

Analogues of two classical

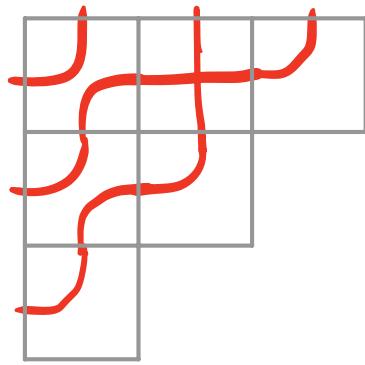
Pipedream results on

Bumpless Pipedreams

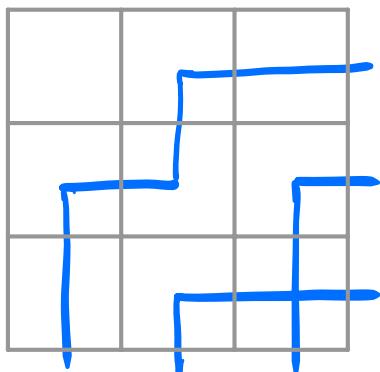
Tianyi Yu

UCSD → UQAM.

Pipedream (PD)



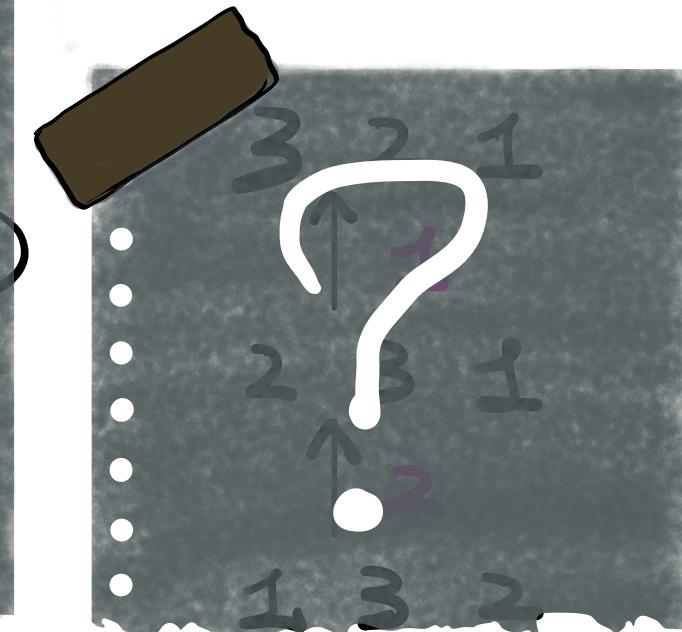
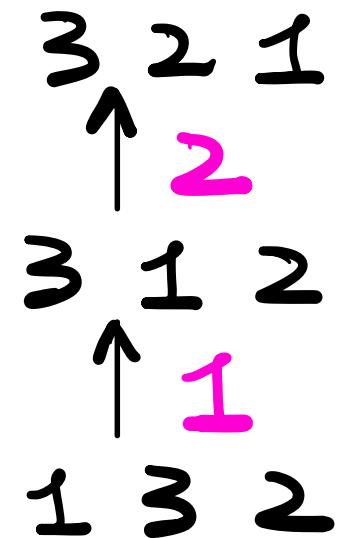
Bumpless Pipedream (BPD)



Fomin - Stanley generating function

$$(1+x_1 u_2) (1+x_1 u_1) \\ (1+x_2 u_2)$$

Lenart - Sottile Bruhat Chain



Schubert Polynomial G_w .

Polynomials indexed by $w \in S_n$.

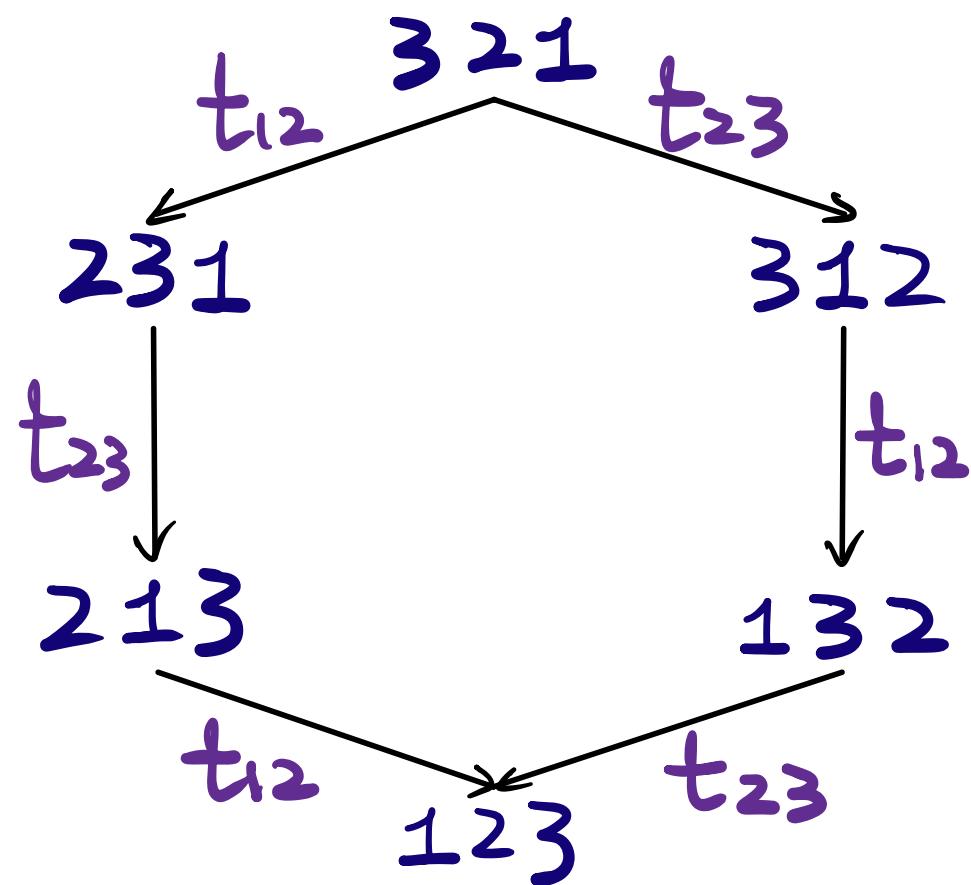
If $w_0 = [n, n-1, \dots, 2, 1]$,

$$G_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-2}^2 x_{n-1}$$

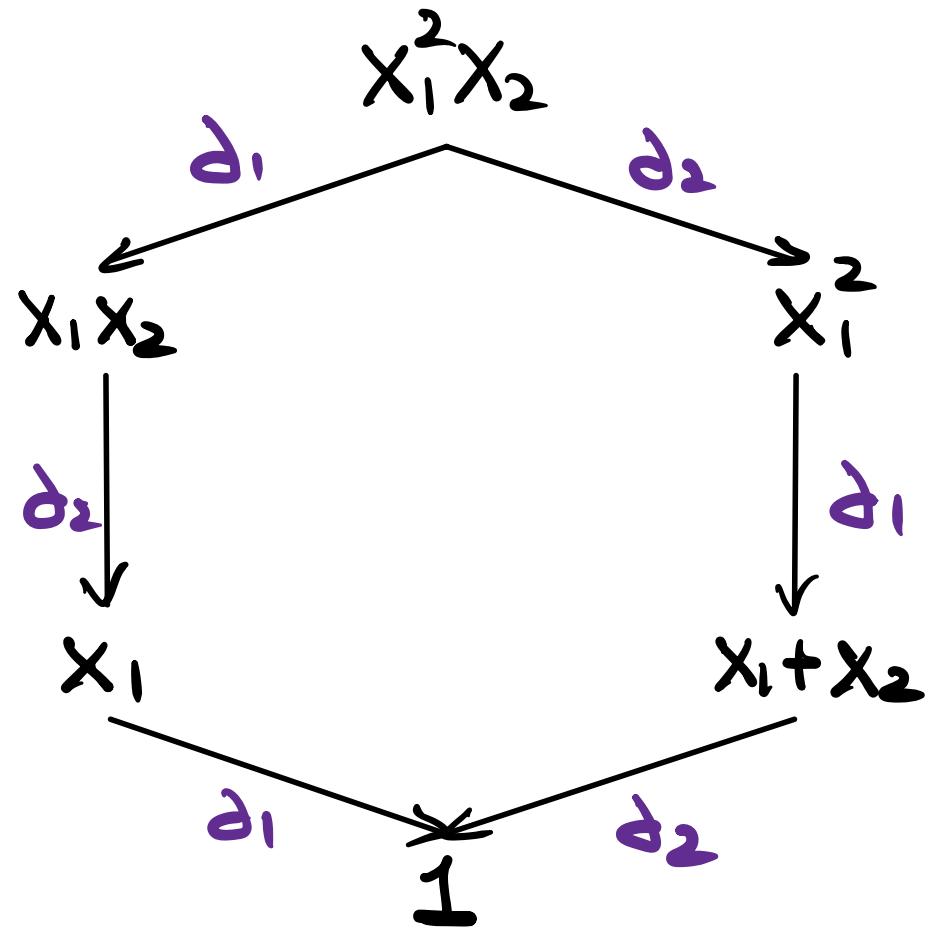
Define ∂_i on polynomials : $\partial_i(f) = \frac{f - s_i f}{x_i - x_{i+1}}$

