# TABLEAU FORMULA FOR VEXILLARY DOUBLE EDELMAN-GREENE COEFFICIENTS

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ABSTRACT. Lam, Lee and Shimozono recently introduced backstable double Grothendieck polynomials to represent K-theory classes of the infinite flag variety. They used them to define double  $\beta$ -Stanley symmetric functions, which expand into double stable Grothendieck functions with polynomial coefficients called double  $\beta$ -Edelman-Greene coefficients. Anderson proved these coefficients are  $\beta$ -Graham positive. For vexillary permutations, this is equivalent to a statement for skew flagged double  $\beta$ -Grothendieck functions. Working in this setting, we give a tableau formula for vexillary double  $\beta$ -Edelman-Greene coefficients that is manifestly  $\beta$ -Graham positive. Our formula demonstrates a finer notion of positivity than was previously known.

## 1. Introduction

The goal of this paper to understand combinatorially certain geometric positivity results in cohomology and K-theory from [1]. Specifically, Anderson shows double  $\beta$ -Stanley symmetric functions and double stable  $\beta$ -Grothendieck functions from [14] represent certain degeneracy loci, with the former expanding into the latter. The coefficients are polynomials called double  $\beta$ -Edelman-Greene coefficients and can include negative terms, but exhibit a finer notion of Graham positivity, as was conjectured in [14] and proved in [1] using geometric methods. We give a combinatorial description of these coefficients for vexillary permutations in terms of tableaux that is manifestly Graham positive. Earlier combinatorial interpretations of double Edelman-Greene coefficients are not obviously Graham positive even in the cohomology case  $\beta = 0$ . Classical Edelman–Greene coefficients from [7] compute the Schur expansion of Stanley symmetric functions from [23]. They generalize Littlewood-Richardson coefficients and have been used to describe several more general families of Schubert structure coefficients [9]. Similarly, while double  $\beta$ -Edelman-Greene coefficients are little studied, they are known to compute double K-homology structure coefficients in the Grassmannian. As such, our work provides a new entry point towards the combinatorial understanding of Schubert calculus. Let  $w \in S_{\mathbb{Z}}$ , the permutations of  $\mathbb{Z}$  fixing all but finitely many values. We make use of the variables  $\mathbf{x} = (x_i)_{i \in \mathbb{Z}}, \ \mathbf{y} = (y_j)_{j \in \mathbb{Z}}, \ \mathbf{x}_+ = (x_i)_{i \geq 1}, \ \text{and} \ \mathbf{x}_- = (x_i)_{i \leq 0}.$  Each double Stanley symmetric function is a specialization of a backstable  $\beta$ -Grothendieck polynomial  $\overleftarrow{\mathfrak{G}}_w(\beta; \mathbf{x}; \mathbf{y})$ , which represents both the connective K-theory for certain degeneracy loci [1] and, with  $\beta = -1$ , the K-theory of Schubert varieties in the infinite flag variety [14]. Each  $\mathfrak{G}_w(\beta; \mathbf{x}; \mathbf{y})$  expands into double stable  $\beta$ -Grothendieck functions  $G_{\lambda}(\beta; \mathbf{x}_{-}; \mathbf{y})$ , which

(1) 
$$\overleftarrow{\mathfrak{G}}_{w}(\beta; \mathbf{x}; \mathbf{y}) = \sum_{\mu} a_{\mu}^{w}(\beta; \mathbf{x}; \mathbf{y}) \cdot G_{\mu}(\beta; \mathbf{x}_{-}; \mathbf{y})$$

represent the connective K-theory of the Grassmannian:

where  $a_{\mu}^{w}(\beta; \mathbf{x}; \mathbf{y})$  is a rational function. Mapping  $\mathbf{x} \mapsto \mathbf{y}$  in these coefficients gives  $j_{\mu}^{w}(\beta; \mathbf{y}) = a_{\mu}^{w}(\beta; \mathbf{y}; \mathbf{y})$ , and the double  $\beta$ -Stanley symmetric function of w is

(2) 
$$F_w(\beta; \mathbf{x}; \mathbf{y}) = \sum_{\mu} j_{\mu}^w(\beta; \mathbf{y}) \cdot G_{\mu}(\beta; \mathbf{x}_{-}; \mathbf{y}).$$

Let  $a \ominus b = \frac{a-b}{1+\beta b}$ . Say  $f(\beta; \mathbf{y})$  is  $\beta$ -Graham positive if  $f(\beta; \mathbf{y}) \in \mathbb{Z}_{\geq 0}[\beta(y_i \ominus y_j) : i \prec j]$  where  $1 \prec 2 \prec \cdots \prec -2 \prec -1 \prec 0$ . In [1], Anderson showed  $\beta^{\ell(w)-|\lambda|}j_{\lambda}^w(\beta; \mathbf{y})$  is  $\beta$ -Graham positive, resolving [14, Conj 8.23]. Our main theorem is a refinement of Anderson's result for vexillary w. A  $\beta$ -Graham positive monomial is a product of terms  $\beta(y_i \ominus y_j)$  with  $i \prec j$ . Define three types of terms  $y_i \ominus y_j$ :

Type 1: 
$$0 < i < j$$
, Type 2:  $i < j < 0$ , Type 3:  $j < 0 < i$ .

**Theorem 1.1.** For  $w \in S_{\mathbb{Z}}$  vexillary,  $\beta^{\ell(w)-|\mu|}j^w_{\mu}(\beta; \mathbf{y})$  is a sum of  $\beta$ -Graham positive monomials indexed by tableaux. Each such monomial contains  $\beta(y_i \ominus y_j)$  at most twice if it is Type 3 and at most once otherwise. Furthermore, for all  $\mu$  these monomials will only have terms of one of the Types 1 or 2.

Our proof is combinatorial and implicitly gives a tableau formula for  $j_{\mu}^{w}(\beta; \mathbf{y})$ .

The condition that  $j_{\mu}^{w}$  can be expressed without using terms of both Type 1 and Type 2 does not appear in Anderson's work, and does not hold for  $j_{\mu}^{w}(\beta; \mathbf{y})$  when w is not vexillary. For example, with  $\overline{k} = -k$  the non-vexillary permutation  $w = s_{2}s_{1}s_{\overline{1}}s_{0}$  has

$$j_{(1)}^w(\beta; \mathbf{y}) = (y_1 \ominus y_2)(y_{\overline{1}} \ominus y_0)(y_1 \ominus y_0),$$

which involves terms of all three types.

As a corollary, setting  $\beta = 0$  recovers the cohomology case. Here  $y_i \ominus y_j = y_i - y_j$ . Define

$$\overleftarrow{\mathfrak{S}}_w(\mathbf{x};\mathbf{y}) = \overleftarrow{\mathfrak{G}}_w(0;\mathbf{x};\mathbf{y}), \quad s_\lambda(\mathbf{x}_-;\mathbf{y}) = G_\lambda(0;\mathbf{x}_-;\mathbf{y}), \quad j_\mu^w(\mathbf{y}) = j_\mu^w(0;\mathbf{y}).$$

Then Theorem 1.1 specializes to a description of the double Edelman–Greene coefficient  $j_{\mu}^{w}(\mathbf{y})$  from [15], providing the first Graham positive combinatorial description of such coefficients as well.

Our approach to proving Theorem 1.1 is based on tableau formulas for vexillary backstable double  $\beta$ -Grothendieck polynomials. By [24], each vexillary permutation w is determined by its associated partition  $\lambda(w)$  and flag  $\phi(w)$ . For w vexillary  $\mathfrak{G}_w(\beta; \mathbf{x}, \mathbf{y}) = G_{\lambda(w)}^{\phi(w)}(\beta; \mathbf{x}; \mathbf{y})$ , which is a flagged double stable  $\beta$ -Grothendieck function. Working in this setting, our proof of Theorem 1.1 has three main steps. We first establish the positivity of vexillary  $j_{\mu}^{w}(\beta; \mathbf{y})$  when  $\phi(w)$  contains only non-negative integers. Then, we use an extension  $\tilde{\omega}$  of the  $\omega$  involution for symmetric functions to cover the case where  $\phi(w)$  has only non-positive integers. Lastly, we combine and refine these previous arguments combine to prove Theorem 4.7, a slight strengthening of Theorem 1.1.

**Organization:** In Section 2, we give the necessary background to state Theorems 1.1 precisely. We prove special cases of Theorem 1.1 in Section 3, then give a complete proof in Section 4. We conclude with some final remarks in Section 5.

## 2. Preliminaries

Write  $\mathbb{Z}_+ = \{i \in \mathbb{Z} : i \geq 1\}$ ,  $\mathbb{Z}_- = \{j \in \mathbb{Z} : j \leq 0\}$  and  $\mathbb{Z}_{\geq 0} = \mathbb{Z}_+ \cup \{0\}$ . Let  $\mathcal{S}$  be the set of non-empty finite subsets of  $\mathbb{Z}$ ,  $\mathcal{S}_+$  the set of non-empty finite subsets of  $\mathbb{Z}_+$ , and  $\mathcal{S}_-$  the set of non-empty finite subsets of  $\mathbb{Z}_-$ . Let  $\overline{k} = -k$  and recall  $\prec$  is the order on  $\mathbb{Z}$  with  $1 \prec 2 \prec \ldots \prec \overline{2} \prec \overline{1} \prec 0$ . Let  $\mathbf{x} = (x_i)_{i \in \mathbb{Z}}$  and  $\mathbf{y} = (y_j)_{j \in \mathbb{Z}}$  be commuting variables. Write  $\mathbf{x}_+ = (x_i)_{i \geq 1}$  and  $\mathbf{x}_- = (x_i)_{i \leq 0}$ . Use  $\mathbf{x} \mapsto \mathbf{y}$  to denote the substitution  $x_i \mapsto y_i$  for  $i \in \mathbb{Z}$ , and  $\mathbf{x} \leftrightarrow \mathbf{y}$  when swapping  $x_j \leftrightarrow y_j$  for  $j \in \mathbb{Z}$ .

2.1. **Tableaux.** A partition is a sequence  $\lambda = (\lambda_1 \ge ... \ge \lambda_\ell)$  of weakly decreasing positive integers. Here  $\lambda$  has size  $|\lambda| = \sum \lambda_i$  and length  $\ell(\lambda) = \ell$ , the number of nonzero parts. Identify each partition  $\lambda$  with its Young diagram

$$\{(r,c): 1 \le r \le \ell(\lambda), 1 \le c \le \lambda_r\}$$

whose elements are called *cells*, which we plot using matrix coordinates. Viewing partitions as Young diagrams, for  $\mu \subseteq \lambda$  the *skew diagram*  $\lambda/\mu$  is the set of cells in  $\lambda$  that are not in  $\mu$ .

