

Answer the questions in the boxes provided on the question sheets. If you run out of room for an answer, add a page to the end of the document.

Related Readings: <http://pages.cs.wisc.edu/~hasti/cs240/readings/>

Name: \_\_\_\_\_ Wisc id: \_\_\_\_\_

## Logic

1. Using a truth table, show the equivalence of the following statements.

(a)  $P \vee (\neg P \wedge Q) \equiv P \vee Q$

**Solution:**

(b)  $\neg P \vee \neg Q \equiv \neg(P \wedge Q)$

**Solution:**

(c)  $\neg P \vee P \equiv \text{true}$

**Solution:**

(d)  $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

**Solution:**

## Sets

2. Based on the definitions of the sets  $A$  and  $B$ , calculate the following:  $|A|$ ,  $|B|$ ,  $A \cup B$ ,  $A \cap B$ ,  $A \setminus B$ ,  $B \setminus A$ .

(a)  $A = \{1, 2, 6, 10\}$  and  $B = \{2, 4, 9, 10\}$

**Solution:**

(b)  $A = \{x \mid x \in \mathbb{N}\}$  and  $B = \{x \in \mathbb{N} \mid x \text{ is even}\}$

**Solution:**

## Relations and Functions

3. For each of the following relations, indicate if it is reflexive, antireflexive, symmetric, antisymmetric, or transitive.

(a)  $\{(x, y) : x \leq y\}$

**Solution:**

(b)  $\{(x, y) : x > y\}$

**Solution:**

(c)  $\{(x, y) : x < y\}$

**Solution:**

(d)  $\{(x, y) : x = y\}$

**Solution:**

4. For each of the following functions (assume that they are all  $f : \mathbb{Z} \rightarrow \mathbb{Z}$ ), indicate if it is surjective (onto), injective (one-to-one), or bijective.

(a)  $f(x) = x$

**Solution:**

(b)  $f(x) = 2x - 3$

**Solution:**

(c)  $f(x) = x^2$

**Solution:**

5. Show that  $h(x) = g(f(x))$  is a bijection if  $g(x)$  and  $f(x)$  are bijections.

**Solution:**

## Induction

6. Prove the following by induction.

(a)  $\sum_{i=1}^n i = n(n+1)/2$

**Solution:**

(b)  $\sum_{i=1}^n i^2 = n(n+1)(2n+1)/6$

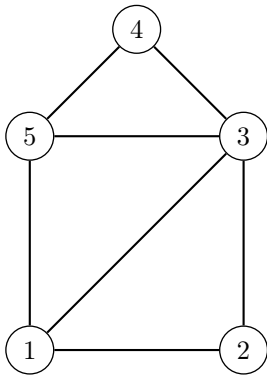
**Solution:**

(c)  $\sum_{i=1}^n i^3 = n^2(n+1)^2/4$

**Solution:**

## Graphs and Trees

7. Give the adjacency matrix, adjacency list, edge list, and incidence matrix for the following graph.



**Solution:**

8. How many edges are there in a complete graph of size  $n$ ? Prove by induction.

**Solution:**

9. Draw all possible (unlabelled) trees with 4 nodes.

**Solution:**

10. Show by induction that, for all trees,  $|E| = |V| - 1$ .

**Solution:**

## Counting

11. How many 3 digit pin codes are there?

**Solution:**

12. What is the expression for the sum of the  $i$ th line (indexing starts at 1) of the following:

1  
2 3  
4 5 6  
7 8 9 10  
⋮

**Solution:**

13. A standard deck of 52 cards has 4 suits, and each suit has card number 1 (ace) to 10, a jack, a queen, and a king. A standard poker hand has 5 cards. For the following, how many ways can the described hand be drawn from a standard deck.

- (a) A royal flush: all 5 cards have the same suit and are 10, jack, queen, king, ace.

**Solution:**

- (b) A straight flush: all 5 cards have the same suit and are in sequence, but not a royal flush.