$$\partial_1(x_1^2 x_2) = \frac{x_1^2 x_2 - x_1 x_2^2}{x_1 - x_2} = x_1 x_2$$

Schubert Polynomial

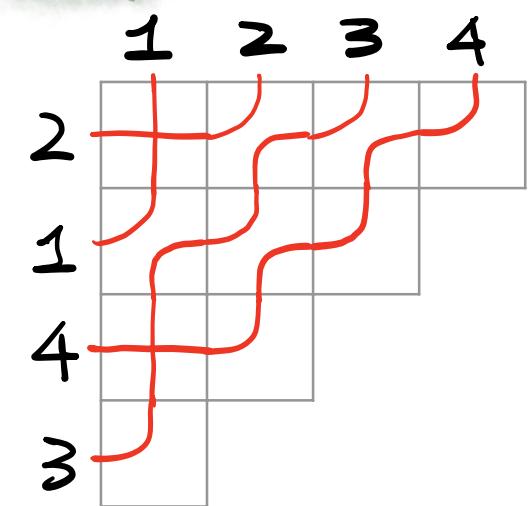
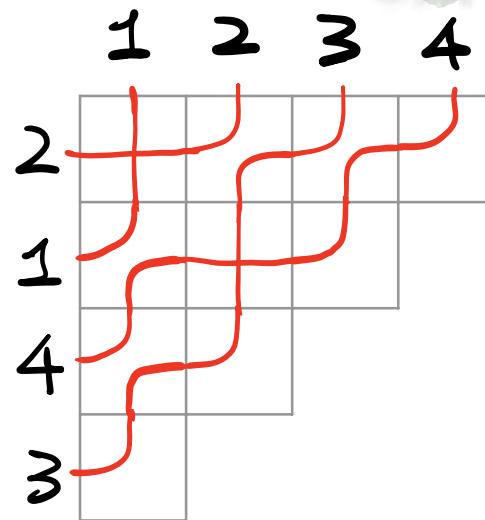
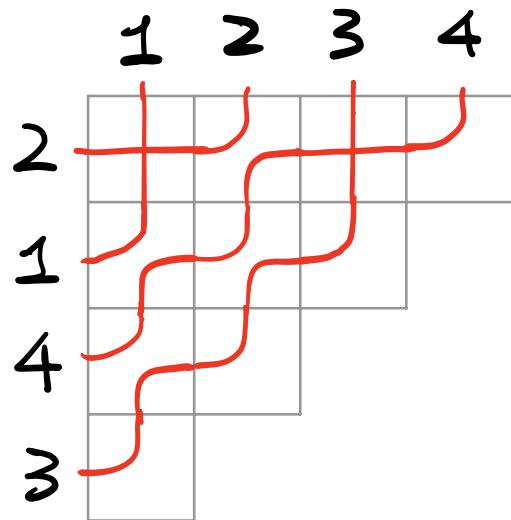
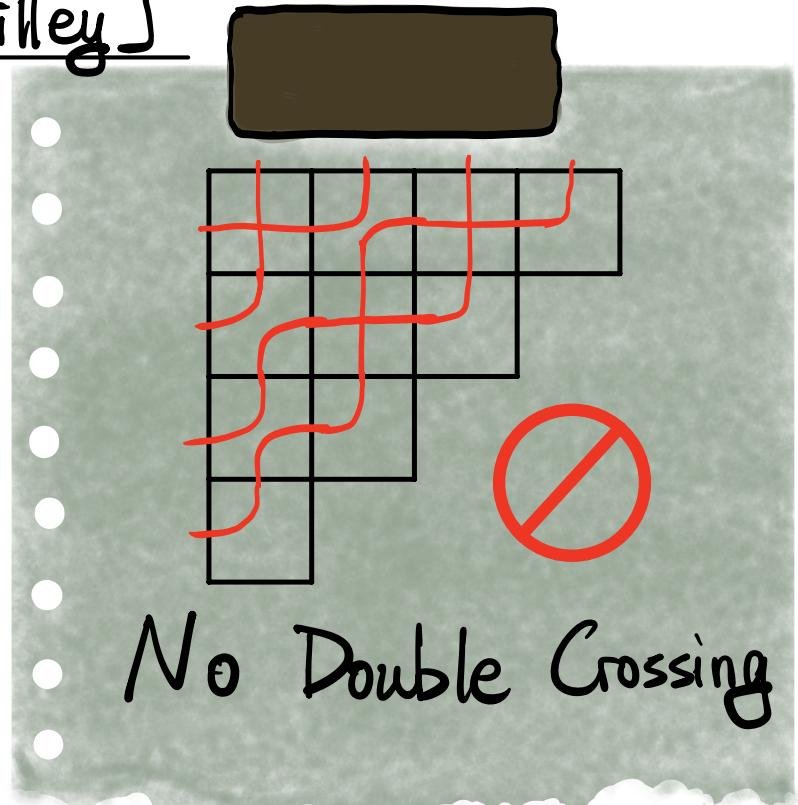
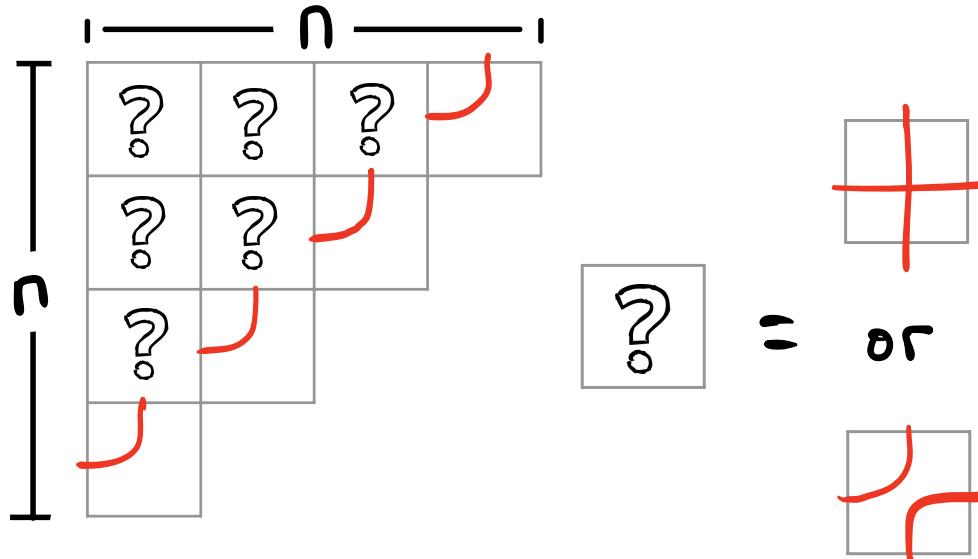


Take $t_{i,i+1}$ down



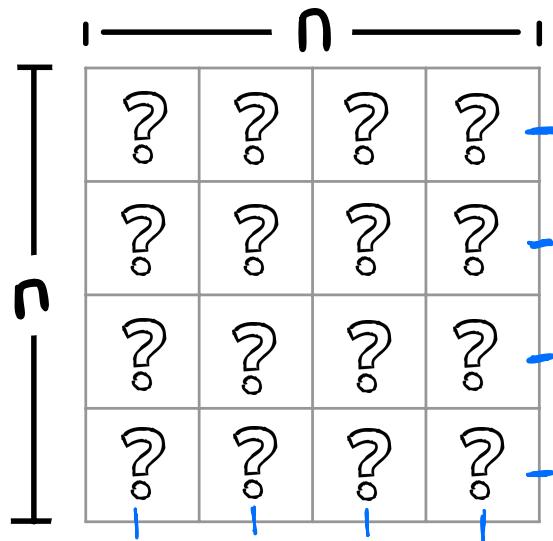
Apply ∂_i

Pipedream (PD) [Bergeron - Billey]