For  $\mu \subseteq \lambda$ , a S-tableau of shape  $\lambda/\mu$  is a filling  $T: \lambda/\mu \to S$  of the cells in  $\lambda/\mu$  by integers. Let  $a \ominus b = \frac{a-b}{1+\beta b}$ . Each S-tableau is assigned the monomial weight

(3) 
$$\operatorname{wt}(T) = \beta^{-|\lambda/\mu|} \prod_{(r,c)\in\lambda/\mu} \prod_{i\in T(r,c)} \beta(x_i \ominus y_{i+c-r}).$$

A S-tableau T of shape  $\lambda/\mu$  is semistandard if

$$\max T(r,c) \le \min T(r,c+1)$$
 and  $\max T(c,r) < \min T(r+1,c)$ 

whenever (r,c), (r,c+1) or (r,c), (r+1,c) are in  $\lambda/\mu$ . This means T increases strictly down columns and weakly across rows. Write SetSSYT $(\lambda/\mu)$  for the set of semistandard S-tableaux of shape  $\lambda/\mu$ . More generally, for T a set of tableaux, let  $T_+$  and  $T_-$  be those with entries in  $S_+$  and  $S_-$  respectively, e.g., SetSSYT $_+(\lambda)$  is the set of semistandard S-tableaux of shape  $\lambda$  whose entries are sets of positive integers.

**Remark 2.1.** For any  $T \in \text{SetSSYT}(\lambda/\mu)$ , the weights of two elements are different.

**Example 2.2.** The following T is a semistandard S-tableau of shape (4,3,2):

$$(4) \quad T = \begin{bmatrix} \hline 3,\overline{2} & \overline{2},0 & 0 & 1 \\ \hline \overline{1} & 1 & 3 \\ \hline 0,2 & 2 \end{bmatrix} \quad \text{has cell weights} \quad \begin{bmatrix} \beta^2(x_{\overline{3}}\ominus y_{\overline{3}})(x_{\overline{2}}\ominus y_{\overline{2}}) & \beta^2(x_{\overline{2}}\ominus y_{\overline{1}})(x_0\ominus y_1) & \beta(x_0\ominus y_2) & \beta(x_1\ominus y_4) \\ \hline \beta(x_1\ominus y_{\overline{2}}) & \beta(x_1\ominus y_1) & \beta(x_3\ominus y_4) \\ \hline \beta^2(x_0\ominus y_{\overline{2}})(x_2\ominus y_0) & \beta(x_2\ominus y_1) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_1) & \beta(x_2\ominus y_1) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_1) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\ \hline \beta(x_1\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) & \beta(x_2\ominus y_2) \\$$

Then wt(T) is the product of  $\beta^{-9}$  and the cell weights.

We often specify the largest entry allowed for each row using a weakly increasing sequence  $\phi = (\phi_1 \le ... \le \phi_\ell)$ , called a flag. The  $\phi$ -flagged semistandard S-tableaux of shape  $\lambda/\mu$  are

$$SetSSYT^{\phi}(\lambda/\mu) = \{ T \in SetSSYT(\lambda/\mu) : \max T(r,c) \le \phi_r \text{ for all } (r,c) \in \lambda/\mu \}.$$

For instance, with T as in (4) we have  $T \in \text{SetSSYT}^{(1,3,3)}((4,3,2))$  and  $T \notin \text{SetSSYT}^{(1,2,3)}((4,3,2))$ .

2.2. **Permutations.** Let  $S_{\mathbb{Z}}$  denote the set of bijections  $w: \mathbb{Z} \to \mathbb{Z}$  whose  $support \operatorname{Supp}(w) = \{k \in \mathbb{Z} : w(k) \neq k\}$  is finite. Write  $S_{\infty}$  for the subset of  $S_{\mathbb{Z}}$  with support contained in  $\mathbb{Z}_+$ . Let  $s_i$  denote the simple transposition (i, i+1), and note that  $S_{\mathbb{Z}}$  is generated by the  $s_i$  for  $i \in \mathbb{Z}$  with relations

(5) 
$$s_i^2 = 1$$
,  $s_i s_j = s_j s_i$  for  $|i - j| > 1$ ,  $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$  for  $i \in \mathbb{Z}$ ,

the latter two called *braid relations*. Therefore, each  $w \in S_{\mathbb{Z}}$  can be expressed as  $w = s_{i_1} \dots s_{i_p}$ . Such an expression is called *reduced* if it is of minimal length with associated *reduced word*  $(i_1, \dots, i_p)$ . The value i is a *descent* in  $w \in S_{\mathbb{Z}}$  if w(i) > w(i+1), and  $Des(w) = \{i : w(i) > w(i+1)\}$  is the *descent set* of w. Say w is k-Grassmannian if it has at most one descent in position k. There is a bijection  $\lambda \mapsto w_{\lambda}$  from partitions to 0-Grassmannian permutations with  $w_{\lambda}(i) = i + \lambda_{1-i}$  for  $i \leq 0$  and  $w(\lambda) = i - \lambda'_i$  for i > 0.

We make extensive use of two automorphisms of  $S_{\mathbb{Z}}$ : the map  $\neg$  is defined by  $\neg s_i = s_{\overline{i}}$  for  $i \in \mathbb{Z}$ , while  $\iota$  is defined by  $\iota(s_i) = s_{i+1}$  for  $i \in \mathbb{Z}$ . The map  $\neg$  is closely related to the better known automorphism conjugation by the long element. Recall  $S_n$  is the set of permutations with support in  $\{1, 2, \ldots, n\}$ . For  $w \in S_n$  and  $w_0 = n \ldots 21$ , recall we obtain  $w_0 w w_0$  from w by mapping  $s_i$  to  $s_{n-i}$  in any of its expressions. Therefore  $w_0 w w_0 = \iota^n \circ \neg(w)$ .

Say  $w \in S_{\mathbb{Z}}$  is vexillary if there are no  $i < j < k < \ell$  with  $w(j) < w(i) < w(\ell) < w(k)$ . Each vexillary permutation  $w \in S_{\mathbb{Z}}$  determines a shape  $\lambda(w)$  and flag  $\phi(w)$  of the same length, as demonstrated by Wachs [24]. Let  $I_k(w) = \{j > k : w(k) > w(j)\}$ ,  $c_k(w) = |I_k(w)|$  and  $p_k(w) = \min I_k(w) - 1$  where  $p_k(w)$  is not defined when  $I_k(w)$  is empty. The shape  $\lambda(w)$  is obtained by sorting the code  $c(w) = (\dots, c_{\overline{1}}(w), c_0(w), c_1(w), \dots)$  in decreasing order, and the flag  $\phi(w)$  is obtained by arranging the  $p_k(w)$ 's in increasing order. We say flags  $\phi$  and  $\phi'$  are equivalent with respect to the partition  $\lambda$  if they are both compatible with  $\lambda$  and  $SSYT^{\phi}(\lambda) = SSYT^{\phi'}(\lambda)$ . For example, w = 345162 is vexillary with  $\lambda(w) = (2, 2, 2, 1)$  and  $\phi(w) = (3, 3, 3, 5)$ . Since  $SetSSYT^{(3,3,3,5)}((2,2,2,1)) = SetSSYT^{(1,2,3,5)}((2,2,2,1))$ , the flags (3,3,3,5) and (1,2,3,5) are equivalent with respect to (2,2,2,1). Other authors, e.g. [25], use the latter as the flag for w, but our choice of conventions is necessary for Lemma 2.4.

The partition  $\lambda(w)$  and flag  $\phi(w)$  obtained from a vexillary  $w \in S_{\mathbb{Z}}$  satisfy

(6) 
$$\phi(w)_{i+1} - \phi(w)_i \le \lambda(w)_i - \lambda(w)_{i+1} + 1 \quad \text{for each} \quad 1 \le i \le \ell(\lambda(w)).$$

When this holds, we say the partition  $\lambda$  and flag  $\phi$  are *compatible*. Let  $w_{\lambda,\phi}$  denote the permutation determined by the subsequent result.

**Proposition 2.3** ([17, (1.37) and (1.38)]). The flag  $\phi$  is compatible with the partition  $\lambda$  if and only if there exists a vexillary permutation  $w \in S_{\mathbb{Z}}$  so that  $\lambda = \lambda(w)$  and  $\phi$  is equivalent to  $\phi(w)$  with respect to  $\lambda$ .

The above result is implicit in work of Wachs [24] and proved in Macdonald [17] only for positive flags. To see the result holds for  $w \in S_{\mathbb{Z}}$  vexillary, note that  $\iota(w)$  is vexillary and  $\phi(\iota(w))$  is obtained from  $\phi(w)$  by incrementing each entry by one. Since this process is reversible, by applying  $\iota$  sufficiently many times we can reduce to the case where  $\phi$  has all positive entries.

Recall  $\lambda'$  is the transpose of  $\lambda$ . The following result translates [17, (1.42)] from conjugation by  $w_0$  to  $\neg$ .

**Lemma 2.4.** The permutation  $\neg w_{\lambda,\phi}$  is vexillarly with shape  $\lambda'$  and flag  $\psi = \phi(\neg w_{\lambda,\phi})$ , where  $\psi_i = -\phi_{\lambda'}$ .

Proof. Let  $w \in S_n$  be vexillary with  $\phi(w) = (f_1^{m_1}, \dots, f_k^{m_k})$ , so  $f_i$  occurs  $m_i$  times. Then [17, (1.42)] asserts  $\lambda(w_0ww_0) = \lambda(w)'$  and  $\phi(w_0ww_0) = (g_1^{n_1}, \dots, g_k^{n_k})$  where  $g_i = n - f_{k+1-i}$ . Since  $\neg(w) = \iota^{-n}(w_0ww_0)$ , we see  $\phi(\neg(w))$  has ith distinct term  $g_i - n = -f_{k+1-i}$ . Since our conventions give  $\phi_i = \phi_j$  when  $\lambda_i = \lambda_j$ , we see for  $\psi_j = g_i$  that  $\phi_{\lambda'_i} = f_{k+1-i}$ .

2.3. **Functions.** There are several equivalent conventions for defining  $(\beta-)$ Grothendieck polynomials. Ours are equivalent to those from [14] after setting  $\beta = -1$ . To begin, we define the *double stable*  $\beta$ -Grothendieck and  $\phi$ -flagged double stable  $\beta$ -Grothendieck functions as

(7) 
$$G_{\lambda}(\beta; \mathbf{x}_{-}; \mathbf{y}) = \sum_{T \in \text{SetSSYT}_{-}(\lambda)} \text{wt}(T) \quad \text{and} \quad G_{\lambda}^{\phi}(\beta; \mathbf{x}; \mathbf{y}) = \sum_{T \in \text{SetSSYT}^{\phi}(\lambda)} \text{wt}(T).$$

To define backstable Grothendieck polynomials, we first define the usual double Grothendieck polynomial. Each  $\sigma \in S_{\mathbb{Z}}$  acts on the formal power series  $f(\mathbf{x}, \mathbf{y})$  by permuting the **x**-variables, denoted  $f(\sigma \cdot \mathbf{x}; \mathbf{y})$ . Let  $\partial_i$  be the *i*th *divided difference operator*, which acts on the polynomials by

$$\partial_i(f(\mathbf{x})) = \frac{f(\mathbf{x}) - f(s_i \cdot \mathbf{x})}{x_i - x_{i+1}},$$

and define the *i*th  $\beta$ -isobaric divided difference operator  $\pi_i(f(\mathbf{x})) = \partial_i((1 + \beta x_i)f(\mathbf{x}))$ . The  $\beta$ -isobaric divided difference operators satisfy  $\pi_i^2 = \beta \pi_i$  and the braid relations from (5). Therefore, we can define  $\pi_w = \pi_{i_1} \dots \pi_{i_p}$  where  $(i_1, \dots, i_p)$  is any reduced word for w. The double  $\beta$ -Grothendieck polynomial for a permutation  $w \in S_n$  is

(8) 
$$\mathfrak{G}_w(\beta; \mathbf{x}_+; \mathbf{y}_+) = \pi_{w^{-1}w_0} \prod_{\substack{1 \le i, j \le n-1, \\ i+j \le n}} x_i \ominus y_j$$

where  $w_0 = n \dots 21$ . It is a remarkable consequence of this definition that  $\mathfrak{G}(\beta; \mathbf{x}_+; \mathbf{y}_+)$  is independent of n, hence we can view w as an element of  $S_{\infty}$ .

