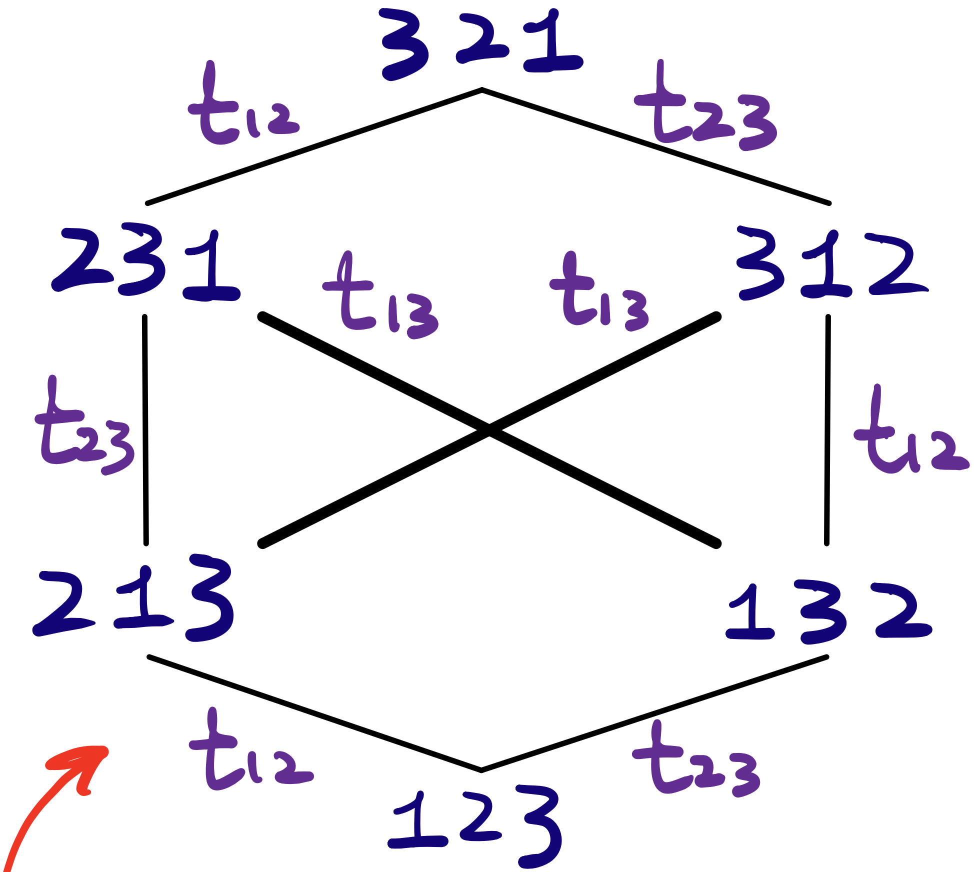
This is a bijection between BPD of w and fillings of the "staircase" such that

- Row i filled w/ #s $\leq i$
- Correspond to a chain from w to w_0 in Bruhat order.





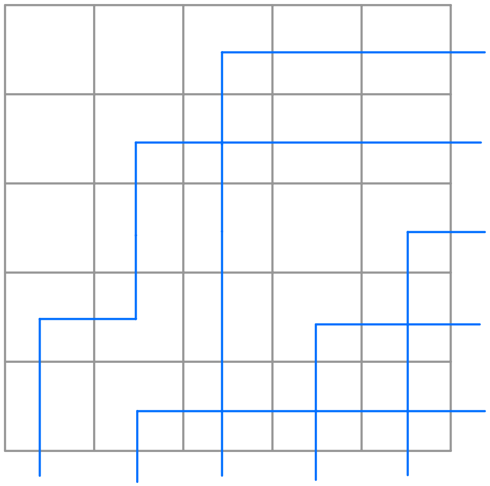
Weak Bruhat Order



(Strong) Bruhat Order

When you swap, #s in between are not in between.

Read the permutation



5	4	3	2	1
		1	1	
	2	2		
2				

$$54321 \xrightarrow{t_{12}} 45321 \xrightarrow{t_{13}} 35421 \xrightarrow{t_{23}} 34521 \xrightarrow{t_{24}} 32541 \xrightarrow{t_{25}} 31542$$

5	4	3	2	1
		1	1	
2		2		
4				

(BAD)

$$54321 \xrightarrow{t_{12}} 45321 \xrightarrow{t_{13}} 35421 \xrightarrow{t_{23}} 34521 \xrightarrow{t_{25}} 31524$$