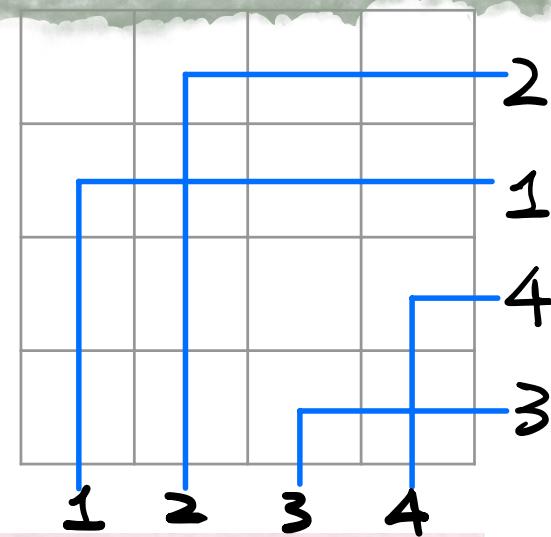
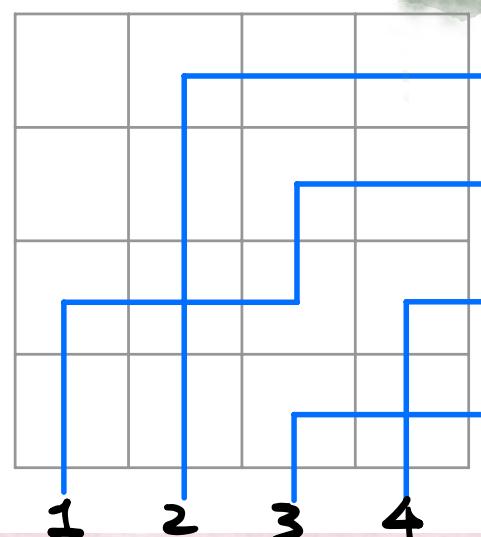
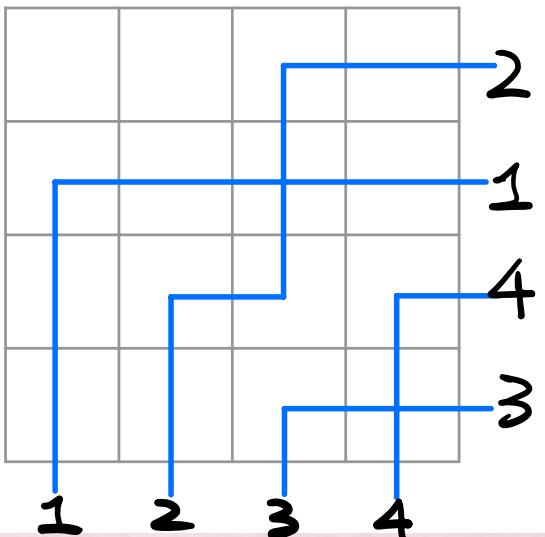
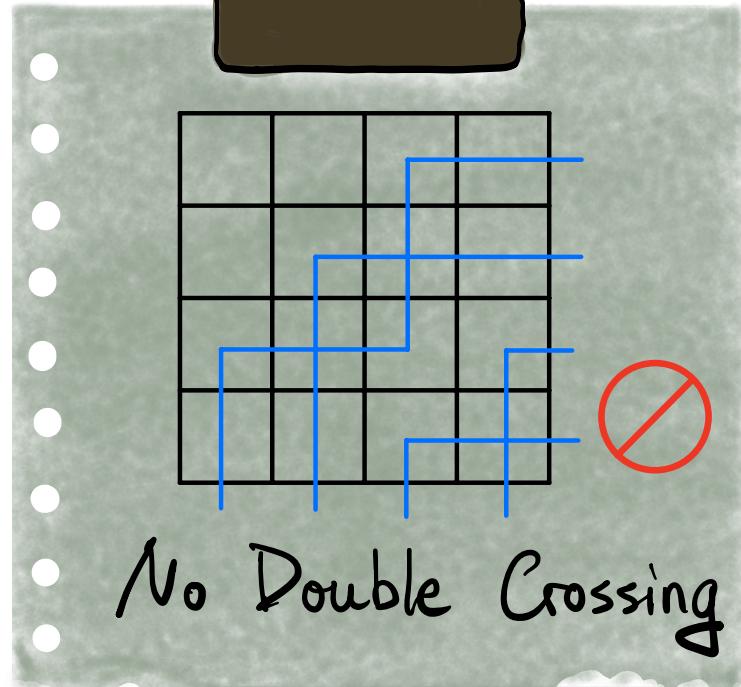


$$G_{2143} = x_1^2 + x_1 x_2 + x_1 x_3$$

Bumpless Pipedreams (BPD) [Lam - Lee - Shimozono]



The diagram illustrates the decomposition of a question mark shape into two rectangles. On the left, a gray square contains a black question mark. A blue horizontal line extends from the right side of this square to the right edge of the entire figure. To the right of this line, the figure is divided into two gray rectangles by a vertical blue line. The top rectangle is taller than the bottom one. Below this, another vertical blue line creates a second rectangle, which is shorter than the first. This visualizes how the question mark shape can be represented as the union of these two rectangles.



$$G_{2143} = x_1^2 + x_1x_2 + x_1x_3$$

Fomin - Stanley

- Use one expression to include all possible pipedreams.

$$\begin{aligned} & \left(\text{Hamburger} + \text{Pizza slice} + \text{Hotdog} \right) \times \left(\text{French fries} + \text{Popcorn} \right) \times \left(\text{Ice cream cone} + \text{Drink} \right) \\ = & \quad \text{Hamburger} \quad \text{French fries} \quad \text{Ice cream cone} \quad + \quad \text{Hamburger} \quad \text{French fries} \quad \text{Drink} \quad + \quad \dots \end{aligned}$$

?	?	
?		
	x_1	
	x_1	

$$\left(\text{ } + x_1 \text{ } \right) \times \left(\text{ } + x_1 \text{ } \right) \times \left(\text{ } + x_2 \text{ } \right)$$

- How to read permutations ?
- How to make sure no double crossings ?

[Food stickers by gabby-scarball]