Recall  $\iota: S_{\mathbb{Z}} \to S_{\mathbb{Z}}$  where  $\iota(s_i) = s_{i+1}$  for all  $i \in \mathbb{Z}$ , or equivalently  $\iota(w)(i+1) = w(i) + 1$ . Let  $\gamma$  shift variables up by one:  $\gamma(x_i) = x_{i+1}$  and  $\gamma(y_i) = y_{i+1}$ , and note  $\gamma$  is invertible. The backstable double  $\beta$ -Grothendieck polynomial for  $w \in S_{\mathbb{Z}}$  is

$$\overleftarrow{\mathfrak{G}}_{w}(\beta; \mathbf{x}; \mathbf{y}) = \lim_{n \to \infty} \gamma^{-p} \mathfrak{G}_{\iota^{p}(w)}(\beta; \mathbf{x}_{+}; \mathbf{y}_{+}).$$

Note this definition does not depend on  $\mathfrak{G}_w(\beta; \mathbf{x}; \mathbf{y})$  when  $w \notin S_\infty$  as for p sufficiently large  $\iota^p(w) \in S_\infty$ . By setting  $\beta = 0$ , we recover the double Schubert polynomial  $\mathfrak{S}_w(\mathbf{x}; \mathbf{y}) = \mathfrak{G}_w(0; \mathbf{x}; \mathbf{y})$  and backstable double Schubert polynomial  $\mathfrak{S}(\mathbf{x}; \mathbf{y}) = \mathfrak{G}_w(0; \mathbf{x}; \mathbf{y})$ .

For w vexillary, the following is a straightforward consequence of [12, Thm 5.8].

**Proposition 2.5.** For 
$$w \in S_{\mathbb{Z}}$$
 vexillary,  $\overleftarrow{\mathfrak{G}}_w(\beta; \mathbf{x}; \mathbf{y}) = G_{\lambda(w)}^{\phi(w)}(\beta; \mathbf{x}; \mathbf{y})$ .

*Proof.* By Proposition 2.3 we know  $\phi$  and  $\lambda$  are compatible if and only if there exists  $w \in S_{\infty}$  vexillary with  $\lambda(w) = \lambda$  and  $\phi(w) = \phi$ . For such w the double Schubert polynomial equality  $\mathfrak{G}_w(\beta; \mathbf{x}_+; \mathbf{y}_+) = G_{\lambda}^{\phi}(\beta; \mathbf{x}_+; \mathbf{y}_+)$  is [12, Thm 5.8]. Note  $\phi(\iota(w))$  is obtained from  $\phi(w)$  by incrementing each entry by one and  $\gamma$  decrements every variable index by one. The result now follows from the definition of  $\mathfrak{G}_w(\beta; \mathbf{x}; \mathbf{y})$ .

An important special case is that of the 0-Grassmannian permutations.

**Proposition 2.6** ([14, Lem 5.32]). For  $\lambda$  a partition,  $G_{\lambda}(\beta; \mathbf{x}_{-}; \mathbf{y}) = \overleftarrow{\mathfrak{G}}_{w_{\lambda}}(\beta; \mathbf{x}; \mathbf{y})$ .

For variables  $\mathbf{z} = (z_i)_{i \in I}$ , define

$$R(\beta; \mathbf{z}) = \mathbb{Z}(\beta) \left[ z_i, \frac{1}{1 + \beta z_i} : i \in I \right],$$

so  $\beta, \beta^{-1} \in R(\beta; \mathbf{z})$  independent of  $\mathbf{z}$ . The double stable  $\beta$ -Grothendieck functions are closed under multiplication over  $R(\beta; \mathbf{y})$  [21] so they form a basis for an algebra, which we denote  $\Gamma(\beta; \mathbf{x}_-; \mathbf{y})$ .

**Proposition 2.7.** For  $w \in S_{\mathbb{Z}}$ ,  $\overleftarrow{\mathfrak{G}}_w(\beta; \mathbf{x}; \mathbf{y}) \in R(\beta; \mathbf{x}; \mathbf{y}) \otimes \Gamma(\beta; \mathbf{x}; \mathbf{y})$ .

Proposition 2.7 is arguably implicit in [14] and implies (1); we give a proof in Appendix A. As a consequence, we have

$$\overleftarrow{\mathfrak{G}}_{w}(\beta; \mathbf{x}; \mathbf{y}) = \sum_{\mu} a_{\mu}^{w}(\beta; \mathbf{x}; \mathbf{y}) \cdot G_{\mu}(\beta; \mathbf{x}_{-}; \mathbf{y})$$

with  $a_{\lambda}^{w}(\beta; \mathbf{x}; \mathbf{y}) \in R(\beta; \mathbf{x}; \mathbf{y})$ . By Proposition 2.3, for  $\lambda$  and  $\phi$  compatible this implies

$$G_{\lambda}^{\phi}(\beta; \mathbf{x}; \mathbf{y}) = \sum_{\mu} a_{\mu}^{\lambda, \phi}(\beta; \mathbf{x}; \mathbf{y}) \cdot G_{\mu}(\beta; \mathbf{x}_{-}; \mathbf{y}),$$

again with  $a_{\mu}^{\lambda,\phi}(\beta;\mathbf{x};\mathbf{y}) \in R(\beta;\mathbf{x};\mathbf{y})$ . The double  $\beta$ -Stanley symmetric function of w is

$$F_w(\beta; \mathbf{x}_-; \mathbf{y}) = \sum_{\lambda} a_{\lambda}^w(\beta; \mathbf{y}; \mathbf{y}) \cdot G_{\lambda}(\beta; \mathbf{x}_-; \mathbf{y}).$$

We call  $j_{\lambda}^{w}(\beta; \mathbf{y}) := a_{\lambda}^{w}(\beta; \mathbf{y}; \mathbf{y})$  the double  $\beta$ -Edelman-Greene coefficient. With  $j_{\mu}^{\lambda}(\beta; \mathbf{y}) := a_{\mu}^{\lambda}(\beta; \mathbf{x}; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}$ , for w vexillary note

(9) 
$$j_{\mu}^{w}(\beta; \mathbf{y}) = j_{\mu}^{\lambda(w), \phi(w)}(\beta; \mathbf{y}).$$

Define the map  $\omega_1: R(\beta; \mathbf{x}; \mathbf{y}) \to R(\beta; \mathbf{x}; \mathbf{y})$  by  $\omega_1(x_i) = \ominus x_{1-i}$  and  $\omega_1(y_i) = \ominus y_{1-i}$  and the map  $\omega_2: \Gamma(\beta; \mathbf{x}_-; \mathbf{y}) \to \Gamma(\beta; \mathbf{x}_-; \mathbf{y})$  by  $\omega_2(G_{\lambda}(\beta; \mathbf{x}_-; \mathbf{y})) = G_{\lambda'}(\beta; \mathbf{x}_-; \mathbf{y})$ . Then  $\tilde{\omega} = \omega_1 \otimes \omega_2$  acts on backstable double  $\beta$ -Grothendieck functions.

**Lemma 2.8** ([14, Prop 5.23]). For  $w \in S_{\mathbb{Z}}$ ,

$$\tilde{\omega}\left(\overleftarrow{\mathfrak{G}}_{w}(\beta;\mathbf{x};\mathbf{y})\right) = \overleftarrow{\mathfrak{G}}_{\neg w}(\beta;\mathbf{x};\mathbf{y}).$$

Combined with Lemma 2.4 and Proposition 2.5, this implies:

Corollary 2.9. Consider compatible  $\lambda$  and  $\phi$ . Then  $j_{\mu}^{\lambda,\phi}(\beta;\mathbf{y}) = \omega_1(j_{\mu'}^{\lambda',\xi}(\beta;\mathbf{y}))$ , where  $\xi_i = -\phi_{\lambda_i'}$ . In particular, if  $\phi$  has only non-positive entries, then  $\xi$  has only non-negative entries.

## 3. Main Result

3.1. Edelman–Green coefficients via tableaux. We start with a way to decompose a semistandard S-tableau into two tableaux.

**Definition 3.1.** Let  $\lambda$  be a partition and take  $T \in \text{SetSSYT}(\lambda)$ . The non-positive values in cells of T form a tableau  $T_-$  of shape  $\nu \subseteq \lambda$  in  $\text{SetSSYT}_-(\nu)$ . The positive values in cells of T form a tableau  $T_+$  of shape  $\lambda/\mu$  in  $\text{SSYT}(\lambda/\mu)_+$ .

For example, the tableau

We now describe the image of SetSSYT<sup> $\phi$ </sup>( $\lambda$ ) under this decomposition. Say a skew shape  $\lambda/\mu$  is disconnected if it contains no adjacent cells. For  $\phi = (\phi_1, \dots, \phi_\ell)$  a flag, define  $\phi_-$  by  $(\phi_-)_i = \min\{\phi_i, 0\}$ . Note that  $\phi_-$  is weakly increasing and has only non-positive numbers.

**Lemma 3.2.** Let  $\lambda$  be a partition and  $\phi$  a flag. For T a set-valued tableau, the map  $T \mapsto (T_-, T_+)$  defined on

(10) 
$$\operatorname{SetSSYT}^{\phi}(\lambda) \to \bigsqcup_{\nu \subseteq \lambda} \operatorname{SetSSYT}^{\phi^{-}}(\nu) \times \left( \bigsqcup_{\mu \subseteq \nu: \ \nu/\mu \ \operatorname{disconnected}} \operatorname{SetSSYT}^{\phi^{+}}(\lambda/\mu) \right),$$

 $is\ a\ bijection.$ 

*Proof.* Take  $T \in \operatorname{SetSSYT}^{\phi}(\lambda)$ . Say  $T_{-}$  and  $T_{+}$  have shapes  $\nu$  and  $\lambda/\mu$  respectively. We know  $\mu$  (resp.  $\nu$ ) consists of all  $(r,c) \in \lambda$  so that  $T(r,c) \subseteq \mathbb{Z}_{-}$  (resp.  $T(r,c) \cap \mathbb{Z}_{-} \neq 0$ ). Thus,  $\mu \subseteq \nu$ .