Encoding a BPD

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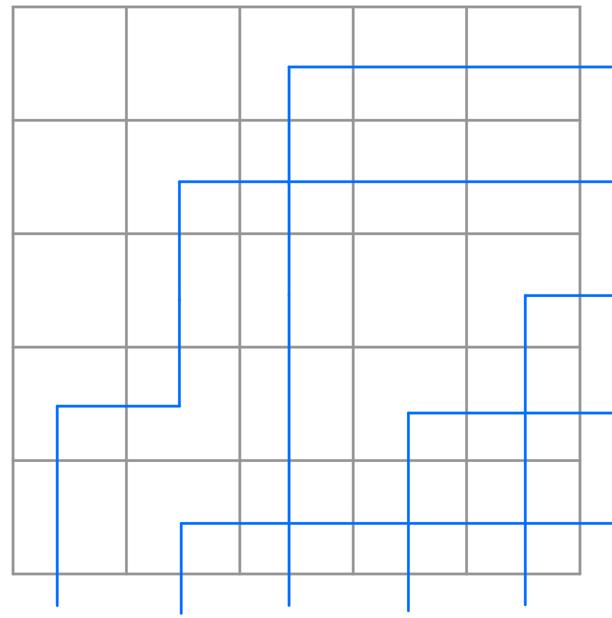
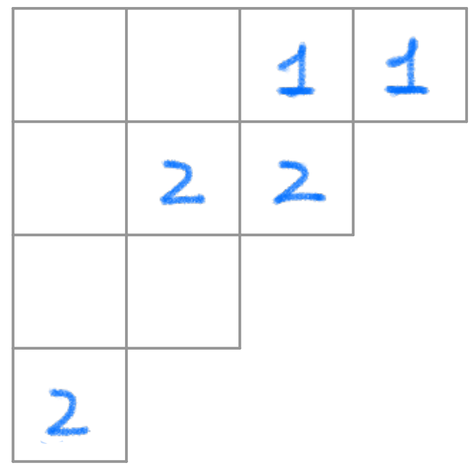




Fomin-Kirillov Algebra

$\mathcal{E}_n$ is generated by d_{ij}, $1 \le i < j \le n$.

- $d_{ij} d_{ij} = 0$
- $d_{ij} d_{jk} = d_{ik} d_{ij} + d_{jk} d_{ik}$
- $d_{jk} d_{ij} = d_{ij} d_{ik} + d_{ik} d_{jk}$
- $d_{ij} d_{kl} = d_{kl} d_{ij}$

if i, j, k, l distinct.

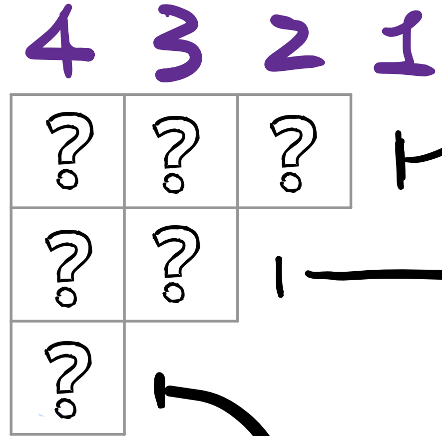
Fomin-Kirillov Algebra

$\mathcal{E}_n$ is generated by d_{ij}, $1 \le i < j \le n$.

$\mathcal{E}_n$ acts on $\mathbb{Q}[S_n]$

$$w \odot d_{ij} = \begin{cases} wt_{ij} & \text{if } w > wt_{ij} \\ 0 & \text{otherwise.} \end{cases}$$

BPD Analogue



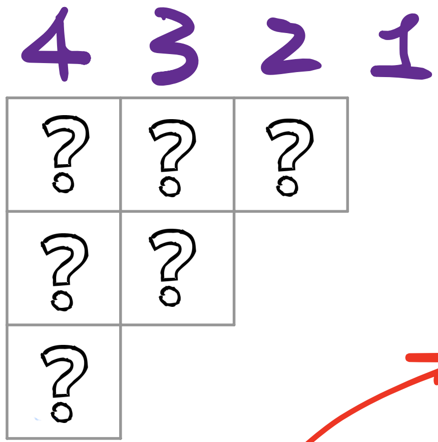
$$(x_1 + \overset{2}{\boxed{1}})(x_1 + \overset{3}{\boxed{1}})(x_1 + \overset{4}{\boxed{1}})$$
$$= (x_1 + d_{12})(x_1 + d_{13})(x_1 + d_{14})$$

$$(x_2 + \overset{3}{\boxed{1}} + \overset{3}{\boxed{2}})(x_2 + \overset{4}{\boxed{1}} + \overset{4}{\boxed{2}})$$
$$= (x_2 + d_{13} + d_{23})(x_2 + d_{14} + d_{24})$$

$$(x_3 + \overset{4}{\boxed{1}} + \overset{4}{\boxed{2}} + \overset{4}{\boxed{3}})$$
$$= (x_3 + d_{14} + d_{24} + d_{34})$$

BPD Analogue of Fomin – Stanley

Definition [Y, 23+]



$$\mathfrak{G} = w_0 \odot (x_1 + d_{12})(x_1 + d_{13})(x_1 + d_{14})$$
$$(x_2 + d_{13} + d_{23})(x_2 + d_{14} + d_{24})$$
$$(x_3 + d_{14} + d_{24} + d_{34})$$

This is a generating function of BPDs.

To show:

$$\mathfrak{G}_w = \sum_{D \in BPD(w)} x^{wt(D)}$$

Just need:

$$\mathfrak{G} = \sum_{w \in S_n} \mathfrak{G}_w \, w$$

As before, it reduces to [Y, 24+]

$$\partial_i(\mathfrak{G}) = \mathfrak{G} u_i .$$

Thank Yibo Gao for telling me this problem

Thank Yibo Gao and Zachary Hamaker for valuable guidances.

Thank you for listening!