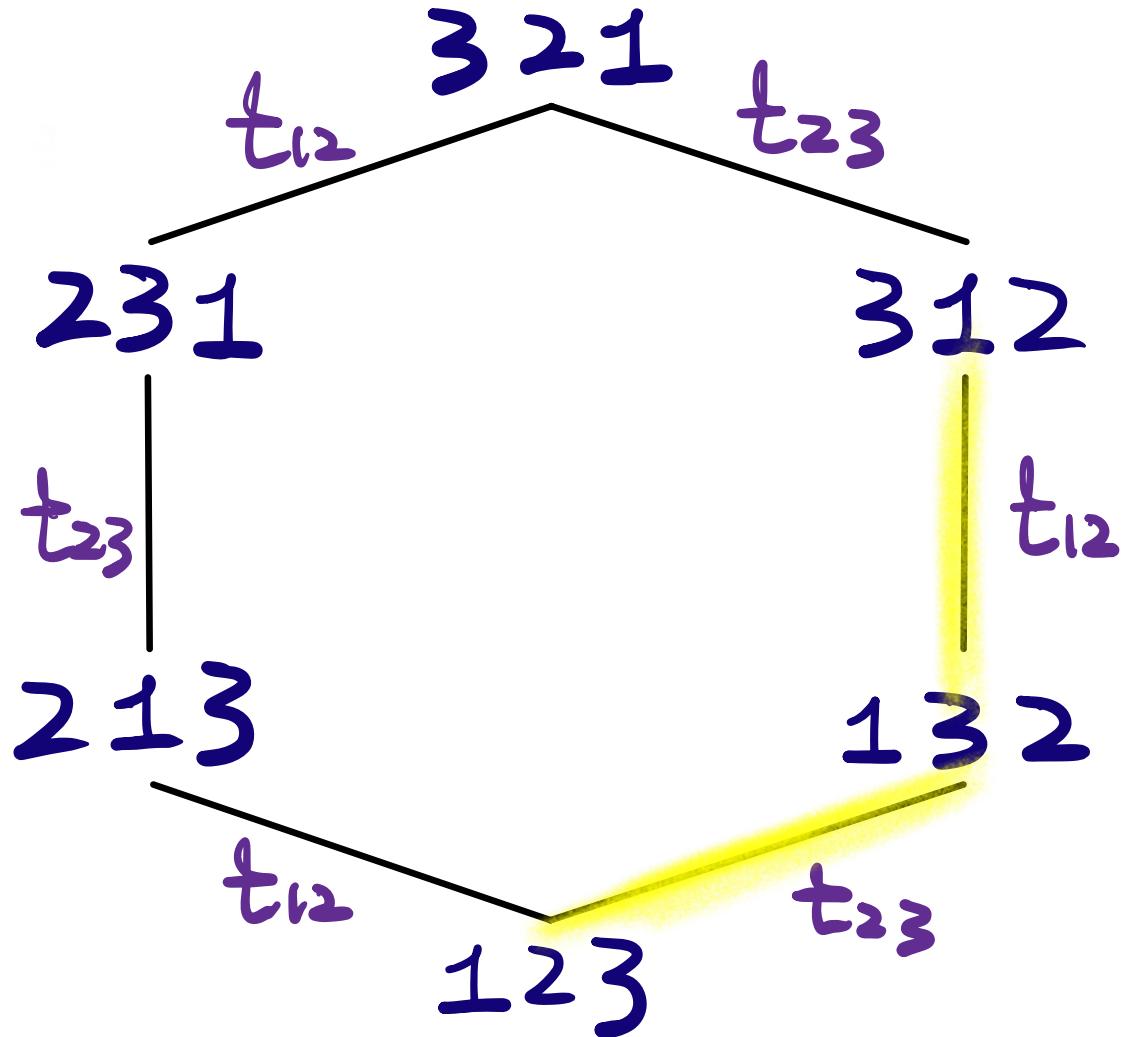
Nil-Coxeter Algebra

Generated by

u_1, u_2, \dots, u_{n-1}

- $u_i u_i = 0$
- $u_i u_j = u_j u_i$ if $|i - j| > 1$.
- $u_i u_i + u_i = u_i + u_i u_i + u_i$

Each monomial is zero or corresponds to some $w \in S_n$



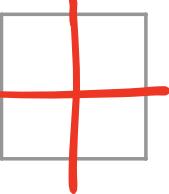
$$u_2 u_1 \sim 312$$

$$u_2 u_2 = 0$$

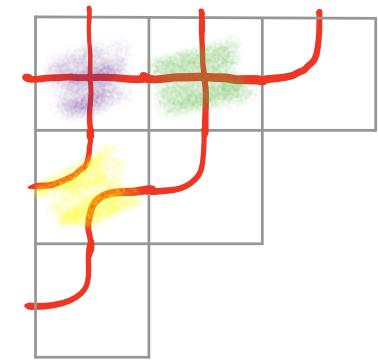
Fomin - Stanley

$\nearrow \text{ or } +$ $l + x_1 u_1$	$\nearrow \text{ or } +$ $l + x_1 u_2$	$\nearrow \text{ or } +$ $l + x_1 u_3$	$\nearrow \text{ or } +$ $l + x_1 u_4$	$\nearrow \text{ or } +$ $l + x_1 u_5$	
$\nearrow \text{ or } +$ $l + x_2 u_2$	$\nearrow \text{ or } +$ $l + x_2 u_3$	$\nearrow \text{ or } +$ $l + x_2 u_4$	$\nearrow \text{ or } +$ $l + x_2 u_5$		
$\nearrow \text{ or } +$ $l + x_3 u_3$	$\nearrow \text{ or } +$ $l + x_3 u_4$	$\nearrow \text{ or } +$ $l + x_3 u_5$			

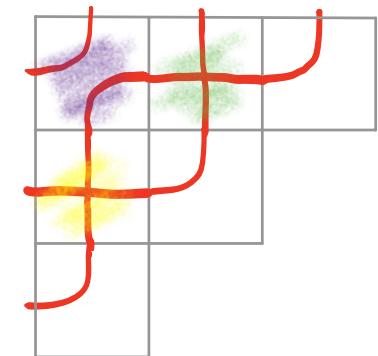
• • •

IDEA:  in diagonal j behaves as u_j

Fomin - Stanley



$$(1 + x_1 u_2) \times (1 + x_1 u_1) \times (1 + x_2 u_2)$$
$$x_1^2 u_2 u_1$$



$$(1 + x_1 u_2) \times (1 + x_1 u_1) \times (1 + x_2 u_2)$$
$$x_1 x_2 u_2 u_1 = 0$$

Fomin - Stanley

$$G := (1 + X_1 u_2) \times (1 + X_1 u_1) \times (1 + X_2 u_2)$$
$$= \sum_{\text{Pipedream } P} x^{\text{wt}(P)} w(P)$$

To prove the PD formula for G_w ,
it's enough to show:

Thm [Fomin - Stanley]

$$\partial_i(G) = G u_i.$$

Fomin - Stanley on BPD?

?	?	?	?	-
?	?	?	?	-
?	?	?	?	-
?	?	?	?	-

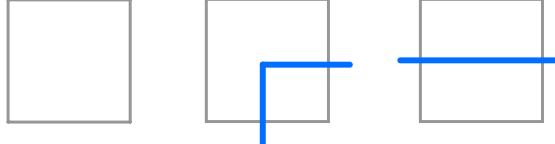
$$\begin{aligned} & \left(\begin{array}{|c|c|} \hline \text{---} & +x_1 \end{array} \right) + \left(\begin{array}{|c|c|} \hline \text{---} & +x_1 \end{array} \right) + \left(\begin{array}{|c|c|} \hline \text{---} & +x_1 \end{array} \right) \\ & \times \left(\begin{array}{|c|c|} \hline \text{---} & +x_1 \end{array} \right) + \left(\begin{array}{|c|c|} \hline \text{---} & +x_1 \end{array} \right) + \left(\begin{array}{|c|c|} \hline \text{---} & +x_1 \end{array} \right) \\ & \times \left(\begin{array}{|c|c|} \hline \text{---} & +x_1 \end{array} \right) + \left(\begin{array}{|c|c|} \hline \text{---} & +x_1 \end{array} \right) + \left(\begin{array}{|c|c|} \hline \text{---} & +x_1 \end{array} \right) \\ & \times \dots \end{aligned}$$

Challenges :

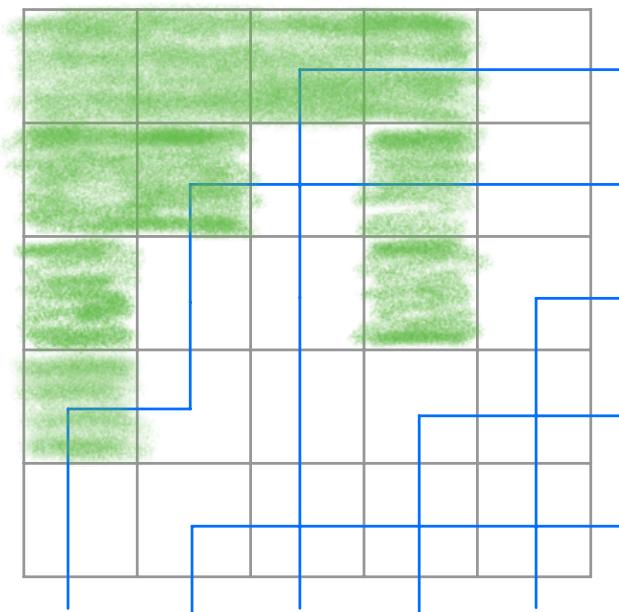
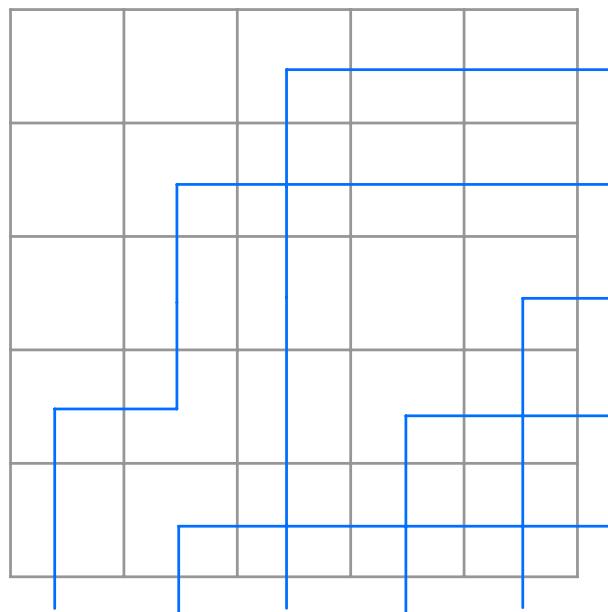
- Too many terms
- Too many "bad" terms
- Hard to read permutation



Encoding a BPD

Look at  , but ignore the rightmost such tile in each row.