We clearly have  $T_- \in \operatorname{SetSSYT}^{\phi^-}(\nu)$  and  $T_+ \in \operatorname{SetSSYT}^{\phi^+}(\lambda/\mu)$ . To show  $\nu/\mu$  is disconnected, we consider  $(r,c) \in \nu/\mu$ . Then T(r,c) has both positive and non-positive entries. Thus, should they exist,

T(r-1,c) and T(r,c-1) (resp. T(r+1,c) and T(r,c+1)) cannot have positive (resp. non-negative) entries, so they cannot live in  $\nu/\mu$ .

To see our map is a bijection, take  $T' \in \operatorname{SetSSYT}^{\phi^-}(\nu)$  and  $T'' \in \operatorname{SetSSYT}^{\phi^+}(\lambda/\mu)$ . Their cell-wise union  $T = T' \cup T''$  is a tableau of shape  $\lambda$ . It is easy to check that  $T \in \operatorname{SetSSYT}^{\phi}(\lambda)$ ,  $T_+ = T'$ ,  $T_- = T''$ . Both directions are clearly injective, completing our proof.

We define the flag  $\phi^+$  as  $\phi_i^+ = \max(\phi_i, 0)$ . Clearly, SetSSYT $_{\phi}^{\phi^+}(\lambda/\mu) = \text{SetSSYT}_{+}^{\phi^+}(\lambda/\mu)$ . Next we translate (10) from the language of tableaux to the language of polynomials:

Corollary 3.3. Suppose the partition  $\lambda$  and flag  $\phi$  are compatible. Then

$$G^\phi_\lambda(\beta;\mathbf{x};\mathbf{y}) = \sum_{\nu \subseteq \lambda} G^{\phi^-}_\nu(\beta;\mathbf{x};\mathbf{y}) \sum_{\mu \subseteq \nu: \ \nu/\mu \ \mathrm{disconnected}} \beta^{|\nu| - |\mu|} G^\phi_{\lambda/\mu}(\beta;\mathbf{x}_+;\mathbf{y}).$$

*Proof.* The left hand side is  $\sum_T \operatorname{wt}(T)$  where  $T \in \operatorname{SetSSYT}^{\phi}(\lambda)$ . On the right hand side, each valid choice of  $\nu, \mu$  contributes  $\sum_{T_1, T_2} \operatorname{wt}(T_1) \operatorname{wt}(T_2)$  where  $T_1 \in \operatorname{SetSSYT}^{\phi^-}(\nu)$  and  $T_2 \in \operatorname{SetSSYT}^{\phi^+}(\lambda/\mu)$ . Then this equation is implied by (10).

We may restrict this corollary to  $\phi$  with only positive entries, obtaining a formula for  $a_{\nu}^{\lambda,\phi}$ .

Corollary 3.4. Suppose the partition  $\lambda$  and the flag  $\phi$  are compatible. Further assume  $\phi$  has only non-negative entries. Then

$$a_{\nu}^{\lambda,\phi}(\beta;\mathbf{x};\mathbf{y}) = \sum_{\mu \subseteq \nu: \ \nu/\mu \ \text{disconnected}} \beta^{|\nu|-|\mu|} G_{\lambda/\mu}^{\phi}(\beta;\mathbf{x}_{+};\mathbf{y}).$$

Consequently,  $a_{\nu}^{\lambda,\phi}(\beta;\mathbf{x};\mathbf{y})$  is 0 if  $\lambda/\nu$  has a cell in row i where  $\phi_i=0$ .

*Proof.* Since  $\phi$  has only positive entries,  $\phi^-$  has only 0s. Therefore, in Corollary 3.3 the terms  $G_{\nu}^{\phi^-}(\beta; \mathbf{x}; \mathbf{y})$  become  $G_{\nu}(\beta; \mathbf{x}; \mathbf{y})$ . The equation in this corollary follows from the definition of  $a_{\nu}^{\lambda,\phi}(\beta; \mathbf{x}; \mathbf{y})$ .

Now suppose  $\lambda/\nu$  has a cell in row i where  $\phi_i = 0$ . Then  $\lambda/\mu$  also has a cell in row i for any  $\mu \subseteq \nu$ . Thus  $G_{\lambda/\mu}^{\phi}(\beta; \mathbf{x}_+; \mathbf{y})$  vanishes, so  $a_{\nu}^{\lambda,\phi}(\beta; \mathbf{x}; \mathbf{y}) = 0$ .

Finally, for general compatible  $\lambda$  and  $\phi$ , we obtain the following tableau formula involving a summation. We define the flag  $\phi^+$  as  $\phi_i^+ = \max(\phi_i, 0)$ .

**Proposition 3.5.** Suppose the partition  $\lambda$  and the flag  $\phi$  are compatible. Then

(11) 
$$G_{\lambda}^{\phi}(\beta; \mathbf{x}; \mathbf{y}) = \sum_{\nu \subseteq \lambda} G_{\nu}^{\phi^{-}}(\beta; \mathbf{x}; \mathbf{y}) a_{\nu}^{\lambda, \phi^{+}}(\beta; \mathbf{x}; \mathbf{y}).$$

Consequently,

(12) 
$$a_{\rho}^{\lambda,\phi}(\beta;\mathbf{x};\mathbf{y}) = \sum_{\nu \subseteq \lambda} a_{\rho}^{\nu,\phi^{-}}(\beta;\mathbf{x};\mathbf{y}) a_{\nu}^{\lambda,\phi^{+}}(\beta;\mathbf{x};\mathbf{y}).$$

By setting  $\mathbf{x}$  into  $\mathbf{y}$ , we obtain:

(13) 
$$j_{\rho}^{\lambda,\phi}(\beta;\mathbf{y}) = \sum_{\nu \subset \lambda} j_{\rho}^{\nu,\phi^{-}}(\beta;\mathbf{y}) j_{\nu}^{\lambda,\phi^{+}}(\beta;\mathbf{y}).$$

*Proof.* We first observe in Corollary 3.3 that we can replace  $G_{\lambda/\mu}^{\phi}(\beta; \mathbf{x}_{+}; \mathbf{y})$  with  $G_{\lambda/\mu}^{\phi^{+}}(\beta; \mathbf{x}_{+}; \mathbf{y})$ : Note  $\phi$  and  $\phi^{+}$  differ on entry i only when  $\phi_{i} < 0$ . If  $\lambda/\mu$  has a cell in such row i, the two polynomials both vanish. Otherwise, the two polynomials clearly agree.

Next, since  $\phi^+$  has only non-negative entries, Corollary 3.4 and Corollary 3.3 give (11).

For (12), we first need to show  $\nu$  and  $\phi^-$  are compatible to ensure the notation  $a_{\mu}^{\nu,\phi^-}(\beta; \mathbf{x}; \mathbf{y})$  makes sense. We check  $\phi_{i+1}^- - \phi_i^- \leq \nu_i - \nu_{i+1} + 1$  by considering three cases.

• If  $\phi_{i+1}$ ,  $\phi_i \leq 0$ , then  $\phi_{i+1}^- - \phi_i^- = \phi_{i+1} - \phi_i$ . By the condition of  $\nu$ ,  $\nu_i - \nu_{i+1} = \lambda_i - \lambda_{i+1}$ . Then the equation to check follows from the compatibility of  $\lambda$  and  $\phi$ .

• If  $\phi_i \leq 0 < \phi_{i+1}$ , then  $\phi_i^- = \phi_i$  and  $\phi_{i+1}^- = 0 < \phi_{i+1}$ . By the condition on  $\nu$ , we know  $\nu_i = \lambda_i$  and  $\nu_{i+1} < \lambda_{i+1}$ . We have

$$\phi_{i+1}^- - \phi_i^- < \phi_{i+1} - \phi_i \le \lambda_i - \lambda_{i+1} + 1 < \nu_i - \nu_{i+1} + 1.$$

• If  $0 < \phi_i, \phi_{i+1}$ , we have  $\phi_{i+1}^- - \phi_i^- = 0 - 0 = 0$ . Our inequality is trivial.

Then (12) follows by expanding each  $G_{\nu}^{\phi^{-}}(\beta; \mathbf{x}; \mathbf{y})$  from (11) into  $G_{\rho}(\beta; \mathbf{x}_{-}; \mathbf{y})$ 's. Finally, we obtain (13) by setting  $\mathbf{x}$  to  $\mathbf{y}$  in (12).

Recall our main objective is to give a tableau formula for  $j_{\rho}^{\lambda,\phi}(\beta;\mathbf{y})$  when  $\lambda$  and  $\phi$  are compatible. By (13), this reduces to identifying formulas for  $j_{\nu}^{\lambda,\phi^-}(\beta;\mathbf{y})$  and  $j_{\nu}^{\lambda,\phi^+}(\beta;\mathbf{y})$ . For the latter, by Corollary 3.4,

(14) 
$$j_{\nu}^{\lambda,\phi^{+}}(\beta;\mathbf{y}) = \sum_{\mu \subseteq \nu: \ \nu/\mu \text{ disconnected}} \beta^{|\nu|-|\mu|} G_{\lambda/\mu}^{\phi^{+}}(\beta;\mathbf{x}_{+};\mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}.$$

Therefore, we take a detour in the next section and study  $G_{\lambda/\mu}^{\phi^+}(\beta; \mathbf{x}_+; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}$ .

- 3.2. Understanding  $G^{\phi}_{\lambda/\mu}(\beta; \mathbf{x}_+; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}} \mathbf{when} \ \phi$  is non-negative. We fix  $\lambda, \phi$  compatible and require  $\phi$  non-negative. Also, fix  $\mu \subseteq \lambda$ . Clearly,  $G_{\lambda/\mu}^{\phi}(\beta; \mathbf{x}_+; \mathbf{y}) = 0$  if  $\lambda/\mu$  has a cell in row i and  $\phi_i = 0$ , so we assume this is not the case. If  $\lambda/\mu$  contains a cell on the diagonal, any tableau T of shape  $\lambda/\mu$  has a cell weight containing a term of the form  $(x_i \ominus y_i)$  for some i. Consequently,  $G_{\lambda/\mu}^{\phi}(\beta; \mathbf{x}_+, \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}$  must vanish. Therefore, we may assume  $\lambda/\mu$  has no cells on the diagonal. Such skew shapes  $\lambda/\mu$  can be separated into two parts by the diagonal. There exist partitions  $\mu^U$  and  $\mu^D$  satisfying:
  - Cells of  $\lambda/\mu^U$  and cells of  $\lambda/\mu^D$  form a partition of cells in  $\lambda/\mu$ .
  - Cells in  $\lambda/\mu^U$  lies above the diagonal while cells in  $\lambda/\mu^D$  lies below the diagonal.