**Solution:**

- (c) A flush: all 5 cards have the same suit, but not a royal or straight flush.

**Solution:**

- (d) Only one pair (2 of the 5 cards have the same number/rank, while the remaining 3 cards all have different numbers/ranks):

**Solution:**



## Proofs

14. Show that  $2x$  is even for all  $x \in \mathbb{N}$ .

(a) By direct proof.

**Solution:**

(b) By contradiction.

**Solution:**

15. For all  $x, y \in \mathbb{R}$ , show that  $|x + y| \leq |x| + |y|$ . (Hint: use proof by cases.)

**Solution:**

## Program Correctness (and Invariants)

16. For the following algorithms, describe the loop invariant(s) and prove that they are sound and complete.

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**Algorithm 1:** findMin

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**Input:**  $a$ : A non-empty array of integers (indexed starting at 1)

**Output:** The smallest element in the array

(a) **begin**  
     $min \leftarrow \infty$   
    **for**  $i \leftarrow 1$  **to**  $len(a)$  **do**  
        **if**  $a[i] < min$  **then**  
             $min \leftarrow a[i]$   
        **end**  
    **end**  
    **return**  $min$   
**end**

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**Solution:**

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**Algorithm 2:** InsertionSort

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**Input:**  $a$ : A non-empty array of integers (indexed starting at 1)  
**Output:**  $a$  sorted from largest to smallest

**begin**  
    **for**  $i \leftarrow 2$  **to**  $\text{len}(a)$  **do**  
         $val \leftarrow a[i]$   
        **for**  $j \leftarrow 1$  **to**  $i - 1$  **do**  
            **if**  $val > a[j]$  **then**  
                shift  $a[j..i - 1]$  to  $a[j + 1..i]$   
                 $a[j] \leftarrow val$   
                **break**  
            **end**  
        **end**  
    **end**  
    **return**  $a$   
**end**

(b)

---

**Solution:**

## Recurrences

17. Solve the following recurrences.

(a)  $c_0 = 1; c_n = c_{n-1} + 4$

**Solution:**

(b)  $d_0 = 4; d_n = 3 \cdot d_{n-1}$

**Solution:**

- (c)  $T(1) = 1; T(n) = 2T(n/2) + n$  (An upper bound is sufficient.)

**Solution:**

- (d)  $f(1) = 1; f(n) = \sum_{i=1}^{n-1} (i \cdot f(i))$   
(Hint: compute  $f(n+1) - f(n)$  for  $n > 1$ )

**Solution:**

## Coding Question

Most assignments will have a coding question. You can code in C, C++, C#, Java, Python, or Rust. You will submit a Makefile and a source code file.

**Makefile:** In the Makefile, there needs to be a build command and a run command. Below is a sample Makefile for a C++ program. You will find this Makefile in assignment details. Download the sample Makefile and edit it for your chosen programming language and code.

```
#Build commands to copy:
#Replace g++ -o HelloWorld HelloWorld.cpp below with the appropriate command.
#Java:
#    javac source_file.java
#Python:
#    echo "Nothing to compile."
#C#:
#    mcs -out:exec_name source_file.cs
#C:
#    gcc -o exec_name source_file.c
#C++:
#    g++ -o exec_name source_file.cpp
#Rust:
#    rustc source_file.rs

build:
    g++ -o HelloWorld HelloWorld.cpp

#Run commands to copy:
#Replace ./HelloWorld below with the appropriate command.
#Java:
#    java source_file
#Python 3:
#    python3 source_file.py
#C#:
#    mono exec_name
#C/C++:
#    ./exec_name
#Rust:
#    ./source_file

run:
    ./HelloWorld
```

**HelloWorld Program Details** The input will start with a positive integer, giving the number of instances that follow. For each instance, there will be a string. For each string  $s$ , the program should output Hello,  $s$ ! on its own line.

A sample input is the following:

```
3
World
Marc
Owen
```

The output for the sample input should be the following:

```
Hello, World!
```

```
Hello, Marc!
```

```
Hello, Owen!
```