Record the pipe in each such cell

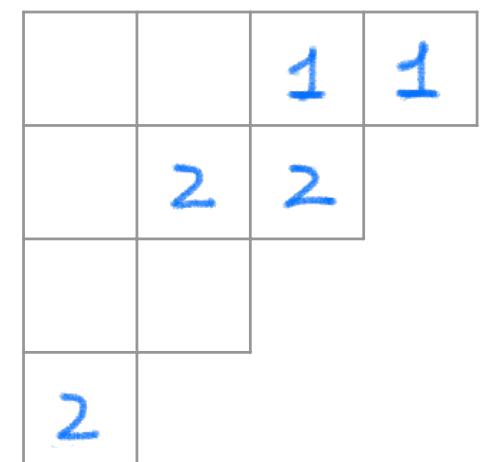
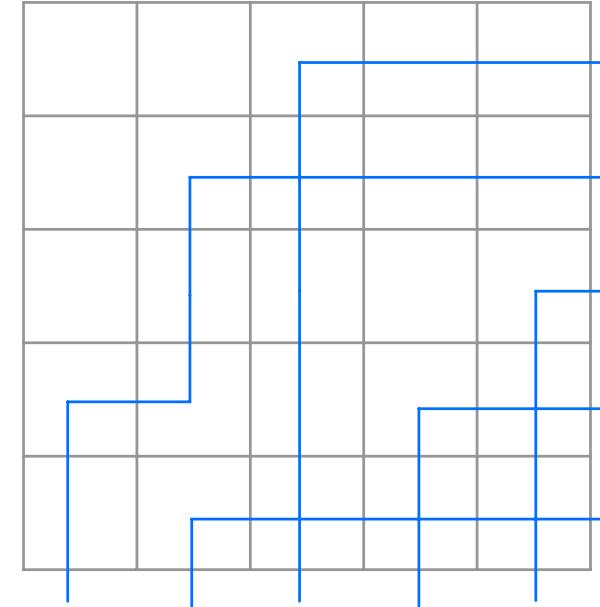


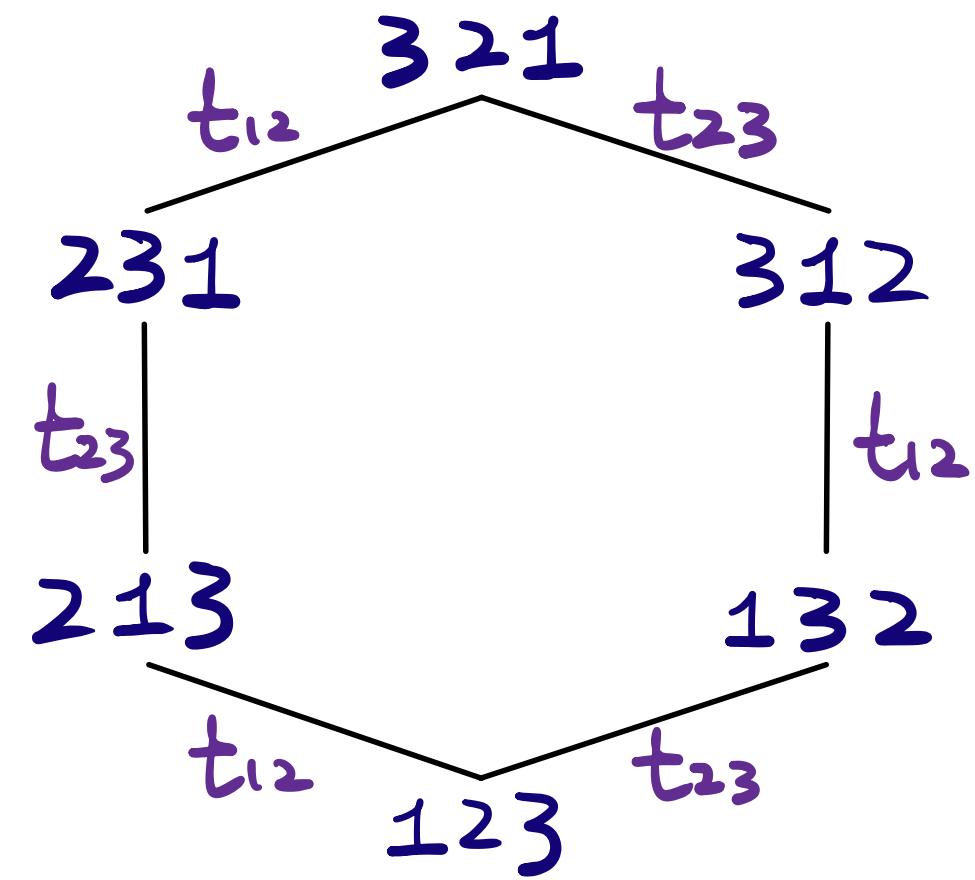
	1	1
2	2	
	2	

Encoding a BPD

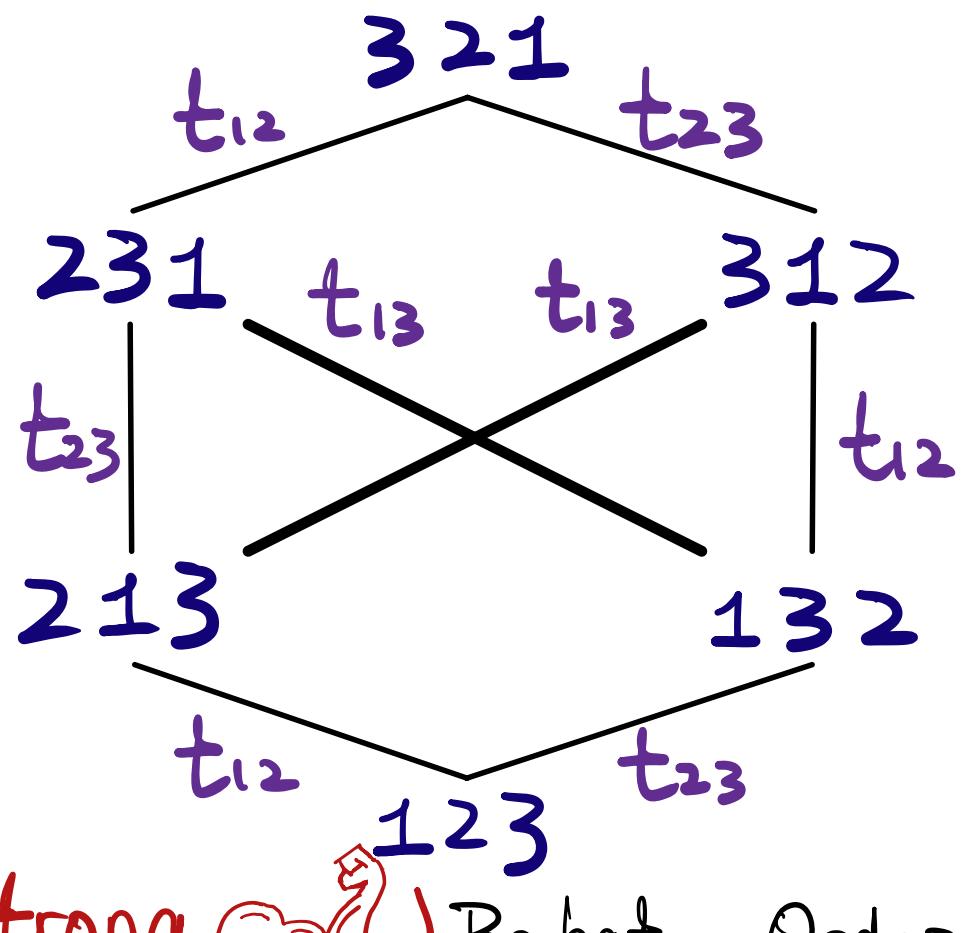
What fillings
can appear?

- Row i filled w/ $\#s \leq i$
 - ? ? ?





weak Bruhat Order

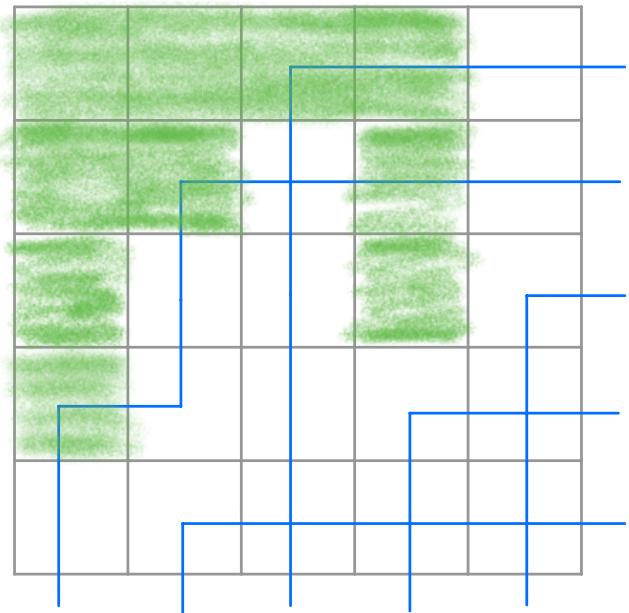


(Strong) Bruhat Order

When you make a swap,

#s in between are not in between.