Consequently,  $G_{\lambda/\mu}^{\phi}(\beta; \mathbf{x}_{+}, \mathbf{y}) = G_{\lambda/\mu^{U}}^{\phi}(\beta; \mathbf{x}_{+}, \mathbf{y}) \cdot G_{\lambda/\mu^{D}}^{\phi}(\beta; \mathbf{x}_{+}, \mathbf{y}).$ For instance, let  $\lambda = (2, 1, 1)$ ,  $\phi = (2, 2, 3)$  and  $\mu = (1)$ . Then  $\mu^{U} = (1, 1, 1)$  and  $\mu^{D} = (2)$ . We have

$$G^{\phi}_{(2,1,1)/(1)}(\beta;\mathbf{x}_{+},\mathbf{y}) = G^{\phi}_{(2,1,1)/(1,1,1)}(\beta;\mathbf{x}_{+},\mathbf{y})G^{\phi}_{(2,1,1)/(2)}(\beta;\mathbf{x}_{+},\mathbf{y}).$$

We now must study the polynomials  $G_{\lambda/\mu^U}^{\phi}(\beta; \mathbf{x}_+; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}$  and  $G_{\lambda/\mu^D}^{\phi}(\beta; \mathbf{x}_+; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}$ . The former is relatively straightforward.

Lemma 3.6. We have

$$G^{\phi}_{\lambda/\mu^U}(\beta; \mathbf{x}_+; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}} = \sum_{T \in \text{SetSSYT}^{\phi^+}(\lambda/\mu^U)} \operatorname{wt}(T) \mid_{\mathbf{x} \mapsto \mathbf{y}}.$$

Moreover, each term  $\operatorname{wt}(T)|_{\mathbf{x}\mapsto\mathbf{y}}$  is a product of distinct Type 1 terms and a power of  $\beta$ .

*Proof.* Let  $T \in \text{SetSSYT}^{\phi}_{+}(\lambda/\mu)$ . We only need to show  $\text{wt}(T) \mid_{\mathbf{x} \mapsto \mathbf{y}}$  can be written as a product of distinct  $(y_i \ominus y_j)$  with 0 < i < j and a power of  $\beta$ . A cell (r,c) in T satisfies r < c and has cell weight  $x_{T(r,c)} \ominus x_{T(r,c)}$  $y_{T(r,c)+c-r}$ . We have 0 < T(r,c) < T(r,c) + c - r. Thus, after setting  $\mathbf{x} \mapsto \mathbf{y}$ , it becomes  $(y_i \ominus y_j)$  with 0 < i < j. Then the proof is finished since other cells cannot have the same cell weight as observed in Remark 2.1.

Our analysis of polynomial  $G^{\phi}_{\lambda/\mu^D}(\beta; \mathbf{x}_+; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}$  is more subtle, requiring us to introduce a new flag. For  $\lambda$  and  $\phi$  compatible, define  $\psi(\lambda, \phi)$  as the sequence with  $\psi(\lambda, \phi)_i = \min(i - \lambda_i, \phi_i)$ . For the remainder of the section,  $\lambda$  and  $\phi$  will be clear, so abusing notation we write  $\psi$ . Note  $\psi$  is weakly increasing since  $\phi_i \leq \phi_{i+1}$ and  $i - \lambda_i < (i+1) - \lambda_{i+1}$ .

**Lemma 3.7.** There exists a permutation  $\pi \in S_{\infty}$  depending on  $\lambda$  and  $\phi$  so that

(15) 
$$G_{\lambda/\mu^D}^{\phi}(\beta; \mathbf{x}_+; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}} = G_{\lambda/\mu^D}^{\psi}(\beta; \pi \cdot \mathbf{x}_+; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}.$$

Here  $\pi$  permutes the subscripts of the x-variables.

We postpone the proof of Lemma 3.7. Let us first use it to obtain the analogue of Lemma 3.6.

**Lemma 3.8.** We can write  $G^{\phi}_{\lambda/\mu^D}(\beta; \mathbf{x}_+; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}$ , as a sum where each summand is a product of distinct Type 3 terms and a power of  $\beta$ .

*Proof.* By Lemma 3.7, we must write the right hand side of (15) as such a sum. Let  $T \in \text{SetSSYT}_+^{\psi}(\lambda/\mu)$ . We first show wt(T) is a product of distinct  $(x_i \ominus y_j)$  with  $j \le 0 < i$  and a power of  $\beta$ . Consider a value  $i \in T(r,c)$  with  $(r,c) \in \lambda/\mu$ . We know  $1 \le i \le \psi_r \le r - \lambda_r$ . Thus,

$$T(r,c) + c - r \le (r - \lambda_r) + c - r = c - \lambda_r \le 0.$$

Then this value contributes the weight  $y_{\pi(i)} \ominus y_j$  to the right hand side of (15) with  $j \le 0 < \pi(i)$ , which is a type 3 term. The proof is finished since cells cannot have the same cell weight, as observed in Remark 2.1.  $\square$ 

We briefly summarize our conclusion in this subsection.

**Remark 3.9.** Consider  $\lambda, \phi$  compatible. Assume entries in  $\phi$  are non-negative. Take  $\mu \subseteq \lambda$ . We can characterize  $G = G^{\phi}_{\lambda/\mu}(\beta; \mathbf{x}_+; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}$  as follows, ignoring the  $\beta$  for clarity:

- (Case Zero): If  $\lambda/\mu$  has a cell on the diagonal, or in row i with  $\phi_i = 0$ , then G = 0.
- (Case Up): If  $\lambda/\mu$  is completely above the diagonal, G is a sum where each summand is a distinct product of Type 1 terms.
- (Case Down): If  $\lambda/\mu$  is completely under the diagonal, G is a sum where each summand is a distinct product of Type 3 terms.
- (Case Combined): Otherwise, G is a sum with each summand a distinct product of Type 1 and Type 3 terms.

In the remainder of this subsection, we will prove Lemma 3.7. We rename  $\mu^D$  as  $\mu$  for simplicity. The first ingredient is a result showing  $G_{\lambda/\mu}^{\phi}(\beta; \mathbf{x}_+; \mathbf{y})$  is symmetric in certain  $\mathbf{x}$ -variables.

**Lemma 3.10.** The polynomial  $G^{\phi}_{\lambda/\mu}(\beta; \mathbf{x}_+; \mathbf{y})$  is symmetric in  $x_i, x_{i+1}, \dots, x_j$  as long as  $i, i+1, \dots, j-1$  do not appear in  $\phi$ .

*Proof.* We use a Bender-Knuth involution argument. It is enough to prove the lemma with j = i + 1. Suppose  $\lambda/\mu$  is a single row of n cells. Then

$$G_{\lambda/\mu}^{\phi}(\beta; x_i, x_{i+1}; \mathbf{y}) = G_{(n)}(\beta; x_i, x_{i+1}; \sigma \cdot \mathbf{y}),$$

where  $\sigma$  is some permutation shifting the **y**-variables. This polynomial is well known to be symmetric in  $x_i$  and  $x_{i+1}$ . One argument for this fact is that, by Proposition 2.6  $\overleftarrow{\mathfrak{G}}_{\iota^{i+1}(w((n)))}(\beta; \mathbf{x}; \mathbf{y})$  is fixed by  $\pi_i$  (up to a factor of  $\beta$ ), which implies it is fixed by  $s_i$  acting on the **x**-variables.

Now consider an arbitrary  $\lambda/\mu$ . Given  $T \in \operatorname{SetSSYT}_+^{\phi}(\lambda/\mu)$ , freeze all values in T that are not i or i+1. Moreover, freeze all pairs i, i+1 with i directly above i+1 in T – necessarily i is the maximum value of its cell while i+1 is the minimum. All other entries are free. The free entries in T will be in consecutive row segments, each with some weakly increasing filling from  $\{i, i+1\}$ . Hence we obtain a partition of  $\operatorname{SSYT}_+^{\phi}(\lambda/\mu)$  into  $A_1 \sqcup A_2 \sqcup \ldots$ , where each  $A_k$  consists of tableaux having the same frozen entries. The contribution of a frozen pair of i and i+1 to a tableau will be  $(x_i \ominus y_j)(x_{i+1} \ominus y_j)$ , which is symmetric in  $x_i$  and  $x_{i+1}$ . Consider each  $g_k = \sum_{T \in A_k} \operatorname{wt}(T)$ . The contribution of each non-frozen row segment is  $G_{\lambda'/\mu'}(\beta; x_i, x_{i+1}; \mathbf{y})$  for some  $\lambda'/\mu'$  that is a single row. This is symmetric in  $x_i$  and  $x_{i+1}$  by the arguments above.  $\square$ 

Recall  $\psi$  is the flag with  $\psi_i = \min(i - \lambda_i, \phi_i)$ . Let  $\ell = \ell(\lambda)$ . Then let  $\Delta$  be the sequence of length  $\ell$  with  $\Delta_i = \phi_i - \psi_i$ .

**Lemma 3.11.** The sequence  $\Delta$  is a weakly decreasing sequence of non-negative numbers.

*Proof.* First, notice that

$$\Delta_i = \phi_i - \psi_i = \phi_i - \min(\phi_i, i - \lambda_i) = \max(0, \phi_i - i + \lambda_i).$$

Thus,  $\Delta_i \geq 0$ . To see  $\Delta$  is weakly decreasing, we only need to show

$$\phi_i - i + \lambda_i \ge \phi_{i+1} - (i+1) + \lambda_{i+1}$$
.

After rearranging, this becomes:

$$\lambda_i - \lambda_{i+1} + 1 > \phi_{i+1} - \phi_i,$$

which follows from the compatibility of  $\lambda$  and  $\phi$ .

Next, we provide an algorithm to compute  $\pi$  using  $\lambda$  and  $\phi$  as applied in Lemma 3.7. Our algorithm is iterative, producing a sequence of permutations  $\pi_0, \ldots, \pi_\ell$  with  $\pi = \pi_\ell$ . While defining these permutations, we make sure that the numbers  $1, \ldots, \Delta_i$  appear consecutively in increasing order in the one-line notation of  $\pi_i$ .

**Algorithm 3.12.** Let  $\pi_0$  be the identity permutation. For  $i \in [\ell]$ :

- if  $\psi_i \leq 0$ , set  $\pi_i := \pi_{i-1}$ ;
- if  $\psi_i > 0$ , construct  $\pi_i$  from  $\pi_{i-1}$  by shifting the values  $1, \ldots, \Delta_i$  to the right until they are at indices  $\psi_i + 1, \ldots, \phi_i$ .

**Example 3.13.** For  $\lambda = (4, 4, 4, 4, 4, 2, 1)$  and  $\phi = (3, 4, 4, 5, 6, 6, 6)$ , we have  $\psi = (-3, -2, -1, 0, 1, 4, 6)$  and  $\Delta = (6, 6, 5, 5, 5, 2, 2)$ ; so  $\pi_0 = \pi_1 = \ldots = \pi_4 = \text{identity}$  and

$$\pi_5 = 6\underline{12345}78$$

$$\pi_6 = 6345\underline{12}78$$

$$\pi_7 = 634578\underline{12}$$

Here, we have underlined the consecutive sequences that have moved.