Chain from encoding



54321

$\downarrow t_{12}$

45321

t_{13}

35421

$\downarrow t_{23}$

34521

\downarrow t_{24}

32541

125

31542

5 4 3 2 1

			1	1
2			2	
4				5

(BAD)

54321

١٢

45321

$\downarrow t_{13}$

35421

↓ t₂₃

34521

Handwriting logo consisting of a stylized orange pencil writing the numbers 1, 2, and 3.

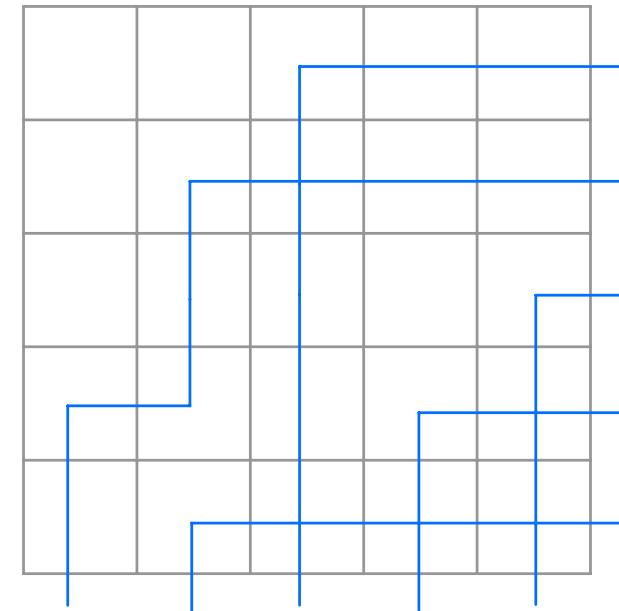
31524

Encoding a BPD

Prop [Y]

This is a bijection between BPD of w and fillings of the "staircase" such that

- Row i filled w/ $\#s \leq i$
- Correspond to a chain from w_0 to w in Bruhat order.



		1	1
	2	2	
	2		

Fomin - Kirillov Algebra

Σ_n is generated by d_{ij} , $1 \leq i < j \leq n$.

- $d_{ij} d_{ij} = 0$
- $d_{ij} d_{jk} = d_{ik} d_{ij} + d_{jk} d_{ik}$
- $d_{jk} d_{ij} = d_{ij} d_{ik} + d_{ik} d_{jk}$
- $d_{ij} d_{kt} = d_{kt} d_{ij}$

if i, j, k, t distinct.

Fomin - Kirillov Algebra

Σ_n is generated by d_{ij} , $1 \leq i < j \leq n$.

- Σ_n acts on $\mathbb{Q}[S_n]$
- $d_{ij} d_{ik} = d_{kj} d_{ij} + d_{jk} d_{ik}$
- $d_{jk} d_{ij} w t_{ij} = d_{ij} d_{jk} + d_{ik} d_{ik}$
- $\bar{w}_{ij} d_{kt} = d_{kt} d_{ij}$
- if i, j, k, l otherwise.

BPD Analogue of Fomin - Stanley

Diagram illustrating the BPD Analogue of Fomin - Stanley for a 3x3 grid.

The grid has row indices 4, 3, 2, 1 and column indices 2, 3, 4. The entries are:

?	?	?
?	?	?
?		

Arrows point from the grid to the corresponding terms in the factorizations:

- Row 1 (index 1) points to the term $(x_1 + \boxed{1})(x_1 + \boxed{1})(x_1 + \boxed{1})$, which is equal to $(x_1 + d_{12})(x_1 + d_{13})(x_1 + d_{14})$.
- Row 2 (index 2) points to the term $(x_2 + \boxed{1} + \boxed{2})(x_2 + \boxed{1} + \boxed{2})$, which is equal to $(x_2 + d_{13} + d_{23})(x_2 + d_{14} + d_{24})$.
- Row 3 (index 3) points to the term $(x_3 + \boxed{1} + \boxed{2} + \boxed{3})$, which is equal to $(x_3 + d_{14} + d_{24} + d_{34})$.

BPD Analogue of Fomin - Stanley

4 3 2 1 Definition [Y]

?	?	?
?	?	
?		

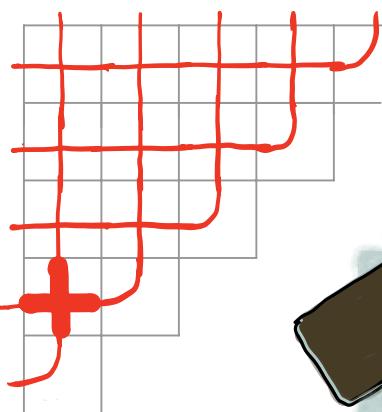
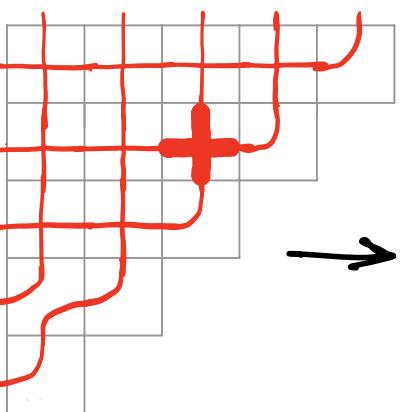
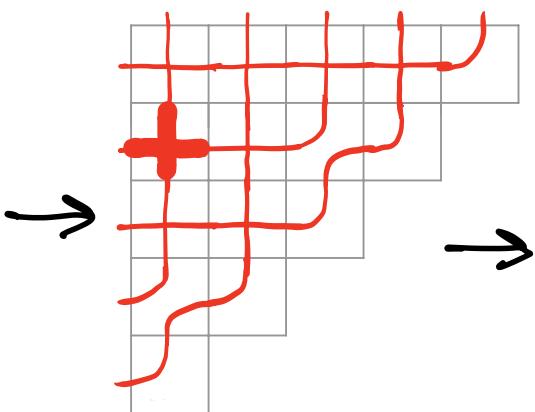
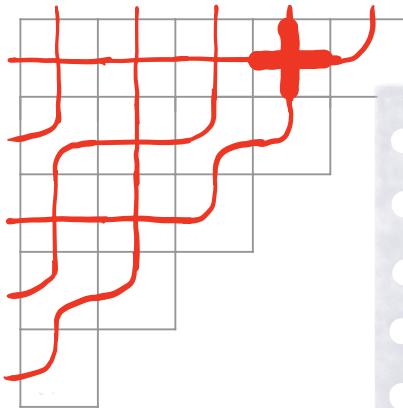
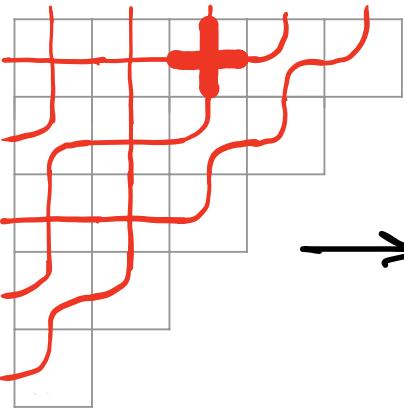
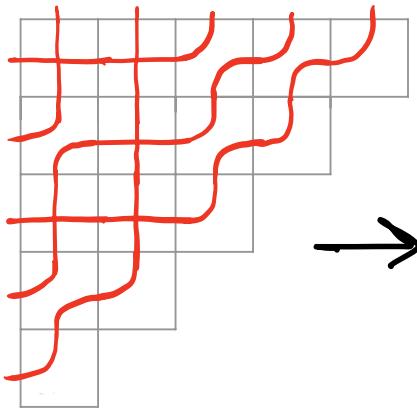
$$G = w_0 \circ (x_1 + d_{12})(x_1 + d_{13})(x_1 + d_{14}) \\ (x_2 + d_{13} + d_{23})(x_2 + d_{14} + d_{24}) \\ (x_3 + d_{14} + d_{24} + d_{34})$$

$$= \sum_{BPD \ D} x^{\text{wt}(D)} w(D)$$

As the PD case, showing the BPD formula for G_w reduces to :

Theorem [Y] $d_i(G) = G \circ u_i$

Lenart - Sottile Chain



Change \nearrow into
+ from top to
bottom, left to
right on each row.