Correspondingly, we define a sequence of flags. For  $i = 0, \dots, \ell$ , let

$$\chi^{(i)} = (\psi_1, \dots, \psi_i, \phi_{i+1}, \dots, \phi_\ell).$$

Each  $\chi^{(i)}$  is a flag since  $\phi$ ,  $\psi$  are both flags and  $\psi_i \leq \psi_{i+1} \leq \phi_{i+1}$ . As demonstrated by the following example, these  $\chi^{(i)}$  characterizes the tableaux whose weights do not vanish after applying the modification  $\pi_i \cdot \mathbf{x}$  and substitution  $\mathbf{x} \mapsto \mathbf{y}$ .

We summarize the role of the  $\chi^{(i)}$  flags in the following lemma.

**Lemma 3.14.** For all  $0 \le i \le \ell$ , each  $G_{\lambda/\mu}^{\chi^{(i)}}(\beta; \pi_i \cdot \mathbf{x}_+; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}$  is the same.

*Proof.* Take  $0 \le i < \ell$ . We need to show

(16) 
$$G_{\lambda/\mu}^{\chi^{(i)}}(\beta; \pi_i \cdot \mathbf{x}_+; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}} = G_{\lambda/\mu}^{\chi^{(i+1)}}(\beta; \pi_{i+1} \cdot \mathbf{x}_+; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}.$$

This is separated into two steps.

## Step 1: We first show

$$G_{\lambda/\mu}^{\chi^{(i)}}(\beta; \pi_i \cdot \mathbf{x}_+; \mathbf{y}) = G_{\lambda/\mu}^{\chi^{(i)}}(\beta; \pi_{i+1} \cdot \mathbf{x}_+; \mathbf{y}).$$

This is immediate if  $\pi_i = \pi_{i+1}$ . Otherwise,  $\pi_{i+1}$  is obtained from  $\pi_i$  by moving  $1, \dots, \Delta_{i+1}$  to the right. In  $\pi_i$ , the number 1 is at index  $\max(\psi_i + 1, 1)$ . In  $\pi_{i+1}$ , the number  $\Delta_{i+1}$  is at index  $\phi_{i+1}$ . As a summary,  $\pi_{i+1}$  is obtained from  $\pi_i$  by permuting the numbers whose indices are between  $\max(\psi_i + 1, 1)$  and  $\phi_{i+1}$  inclusively.

is obtained from  $\pi_i$  by permuting the numbers whose indices are between  $\max(\psi_i + 1, 1)$  and  $\phi_{i+1}$  inclusively. On the other hand, by Lemma 3.10,  $G_{\lambda/\mu}^{\chi^{(i)}}(\beta; \mathbf{x}_+; \mathbf{y})$  is symmetric in variables with subscript between  $\max(\psi_i + 1, 1)$  and  $\phi_{i+1}$  inclusively, so Step 1 follows.

#### Step 2: We then show

$$G_{\lambda/\mu}^{\chi^{(i)}}(\beta; \pi_{i+1} \cdot \mathbf{x}_+; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}} = G_{\lambda/\mu}^{\chi^{(i+1)}}(\beta; \pi_{i+1} \cdot \mathbf{x}_+; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}.$$

This is trivial when  $\psi_{i+1} = \phi_{i+1}$ , which would imply  $\chi^{(i)} = \chi^{(i+1)}$ . Suppose this is not the case, then  $\psi_{i+1} = i - \lambda_{i+1}$ .

Take  $T \in \text{SetSSYT}^{\chi^{(i)}}(\lambda/\mu)_+$  but not in  $\text{SetSSYT}^{\chi^{(i+1)}}(\lambda/\mu)_+$ . It is enough to show wt(T) becomes 0 after we permute its **x**-variables by  $\pi_{i+1}$  and then set  $\mathbf{x} \mapsto \mathbf{y}$ . Consider the cell  $(i+1,\lambda_{i+1})$ . By the assumption of T, this is a cell in T containing a value k with  $\psi_{i+1} < k \le \phi_i$ . Notice that the permutation  $\pi_{i+1}$  sends j to  $j - \psi_{i+1} = j - (i+1) + \lambda_{i+1}$  if  $\psi_{i+1} < j \le \phi_i$ . Thus, after we let  $\pi_{i+1}$  permute the x-variables, the value k in this cell has weight

$$x_{k-(i+1)+\lambda_{i+1}} \ominus y_{k-(i+1)+\lambda_{i+1}},$$

which would be 0 after setting  $\mathbf{x} \mapsto \mathbf{y}$ .

To conclude, observe that (16) follows from the two steps above.

We now prove Lemma 3.7.

*Proof of Lemma 3.7.* Set  $\pi = \pi_{\ell}$ . The statement follows from Lemma 3.14.

We can now recover Anderson's positivity result for vexillary permutations. By Proposition 3.5, the result follows from the  $\beta$ -Graham positivity of  $j_{\mu}^{\lambda,\phi}(\beta;\mathbf{y})$  when  $\phi$  has either all non-negative or all non-positive entries. The former follows by combining Lemmas 3.6 and 3.7. The latter follows from the former and Lemma 2.4. We must now prove our finer positivity with restrictions on types of terms  $y_i \ominus y_j$ .

#### 4. Proving Theorem 1.1

Fix some compatible  $\lambda, \phi$  and  $\rho \subseteq \lambda$ . We say a row i is a positive row if  $\phi_i > 0$ , with negative rows, zero rows, non-positive rows and non-negative rows defined similarly. For  $\nu \subseteq \lambda$ , define  $\xi(\nu)$  as a flag for  $\nu'$  by  $\xi(\nu)_i := -\phi_{\nu'_i}^-$ . Since  $\phi^-$  is non-positive, we know  $\xi(\nu)$  is non-negative. Moreover, Corollary 2.9 says  $j_{\rho}^{\nu,\phi^-}(\beta;\mathbf{y}) = \omega_1(j_{\rho'}^{\nu',\xi(\nu)}(\beta;\mathbf{y}))$ . We deduce a simple lemma regarding  $\xi(\nu)$ .

**Lemma 4.1.** Say  $\nu/\rho$  has a cell on a non-negative row. Then  $\nu'/\rho'$  has a cell on row j with  $\xi(\nu)_j = 0$ .

*Proof.* Find i such that  $(i, \nu_i)$  lies in  $\nu/\rho$  and  $\phi_i \ge 0$ . We deduce  $(\nu_i, i)$  lies in  $\nu'/\rho'$  and  $\phi_i^- = 0$ . It remains to show  $\xi(\nu)_{\nu_i} = 0$ . Notice that  $\nu'_{\nu_i} \ge i$ , so

$$\xi(\nu)_{\nu_i} = -\phi_{\nu'_{\nu_i}}^- \le -\phi_i^- = 0.$$

We must have  $\xi_{\nu_i} = 0$  since  $\xi(\nu)$  is non-negative.

We then describe some situations when  $j_{\nu}^{\lambda,\phi^+}(\beta;\mathbf{y})$  or  $j_{\rho}^{\nu,\phi^-}(\beta;\mathbf{y})$  vanish.

**Lemma 4.2.** The polynomial  $j_{\nu}^{\lambda,\phi^{+}}(\beta;\mathbf{y})$  vanishes if  $\lambda/\nu$  has a cell on the diagonal or a non-positive row. The polynomial  $j_{\rho}^{\nu,\phi^{-}}(\beta;\mathbf{y})$  vanishes if  $\nu/\rho$  has a cell on the diagonal or on a non-negative row.

Proof. Recall (14), which says

$$j_{\nu}^{\lambda,\phi^+}(\beta;\mathbf{y}) = \sum_{\mu \subseteq \nu: \; \nu/\mu \text{ disconnected}} \beta^{|\nu|-|\mu|} G_{\lambda/\mu}^{\phi^+}(\beta;\mathbf{x}_+;\mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}.$$

If  $\lambda/\nu$  has a cell on the diagonal or a non-positive row, so does  $\lambda/\mu$  for every  $\mu \subseteq \nu$ . Therefore, by Remark 3.9, we have the first statement.

For the second statement, recall Corollary 2.9 says  $j_{\rho}^{\nu,\phi^-}(\beta;\mathbf{y}) = \omega_1(j_{\rho'}^{\nu',\xi(\nu)}(\beta;\mathbf{y}))$ . If  $\nu/\rho$  has a cell on the diagonal, so does  $\nu'/\rho'$ . If  $\nu/\rho$  has a cell on a non-negative row, by Lemma 4.1,  $\nu'/\rho'$  has a cell on row j with  $\xi(\nu)_j^- = 0$ . Since  $\xi(\nu)$  is non-negative, similar to the previous paragraph, we have  $j_{\rho'}^{\nu',\xi(\nu)}(\beta;\mathbf{y}) = 0$  in either case. Thus,  $j_{\rho}^{\nu,\phi^-}(\beta;\mathbf{y})$  vanishes.

Consequently, we deduce there is only one non-zero summand in (13).

**Corollary 4.3.** For  $\rho \subseteq \lambda$ ,  $j_{\rho}^{\lambda,\phi}(\beta;\mathbf{y})$  vanishes if  $\lambda/\rho$  has a cell on the diagonal or on row i for some i such that  $\phi_i = 0$ . Otherwise,

$$j_{\rho}^{\lambda,\phi}(\beta;\mathbf{y}) = j_{\rho}^{\nu,\phi^{-}}(\beta;\mathbf{y})j_{\nu}^{\lambda,\phi^{+}}(\beta;\mathbf{y})$$

where

$$\nu_i := \begin{cases} \lambda_i & \text{if } \phi_i < 0, \\ \rho_i = \lambda_i & \text{if } \phi_i = 0, \\ \rho_i & \text{if } \phi_i > 0. \end{cases}$$

*Proof.* We consider the summands in (13). Suppose  $\lambda/\rho$  has a cell on the diagonal or on a zero row. Then for each  $\rho \subseteq \nu \subseteq \lambda$ , this cell is either in  $\nu/\rho$  or  $\lambda/\nu$ , neutralizing either  $j_{\rho}^{\nu,\phi^-}(\beta;\mathbf{y})$  or  $j_{\nu}^{\lambda,\phi^+}(\beta;\mathbf{y})$  by Lemma 4.2.

Now suppose  $\lambda/\rho$  has no such cells. We have  $\lambda_i = \rho_i$  if  $\phi_i = 0$ . By Lemma 4.2 to make  $j_{\nu}^{\lambda,\phi^+}(\beta;\mathbf{y})$  non-zero, we must have  $\nu_i = \lambda_i$  whenever  $\phi_i < 0$ . To make  $j_{\rho}^{\nu,\phi^-}(\beta;\mathbf{y})$  non-zero, we must have  $\nu_i = \rho_i$  whenever  $\phi_i > 0$ . These conditions uniquely determine a partition  $\nu$ .

**Example 4.4.** Suppose  $\lambda=(7,4,2,2,1)$  and  $\phi=(\overline{1},0,1,2,4)$ . To make  $j_{\rho}^{\lambda,\phi}$  non-zero, we know  $\lambda/\rho$  cannot have any cells on the diagonal or in row 2. Suppose  $\mu=(5,4,2,1,1)$ . Then by Corollary 4.3,  $j_{\rho}^{\lambda,\phi}(\beta;\mathbf{y})=j_{\rho}^{\nu,\phi^-}(\beta;\mathbf{y})j_{\nu}^{\lambda,\phi^+}(\beta;\mathbf{y})$ , where  $\nu=(7,4,2,1,1)$ . If we change  $\phi$  into  $(\overline{2},\overline{1},1,2,3)$  and change  $\mu$  into (4,2,2,1), then the  $\nu$  in Corollary 4.3 becomes (7,4,2,1).

Finally, for fixed  $\nu$  we derive the finer version of  $\beta$ -Graham positivity. Assume  $j_{\rho}^{\lambda,\phi} \neq 0$ . Let q be the largest such that (q,q) is a cell in  $\lambda$ . By Corollary 4.3, we know (q,q) is also a cell in  $\rho$ . Thus, cells above the diagonal in  $\lambda/\rho$  are above row q whiles cells under the diagonal in  $\lambda/\rho$  are under row q. Following Remark 3.9, we may characterize  $j_{\nu}^{\lambda,\phi^+}(\beta;y)$ .

**Lemma 4.5.** If we ignore the  $\beta$ , we may describe  $j^+ = j_{\nu}^{\lambda,\phi^+}(\beta;y)$  as follows:

- If  $\phi_a > 0$ , then  $j^+$  is a sum where each summand is a distinct product of Type 1 or Type 3 terms.
- If  $\phi_q \leq 0$ , then  $j^+$  is a sum where each summand is a distinct product of Type 3 terms.

Proof. Recall

$$j^{+} = \sum_{\mu \subseteq \nu: \ \nu/\mu \ \text{disconnected}} \beta^{|\nu| - |\mu|} G_{\lambda/\mu}^{\phi^{+}}(\beta; \mathbf{x}_{+}; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}.$$

By Remark 3.9, each  $G_{\lambda/\mu}^{\phi^+}(\beta; \mathbf{x}_+; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}$  is a sum where each summand is a product of distinct terms of Type 1 and 3. So is  $j^+$ .

Now we further assume  $\phi_q \leq 0$ , so  $\phi_i^+ = 0$  if  $i \leq q$ . If  $\lambda/\mu$  has a cell above the diagonal, then that cell is on a non-positive row, making  $G_{\lambda/\mu}^{\phi^+}(\beta; \mathbf{x}_+; \mathbf{y})$  vanish. Thus, to make the summand non-zero,  $\lambda/\mu$  must be under the diagonal. By (Case Down) of Remark 3.9,  $j^+$  is a sum of products of distinct Type 3 terms.  $\square$ 

We can deduce a similar statement of  $j_{\rho}^{\nu,\phi^{-}}(\beta;y)$ .

**Lemma 4.6.** If we ignore the  $\beta$ , we may describe  $j^- = j_{\rho}^{\nu,\phi^-}(\beta;y)$  as follows:

- If  $\phi_q < 0$ , then  $j^-$  is a sum where each summand is a distinct product of Type 2 or Type 3 terms.
- If  $\phi_q \geq 0$ , then  $j^-$  is a sum where each summand is a distinct product of Type 3 terms.

*Proof.* We define flag  $\xi$  for  $\nu'$  by  $\xi_i := -\phi_{\nu'_i}^-$ , so  $\xi$  has only non-negative entries. Recall

$$j^{-} = \omega(j_{\rho'}^{\nu',\xi(\nu)}) = \omega\left(\sum_{\mu \subseteq \rho: \ \rho/\mu \text{ disconnected}} \beta^{|\rho|-|\mu|} G_{\nu'/\mu'}^{\xi(\nu)}(\beta; \mathbf{x}_{+}; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}\right).$$

By Remark 3.9, each  $G_{\nu'/\mu'}^{\xi(\nu)}(\beta; \mathbf{x}_+; \mathbf{y}) \mid_{\mathbf{x} \mapsto \mathbf{y}}$  is a sum where each summand is a product of distinct terms of Type 1 and 3. Under  $\omega$ , Type 1 terms become Type 2 while Type 3 terms remain Type 3. Thus,  $j^-$  is a sum where each summand is a product of distinct terms of Type 2 and 3.

Now we further assume  $\phi_q \geq 0$ . If  $\nu/\mu$  has a cell under the diagonal, say in row i, we know  $i \geq q$ . Thus, that cell is on a non-negative row. By Lemma 4.1,  $\nu'/\mu'$  has a cell in row j with  $\xi(\nu)_j = 0$ , making  $G^{\xi(\nu)}_{\nu'/\mu'}(\beta; \mathbf{x}_+; \mathbf{y})$  vanish. To make the summand non-zero,  $\nu/\mu$  must be above the diagonal, so  $\nu'/\mu'$  is under the diagonal. By (Case Down) of Remark 3.9,  $j^{\nu',\xi(\nu)}_{\rho'}$  is a sum of products of distinct Type 3 terms. So is  $j^-$ .

Finally, we combine Lemma 4.5 and Lemma 4.6 to obtain the finer posivity of  $j_{\rho}^{\lambda,\phi} = j^+j^-$ .

**Theorem 4.7.** Either (or both) of the following two cases hold.

- Case 1: φ<sub>q</sub> ≤ 0. Here j<sub>ρ</sub><sup>λ,φ</sup>(β; y) is a sum of products of Type 2 and Type 3 terms with a power of β.
  In each summand, every Type 2 (resp. 3) term appears at most once (resp. twice).
- Case 2:  $\phi_q \geq 0$ . Here  $j_{\rho}^{\lambda,\phi}(\beta; \mathbf{y})$  is a sum of products of Type 1 and Type 3 terms with a power of  $\beta$ . In each summand, every Type 1 (resp. 3) term appears at most once (resp. twice).

Proof. Immediate from Lemma 4.5 and Lemma 4.6.

# 5. Final Remarks

We conclude with some brief remarks, including suggestions for further directions.

5.1. An explicit tableaux formula. Recall for compatible  $\lambda$  and  $\phi$  that  $w_{\lambda,\phi}$  is the unique vexillary permutation with shape  $\lambda$  and flag  $\phi$ . The permutation  $\neg w_{\lambda,\phi}$  is also vexillary, e.g., by Lemma 2.4.

We extend the map  $\omega_1$  acting on polynomials to one acting on tableaux. A set-valued tableau T is row-strict-decreasing if entries of T strictly decrease in each row and weakly decrease in each column. The r-weight of an element in cell (r, c) of the row-strict-decreasing tableau T is  $y_{i+c-r} \ominus x_i$ . Then the r-weight of T, denoted  $\operatorname{wt}_r(T)$ , is the product of each element weight.

For  $\lambda$  a partition, sequences of the form  $\phi = (\phi_1 \geq \cdots \geq \phi_{\lambda_1})$  are reverse flags for  $\lambda$ . Let SetRSDT $^{\phi}(\lambda/\mu)$  be the set of row-strict-decreasing tableaux with shape  $\lambda/\mu$  such that entries in column i are weakly larger than  $\phi_i$ . Recall SetRSDT $^{\phi}(\lambda/\mu)$  is the subset of SetRSDT $^{\phi}(\lambda/\mu)$  consisting of tableaux whose entries are subsets of  $\mathbb{Z}_{<}$ .

Let  $\omega_1$  act on set-valued tableaux by conjugating, then replacing each value i with 1-i. To justify our use of the notation  $\omega_1$ , observe:

**Lemma 5.1.** The map  $\omega_1 : \operatorname{SetSSYT}(\lambda/\mu) \to \operatorname{SetRSDT}(\lambda'/\mu')$  is a bijection with

$$\operatorname{wt}_r(\omega_1(T)) = \omega_1(\operatorname{wt}(T)).$$

Moreover,  $\omega_1$  restricts to a bijection from SetSSYT $^{\phi}_{+}(\lambda/\mu)$  to SetRSDT $^{1-\phi}_{-}(\lambda'/\mu')$  where  $1-\phi$  is the reverse flag obtained by replacing every i in  $\phi$  by 1-i.

As a consequence, we see  $a_{\rho}^{\lambda,\phi_{-}}(\beta;\mathbf{x};\mathbf{y})$  can be expressed as a sum over row strict decreasing tableaux with weight function  $\mathrm{wt}_{r}$ . Then (12) can be viewed as a sum over tableaux, with positive entries forming a semistandard tableau weighted according to wt and non-positive entries forming a row strict decreasing tableau weighted according to  $\mathrm{wt}_{r}$ .

5.2. More permutations. A complete combinatorial proof that double  $\beta$ -Edelman-Greene coefficients are  $\beta$ -Graham positive remains an open problem, even for  $\beta = 0$ . Our proof depends critically on having a tableau formula for  $\overleftarrow{\mathfrak{G}}_w(\beta; \mathbf{x}; \mathbf{y})$ , which only exists for w vexillary. To gain a tableau formula for all  $w \in S_{\mathbb{Z}}$ , one could extend the Lascoux decomposition of the Grothendieck polynomial  $\mathfrak{G}_w(\beta; \mathbf{x}_+)$  to  $\overleftarrow{\mathfrak{G}}_w(\beta; \mathbf{x}; \mathbf{y})$ . We have explored a candidate for this expansion in the cohomology case. In principle extending our approach appears plausible, but any extension of Lemma 3.10 would be far more subtle.

A relatively easy extension of Theorem 1.1 relies on a recurrence for  $\beta$ -Grothendieck polynomials called the transition equations [16, 25], instances of which are determined by picking certain boxes from the Rothe diagram. If we can apply transition to a box (r,c) with  $r \prec c$  for  $w \in S_{\mathbb{Z}}$  and the resulting terms in the recurrence have Graham positive double  $\beta$ -Edelman-Greene coefficients, so will w. Transition equations terminate with vexillary permutations, and this approach yields many more  $w \in S_{\mathbb{Z}}$  for which the conclusion of Theorem 1.1 applies, but this set is not easily characterized.

5.3. Non-vanishing coefficients. Recall the Edelman–Greene coefficient of  $\mu$  for  $w \in S_{\mathbb{Z}}$  is  $j_{\mu}^{w} := j_{\mu}^{w}(0)$ . These are the codimension zero double Edelman–Greene coefficients. Note that  $\lambda(w)$  is well-defined even for non-vexillary permutations, and let  $\ll$  denote dominance order for integer partitions. Billey and Pawlowski have given an elegant necessary condition for Edelman–Greene coefficients to be non-zero.

**Lemma 5.2** ([4, Lem 3.11]). For 
$$w \in S_{\mathbb{Z}}$$
,  $j_{\mu}^{w} = 0$  unless  $\lambda(w) \ll \mu \ll \lambda(w^{-1})'$ .

From this perspective, w is vexillary if and only  $\lambda(w) = \lambda(w^{-1})'$ . It is natural to ask:

**Question 5.3.** For  $w \in S_{\mathbb{Z}}$  and  $\mu$  a partition, when is  $j_{\mu}^{w}(\beta; \mathbf{y})$  non-zero?