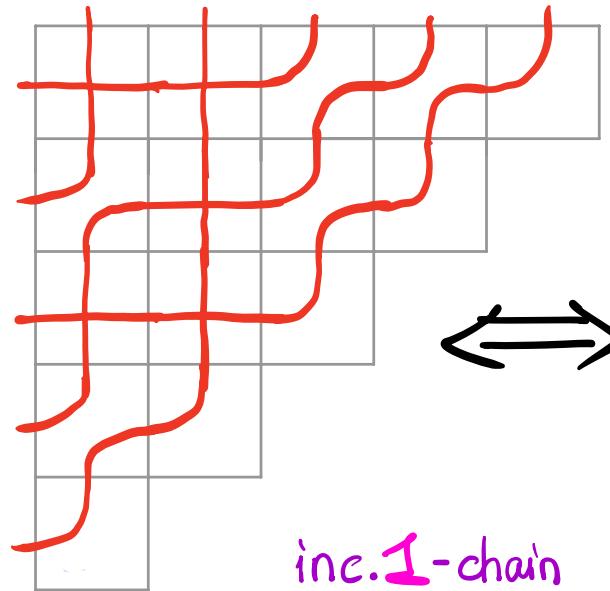
Labels given
by the row
number of the



$$31542 \xrightarrow{1} 41532 \xrightarrow{1} 51432$$

$$\xrightarrow{2} 53412 \xrightarrow{2} 54312 \xrightarrow{4} 54321$$

Lenart - Sottile Chain



inc. 1-chain

$$31542 \xrightarrow{1} 41532 \xrightarrow{1} 51432$$

$$\xrightarrow{2} 53412 \xrightarrow{2} 54312 \xrightarrow{4} 54321$$

inc. 1-chain

inc. 2-chain

inc. 3-chain

inc. 4-chain

$$31542 \rightarrow 51432 \rightarrow 54312 \rightarrow 54312 \rightarrow 54321$$

Thm [Lenart – Sottile]

This is a bijection between $\text{PD}(w)$ and

$$w \xrightarrow{\text{inc. 1-chain}} \bullet \xrightarrow{\text{inc. 2-chain}} \dots \xrightarrow{\text{inc. } n-2\text{-chain}} \bullet \xrightarrow{\text{inc. } n-1\text{-chain}} w_0$$

BPD analogue of Lenart-Sotille

5 4 3 2 1

		1	1	
2	2			
2				

31542 $\xrightarrow{4}$ 32541 $\xrightarrow{2}$ 34521

\Longleftrightarrow $\xrightarrow{2}$ 35421 $\xrightarrow{1}$ 45321 $\xrightarrow{1}$ 54321

inc. 4-chain

inc. 3-chain

inc. 2-chain

inc. 1-chain

31542 \rightarrow 32541 \rightarrow 32541 \rightarrow 35421 \rightarrow 54321

Thm [Y]

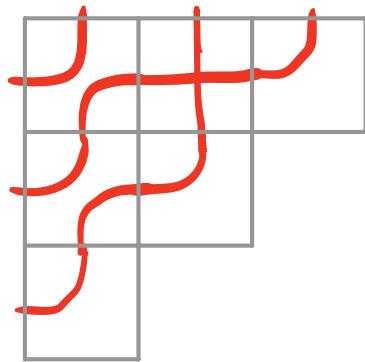
This is a bijection between $\text{BPD}(w)$ and

inc. $n-1$ -chain inc. $n-2$ -chain

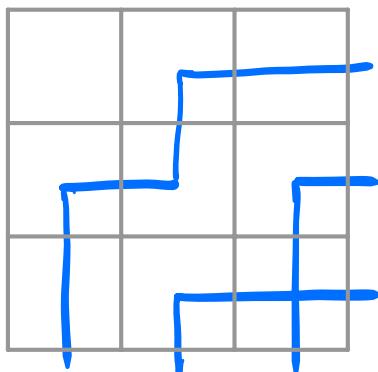
inc. 2-chain inc. 1-chain

$w \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow w_0$

Pipedream (PD)



Bumpless Pipedream (BPD)

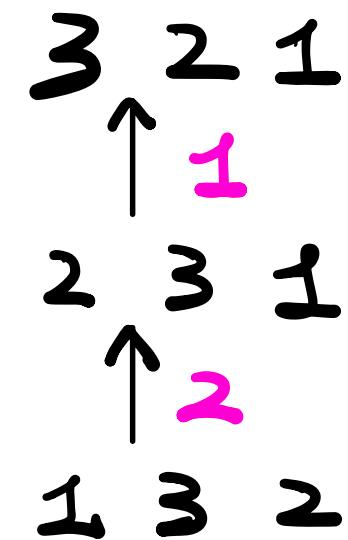
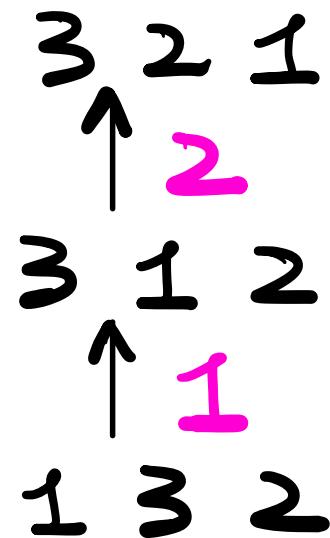


Fomin - Stanley Algebra

$$(1+x_1 u_2) (1+x_1 u_1) \\ (1+x_2 u_2)$$

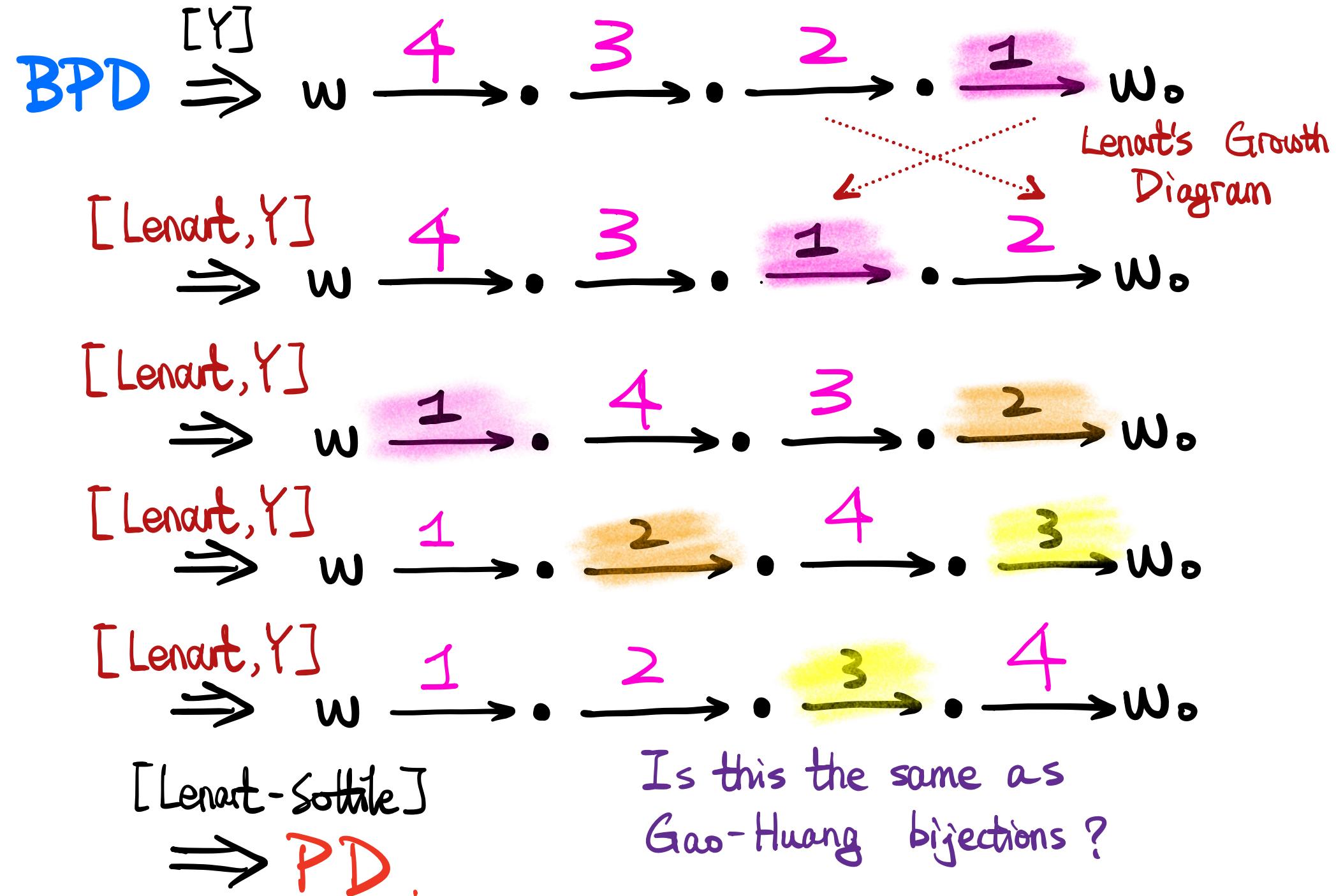
Bijections
 [Gao - Huang]
 [Knutson - Uddell]

Lenart - Sottile Chain



$$(x_1 + d_{12}) (x_1 + d_{13}) \\ (x_2 + d_{13} + d_{23})$$

New bijection between PDs and BPDs



Future Direction

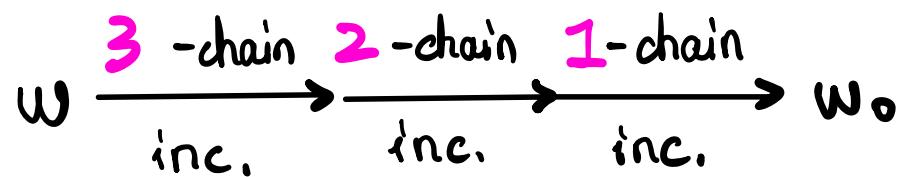
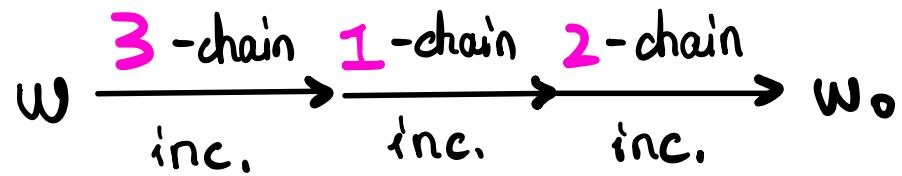
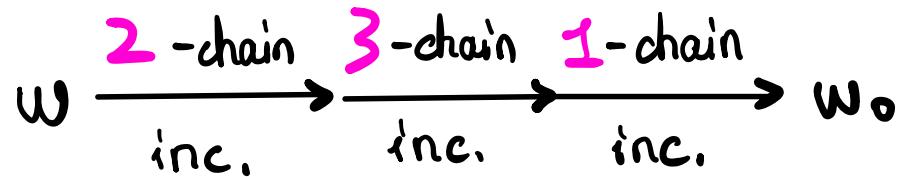
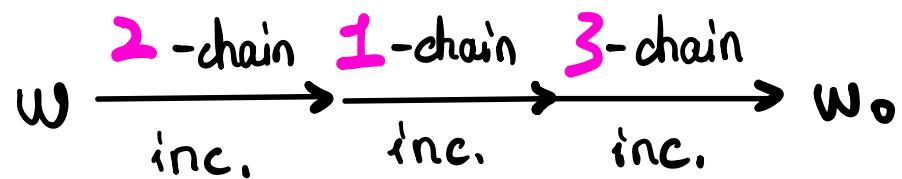
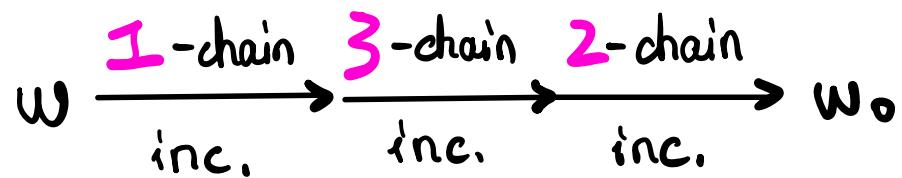
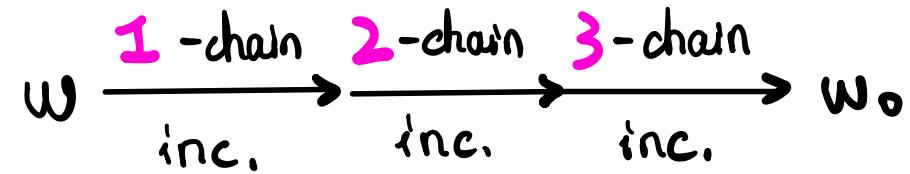
PD:

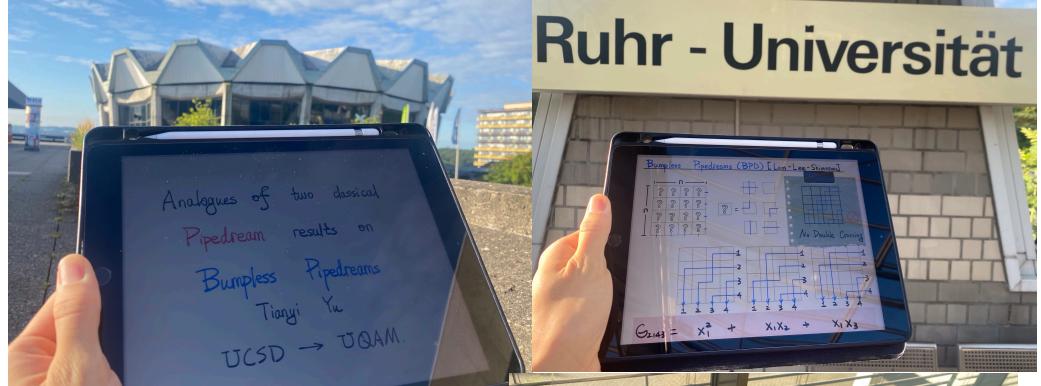
Many interesting stories
on the two ends!

- Crystal graphs
 - Monk's move
 - Double Schubert
-

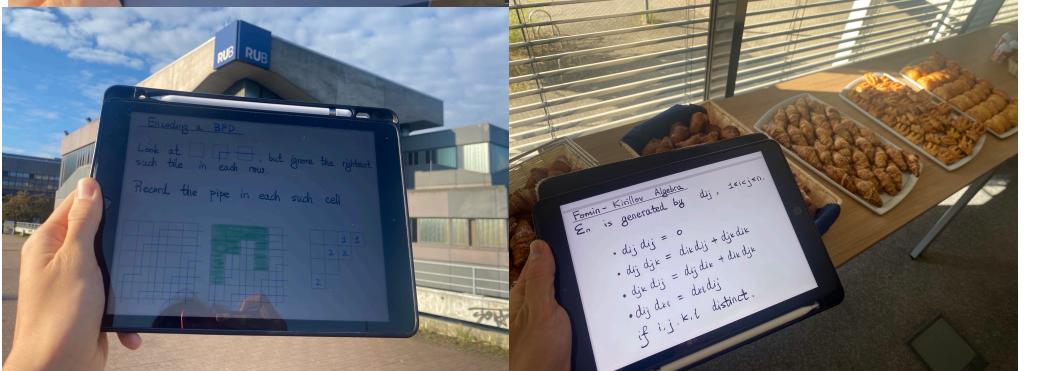
How to generalize to
the other chains?

BPD:

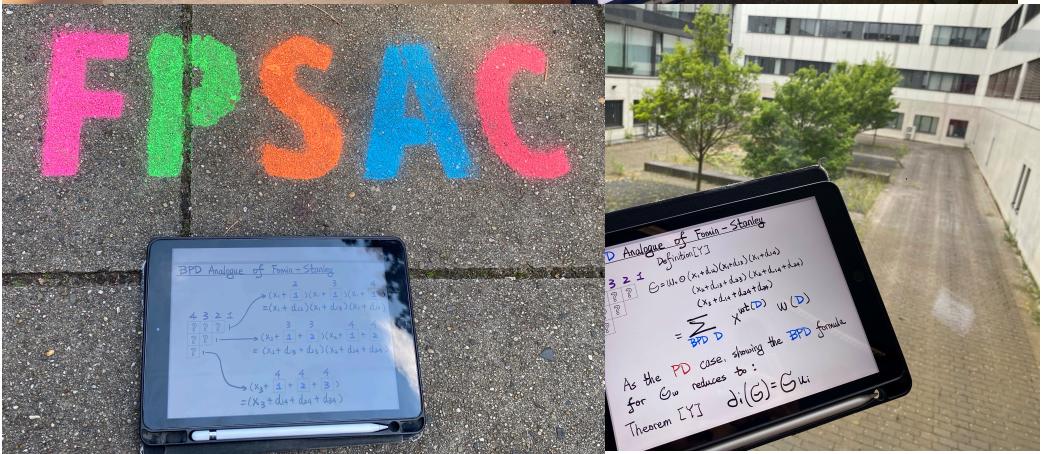




Thank Yibo Gao for telling me this problem.



Thank Yibo Gao and Zachary Hamaker for valuable guidances.



Thank you for listening!