This would be an interesting question to understand even when setting  $\beta=0$ . In the vexillary case for  $\mu\subseteq\lambda(w)$  so that  $\lambda(w)/\mu$  contains no entries on the diagonal, our tableau formula allows us to determine when whether the coefficient vanishes in polynomial time. Otherwise the coefficient vanishes. Additionally, we propose extending Lemma 5.2 to double  $\beta$ -Edelman-Greene coefficients as an interesting problem, and further motivation for understanding such coefficients in general. For this problem, both the  $\beta=0$  and  $\mathbf{y}\mapsto 0$  cases are also of interest.

- 5.4. Variants. One promising avenue is to define backstable Schubert polynomials and double Stanley symmetric functions in the other types. In many ways, the double Schubert polynomials of [10] and double  $\beta$ -Grothendieck polynomials [11] in Types B, C and D are already backstable<sup>1</sup> with formulas akin to Proposition A.2, so such a definition is feasible. As evidence for the existence of these objects and corresponding Graham positivity, one could attempt to extend Theorems 1.1 to flagged double Schur-P and-Q functions and their K-theoretic analogues. Flagged double Schur-P and-Q functions are introduced via Pfaffian formulas in [3, 2] with a tableau formula for the latter in [19].
- 5.5. **Schubert structure coefficients.** One of the most important open problems in algebraic combinatorics is to compute Schubert structure coefficients. The most classical problem of this form is solved by the Littlewood–Richardson rule:

$$s_{\lambda} \cdot s_{\mu} = \sum_{\nu} c^{\nu}_{\lambda \mu} s_{\nu},$$

where  $c_{\lambda\mu}^{\nu}$  has a combinatorial description. This rule has been generalized to the equivariant setting [22, 13], to K-theory [6] and to both simultaneously [21]. A combinatorial rule for the product of Schubert polynomials

(17) 
$$\mathfrak{S}_{u}(\mathbf{x}) \cdot \mathfrak{S}_{v}(\mathbf{x}) = \sum_{w} c_{uv}^{w} \mathfrak{S}_{w}$$

is open. Similarly, there are generalizations to the equivariant setting where  $c_{uv}^w$  is a Graham positive polynomial, and to K-theory, as well as to both simultaneously. Indeed, Graham positivity was introduced to understand the equivariant setting of this problem [8]. As mentioned, the coefficients for single Stanley symmetric functions and their K-theoretic analogues have been used to prove rules for special cases of (17). This raises:

**Question 5.4.** Can equivariant (K-theoretic) Schubert structure coefficients be computed using double ( $\beta$ -)Edelman-Greene coefficients?

It would be an accomplishment to recover equivariant Grassmannian cohomology [13] or K-theory structure coefficients [21] in this way. One piece of evidence that this question could prove fruitful is that the proof from [1] that the  $\overleftarrow{\mathfrak{G}}_w(\beta;\mathbf{x};\mathbf{y})$ 's are closed under multiplication relies on the Graham positivity of double Stanley symmetric functions. Additionally, Thomas Lam has informed us that double  $\beta$ -Edelman-Greene coefficients should compute equivariant K-homology for the Grassmannian, first computed in [26]. Additionally, we suspect these coefficients should generalize results from [20] to double Grothendiecks.

**Acknowledgements:** We thank Dave Anderson for helping us better understand the geometric setting for our work and Thomas Lam for sharing his proof outline for Proposition 2.7.

Appendix A. Details on double symmetric functions and K-theoretic extensions

The main goal of this Appendix is to prove Proposition 2.7. We begin by remarking that, while [14] works with the specialization  $\beta = -1$ , by inserting powers of  $\beta$  so that all terms are homogeneous with the convention deg  $\beta = -1$ , we recover our conventions. This is explained in full detail in [1, Rem 7.2].

Let  $p_k(\mathbf{x}_-; \mathbf{y}) = p_k(\mathbf{x}_-/\mathbf{y}) = \sum_{i \leq 0} x_i^k - y_i^k$ , and define  $\Lambda(\mathbf{x}_-; \mathbf{y})$  as the  $\mathbb{C}[\mathbf{y}]$ -algebra generated by the  $p_k(\mathbf{x}; \mathbf{y})$ 's for all k. For any symmetric function  $f(\mathbf{x}_-)$ , we can expand f into the power sum basis and apply the map  $p_k(\mathbf{x}_-) \mapsto p_k(\mathbf{x}_-/\mathbf{y})$  to construct a new symmetric function  $f(\mathbf{x}_-/\mathbf{y})$ . For M a graded algebra, let  $M^{\beta}$  be the homogenized graded completion of M, that is the set of formal power series

$$\left\{ \sum_{i=k}^{\infty} \beta^{i-k} m_i : k \in \mathbb{Z}_+, \deg(m_i) = i \text{ in } M \right\}$$

with  $\deg(\beta) = -1$ . Then  $\Gamma(\beta; \mathbf{x}_-; \mathbf{y}) = \operatorname{span}_{R(\beta; \mathbf{y})}(\{G_{\lambda}(\beta; \mathbf{x}_-; \mathbf{y})\})$  is a subset of  $\Lambda^{\beta}(\mathbf{x}_-; \mathbf{y})$ . In [14], it is shown for  $w \in S_{\mathbb{Z}}$  that  $\overleftarrow{\mathfrak{G}}_w(\beta; \mathbf{x}; \mathbf{y}) \in R(\beta; \mathbf{x}; \mathbf{y}) \otimes \Lambda^{\beta}(\mathbf{x}_-; \mathbf{y})$ . Our goal is to prove:

**Proposition A.1** (Proposition 2.7). For  $w \in S_{\mathbb{Z}}$ ,  $\overleftarrow{\mathfrak{G}}_{w}(\beta; \mathbf{x}; \mathbf{y})$  lies in  $R(\beta; \mathbf{x}; \mathbf{y}) \otimes \Gamma(\beta; \mathbf{x}_{-}; \mathbf{y})$ .

<sup>&</sup>lt;sup>1</sup>This observation was first made in [18], where the single backstable  $\beta$ -Grothendieck polynomial was first defined.

Recall the Grothendieck polynomial of  $w \in S_{\infty}$  is  $\mathfrak{G}_w(\beta; \mathbf{x}_+) = \mathfrak{G}_w(\beta; \mathbf{x}_+; \mathbf{y}_+) \mid_{\mathbf{y} \to 0}$ . The (single) stable Grothendieck polynomial of  $\lambda$  is  $G_{\lambda}(\beta; \mathbf{x}_-) = G_{\lambda}(\beta; \mathbf{x}_-; \mathbf{y}) \mid_{\mathbf{y} \to 0}$ . The (single)  $\beta$ -Stanley symmetric function<sup>2</sup> of  $w \in S_{\mathbb{Z}}$  is  $F_w(\beta; \mathbf{x}_-) = F_w(\beta; \mathbf{x}_-; \mathbf{y}) \mid_{\mathbf{y} \to 0}$ . As shown in [5]

$$F_w(\beta; \mathbf{x}_-) \in \mathbb{Z}_{\geq 0}[\beta^{m-\ell(w)}G_\lambda(\beta; \mathbf{x}_-) : \lambda \vdash m \geq \ell(w)].$$

Define the 0-Hecke monoid  $(S_{\mathbb{Z}}, *)$  where u \* v = w when  $\pi_u * \pi_v = \beta^{\ell(u) + \ell(v) - \ell(w)} \pi_w$ . Let  $S_{-\infty} = \omega(S_{\infty})$ , and for  $w \in S_{-\infty}$  define  $\mathfrak{G}_w(\beta; \mathbf{x}) = \tilde{\omega}(\mathfrak{G}_{\neg w}(\beta; \mathbf{x}))$ . Let  $S_{\neq 0} = S_{-\infty} \times S_{\infty}$ , and for  $w = u \times v \in S_{\neq 0}$  with  $u \in S_{-\infty}$ ,  $v \in S_{\infty}$  define  $\mathfrak{G}_w(\beta; \mathbf{x}) = \mathfrak{G}_u(\beta; \mathbf{x}) \cdot \mathfrak{G}_v(\beta; \mathbf{x})$ . Note  $S_{\neq 0}$  is generated by  $\{s_i : i \in \mathbb{Z} - \{0\}\}$ .

**Proposition A.2** ([14, Prop 5.17]). For  $w \in S_{\mathbb{Z}}$ ,

$$\overleftarrow{\mathfrak{G}}_w(\beta;\mathbf{x};\mathbf{y}) = \sum_{\substack{u*v*z=w\\u,z\in S_{\neq 0}}} \mathfrak{G}_{u^-1}(\ominus \mathbf{y}) \cdot F_v(\beta;\mathbf{x}_-/\mathbf{y}) \cdot \mathfrak{G}_z(\beta;\mathbf{x}).$$

Proof of Proposition 2.7. By Propositions 2.6 and A.2, we have

$$G_{\lambda}(\beta;\mathbf{x};\mathbf{y}) = \overleftarrow{\mathfrak{G}}_{w_{\lambda}}(\beta;\mathbf{x};\mathbf{y}) = \sum_{\substack{u * v = w_{\lambda} \\ u \in S_{\neq 0}}} \mathfrak{G}_{u^{-1}}(\beta;\ominus\mathbf{y}) \cdot F_{v}(\beta;\mathbf{x}_{-}/\mathbf{y}).$$

Here the  $\mathfrak{G}_z(\beta; \mathbf{x})$  term is omitted since every reduced word for  $w_\lambda$  ends in  $s_0$ , hence any z from Proposition A.2 must be the identity, and  $\mathfrak{G}_1(\beta; \mathbf{x}) = 1$ . This equation implies  $G_\lambda(\beta; \mathbf{x}; \mathbf{y}) \in \Lambda(\beta; \mathbf{x}; \mathbf{y})$ . To complete our proof, move all terms with  $v \neq w_\lambda$  to the left hand side, giving

$$F_{\lambda}(\beta; \mathbf{x}_{-}/\mathbf{y}) \cdot \sum_{\substack{u * w_{\lambda} = w_{\lambda} \\ u \in S_{\neq 0}}} \mathfrak{G}_{u^{-1}}(\beta; \ominus \mathbf{y}) = G_{\lambda}(\beta; \mathbf{x}_{-}; \mathbf{y}) - \sum_{\substack{u * v = w_{\lambda} \\ u \in S_{\neq 0}, \ v \neq w_{\lambda}}} \mathfrak{G}_{u^{-1}}(\beta; \ominus \mathbf{y}) F_{v}(\beta; \mathbf{x}_{-}/\mathbf{y}).$$

Note v is also 0-Grassmannian with  $\ell(v) < |\lambda|$ , so by induction on  $|\lambda|$ , we can assume the right hand side belongs to  $\Gamma(\beta; \mathbf{x}; \mathbf{y})$ . The sum in the left side is finite and  $\mathfrak{G}_w(\beta; \ominus \mathbf{y}) \in R(\beta; \mathbf{y})$  for all  $w \in S_{\neq 0}$ , so the coefficient of  $G_{\lambda}(\beta; \mathbf{x}_{-}/\mathbf{y})$  is in  $R(\beta; \mathbf{y})$  as desired.

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<sup>&</sup>lt;sup>2</sup>More commonly referred to as the stable Grothendieck polynomial of w.

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